Minimum-weight combinatorial structures under random cost-constraints

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Abstract

Recall that Janson showed that if the edges of the complete graph K_n are assigned exponentially distributed independent random weights, then the expected length of a shortest path between a fixed pair of vertices is asymptotically equal to $(\log n)/n$. We consider analogous problems where edges have not only a random length but also a random cost, and we are interested in the length of the minimum-length structure whose total cost is less than some cost budget. For several classes of structures, we determine the correct minimum length structure as a function of the cost-budget, up to constant factors. Moreover, we achieve this even in the more

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general setting where the distribution of weights and costs are arbitrary, so long as the density $f(x)$ as $x \to 0$ behaves like cx^γ for some $\gamma \geqslant 0$; previously, this case was not understood even in the absence of cost constraints. We also handle the case where each edge has several independent costs associated to it, and we must simultaneously satisfy budgets on each cost. In this case, we show that the minimum-length structure obtainable is essentially controlled by the product of the cost thresholds.

Mathematics Subject Classifications: 05C80, 05C85, 90C27

1 Introduction

Let the edges of the complete graph be given independent random edge weights $w(e)$ and a random cost $c(e)$ for $e \in E(K_n)$. We are interested in the problem of estimating the minimum weight of a combinatorial structure S where the total cost of S is bounded by some value C. More generally, we allow r costs $\mathbf{c}(e) = (c_i(e), i = 1, 2, \dots, r)$ for each edge. The distribution of weights $w(e)$ will be independent copies of Z_E^α where Z_E denotes the exponential rate one random variable and $\alpha \leqslant 1$. The distribution of costs $c_i(e)$ will be independent copies of Z_E^β where $\beta \leqslant 1$. In Section 6 we will see that, since we are allowing powers of exponentials, a simple coupling argument will allow us to model a very general class of independent weights and costs, where we require just that the densities satisfy $f(x) \approx cx^\gamma, \gamma \geqslant 0$ as $x \to 0$; here we mean that $cx^\gamma/f(x) \to 1$ as $x \to 0$.

Suppose we are given cost budgets of $\mathbf{C} = (C_i, i = 1, 2, \dots, r)$ and we consider the following problem: let $\mathcal{S}$ denote some collection of combinatorial strutures such as paths, matchings, Hamilton cycles, we would like to solve

$$Opt(\mathcal{S}, \mathbf{C}) : \text{Minimise } w(S) \text{ subject to } S \in \mathcal{S} \text{ and } c_i(S) \leqslant C_i, i = 1, 2, \dots, r,$$

and let

$$w^*(\mathbf{C}) \text{ denote the minimum value in } Opt(\mathcal{S}, \mathbf{C}).$$

We remark that Frieze and Tkocz [9], [10] have considered finding minimum weight spanning trees or arborescences in the context of a single cost constraint and uniform $[0, 1]$ weights and costs. I.e. the case where $\mathcal{S}$ is the set of spanning trees of K_n and the case where $\mathcal{S}$ is the set of spanning arborescences of $\vec{K}_n$. In these cases, they obtain asymptotically optimal estimates for those problems, whereas for the problems in the present paper we have only obtained estimates that are correct to within a constant factor.

The first problem we study involves paths, here denoted as minimum weight paths for consistency with the remainder of the paper. Let $\mathcal{P}(i, j)$ denote the set of paths from vertex i to vertex j in K_n.

Constrained Minimum Weight Path (CMWP): $Opt(\mathcal{P}(1, n), \mathbf{C})$.
Without the constraint $\mathbf{c}(P) \leqslant \mathbf{C}$, there is a beautiful result of Janson [13] that gives a precise value for the expected minimum weight of a path, when the $w(e)$'s are independent exponential mean one. With the constraints, we are only able to estimate the expected minimum weight up to a constant (but can do so for a more general class of distributions).

Throughout the paper we let

$$\Upsilon = \prod_{i=1}^{r} C_i$$

be the product of the cost thresholds. Our results show that for the structures we consider, this product of the cost thresholds controls the dependency of the minimum-weight structure on the vector of cost constraints. In particular, for the minimum weight path problem, we have:

Theorem 1. *If* $\frac{n\Upsilon^{1/\beta}}{\log^{r/\beta} n} \to \infty$ *and* $C_i \leqslant 10 \log n$, $i = 1, 2, \ldots, r$ *then w.h.p.*

$$w^*(\mathbf{C}) = \Theta\left(\frac{\log^{r\alpha/\beta+1} n}{n^\alpha \Upsilon^{\alpha/\beta}}\right).$$

$A = \Theta(B)$ *denotes* $A = O(B)$ *and* $B = O(A)$*. And here, the hidden constants depend only on* r, α, β*.*

For the unconstrained problem, see Hassin and Zemel [11], Janson [13] and Bhamidi and van der Hofstad [2], [3].

Now consider the case of perfect matchings in the complete bipartite graph $K_{n,n}$. Let $\mathcal{M}_2$ denote the set of perfect matchings in $K_{n,n}$.

Constrained Assigment Problem (CAP): $Opt(\mathcal{M}_2, \mathbf{C})$.

Theorem 2. *If* $\Upsilon^{1/\beta} \gg n^{r/\beta-1} \log n$[1] *and* $C_i \leqslant n$, $i = 1, 2, \ldots, r$ *then w.h.p.*

$$w^*(\mathbf{C}) = \Theta\left(\frac{n^{1+r\alpha/\beta-\alpha}}{\Upsilon^{\alpha/\beta}}\right).$$

We note that requiring a lower bound on Υ is necessary in Theorem 2. Indeed, if $\Upsilon^{1/\beta} \leqslant e^{-(r+1)}\beta n^{r/\beta-1}$ then the optimization problem is infeasible w.h.p. To see this we bound the expected number of feasible solutions as follows: let $Z_1, Z_2, \ldots, Z_r$ be independent sums of n independent copies of $Z_E^{1/\beta}$. Then,

$$n! \prod_{i=1}^{r} \mathbb{P}\left(Z_i \leqslant C_i\right) \leqslant n! \prod_{i=1}^{r} \frac{C_i^{n/\beta}}{\beta^n n! n^{n(1/\beta-1)}} \leqslant \left(\frac{e^r \Upsilon^{1/\beta}}{\beta n^{r/\beta-1}}\right)^n = o(1).$$

We use Lemma 7 here to bound $\mathbb{P}(Z_i \leqslant C_i)$. We note that a problem similar to this was studied by Arora, Frieze and Kaplan [1] with respect to the worst-case.

