All minor-minimal apex obstructions with connectivity two

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Abstract

A graph is an *apex graph* if it contains a vertex whose deletion leaves a planar graph. The family of apex graphs is minor-closed and so it is characterized by a finite list of minor-minimal non-members. The long-standing problem of determining this finite list of apex obstructions remains open. This paper determines the 133 minor-minimal, non-apex graphs that have connectivity two.

Mathematics Subject Classifications: 05C10, 05C83

1 Introduction

A graph is an *apex graph* if it contains a vertex, called an *apex*, whose deletion leaves a planar graph. The family of apex graphs is minor-closed and so Robertson and Seymour's graph minor theorem [16] implies that this family has a finite list of minor-minimal non-members (also known as obstructions). Despite considerable efforts for decades, the problem of determining the list of planar apex obstructions remains open. In this paper we determine the 133 minor-minimal non-apex graphs that have connectivity two.

Apex graphs play a key role in what is commonly now referred to as the "weak structure theorem" of the Robertson and Seymour's graph minors project (recently this "weak structure theorem" has been optimized by Giannopoulou and Thilikos [10]). Apex graphs have also featured prominently in the resolution of Hadwiger's conjecture for K_6-free graphs [17] and the characterization of linklessly embeddable graphs. The latter problem highlights the Petersen family of graphs [19], a significant collection of apex obstructions. Even with advances in algorithmic refinements of the graph minors project that are focused on determining obstruction sets for apex families [2, 14], the very general and theoretical approaches are frustratingly impractical. Disconcertingly the problem of determining all apex-planar obstructions remains open, despite classical linear-time planarity testing algorithms and dramatic increases in computing power.

Researchers have made progress characterizing families of graphs that are closely related to the apex graphs. An old result of Wagner [4] characterizes the "almost planar" graphs; a graph that is non-planar but the deletion of any vertex makes it planar. The terms "almost planar" and "nearly planar" appear in many articles with a dizzying variety of meaning. Gubser [11] characterizes another family of graphs with another notion of "almost planar"—a non-planar graph such that for any edge, either the contraction or the deletion of that edge makes the graph planar (see also [6]). Ding and Dzobiak [5, 7] determined the 57 graphs that are the obstructions for the minor-closed family of apex-outerplanar graphs.

It has been a long standing open problem to determine the apex obstructions. At a conference in 1993, the second author discussed the apex obstruction problem with Robin Thomas. Together with Daniel Sanders we decided to share lists of known apex obstructions; the combined list contained 123 graphs at that time. Twenty years later we reported 396 known apex obstructions [13]. Our list has since grown to 401 non-isomorphic apex obstructions. Other groups have relayed the problem or worked to find obstructions [1, 3, 8, 9, 12, 15]. Indeed we credit David Eppstein (see bottom entry of [9]) for finding one particularly beautiful 16-vertex apex obstruction. He describes it this way: "start with a cube, find a four-vertex independent set, and make three copies of each of its vertices. The resulting 16-vertex graph has four $K_{3,3}$ subgraphs, one for each tripled vertex." We refer to this obstruction as the *Eppstein graph*.

In this paper we determine the 133 minor-minimal non-apex graphs that have connectivity two. It is straightforward to prove there are three disconnected apex obstructions and no apex obstructions with connectivity one (see the beginning of section 3 for details). Determining all apex obstructions that have connectivity greater than two is the focus of future research. Lipton et al. [12] have shown that apex obstructions have connectivity at most five. Our proof here presents the connectivity-2 apex obstructions in five groups (see Figure 1); this is essentially the same argument we presented at the AMS Meeting in 2013 [13]. Section 2 introduces basic notation and definitions. Section 3 presents elementary observations about apex obstructions. Sections 4–7 present a general outline and resolve four of the five groups of obstructions.

