The threshold for the full perfect matching color profile in a random coloring of random graphs

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Abstract

Consider a graph G with a coloring of its edge set $E(G)$ from a set $Q = \{c_1, c_2, \ldots, c_q\}$. Let Q_i be the set of all edges colored with c_i. Recently, Frieze defined a notion of the perfect matching color profile denoted by $\mathrm{mcp}(G)$, which is the set of vectors $(m_1, m_2, \ldots, m_q)$ such that there exists a perfect matching M in G with $|Q_i \cap M| = m_i$ for all i. Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be positive constants such that $\sum_{i=1}^{q} \alpha_i = 1$. Let G be the random bipartite graph $G_{n,n,p}$. Suppose the edges of G are independently colored with color c_i with probability α_i. We determine the threshold for the event $\mathrm{mcp}(G) = \{(m_1, \ldots, m_q) \in [0, n]^q : m_1 + \cdots + m_q = n\}$, answering a question posed by Frieze. We further extend our methods to find the threshold for the same event in a randomly colored random graph $G_{n,p}$.

Mathematics Subject Classifications: 05C80

1 Introduction

Randomly colored random graphs have been extensively studied in various contexts throughout the last two decades. A few examples include (i) rainbow spanning graphs such as matchings and Hamilton cycles, see e.g., [2], [8], [10], [11], [14]; (ii) rainbow connection, see e.g., [4], [13], [15], [16]; (iii) pattern colored Hamilton cycles, see e.g., [1], [5], [12]; (iv) packing problems, see e.g., [9]. Continuing the research in this line, Frieze defined an elegant notion of a color profile in [6] and gave bounds on the matching color profile for randomly colored random bipartite graphs.

Throughout this paper, we have the following setting: We are given a graph G, and positive constants $\alpha_1, \alpha_2, \ldots, \alpha_q$ with $\sum_{i=1}^{q} \alpha_i = 1$. Suppose each of the edges of G

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are independently colored with a random color from the set $Q = \{c_1, c_2, \ldots, c_q\}$ with probability $\mathbb{P}(c(e) = c_i) = \alpha_i$, where $c(e)$ denotes the color of the edge $e \in E(G)$. Define the color class $Q_i = \{e \in E(G) : c(e) = c_i\}$. The perfect matching color profile $\mathrm{mcp}(G)$ is defined to be the set of vectors $(m_1, m_2, \ldots, m_q)$ such that there exists a perfect matching M in G with $|Q_i \cap M| = m_i$ for all i.

We first consider G to be the random bipartite graph $G_{n,n,p}$. For an event E_n, we say that E_n occurs with high probability (in short, w.h.p.) if $\mathbb{P}(E_n) \to 1$ as $n \to \infty$. Erdős and Rényi [3] proved that $G_{n,n,p}$ has a perfect matching w.h.p. when $p = \frac{\log n + \omega}{n}$ for any $\omega = \omega(n) \to \infty$. Moreover, for the same value of p, Frieze [6] proved that if the edges of $G = G_{n,n,p}$ are independently colored with q colors with constant probabilities, then most of the elements $(m_1, m_2 \ldots m_q) \in [0, n]^q$ such that $\sum_{i=1}^q m_i = n$ are present in $\mathrm{mcp}(G)$ w.h.p.

Theorem 1 (Frieze)**.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_q, \beta$ be positive constants such that $\alpha_1 + \alpha_2 + \cdots + \alpha_q = 1$ and $\beta < 1/q$. Let G be the random bipartite graph $G_{n,n,p}$ where $p = \frac{\log n + \omega}{n}$, $\omega = \omega(n) \to \infty$. Suppose that the edges of G are independently colored with colors from $Q = \{c_1, c_2, \ldots, c_q\}$ where $\mathbb{P}(c(e) = c_i) = \alpha_i$ for $e \in E(G), i \in [q]$. Let $m_1, m_2, \ldots, m_q$ satisfy: (i) $m_1 + \cdots + m_q = n$ and (ii) $m_i \geqslant \beta n, i \in [q]$. Then w.h.p., there exists a perfect matching M in which exactly m_i edges are colored with $c_i, i = 1, 2, \ldots, q$.*

It is not hard to check that w.h.p. $(n, 0, \ldots, 0) \notin \mathrm{mcp}(G)$, in view of the fact that the bipartite graph induced by the first color is distributed as $G_{n,n,\alpha_1 p}$ and has isolated vertices w.h.p. Frieze posed the natural problem of determining the threshold for $\mathrm{mcp}(G) = \{(m_1, \ldots, m_q) \in [0, n]^q : m_1 + \cdots + m_q = n\}$. In this paper, we determine that threshold.