Now consider the case of perfect matchings in the complete graph K_n. Let $\mathcal{M}_1$ denote the set of perfect matchings in K_n.

Constrained Matching Problem (CMP) $Opt(\mathcal{M}_1, \mathbf{C})$.

Theorem 3. *If* $\Upsilon^{1/\beta} \gg n^{r-1} \log n$ *and* $C_i \leqslant n$, $i = 1, 2, \ldots, r$ *then w.h.p.*

$$w^*(\mathbf{C}) = \Theta\left(\frac{n^{1+r\alpha/\beta-\alpha}}{\Upsilon^{\alpha/\beta}}\right).$$

[1] Here $A = A(n) \gg B = B(n)$ if $A/B \to \infty$ as $n \to \infty$.

Now consider the following set of problems: Minimum Spanning Tree (MST), Minimum Spanning Arborescence (MSA), Travelling Salesperson Problem (TSP). Let $\mathcal{T}$ denote the set of Hamilton cycles in K_n or the set of directed Hamilton cycles in $\vec{K}_n$.

Constrained Spanning Tree/Arborescence/Travelling Salesperson Problem (CTSP+) $Opt(\mathcal{T}, \mathbf{C})$.

Theorem 4. *If $\Upsilon^{1/\beta} \gg n^{r-1} \log n$ and and $C_i \leqslant n$, $i = 1, 2, \ldots, r$ then w.h.p.*

$$w^*(\mathbf{C}) = \Theta\left(\frac{n^{1+r\alpha/\beta-\alpha}}{\Upsilon^{\alpha/\beta}}\right).$$

2 Structure of the paper

We prove the above theorems in their order of statement. The upper bounds are proved as follows: we consider the random graph $G_{n,p}$ (or bipartite graph $G_{n,n,p}$ or digraph $D_{n,p}$) for suitably chosen p associated with the random costs. We then seek minimum weight objects contained in these random graphs. The definition of p is such that objects, if they exist, automatically satisfy the cost constraints. For minimum weight paths we adapt the methodology of [13]. For the remaining problems we use theorems in the literature stating the high probability existence of the required objects when each vertex independently chooses a few (close) random neighbors.

In Section 6 we consider more general distributions. We are able to extend the above theorems under some extra assumptions about the C_i.

3 CSP

3.1 Upper Bound for CSP

In the proof of the upper bound, we first consider weights $\widehat{w}(e)$ where the $\widehat{w}(e)$ are independent exponential mean one random variables. The costs will remain independent copies of Z_E^β. We will then use Holder's inequality to obtain the final result.

3.2 $\frac{\log^{2+r/\beta} n}{n} \leqslant \Upsilon^{1/\beta}$ and $C_i \leqslant 10 \log n$, $i = 1, 2, \ldots, r$

Suppose now that we let $L = 10 \log n$ and

$$E_0 = \left\{e : c_i(e) \leqslant \frac{C_i}{L}, i = 1, 2, \ldots . r\right\}.$$

The proof in this case goes as follows:

(i) We search for short paths that only use edges in E_0 and note that the graph $([n], E_0)$ is distributed as $G_{n,p}$.

(ii) Observe that any path using fewer than L edges of E_0 automatically satifies the cost constraints.

(iii) A simple calculation shows that w.h.p. the number of edges between a set S of size k and the remaining vertices is close to the expectation $k(n-k)p$ for all sets of vertices S, see (3) and (4).

(iv) We run Dijkstra's algorithm for finding shortest (now minimum weight) paths from vertex 1. We use Janson's argument [13] to bound the distance to the $m = n/3$ closest vertices V_1. We need the claim in item (iii) here.

(v) We repeat (iv), starting from vertex set n, to obtain the $m = n/3$ closest vertices V_2. If $V_1 \cap V_2 \neq \emptyset$ we will have found a path of low enough weight, otherwise we claim that w.h.p. there will be a low enough weight edge joining V_1, V_2.

(vi) We then argue that the trees constructed by the Dijkstra algorithm are close to being Random Recursive Trees and we can easily bound their height. Showing that we can use item (ii).

(vii) We finally use Holder's inequality to switch from $\widehat{w}$ to w.

We first bound the value of $\mathbb{P}(e \in E_0)$.

$$p = \mathbb{P}(e \in E_0) = \prod_{i=1}^{r} \left(1 - \exp\left\{-\left(\frac{C_i}{3L}\right)^{1/\beta}\right\}\right), \tag{1}$$

where e is an arbitrary edge.

We note that if $0 < x \leqslant 1$ then $x/2 \leqslant 1 - e^{-x} \leqslant x$. This implies that

$$\frac{\Upsilon^{1/\beta}}{2^r (3L)^{r/\beta}} \leqslant p \leqslant \frac{\Upsilon^{1/\beta}}{(3L)^{r/\beta}}. \tag{2}$$

We consider the random graph $G_{n,p}$ where edges have weight given by $\widehat{w}$ and costs $c_i(e) \leqslant C_i/3L, i = 1, 2, \ldots, r$. We modify Janson's argument [13].

We now deal with item (iii). We observe that w.h.p. for every set S of size k, $e(S : \bar{S}) \approx k(n-k)p$ where $e(S : T)$ is the number of edges $\{v, w\}$ with one end in S and the other in T. We only need to check the claim for $|S| \leqslant n/2$. Let $\varepsilon = \frac{1}{\log^{1/3} n}$ and

$$\mathcal{E}_S = \left\{e(S : \bar{S}) \notin (1 \pm \varepsilon)k(n-k)p\right\} \text{ and } \mathcal{E} = \bigcup_{|S| \leqslant n/2} \mathcal{E}_S. \tag{3}$$

Then, using the Chernoff bounds for the binomial distribution,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}) &\leqslant \sum_{k=1}^{n/2} \binom{n}{k} \mathbb{P}(Bin(k(n-k),p) \notin (1\pm\varepsilon)k(n-k)p) \\
&\leqslant 2\sum_{k=1}^{n/2} \left(\frac{ne}{k}\right)^k e^{-\varepsilon^2 k(n-k)p/3} \\
&= 2\sum_{k=1}^{n/2} \left(\frac{ne^{1-\varepsilon^2(n-k)p/3}}{k}\right)^k \\
&\leqslant 2\sum_{k=1}^{n/2} \left(\frac{ne^{-\Omega(\log^{4/3} n)}}{k}\right) \\
&= o(1),
\end{aligned} \tag{4}$$

where we have used (2) to get the last inequality.