Section 8 considers the last group of obstructions, the 72 connectivity-2 apex obstructions with a unique 2-cut having a planar heavy component (defined in the paragraph before Lemma 9). These obstructions are considerably more difficult to characterize because they are very close to 3-connected. One aspect of connectivity-2 apex obstructions is that they may contain vertices that are not branch vertices of any Kuratowski subdivision. There are seven obstructions exhibiting this phenomenon, three appear at the bottom of Figure 5 and four appear at the bottom of Figure 8. This is not a phenomenon encountered in general 3-connected graphs that contain a subdivision of $K_{3,3}$ (see statement (6.2) of [19]). The final 72 connectivity-2 apex obstructions behave similarly to the 3-connected graphs that contain a subdivision of $K_{3,3}$: all vertices are branch vertices of a Kuratowski subdivision. Proposition 21 is the main tool used to prove this claim; it highlights the significance of "close" Kuratowski subdivisions (which we do not define in this paper), a notion remotely reminiscent to 'communicating Kuratowski subgraphs' (introduced in [18]). The most important consequence of Proposition 21 is that the apex obstructions having a unique 2-cut with a planar heavy component contain three Kuratowski subdivisions whose branch vertices cover the entire vertex set. This property is key in our proof that the obstruction list is complete. It appears likely that the study of apex obstructions with higher connectivity will require similar collections of Kuratowski subdivisions whose branch vertices cover the entire vertex set, though the Eppstein graph shows that as many as four such Kuratowski subdivisions are needed. Unfortunately connectivity appears to be a poor proxy for a still missing notion related to close complexes of Kuratowski subdivisions. After reducing to small graphs the computation of the 72 connectivity-2 apex obstructions having a unique 2-cut with a planar heavy component, the final subsection discusses the computer work applied to show the list of connectivity-2 apex obstructions is complete (see Appendix A for graph6 presentations of all 133 connectivity-2 apex obstructions).

Because it may be of independent interest, Section 9 presents another interpretation of the characterization of connectivity-2 apex obstructions in terms of double apex graphs.

2 Preliminaries

All graphs in this paper are finite, simple, and undirected. The set of vertices of the graph G is denoted $V(G)$, which is often abbreviated to V. Similarly $E(G)$, or simply E, denotes the set of edges of G. Edges are unordered pairs of vertices, but following standard notation, the edge $e = \{a, b\}$ is abbreviated ab. The vertices a and b are the *endpoints* of the edge $e = ab$. The edge $e = ab$ is *incident* to a and b. Incidence is often written as membership, as when $a \in e$ signals that edge e is incident with vertex a. The *neighbors* of a vertex u in the graph G is the set $N(u) = \{v \in V : uv \in E\}$, which is sometimes denoted $N_G(u)$ to emphasize the graph. For any $v \in V(G)$, the set of edges incident to v is denoted $E_v = \{e \in E(G) : v \in e\}$. Following standard notation, $d_G(v)$, $\delta(G)$, $\kappa(G)$ denote the *degree* of the vertex v in G, the *minimum degree* of a vertex in G, and the vertex *connectivity* of G, respectively. A uv-path in G is a path whose endpoints are u and v. If S is collection of vertices, then $G[S]$ denotes the subgraph of G induced by S.

Often, when the meaning is clear, mathematical elements are recast and operations are overloaded for notational convenience. So, for example, the graph induced by $V(G) - S$ is denoted simply $G - S$, which replaces the more syntactically accurate but cumbersome $G[V(G) - S]$, thereby overloading the subtraction operator. Furthermore, in the case that v is a vertex, $G - \{v\}$ is abbreviated to $G - v$, which implicitly recasts a vertex as a set. Recasting graphs as their vertex sets (and vice versa) appears frequently. For example, if A and B are subgraphs of G, then $A \subseteq V(G) - B$ tacitly recasts A and B as their vertex sets and is short for $V(A) \subseteq V(G) - V(B)$ since it does not make sense to subtract a graph from a vertex set. But note carefully that $A \subseteq V(G) - B$ is very different in meaning from $A \subseteq G - B$; in the latter expression no recasting takes place because all variables represent graphs and subtraction makes sense. Other operators are overloaded naturally as needed. For example, following standard notation, the addition operator can be applied to (possibly recast) sets of edges or (possibly recast) sets of vertices. For example, $G+e$ represents the graph resulting from the addition of the edge e to the graph G. Similar conveniences appear throughout.