Theorem 2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be positive constants such that $\alpha_1 + \alpha_2 + \cdots + \alpha_q = 1$. Let*

$$\alpha_{\min} = \min\{\alpha_i : i \in [q]\}.$$

Let G be the random bipartite graph $G_{n,n,p}$ where $p = \frac{\log n + \omega}{\alpha_{\min} n}$, $\omega = \omega(n) \to \infty$. Suppose that the edges of G are independently colored with colors from $C = \{c_1, c_2, \ldots, c_q\}$ where $\mathbb{P}(c(e) = c_i) = \alpha_i$ for $e \in E(G), i \in [q]$. Then, w.h.p. for each $m_1, m_2, \ldots, m_q$ satisfying $m_1 + \cdots + m_q = n$, there exists a perfect matching M in which exactly m_i edges are colored with $c_i, i = 1, 2, \ldots, q$. In other words,

$$\mathrm{mcp}(G) = \{(m_1, \ldots, m_q) \in [0, n]^q : m_1 + \cdots + m_q = n\}.$$

Let us first determine the lower bound on the threshold. Assume that $\alpha_{\min} = \alpha_i$. To prove the lower bound, note that it is enough to show that the same threshold holds even for the event that G contains a perfect matching in color c_i. To see this, remember that the bipartite graph induced by the color c_i is distributed as $G_{n,n,\alpha_i p}$. The claim now follows from the known thresholds of the random bipartite graph to have a perfect matching, see e.g., Theorem 6.1 of [7]. The general strategy to prove the upper bound on the threshold in Theorem 2 is to do the following modification iteratively. For each $i \neq j$, if G contains a perfect matching M using $m_i \geqslant \frac{n}{q}$ edges with color c_i, then we can find a perfect matching M' consisting of one fewer edge of color c_i and one more edge of color c_j. Frieze [6] also suggested studying the same problem for the random graph $G_{n,p}$. A simple extension of our techniques establishes the threshold for $G_{n,p}$ as well.

Theorem 3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be positive constants such that $\alpha_1 + \alpha_2 + \cdots + \alpha_q = 1$. Let*

$$\alpha_{\min} = \min \{\alpha_i : i \in [q]\}.$$

Let G be the random graph $G_{n,p}$ where $p = \frac{\log n + \omega}{\alpha_{\min} n}$, $\omega = \omega(n) \to \infty$. Suppose that the edges of G are independently colored with colors from $C = \{c_1, c_2, \ldots, c_q\}$ where $\mathbb{P}(c(e) = c_i) = \alpha_i$ for $e \in E(G), i \in [q]$. Then, w.h.p. for each $m_1, m_2, \ldots, m_q$ satisfying $m_1 + \cdots + m_q = \lfloor \frac{n}{2} \rfloor$, there exists a perfect matching M in which exactly m_i edges are colored with $c_i, i = 1, 2, \ldots, q$. In other words, $\mathrm{mcp}(G) = \left\{(m_1, \ldots, m_q) \in [0, n]^q : m_1 + \cdots + m_q = \lfloor \frac{n}{2} \rfloor \right\}$.

Similar to Theorem 2, the lower bound on the threshold for Theorem 3 follows from the known thresholds of the random graph to have a perfect matching (see, e.g., Theorem 6.2 of [7]).

This short note is organized as follows. The next section is devoted to stating a few simple structural lemmas about random bipartite graphs and random graphs. Section 3 contains the proof of Theorem 2 and Theorem 3. Finally, we finish with a few concluding remarks.

2 Structural lemmas

Let $\alpha_i, 1 \leqslant i \leqslant q$, and $\alpha_{\min}$ be as in Theorems 2 and 3. Throughout this section, the graph G will be either the random bipartite graph $G_{n,n,p}$ or the random graph $G_{n,p}$, where the probability $p = \frac{\log n + \omega}{\alpha_{\min} n}$, for some $\omega = \omega(n) \to \infty$. The edges of G are randomly colored as in Theorems 2 and 3.