We now continue with item (iv). We set $S_1 = \{1\}$ and $d_1 = 0$ and consider running Dijkstra's algorithm [6]. At the end of Step k, we will have computed $S_k = \{1 = v_1, v_2, \ldots, v_k\}$ and $0 = d_1, d_2, \ldots, d_k$ where d_i is the minimum weight of a path from 1 to $i, i = 1, 2, \ldots, k$. Let there be ν_k edges from S_k to $[n] \setminus S_k$. Arguing as in [13] we see that $d_{k+1} - d_k = Z_k$ where Z_k is the minimum of ν_k independent exponential mean one random variables. Also, the memoryless property of the exponential distribution implies that Z_k is independent of d_k. It follows that for $k < n/2$,

$$\begin{aligned}
\mathbb{E}(d_k \mid \neg\mathcal{E}) = \mathbb{E}\left(\sum_{i=1}^{k} \frac{1}{\nu_i}\middle|\neg\mathcal{E}\right) = \sum_{i=1}^{k} \frac{1+o(1)}{i(n-i)p} = \frac{1+o(1)}{np} \sum_{i=1}^{k} \left(\frac{1}{i} + \frac{1}{n-i}\right) \\
= \frac{1+o(1)}{np}\left(H_k + H_{n-1} - H_{n-k+1}\right),
\end{aligned} \tag{5}$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

By the same token,

$$\mathbf{Var}(d_k \mid \neg\mathcal{E}) = \sum_{i=1}^{k} \mathbf{Var}(Z_i \mid \neg\mathcal{E}) = \sum_{i=1}^{k} \frac{1+o(1)}{(i(n-i)p)^2} = O((np)^{-2}). \tag{6}$$

We only pursue the use of Dijkstra's algoritm from vertex 1 for $m = n/3$ iterations. It follows from (5) and (6) and the Chebyshev inequality that we have w.h.p.

$$d_m \approx \frac{\log n}{np}. \tag{7}$$

We next deal with item (vi). The tree built by Dijkstra's algorithm is close in distribution to a random recursive tree i.e. vertex v_{k+1} attaches to a near uniformly random member of $\{v_1, v_2, \ldots, v_k\}$. Indeed, assuming $\mathcal{E}$ does not occur,

$$\mathbb{P}(v_{k+1} \text{ attaches to } v_i) = \frac{e(v_i : \bar{S}_k)}{\nu_k} \leqslant \frac{(1+\varepsilon)(n-1)p}{(1-\varepsilon)k(n-k)p}.$$

Hence, if T is the tree constructed in the first m rounds of Dijkstra's algorithm, then

$$\begin{aligned}\mathbb{P}(height(T) \geqslant L) &\leqslant \sum_{1<t_1<\cdots<t_L<m} \prod_{i=1}^{L} \frac{3(1+\varepsilon)}{2(1-\varepsilon)t_i} \\ &\leqslant \frac{1}{L!}\left(\frac{3(1+\varepsilon)}{2(1-\varepsilon)}\right)^L \left(\sum_{i=1}^{n} \frac{1}{i}\right)^L \\ &\leqslant \left(\frac{3(\log n+1)e^{1+o(1)}}{2L}\right)^L = o(1).\end{aligned} \tag{8}$$

It follows from (2), (7) and (8) that w.h.p., for every $v \in V_1 = S_m$, there exists a path P from 1 to v of weight at most

$$\lambda \approx \lambda_0 = \frac{\log n}{np} \lesssim \frac{30^{r/\beta} \log^{r/\beta+1} n}{n\Upsilon^{1/\beta}}$$

and costs $c_i(P) \leqslant LC_i/3L \leqslant C_i/3$.[2]

We now deal with item (v). We next consider applying Dijkstra's algorithm to find a minimum weight path from vertex n to other vertices. Using the same argument as above, we see that we can find m vertices V_2 that are within distance λ_0 of vertex n. If $V_1 \cap V_2 \neq \emptyset$ then we have found a path of weight at most $2\lambda_0$ between vertex 1 and vertex n.

If V_1, V_2 are disjoint then w.h.p. there is an edge of weight $20/np$ between them. Indeed,

$$\mathbb{P}(\exists V_1, V_2 \text{ with no such edge}) \leqslant \binom{n}{m}^2 (e^{-20/np})^{n^2/9} = o(1).$$

This yields a path P with

$$\widehat{w}(P) \leqslant 2\lambda_0 + \frac{20}{np} \leqslant \frac{3 \cdot 30^{r/\beta} \log^{r/\beta+1} n}{n\Upsilon^{1/\beta}}. \tag{9}$$

$$c_i(P) \leqslant \frac{2C_i}{3} + \frac{C_i}{3} = C_i, \quad i = 1, 2, \ldots, r. \tag{10}$$

(Here we have used $C_i \geqslant \Upsilon^{1/\beta}/L^{r-1} \gg p$.)

We now deal with item (vii). We use Holder's inequality to yield

$$w(P) = \sum_{e\in P} \widehat{w}(e)^\alpha \leqslant \left(\sum_{e\in P} \widehat{w}(e)\right)^\alpha L^{1-\alpha} = O\left(\frac{\log^{r\alpha/\beta+1} n}{n^\alpha \Upsilon^{\alpha/\beta}}\right). \tag{11}$$

This completes the proof of Theorem 1 for this case.

[2] Here we write $A = A(n) \lesssim B = B(n)$ if $A \leqslant (1+o(1))B$.

3.3 $\frac{3\omega \log^{r/\beta} n}{n} \leqslant \Upsilon^{1/\beta} \leqslant \frac{\log^{r/\beta+2} n}{n}$ and $C_i \leqslant 10 \log n,\ i = 1, 2, \ldots, r$

The proof is similar to that of Section 3.2, but requires some changes in some places. The problem is that we cannot now assume the non-occurrence of $\mathcal{E}$. Other than this, the proof will follow the same strategy. Our problem therefore is to argue that w.h.p. $e(S_k : \bar{S}_k)$ is sufficiently large.