The number of components in $G - S$ is $c(G - S)$. A vertex set S is a *cutset* for a graph G if $c(G) < c(G - S)$. A cutset with k vertices is called a *k-cut*. A cutset S for G is said to *separate* vertices u and v if u and v are in the same component of G but different components of $G-S$. A path of G *crosses* a cutset S if it contains vertices from two different components of $G - S$. If $S = \{v\}$ is a cutset for G, then v is a *cut vertex*, also known as a 1-cut. If H is a subgraph of a component of $G - S$, then the *H-side* (of $G - S$) is the component of $G - S$ that contains H.

A subdivision of K_5 or $K_{3,3}$ is called a *Kuratowski subgraph*. Kuratowski proved that a graph is non-planar if and only if it contains a Kuratowski subgraph. A vertex of a Kuratowski subgraph is a *branch* vertex if its degree in the Kuratowski subgraph is at least 3. A vertex of a Kuratowski subgraph that is not a branch vertex is also called a *subdividing vertex* or a *non-branch* vertex.

The contraction of an edge e in the graph G produces a graph denoted G/e. A graph G contains H as a *minor*, denoted $H \leqslant_m G$, if a subgraph isomorphic to H can be obtained from a subgraph of G by a sequence of edge contractions. Observe that the minor order is transitive. A family $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ and $H \leqslant_m G$ implies that $H \in \mathcal{F}$. If $\mathcal{F}$ is a minor-closed family, then the minor-minimal graphs that are not in $\mathcal{F}$ are called *obstructions*; so an obstruction is a graph $G \notin \mathcal{F}$ such that $H \in \mathcal{F}$, for all $H \leqslant_m G$ and $H \neq G$.

The disjoint union of graphs G and H is denoted $G + H$. Also $2G$ abbreviates $G + G$.

3 Simple observations

A graph is an *apex graph* if it contains a vertex, called an *apex*, whose deletion produces a planar graph. Every planar graph is an apex graph. The family of apex graphs is minor-closed and so it has finite list of minor-minimal non-members, also known as forbidden minors or obstructions. Let $\mathcal{F}$ denote the finite set of obstructions for apex graphs.

We begin with a few simple observations about graphs in $\mathcal{F}$.

Lemma 1. *If $G \in \mathcal{F}$, then $\delta(G) \geqslant 3$.*

Proof. Suppose, to the contrary, that there exists $G \in \mathcal{F}$ and $v \in V(G)$ with $d_G(v) \leqslant 2$. If $d_G(v) \leqslant 1$, then $G - v$ is apex with an apex, w. Note that $G - v - w$ is planar which implies that $G - w$ is also planar, contradicting $G \in \mathcal{F}$. If $d_G(v) = 2$, then consider an edge e incident to v. The contraction of e produces an apex graph G/e with an apex vertex, w. Now $G/e - w$ is planar which implies that $G - w$ is planar, contradicting $G \in \mathcal{F}$. □

Lemma 2. *The disconnected graphs in $\mathcal{F}$ are $2K_5, 2K_{3,3}$, and $K_5 + K_{3,3}$.*

Proof. The reader can easily verify that $2K_5, 2K_{3,3}$, and $K_5+K_{3,3}$ are disconnected graphs in $\mathcal{F}$. It suffices to show there are no others. Consider a disconnected graph $G \in \mathcal{F}$. Each component of G must be an apex graph since removing any one edge from G produces an apex graph. There must be at least two non-planar components since otherwise the whole graph is an apex graph. It follows that there are exactly two components and each is non-planar apex. Consider an arbitrary component of G, call it C. Let H be a Kuratowski subgraph in C. If there is an edge $e \in E(C) - E(H)$, then $G - e$ is not apex, contradicting that $G \in \mathcal{F}$; hence, $E(H) = E(C)$. Lemma 1 guarantees $\delta(G) \geqslant 3$, from which it follows that all the vertices of C are branch vertices of H; consequently, $C \cong K_5$ or $C \cong K_{3,3}$. □