Lemma 4. *Let G be the random bipartite graph $G_{n,n,p}$ with the vertex bipartition $A \cup B$. Suppose that the edges of G are independently colored with colors from $C = \{c_1, c_2, \ldots, c_q\}$ where each edge is colored with c_i by probability α_i. Then, w.h.p. for each $i \in [q]$, and any $X \subseteq A, Y \subseteq B$ with $|X|, |Y| \geqslant \frac{n}{4q}$, there is an edge with color c_i between X and Y in G.*

Proof. Note that it is enough to prove this lemma with $|X| = |Y| = \frac{n}{4q}$. Now by a simple union bound, we have the following:

$$\begin{aligned}
\mathbb{P}(\exists X, Y \text{ s.t. condition is not satisfied}) &\leqslant \binom{n}{n/4q}^2 \sum_{i=1}^{q} (1 - p\alpha_i)^{\frac{n^2}{16q^2}} \\
&\leqslant q \left(\frac{ne}{n/4q}\right)^{n/2q} \left(1 - \frac{\log n}{n}\right)^{\frac{n^2}{16q^2}} \\
&\leqslant q \left((4eq)^{1/2q} \cdot e^{-\frac{\log n}{16q^2}}\right)^n \\
&= o(1).
\end{aligned}$$

□

Lemma 5. *Let G be the random graph $G_{n,p}$. Suppose that the edges of G are colored in the exact same way as in Lemma 4. Then, w.h.p. for each $i \in [q]$, and any disjoint sets $X, Y \subseteq V(G)$ with $|X|, |Y| \geqslant \frac{n}{8q}$, there is an edge with color c_i between X and Y in G.*

Proof. This follows very similarly to the proof of Lemma 4. □

Lemma 6. *Let G be the random bipartite graph $G_{n,n,p}$ or the random graph $G_{n,p}$. Then, w.h.p. for each $i \in [q]$, the graph G contains a perfect matching in color c_i.*

Proof. This is an easy consequence of Theorems 6.1 and 6.2 of [7]. □

3 Proof of the main results

Proof of Theorem 2. Suppose that we are given a bipartite graph G for which the high probability properties (Lemmas 4 and 6) of the random bipartite graph $G_{n,n,p}$ mentioned in the last section hold. The proof mainly consists of showing that the following can be done. For each $i \neq j$, if G contains a perfect matching M with at least $\frac{n}{q}$ edges with color c_i, then G contains a perfect matching with the same color profile as M but with one fewer edge of color c_i and one more edge of color c_j. We next show how we can iteratively apply this modification to obtain a perfect matching with any given color profile.

Fix $(m_1, m_2, \ldots, m_q) \in [0, n]^q$ such that $\sum_{i=1}^{q} m_i = n$. Our goal is to show that G has a perfect matching M such that $|M \cap Q_i| = m_i$ for all i. Without loss of generality we can assume that $m_1 = \max\{m_i : i \in [q]\}$. This implies that $m_1 \geqslant \frac{n}{q}$. By Lemma 6, we know that there is a perfect matching in the subgraph induced by color c_1 in G. We proceed in the following way: starting with a perfect matching with color profile $(n, 0, \ldots, 0)$, for any fixed color c_j with $j \neq 1$ we show the existence of a perfect matching with one fewer edge in color c_1 and one more edge in color c_j. We keep doing this process until we get a matching with m_i edges with color c_i for all i. Note that we need $n - m_1$ steps to reach a matching with the color profile $(m_1, m_2, \ldots, m_q)$, because in every step, we find a matching with one fewer edge in color c_1. So, it is enough to show that for any perfect matching M in G with $|M \cap Q_i| = \mu_i$ for each $i \in [q]$ and $\mu_1 \geqslant \frac{n}{q}$, there is a matching M' in G with $|M' \cap Q_1| = \mu_1 - 1$, $|M' \cap Q_2| = \mu_2 + 1$ and $|M' \cap Q_i| = \mu_i$ for all other i.