(a) We now have to keep track of the size of $e(S_k : \bar{S}_k)$ as a random process. This is equation (12).

(b) The term η_k is the number of edges between $v \notin S_k$ and S_k. We don't want this to be large, as it reduces $e(S_{k+1} : \bar{S}_{k+1})$. So, we do not add vertices to S_k if $\eta_k \geqslant 2np$, which only happends rarely.

(c) Finally, we have to work harder in the case where V_1, V_2 are disjoint. We need to use edges of slightly higher cost in order to get a low weight edge in $e(V_1 : V_2)$.

Let p be as in (1) where $L = 20 \log n$. Note that from (2) we see that

$$p \leqslant \frac{\log^2 n}{n}.$$

We again consider the random graph $G_{n,p}$ where edges have weight given by $\widehat{w}$ and costs at most $C_i/3L$ and again modify Janson's argument [13]. We also restrict our search for paths, avoiding vertices of high degree.

We set $S_1 = \{1\}$ and $d_1 = 0$. At the end of Step k we will have computed $S_k = \{1 = v_1, v_2, \ldots, v_k\}$ and $0 = d_1, d_2, \ldots, d_k$ where d_i is the minimum weight of a path from 1 to $i, i = 1, 2, \ldots, k$. Let there be ν_k edges from S_k to $[n] \setminus S_k$. We cannot rely on $\mathcal{E}$ of (4) not to occur and so we need to modify the argument here.

Assumption: $1 \leqslant k \leqslant n_0 = 1/3p$

Modification: if our initial choice v for v_{k+1} satisfies $e(v : \bar{S}_k) \geqslant 2np$ then we reject v permanently from the construction of paths from vertex 1.

The initial aim is roughly the same, we want to show that w.h.p.

$$\sum_{\ell \leqslant k} \nu_\ell \geqslant (1 - o(1))knp. \tag{12}$$

For $v \notin S_k$, let $\eta_{k,v} = e(S_k : \{v\})$ and $\eta_k = \eta_{k,v_{k+1}}$. Then, w.h.p.

$$\nu_{k+1} \geqslant \nu_k - \eta_k + B_k \text{ where } B_k = Bin(n_1, p) 1_{Bin(n,p) \leqslant 2np}, \tag{13}$$

where $n_1 = n - 2n_0$.

The binomials are independent here. This is because the edges between v_{k+1} and $\bar{S}_k$ have not been exposed by the algorithm to this point. The number of trials n_1 comes from the following: we know from the Chernoff bounds that

$$\mathbb{P}(Bin(n, p) \geqslant 2np) \leqslant e^{-np/3}. \tag{14}$$

It follows from the Markov inequality that w.h.p. there are at most $ne^{-np/4}$ instances where the modification is invoked. This means that w.h.p. the initial choice for v_k has at least $n - n_0 - ne^{-np/4} \geqslant n_1$ possible neighbors. We now define

$$S_k = \sum_{\ell=1}^{k} B_k.$$

We need a lower bound for B_k and an upper bound for η_k. We next observe that if

$$\varepsilon = (np)^{-1/3}$$

then

$$\mathbb{P}(B_k \leqslant (1-\varepsilon)np) = \mathbb{P}(Bin(n_1,p) \geqslant 2np) + \mathbb{P}(Bin(n_1 \leqslant (1-\varepsilon)np)) \leqslant (1+o(1))e^{-\varepsilon^2 np/3}. \tag{15}$$

It follows that if $k_0 = \min\left\{n_0, e^{\varepsilon^2 np/4}\right\}$ then w.h.p.

$$\mathbb{P}(\exists 0 \leqslant k \leqslant k_0 : B_k \leqslant (1-\varepsilon)np) \leqslant (1+o(1))k_0 e^{-\varepsilon^2 np/3} \leqslant e^{-\varepsilon^2 np/12}. \tag{16}$$

For $k \geqslant k_0$, we use the fact that S_k is the sum of bounded random variables. Hoeffding's inequality [12] gives that

$$\mathbb{P}(S_k \leqslant \mathbb{E}(S_k) - t) \leqslant \exp\left\{-\frac{2t^2}{4kn^2p^2}\right\}.$$

Now $\mathbb{E}(B_k) \geqslant (1-\varepsilon)np$ and so putting $t = k^{2/3}np$ we see that

$$\mathbb{P}(S_k \leqslant (1-\varepsilon)knp - k^{2/3}np) \leqslant e^{-k^{1/3}/2}.$$

So

$$\mathbb{P}(\exists k \geqslant k_0 : \; S_k \leqslant (1-\varepsilon)knp - k^{2/3}np) \leqslant \sum_{k \geqslant k_0} e^{-k^{1/3}/2} = o(1). \tag{17}$$

We next observe that

$$\begin{aligned}\mathbb{P}(\exists S : \; |S| = s \leqslant 1/3p, e(S:S) \geqslant s+r) &\leqslant \sum_{s=1}^{1/3p} \binom{n}{s} \binom{s(s-1)/2}{s+r} p^{s+r} \\ &\leqslant \sum_{s=1}^{1/3p} \left(\frac{e^2np}{2}\right)^s \left(\frac{sep}{2}\right)^r. \end{aligned} \tag{18}$$

Putting $r = s(np)^{1/2}$, the RHS of (18) becomes

$$\sum_{s=1}^{1/3p} \left(\frac{e^2np}{2}\left(\frac{sep}{2}\right)^{(np)^{1/2}}\right)^s \leqslant \sum_{s=1}^{1/3p} \left(\frac{e^2np}{2}\left(\frac{e}{6}\right)^{(np)^{1/2}}\right)^s = o(1).$$

It follows that w.h.p.,

$$\sum_{\ell=1}^{k} \eta_\ell = e(S_k) \leqslant 2((np)^{1/2}+1)k. \tag{19}$$

It then follows from (13) and (16) and (17) and (19) that w.h.p.

$$\nu_k \geqslant (1-o(1))knp - 2((np)^{1/2}+1)k \geqslant (1-o(1))knp. \tag{20}$$

Arguing as in [13] we see that $d_{k+1}-d_k = Z_k$ where Z_k is the minimum of ν_k independent exponential mean one random variables. Also, Z_k is independent of d_k. It follows that for $k<n$,

$$\mathbb{E}(d_k) = \mathbb{E}\left(\sum_{i=1}^{k}\frac{1}{\nu_i}\right) \leqslant \sum_{i=1}^{k}\frac{1+o(1)}{inp} = \frac{1+o(1)}{np}\sum_{i=1}^{k}\frac{1}{i} = \frac{1+o(1)}{np}H_k, \tag{21}$$

where $H_k = \sum_{i=1}^{k}\frac{1}{i}$.