Lemma 3. *Every connected graph in $\mathcal{F}$ has connectivity at least two.*

Proof. Suppose, to the contrary, that $G \in \mathcal{F}$ and $\kappa(G) = 1$. Let v be a cut vertex of G. Now $G - v$ is non-planar; therefore, there exists a Kuratowski subgraph, H, in one of the components, say component C, of $G - v$. Define $T = \{uv \in E(G) : u \in V(C)\}$. Because $G - T$ is apex with apex vertex in $V(H) \subseteq V(C)$, it follows that $G - C$ is planar. Let e be an edge incident to v that is also incident to a vertex in a component of $G - v$ other than C. Because $G - e$ is apex, there is an apex of $G - e$, say w, that is in $V(H) \subseteq V(C)$. Because $G[V(C) \cup \{v\}]$ is apex (with apex w) and $G - C$ is planar, it follows that $G-w$ is planar: embed $G[V(C) \cup \{v\}] - w$ in the plane with v on the exterior face, separately embed $G - C$ in the plane with v on the exterior face, and identify v's in these embeddings, producing a planar embedding of $G - w$. This contradicts that G is not an apex graph. □

Because of Lemmas 2 and 3, we now consider only obstructions in $\mathcal{F}$ that have connectivity at least two. Indeed, in this paper we determine all graphs $G \in \mathcal{F}$ such that $\kappa(G) = 2$; there are precisely 133 of them.

For $G \in \mathcal{F}$ and $v \in V(G)$, let H_v denote a Kuratowski subgraph in $G - v$. Note that H_v witnesses that G is not apex; it is a *Kuratowski witness* for v.

If $S \subseteq V$ is a cutset for G and C is a component of $G - S$, then the *augmentation of* C is

$$C^+ = G[V(C) \cup S] + \{uv : u, v \in S\}.$$

This is also referred to as the *augmented component* of $G - S$ obtained from C.

The next lemma gathers several elementary properties of Kuratowski witnesses.

Lemma 4. *Suppose $G \in \mathcal{F}$ and $\kappa(G) \geqslant 2$. For all $u, v, w \in V(G)$, and all Kuratowski witnesses H_u, H_v and H_w for u, v and w, respectively,*

i) $V(H_u \cap H_v) \neq \varnothing$,

ii) *if* $V(H_u \cap H_v \cap H_w) = \varnothing$, *then* $E(G) = E(H_u \cup H_v \cup H_w)$, *iii) if $V(H_u \cap H_v \cap H_w) = \varnothing$ and $x \in V(H_u) - V(H_v \cup H_w)$, then x is a branch vertex of H_u,*

iv) if $\kappa(G) = 2$, S is a 2-cut of G, and $s \in S$, then for any Kuratowski witness H_s, there exists an augmented component C^+ of $G - S$ such that $V(H_s) \subseteq V(C^+)$.

Proof. i) If $V(H_u \cap H_v) = \varnothing$, then $G = H_u + H_v$ is a disconnected obstruction in $\mathcal{F}$. *ii)* Assume $V(H_u \cap H_v \cap H_w) = \varnothing$. If there were an edge $e \in E(G) - E(H_u \cup H_v \cup H_w)$, then $G - e$ would have no apex. *iii)* Suppose $V(H_u \cap H_v \cap H_w) = \varnothing$ and $x \in V(H_u) - V(H_v \cup H_w)$. Assume, to the contrary, that x is not a branch vertex of H_u; so $d_H(x) = 2$. Lemma 1 implies $d_G(x) \geqslant 3$. Consequently there is at least one edge e incident to x that does not belong to $E(H_u \cup H_v \cup H_w)$, contradicting that $G - e$ has an apex. *iv)* Assume $G \in \mathcal{F}$, $\kappa(G) = 2$, S is a 2-cut of G, and $s \in S$. Consider a Kuratowski witness H_s. Observe that some augmented component of $G - S$ must contain all the vertices in $V(H_s)$ because no path of the 2-connected subgraph H_s can cross the 1-cut $S - s$. □