We show the above statement by finding an appropriate alternating cycle. More precisely, we find a cycle C with vertex sequence $(x_1 \in A, y_1 \in B, x_2 \in A, y_2 \in B, \ldots, x_\ell \in A, y_\ell \in B, x_1)$ such that (i) $(x_i, y_i) \notin M$, (ii) $(y_i, x_{i+1}) \in M$, (iii) $(x_1, y_1) \in Q_2$, and (iv) $E(C) \setminus \{(x_1, y_1)\} \subseteq Q_1$. For the convenience of writing the proof, we introduce some notation. Label vertices so that the edges $v_i^+ v_i^-$, $i \in [\frac{n}{q}]$, with $v_i^+ \in A$ and $v_i^- \in B$ are distinct edges with color c_1 in M. Create a directed graph D on vertex set $\{v_1, \ldots, v_{n/q}\}$, where there is a directed edge $v_i v_j$ in D if there is an edge with color c_1 between v_i^- and v_j^+ in G.

Note that if there is an edge with color c_2 between v_i^+ and v_j^- in G and a directed path from v_i to v_j in D, then this gives exactly the alternating cycle C which we discussed in the last paragraph. Moreover, by using Lemma 4, we have the following property in D.

1. For each $X, Y \subseteq V(D)$ with $|X|, |Y| \geqslant \frac{n}{4q}$, there is an edge from X to Y in D.

For each $v \in V(D)$, let $B^+(v)$ be the set of vertices reachable by a directed path from v in D (including v), and let $B^-(v)$ be the set of vertices in $V(D)$ from which you can reach v in D with a directed path (including v). Let $V_1 = \left\{v \in V(D) : |B^+(v)| \leqslant \frac{n}{4q}\right\}$ and $V_2 = \left\{v \in V(D) : |B^-(v)| \leqslant \frac{n}{4q}\right\}$.

Now, claim that $|V_1| \leqslant \frac{n}{4q}$. If not, then we can pick a minimal set $V_1' \subseteq V_1$ such that $|\cup_{v \in V_1'} B^+(v)| \geqslant \frac{n}{4q}$, and note that $|\cup_{v \in V_1'} B^+(v)| \leqslant \frac{2n}{4q}$. There are no edges from $\cup_{v \in V_1'} B^+(v)$ into $V(D) \setminus \left(\cup_{v \in V_1'} B^+(v)\right)$, and the latter set has size at least $|D| - \frac{2n}{4q} \geqslant \frac{n}{4q}$, contradicting the property (1). Therefore, $|V_1| \leqslant \frac{n}{4q}$. Similarly, $|V_2| \leqslant \frac{n}{4q}$. Thus, $|V(D) \setminus (V_1 \cup V_2)| \geqslant \frac{n}{2q}$.

By Lemma 4 there is an edge in G with color c_2 between $\left\{v_i^+ : v_i \in V(D) \setminus (V_1 \cup V_2)\right\}$ and $\left\{v_i^- : v_i \in V(D) \setminus (V_1 \cup V_2)\right\}$. Say this is the edge $v_i^+ v_j^-$ and note that $i \neq j$. As $v_i, v_j \in V(D) \setminus (V_1 \cup V_2)$, we have that $|B^+(v_i)|, |B^-(v_j)| \geqslant \frac{n}{4q}$. Thus, there is an edge from $B^+(v_i)$ into $B^-(v_j)$ in D by (1), and therefore there is a directed path from v_i to v_j in D. This finishes the proof of Theorem 2. □

Proof of Theorem 3. The proof of Theorem 2 extends straightforwardly to a proof of Theorem 3. By Lemma 6, we know that $G = G_{n,p}$ has a perfect matching in each color. Now, if a color profile $(m_1, \ldots, m_q)$ is required (say m_1 is the largest of these), then start with a perfect matching in color c_1, and split $V(G)$ into A and B arbitrarily so that M is a matching between A and B. The same arguments as in the proof of Theorem 2 can now be used due to Lemma 5, which is the replacement of Lemma 4 we used before. More precisely, to modify a perfect matching M to another matching M' with the same color profile but one fewer edge of color c_1 and one more edge of color c_j, we choose an arbitrary bipartition $V(G) = A \cup B$ with M being a matching between A and B, and then implement the exact same argument as before. □

Concluding remarks

In this short note, we consider the random bipartite graph $G = G_{n,n,p}$ and the random graph $G = G_{n,p}$, and determine the threshold on the parameter p for the event that G contains perfect matchings of all color profiles. Some interesting directions of future research would be to determine $\mathrm{mcp}(G)$ for Hamilton cycles, spanning trees etc. or to consider deterministic host graphs (e.g., Dirac graphs) instead of random graphs.

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