By the same token,

$$\mathbf{Var}(d_k) = \sum_{i=1}^{k}\mathbf{Var}(Z_i) = \sum_{i=1}^{k}\frac{1+o(1)}{(inp)^2} = O((np)^{-2}). \tag{22}$$

It follows from (21) and (22) and the Chebyshev inequality that w.h.p. we have $d_{n_0} \lesssim \frac{\log n}{np}$. Let V_1 denote the n_0 vertices at this distance from vertex 1.

We next consider applying Dijkstra's algorithm to find a minimum weight path from vertex n to other vertices. Using the same argument as above, we see that we can find n_0 vertices V_2 that are within distance $\frac{(1+o(1))\log n}{np}$ of vertex n. If $V_1 \cap V_2 \neq \emptyset$ then we have found a path of weight at most $\frac{(2+o(1))\log n}{np}$ between vertex 1 and vertex n.

If V_1, V_2 are disjoint then we will use the edges

$$E_1 = \left\{e : c_i(e) \in \left[\frac{C_i}{L}, \frac{2C_i}{L}\right], i = 1,2,\ldots.r\right\}.$$

Given $e = \{x,y\} \in V_1 : V_2$, then given the history of Dijkstra's algorithm so far, either $e \in E_0$ or we can say that

$$\mathbb{P}(e \in E_1 \mid e \notin E_0) \geqslant \mathbb{P}(e \in E_1) = (1-e^{-(2^{1/\beta}-1)p^{1/\beta}})^r. \tag{23}$$

For the equation in (23) we use

$$\begin{aligned}\mathbb{P}(p \leqslant Z_E^\beta \leqslant 2p) &= \mathbb{P}(Z_E^\beta \geqslant p)(1-\mathbb{P}(Z_E^\beta \geqslant 2p \mid Z_E \geqslant p)) \\ &= e^{-p^{1/\beta}}\left(1-\frac{\mathbb{P}(Z_E^\beta \geqslant 2p)}{\mathbb{P}(Z_e^\beta \geqslant p)}\right) \\ &= e^{-p^{1/\beta}}\left(1-e^{-(2^{1/\beta}-1)p^{1/\beta}}\right) \\ &= e^{-p^{1/\beta}} - e^{-(2p)^{1/\beta}} \geqslant \frac{(2^{1/\beta}-1)p^{1/\beta}}{2}.\end{aligned} \tag{24}$$

For the inequality in (24) we use the fact that we now have $p \leqslant \frac{\log^2 n}{n}$.

Then we search for an edge in $E_2 = \{e \in E_1 : \widehat{w}(e) \leqslant 1/np\}$. And,

$$\begin{aligned}\mathbb{P}(E_2 \cap (V_1 : V_2) = \emptyset) &\leqslant \left(1 - \left(\frac{(2^{1/\beta}-1)p^{1/\beta}}{2}\right)(1-e^{-1/np})\right)^{1/9p^2} \\ &\leqslant \left(1 - \frac{(2^{1/\beta}-1)p^{1/\beta}}{2np}\right)^{1/9p^2} = o(1).\end{aligned}$$

This yields a path of weight at most $\frac{(2+o(1))\log n}{np} + \frac{1}{np} = \frac{(2+o(1))\log n}{np}$.

We deal with the height of the Dijkstra trees. Let T be the tree constructed by Dijkstra's algorithm and let $\xi_i, i \leqslant k$ denote the number of edges from v_i to $V_1 \setminus S_i$.

$$\begin{aligned}\mathbb{P}(height(T) \geqslant L) &\leqslant \mathbb{E}\left(\sum_{1<t_1<\cdots<t_L<n_0} \prod_{i=1}^{L} \frac{\xi_{t_i}}{\nu_{t_{i+1}-1}}\right) \\ &\leqslant \mathbb{E}\left(\sum_{1<t_1<\cdots<t_L<n_0} \prod_{i=1}^{L} \frac{2np}{\nu_{t_{i+1}-1}}\right) \\ &\leqslant \mathbb{E}\left(\frac{1}{L!}\left(\sum_{i=1}^{n_0} \frac{2np}{\nu_i}\right)^L\right) \\ &\leqslant o(1) + \frac{(2enp)^L}{(np)^L L!}\left(\sum_{i=1}^{n_0} \frac{1+o(1)}{i}\right)^L = o(1), \end{aligned} \tag{25}$$

since $L \geqslant 20 \log n$.

The first $o(1)$ term in (25) is the probability that there is a small ν_k and this is covered by (20).

It follows from the above that w.h.p. there exists a path P

$$\text{where } \widehat{w}(P) \lesssim \frac{2\log n}{np} \text{ and } c_i(P) \leqslant \frac{(2L+2)C_i}{3L} < C_i, i = 1, 2, \ldots, r. \tag{26}$$

Arguing as for (11) we see that

$$w(P) \leqslant \widehat{w}(P)^\alpha L^{1-\alpha} = O\left(\frac{\log^{r\alpha/\beta+1} n}{n^\alpha \Upsilon^{\alpha/\beta}}\right). \tag{27}$$