Lemma 5. *Suppose that $G \in \mathcal{F}$, $\kappa(G) = 2$, and $S = \{a, b\}$ is a 2-cut of G. Every augmented component C^+ of $G - S$ contains an edge $e \in E(C^+) - ab$ such that there exists an ab-path in $C^+ - e - ab$.*

Proof. Consider an arbitrary component of $G - S$, call it C, and its augmentation C^+. Let P be an ab-path in $C^+ - ab$ (P exists because S is a minimum cutset of G). The path P contains a vertex $v \notin S$. Let e be an edge incident to v that is not in P; the existence of such an edge follows from Lemma 1. Now e is the desired edge in $E(C^+) - ab$. □

Next we present a technical lemma that will be applied in several upcoming arguments.

Lemma 6. *Suppose that $G \in \mathcal{F}$, $\kappa(G) = 2$, $S = \{a, b\}$ is a 2-cut of G, and C is a component of $G - S$. If $e \in E(C^+) - ab$ and there exists an ab-path in $C^+ - e - ab$, then for any apex w of $G - e$ and any Kuratowski subgraph H_w in G avoiding w, the branch vertices of H_w are all in $V(C^+)$.*

Proof. Because $G - e - w$ is planar, it follows that $e \in E(H_w)$. All of the branch vertices of H_w must be in the same augmented component of $G - S$ because $|S| = 2$ but H_w is a subdivision of K_5 or $K_{3,3}$ which are 3-connected. If the branch vertices of H_w are not in $V(C^+)$, then e appears in H_w only as an edge along a path connecting a and b. Consequently, a Kuratowski subgraph in $G - w - e$ would exist by replacing this ab-path in C^+ by another ab-path from $C^+ - e - ab$ (which exists by assumption), contradicting that w is an apex vertex for $G - e$. So the branch vertices of H_w must be in $V(C^+)$. □

Recall that, if $S \subseteq V$ is a cutset for G, then $c(G - S)$ is equal to the number of components in $G - S$.

Lemma 7. *If $G \in \mathcal{F}$, $\kappa(G) = 2$, and S is a 2-cut of G, then $c(G - S) = 2$.*

Proof. Assume, to the contrary, that $c(G-S) > 2$. Let $S = \{a, b\}$ and let C_1, C_2, C_3 be three components of $G-S$. Applying Lemma 4 part *iv)*, we may assume that $V(H_a) \subseteq C_1^+$. It follows from Lemma 4 parts *i)* and *iv)* that $V(H_b) \subseteq C_1^+$. Lemma 5 guarantees there exists an edge e in $E(C_3^+) - ab$ such that an ab-path remains in $C_3^+ - ab - e$. Because $G \in \mathcal{F}$, the graph $G - e$ is an apex graph; let w be an apex for $G - e$. Consider H_w, a Kuratowski subgraph avoiding w. By Lemma 6 the branch vertices of H_w must be in $V(C_3^+)$.