3.4 Lower Bound for CSP

This is a straightforward use of the first moment method. Suppose that

$$\Upsilon^{1/\beta} = \frac{\omega \log^{r/\beta} n}{n}, \quad L = \frac{\varepsilon \log^{r\alpha/\beta+1} n}{n^\alpha \Upsilon^{\alpha/\beta}},$$

where

$$\varepsilon = \left(\alpha\beta^r e^{-2}\left(10\left(\frac{r}{\beta}+\frac{1}{\alpha}\right)\right)^{-(r/\beta+1/\alpha)}\right)^{\alpha},$$

then

$$\begin{aligned}
&\mathbb{P}\left(\exists P : w(P) \leqslant L, c_i(P) \leqslant C_i, i = 1,2,\ldots,r\right)\\
&\leqslant \sum_{k=1}^{n-2} n^{k-1}\left(\frac{L^{k/\alpha}}{\alpha^k k! k^{k(1/\alpha-1)}}\right)\prod_{i=1}^{r}\frac{C_i^{k/\beta}}{\beta^k k! k^{k(1/\beta-1)}} \qquad (28)\\
&\leqslant \frac{1}{n}\sum_{k=1}^{n-1}\left(n\cdot\frac{e\varepsilon^{1/\alpha}\log^{r/\beta+1/\alpha} n}{\alpha n k^{1/\alpha}\Upsilon^{1/\beta}}\cdot\frac{e\Upsilon^{1/\beta}}{\beta^r k^{r/\beta}}\right)^k\\
&= \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{e^2\varepsilon^{1/\alpha}\log^{r/\beta+1/\alpha} n}{\alpha\beta^r k^{r/\beta+1/\alpha}}\right)^k\\
&= \frac{1}{n}\sum_{k=1}^{\frac{1}{2}\log n}\left(\frac{e^2\varepsilon^{1/\alpha}\log^{r/\beta+1/\alpha} n}{\alpha\beta^r k^{r/\beta+1/\alpha}}\right)^k + \frac{1}{n}\sum_{k=\frac{1}{2}\log n}^{n-1}\left(\frac{e^2\varepsilon^{1/\alpha}\log^{r/\beta+1/\alpha} n}{\alpha\beta^r k^{r/\beta+1/\alpha}}\right)^k\\
&\leqslant \frac{1}{n}\sum_{k=1}^{\frac{1}{2}\log n}\left(\frac{\log n}{10(r/\beta+1/\alpha)k}\right)^{(r/\beta+1/\alpha)k} + \frac{1}{n}\sum_{k=\frac{1}{2}\log n}^{n-1} 10^{-k}\\
&= o(1).
\end{aligned}$$

Explanation for (28): we choose a path of length k from 1 to n in at most n^{k-1} ways. Then we use Lemma 7 $r+1$ times. Then we use the union bound.

4 Upper Bounds

4.1 Upper Bound for CAP

Let G denote the subgraph of $K_{n,n}$ induced by the edges that satisfy $c_i(e) \leqslant C_i/n$ for $i = 1,2,\ldots,r$. Let

$$p = \mathbb{P}\left(c_i(e) \leqslant \frac{C_i}{n}, i = 1,2,\ldots,r\right) = \prod_{i=1}^{r}\left(1-\exp\left\{-\left(\frac{C_i}{n}\right)^{1/\beta}\right\}\right).$$

and note that

$$\frac{\log n}{n} \ll \frac{\Upsilon^{1/\beta}}{2^r n^{r/\beta}} \leqslant p \leqslant \frac{\Upsilon^{1/\beta}}{n^{r/\beta}}.$$

The approach for this and the remaining problems is

(i) Look for a small weight structure in an edge weighted random graph G. In this case the random bipartite graph $G_{n,n,p}$.

(ii) Use an idea of Walkup [15] to construct a random subgraph H of G that only uses edges of low weight.

(iii) Use a result from the literature that states that w.h.p. the edges of H contain a copy of the desired structure.

G is distributed as $G = G_{n,n,p}$. Note that by construction, a perfect matching M of G satisfies $c_i(M) \leqslant C_i, i = 1, 2, \dots, r$.

Let $d = np$ and note that because $dnp \gg \log n$ the Chernoff bounds imply that w.h.p. every vertex has degree $\approx d$. Now each edge of G has a weight uniform in $[0, 1]$. Following Walkup [15] we replace $w(e), e = (x, y)$ by $\min\{Z_1(e), Z_2(e)\}$ where

$$Z_1, Z_2 \text{ are independent copies of } Z_W \text{ where } \mathbb{P}(Z_W \geqslant x)^2 = \mathbb{P}(Z_E^\alpha \geqslant x). \tag{29}$$

We assign $Z_1(e)$ to x and $Z_2(e)$ to y.

Let X, Y denote the bipartition of the vertices of G. Now consider the random bipartite graph H where each $x \in X$ is incident to the two Z_1-smallest edges incident with x. Similarly, $y \in Y$ is incident to the two Z_2-smallest edges incident with y. Walkup [16] showed that H has a perfect matching w.h.p. The expected weight of this matching is asymptotically at most

$$\left(\frac{2^\alpha n}{d^\alpha}\right)\left(\Gamma\left(1 + \frac{1}{\alpha}\right) + \Gamma\left(2 + \frac{1}{\alpha}\right)\right) \times \frac{1}{2} = O\left(\frac{n^{1+r\alpha/\beta-\alpha}}{\Upsilon^{\alpha/\beta}}\right). \tag{30}$$

This follows from (i) the expression given in Corollary 6 for the expected minimum and second minimum of d copies of Z and (ii) the matching promised in [16] is equally likely to select a minimum or a second minimum weight edge.

The selected matching is the sum of independent random variables with exponential tails and so will be concentrated around its mean.

4.2 Upper Bound for CMP

We let p, d be as in Section 4.1. We replace Walkup's result [16] by Frieze's result [8] that the random graph G_{2-out} contains a perfect matching w.h.p. The random graph G_{k-out} has vertex set $[n]$ and each vertex $v \in [n]$ independently chooses k random edges incident with v. We again replace $c(e), e = (x, y)$ by $\min\{Z_1(e), Z_2(e)\}$ where Z_1, Z_2 are independent copies of Z_W and associate one copy with each endpoint of the edge. We consider the random graph H where each $v \in [n]$ is incident to the two Z_W-smallest edges incident with x. This is distributed as G_{2-out} and we obtain an expression similar to that in (30).

We have concentration around the mean as in Section 4.1.

4.3 Upper bound for CTSP+

We first consider the TSP. For the symmetric case, we replace the weight $w(e), e = \{x, y\}$ by $\min\{Z_1(e), Z_2(e)\}$ for each edge of K_n and for the asymmetric case we replace $w(e), e =$

(x,y) by $\min\{Z_1(e),Z_2(e)\}$ for each directed edge of $\vec{K}_n$. In both cases we associate one copy of Z_W to each endpoint of e. We define p,d as in Section 4.1 and consider either the random graph $G_{n,p}$ or the random digraph $D_{n,p}$.

For the symmetric case, we consider the random graph H that includes the 3 cheapest edges associated with each vertex, cheapest with respect to $Z_W(e)$. This will be distributed as G_{3-out} which was shown to be Hamiltonian w.h.p. by Bohman and Frieze [4]. For the asymmetric case, we consider the random digraph H that includes the 2 cheapest out-edges and the 2 cheapest in edges associated with each vertex, cheapest with respect to $Z_W(e)$. This will be distributed as $D_{2-in,2-out}$ which has vertex set $[n]$ and where each vertex v independently chooses 2 out- and in-neighbors. The random digraph $D_{2-in,2-out}$ was shown to be Hamiltonian w.h.p. by Cooper and Frieze [5].

The expected weight of the tour promised by [4] or by [5] is asymptotically equal to $O(n^{1+r\alpha/\beta-\alpha}/\Upsilon^{\alpha/\beta})$ as in Section 4.1. We have concentration around the mean as in Section 4.1.

We obtain an upper bound for the MST through the fact that a Hamilton path is also a spanning tree. The same is true for the asymmetric case, since an arborescence is a tree with edges oriented away from the root.

5 Lower Bounds

We proceed as in Section 3.4. Suppose that $\Upsilon=\omega n^{r/\beta-1}\log n$ and $L=\frac{\varepsilon n^{1+r\alpha/\beta-\alpha}}{\Upsilon^{\alpha/\beta}}$ where ε will be a sufficiently small constant. Let Λ denote the relevant structure, matching or cycle. Then, by the union bound and Lemma 7, we have for CAP,CSTSP,CATSP, MST, MSA

$$\begin{aligned}\mathbb{P}\left(\exists\Lambda: w(\Lambda)\leqslant L \text{ and } c_i(\Lambda)\leqslant C_i, i=1,2,\ldots,r\right) &\leqslant n^n\cdot\frac{L^{n/\alpha}}{\alpha^n n! n^{n(1/\alpha-1)}}\cdot\prod_{i=1}^r\frac{C_i^{n/\beta}}{\beta^n n! n^{n(1/\beta-1)}}\\ &\leqslant\left(\frac{e\varepsilon^{1/\alpha}n^{1/\alpha+r/\beta-1}}{\alpha n^{1/\alpha-1}\Upsilon^{1/\beta}}\cdot\frac{e^r\Upsilon^{1/\beta}}{\beta n^{r/\beta}}\right)^n=o(1),\end{aligned}$$

for ε sufficiently small.

For CMP, assuming that $n=2m$,

$$\begin{aligned}&\mathbb{P}\left(\exists\Lambda: w(\Lambda)\leqslant L \text{ and } c_i(\Lambda)\leqslant C_i, i=1,2,\ldots,r\right)\\ &\qquad\leqslant\frac{n!}{m!2^m}\cdot\frac{L^{m/\alpha}}{\alpha^m m! m^{m(1/\alpha-1)}}\cdot\prod_{i=1}^r\frac{C_i^{m/\beta}}{\beta^m m! m^{m(1/\beta-1)}}\\ &\qquad\leqslant\left(\frac{\varepsilon^{1/\alpha}m^{1/\alpha+r/\beta-1}}{2\alpha m^{1/\alpha-1}\Upsilon^{1/\beta}}\cdot\frac{e^r\Upsilon^{1/\beta}}{\beta m^{r/\beta}}\right)^m=o(1),\end{aligned}$$

for ε sufficiently small.

6 More general distributions

We follow an argument from Janson [13]. We will asssume that $w(e)$, has the distribution function $F_w(t) = \mathbb{P}(X \leqslant t)$, of a random variable X, that satisfies $F_w(t) \approx at^{1/\alpha}, \alpha \leqslant 1$ as $t \to 0$. For the costs $c_i(e)$ we have $F_c(t) \approx bt^{1/\beta}, \beta \leqslant 1$. The constants $a, b > 0$ can be dealt with by scaling and so we assume that $a = b = 1$ here. For a fixed edge and say, $w(e)$, we consider random variables $w_<(e), w_>(e)$ such that $w_<(e)$ is distributed as $Z_E^{\alpha+\varepsilon_n}$ and $w_>(e)$ is distributed as $Z_E^{\alpha-\varepsilon_n}$, where $\varepsilon_n = 1/10\log n$. (This choice of ε_n means that $n^{\alpha+\varepsilon_n} = e^{1/10}n^\alpha$.) Then let $U(e)$ be a uniform $[0,1]$ random variable and suppose that X has the distribution $F^{-1}(U)$. We couple $X, w_<, w_>$ by generating $U(e)$ and then $w_<(e) = F_<^{-1}(U) = \log\left(\frac{1}{1-u}\right)^{\alpha-\varepsilon_n}$ and $F_>$ is defined similarly. The coupling ensures that $w_<(e) \leqslant w(e) \leqslant w_>(e)$ as long as $w(e) \leqslant \varepsilon_n$.

Given the above set up, it only remains to show that w.h.p. edges of length $w(e) > \varepsilon_n$ or cost $c_i(e) > \varepsilon_n$ are not needed for the upper bounds proved above. We can ignore the lower bounds, because they only increase if we exclude long edges.

Assumptions for CMWP. For the minimum weight path problem we will assume that $\Upsilon^{1/\beta} \gg \frac{\log^{1+r/\beta} n}{n}$, which is a $\log n$ factor larger than required for Theorem 1. We will assume that $C_i = o(1)$ and then we only use edges of cost of order $C_i/\log n \ll \varepsilon_n$.

Observe that the minimum weight of a path from 1 to n is at most $\frac{4\log n}{np}$ w.h.p. and this is less than ε_n because of the assumption $\frac{\log^{1+r/\beta} n}{n} \ll \Upsilon^{1/\beta}$ and the definition of p (see (2)).

Assumptions for the other problems, We deal with costs by assuming that $C_i = o(n/\log n), i = 1, 2, \dots r$. It is then a matter of showing that w.h.p. the first few order statistics of Z_W are very unlikely to be greater than ε_n. (Z_W is defined in (29).) But in all cases this can be bounded as follows: let $W_1, W_2, \dots, W_m, m \geqslant n/2$ be independent copies of Z_W. Then,

$$\mathbb{P}(|\{i : W_i \leqslant \varepsilon_n\}| \leqslant 3) \leqslant m^3(1-(1-e^{-\varepsilon_n^{1/\alpha}})^{1/2})^{m-3} = m^3 e^{-m^{1-o(1)}}.$$

This bounds the probability of using a heavy edge at any one vertex and inflating by n gives us the result we need.