Now Lemma 5 guarantees there exists an edge $f \in E(C_2^+) - ab$ such that an ab-path remains in $C_2^+ - ab - f$. Notice that if f is an edge in H_w then, because the branch vertices of H_w must be in $V(C_3)$, the edge f appears in H_w only as an edge along an ab-path in $C_2^+ - ab$. Consequently there is a Kuratowski subgraph H_w^* in $G[V(C_2 \cup C_3) \cup \{a, b\}]$ that avoids the edge f; it is obtained from H_w by replacing, if necessary, any ab-path in $H_w \cap (C_2^+ - ab)$ by an ab-path in $C_2^+ - ab - f$. This implies that H_a, H_b and H_w^* share no common vertex, contradicting $G - f$ is apex. $\square$

Lemma 8. *If $G \in \mathcal{F}$, $\kappa(G) = 2$, and S is any 2-cut of G, then the augmentation of any component of $G - S$ is non-planar.*

Proof. Let $S = \{a, b\}$ be an arbitrary 2-cut of G. Lemma 7 guarantees that $G - S$ has exactly two components, call them C_1 and C_2. Applying Lemma 4 part *iv)*, we may assume that $H_a \subseteq C_1^+$. It follows from Lemma 4 parts *i)* and *iv)* that $H_b \subseteq C_1^+$. It suffices to prove that C_2^+ is non-planar. Lemma 5 guarantees there exists an edge $e \in E(C_2^+) - ab$ such that an ab-path remains in $C_2^+ - ab - e$. Because $G \in \mathcal{F}$, the graph $G - e$ is an apex graph; let w be an apex for $G - e$ and let H_w be a Kuratowski subgraph of G avoiding w. By Lemma 6 the branch vertices of H_w must be in $V(C_2^+)$. This implies that $H_w \subseteq C_2^+$, so C_2^+ is non-planar. $\square$

Consider $G \in \mathcal{F}$ with $\kappa(G) = 2$ and $S = \{a, b\}$ any 2-cut of G. By Lemma 7, there are only two components of $G - S$, call them C_1 and C_2. We may assume that $H_a \subseteq C_1^+$ and $H_b \subseteq C_1^+$. We call C_1 the *heavy* component of $G - S$ because $H_a, H_b \subseteq C_1^+$; C_2 is the *light* component. A 2-cut of G is *basic* if the vertex set of its heavy component is minimal (with respect to set inclusion); that is, no 2-cut produces a heavy component whose vertex set is properly contained in this one.

Lemma 9. *If $G \in \mathcal{F}$, $\kappa(G) = 2$, and S is any basic 2-cut of G, then the augmentation of the light component of $G - S$ is isomorphic to K_5, $K_{3,3}$, or $K_{3,3} + e$.*

Proof. Let $S = \{a, b\}$ be an arbitrary basic 2-cut of G. Let C_1 and C_2 be the two components of $G - S$, with C_1 the heavy component and C_2 the light one. Lemma 5 guarantees there exists an edge $e \in E(C_2^+) - ab$ such that an ab-path remains in $C_2^+ - ab - e$. Because $G \in \mathcal{F}$, the graph $G - e$ is an apex graph. Let w be an apex for $G - e$ and let H_w be a Kuratowski subgraph of G avoiding w. Observe $w \in V(H_a \cap H_b) \subseteq V(C_1)$. By Lemma 6 the branch vertices of H_w must be in $V(C_2^+)$. Furthermore, there must be some part of H_w that is not in C_2 since $V(H_w \cap H_a) \neq \varnothing$ and $V(H_w \cap H_b) \neq \varnothing$, so in particular, $\{a, b\} \subseteq V(H_w)$.

Claim: w is an apex for $G - f$ and G/f, for all $f \in E(C_2^+) - \{ab\}$.

Suppose, to the contrary, there is an edge $f \in E(C_2^+) - \{ab\}$ such that w is not an apex for $G - f$ or G/f. There must be an apex vertex for $G - f$ and G/f. Let z be an apex for $G - f$ if w is not an apex for it; otherwise let z be an apex for G/f. Clearly $z \in V(H_a \cap H_b \cap H_w) \subseteq V(C_1)$ and $z \notin \{a, b, w\}$. Because the branch vertices of H_w are in C_2^+ and $z \notin \{a, b\}$, it follows that $H_w \cap G[V(C_1) \cup \{a, b\}]$ is an ab-path through z and $ab \notin E(G)$. Indeed z must be a cut vertex in $G[V(C_1) \cup \{a, b\}]$ that separates a from b since otherwise an ab-path in $G[V(C_1) \cup \{a, b\}]$ that avoids z could substitute in H_w to create a Kuratowski subgraph in $G - f - z$ or $G/f - z$.