7 Conclusion

We have given upper and lower bounds that hold w.h.p. for constrained versions of some classical problems in Combinatorial Optimization. They are within a constant factor of one another, unlike the situation with respect to spanning trees and arborescences, [9], [10], where the upper and lower bounds are asymptotically equal. It is a challenge to find tight bounds for the problems considered in this paper and to allow correlation between length and cost. It would also be interesting to determine how feasiblity depends on $C_1, C_2, \dots, C_r$. This presumably involves lower bounds on their values. If the C_i are very large then w.h.p. they do not affect the optimum solution. The natural question is as to how large can they be so that the constraints are tight and yet very simple algorithms suffice to solve the optimisation problem.

We have not made any claims about $\mathbb{E}(w^*(\mathbf{C}))$ because there is always the (small) probability that the problem is infeasible. It is not difficiult to similarly bound the expectation conditional on feasibility.

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A Auxilliary Lemmas

Lemma 5. *Let $\alpha > 0$ and let $Y_1, Y_2, \ldots$ be i.i.d. copies of $Z = Z_E^\alpha$. For a positive integer m and $1 \leqslant k \leqslant m$, let $X_m^{(k)}$ be the kth minimum of $Y_1, \ldots, Y_m$. Then*

$$\mathbb{E}X_m^{(k)} = \Gamma(1+\alpha) \sum_{j=0}^{k-1} \sum_{i=0}^{j} \binom{m}{j} \binom{j}{i} (-1)^i (m+i-j)^{-\alpha}.$$

In particular, if k is a constant as $m \to \infty$, then

$$\mathbb{E}X_m^{(k)} \approx \frac{1}{(k-1)!} \Gamma\left(k + \frac{1}{\alpha}\right) m^{-\alpha}.$$

Proof. Note that

$$\mathbb{P}(X_m^{(k)} > t) = \sum_{j=0}^{k-1} \binom{m}{j} \mathbb{P}(Y_1 \leqslant t)^j \mathbb{P}(Y_1 > t)^{n-j}$$

(for the kth minimum to be larger than t, we need exactly j variables to be at most t and $m-j$ larger than t, $j = 0, 1, \ldots, k-1$). Integrating gives

$$\mathbb{E}X_m^{(k)} = \int_0^\infty \mathbb{P}(X_m^{(k)} > t) \mathrm{d}t = \sum_{j=0}^{k-1} \binom{m}{j} \int_0^\infty \left(1 - e^{-t^\alpha}\right)^j e^{-(m-j)t^\alpha} \mathrm{d}t.$$

It remains to expand $\left(1 - e^{-t^\alpha}\right)^j$ and use $\int_0^\infty e^{-\lambda t^\alpha} \mathrm{d}t = \Gamma(1+\alpha) \lambda^{-\alpha}$. The asymptotic statements follow by writing $(m+i-j)^{-\alpha} = m^{-\alpha}(1 + \frac{i-j}{m})^{-\alpha}$ and applying the binomial series. $\square$

Corollary 6. *Let $\alpha > 0$ and let $\widehat{Y}_1, \widehat{Y}_2, \ldots$ be i.i.d. copies of Z_1, where Z_1 is as defined in* (29)*. For a positive integer m and $1 \leqslant k \leqslant m$, let $\widehat{X}_m^{(k)}$ be the kth minimum of $\widehat{Y}_1, \ldots, \widehat{Y}_m$. Then*

$$\mathbb{E}\widehat{X}_m^{(k)} = 2^\alpha \Gamma(1+\alpha) \sum_{j=0}^{k-1} \sum_{i=0}^{j} \binom{m}{j} \binom{j}{i} (-1)^i (m+i-j)^{-\alpha}.$$

In particular, if k is a constant as $m \to \infty$, then

$$\mathbb{E}\widehat{X}_m^{(k)} \approx 2^\alpha \frac{1}{(k-1)!} \Gamma(k+\alpha) m^{-\alpha}.$$

Proof. This follows from Lemma 5 and the fact that Z_1 has the same distribution as $2^\alpha Z$ because we have $\mathbb{P}(Z_1 \geqslant x) = \mathbb{P}(Z \geqslant x)^{1/2} = \mathbb{P}(2^\alpha Z \geqslant x)$. (Here $Z = Z_E^\alpha$.) $\square$

Lemma 7. *Let $\alpha \leqslant 1$ and let $Y_1, Y_2, \dots$ be i.i.d. copies of Z_E^α. Then for $t \geqslant 0$, we have*

$$\mathbb{P}(Y_1 + \cdots + Y_n \leqslant t) \leqslant \frac{t^{n/\alpha}}{\alpha^n n! n^{n(1/\alpha-1)}}.$$

Proof. Using the density,

$$\mathbb{P}(Y_1 + \cdots + Y_n \leqslant t) = \int_{x_1,\dots,x_n \geqslant 0, \sum x_i \leqslant t} \prod_{i=1}^n \alpha^{-1} x_i^{1/\alpha-1} e^{-x_i^{1/\alpha}} \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

By the AM-GM inequality,

$$\prod_{i=1}^n x_i \leqslant \left(\frac{\sum_{i=1}^n x_i}{n}\right)^n,$$

and trivially $e^{-x_i^{1/\alpha}} \leqslant 1$, so the integrand can be pointwise bounded as follows

$$\prod_{i=1}^n \alpha^{-1} x_i^{1/\alpha-1} e^{-x_i^{1/\alpha}} \leqslant \alpha^{-n} \left(\frac{\sum_{i=1}^n x_i}{n}\right)^{n(1/\alpha-1)} \leqslant \alpha^{-n} \frac{t^{n(1/\alpha-1)}}{n^{n(1/\alpha-1)}}$$

Thus,

$$\mathbb{P}(Y_1 + \cdots + Y_n \leqslant t) \leqslant \alpha^{-n} \frac{t^{n(1/\alpha-1)}}{n^{n(1/\alpha-1)}} \cdot \mathrm{vol}\left\{x_1, \dots, x_n \geqslant 0,\ \sum_{i=1}^n x_i \leqslant t\right\} = \alpha^{-n} \frac{t^{n/\alpha}}{n! n^{n(1/\alpha-1)}}.$$

$\square$