Now consider the two components of $G[V(C_1) \cup \{a, b\}] - z$; call them U_a and U_b, where $a \in V(U_a)$ and $b \in V(U_b)$. Set $U_a^+ = G[V(U_a) \cup \{z\}]$ and $U_b^+ = G[V(U_b) \cup \{z\}]$.

Note that H_a must be a subgraph of U_a^+ or U_b^+, since H_a is a subgraph of $G[V(C_1) \cup \{a, b\}]$ and z is a cutvertex for this graph. If H_a is a subgraph of U_a^+, then H_z is also a subgraph of U_a^+ since Lemma 4 part (i) shows that $V(H_a \cap H_z) \neq \varnothing$. This implies that $\{z, a\}$ is a 2-cut of G with a heavy component properly contained in C_1, contradicting that S is basic. So H_a is a subgraph of U_b^+. Similarly H_b is a subgraph of U_a^+. Lemma 4 part (i) states that $V(H_a \cap H_b) \neq \varnothing$, so $V(U_a \cap U_b) = \{z\}$ implies $V(H_a \cap H_b) = \{z\}$. But recall that $w \in V(H_a \cap H_b)$; so $w = z$, a contradiction.

Because w is an apex for $G-f$, for all $f \in E(C_2^+) - \{ab\}$, it follows that $E(C_2^+) - \{ab\} \subseteq E(H_w)$. Therefore there are no vertices in $V(C_2)$ that have degree two in H_w since $\delta(G) \geqslant 3$ would otherwise guarantee an edge in $E(C_2^+) - \{ab\} - E(H_w)$. Consequently, all the vertices of C_2 are branch vertices of H_w. If a is not a branch vertex of H_w, then it is a subdividing vertex of H_w so there is an edge of H_w incident to a whose other endpoint is in C_2 (because the branch vertices of H_w are all in $C_2 \cup \{a, b\}$); call this edge f. Now G/f contains H_w/f, contradicting the claim that established w is an apex for G/f. So a is a branch vertex of H_w. Symmetrically b is a branch vertex of H_w. Because all of the vertices in $C_2 \cup \{a, b\}$ are branch vertices of H_w and the branch vertices of H_w are all in $C_2 \cup \{a, b\}$, it now follows that C_2^+ is isomorphic to K_5, $K_{3,3}$, or $K_{3,3} + e$. □

The next lemma, a consequence of Lemma 8 and Lemma 9, is a powerful tool to analyze obstructions with more than one 2-cut.

Lemma 10. *Suppose that $G \in \mathcal{F}$ and $\kappa(G) = 2$. Every 2-cut of G is basic.*

Proof. Suppose that $S = \{a, b\}$ is an arbitrary 2-cut of $G \in \mathcal{F}$. Assume, to the contrary, that S is not basic. Let C_S (resp. L_S) denote the heavy (resp. light) component of $G - S$. Because S is not basic, there exists a basic 2-cut Q with heavy (resp. light) component C_Q (resp. L_Q) such that $V(C_Q) \subsetneq V(C_S)$, or equivalently, $\overline{V(C_S)} \subsetneq \overline{V(C_Q)}$. Now

$$V(L_S^+) = V(L_S) \cup S = \overline{V(C_S)} \subsetneq \overline{V(C_Q)} = V(L_Q) \cup Q = V(L_Q^+),$$

so there exists a vertex $z \in V(L_Q^+) - V(L_S^+)$. Because Q is basic, Lemma 9 implies that L_Q^+ is isomorphic to K_5, $K_{3,3}$, or $K_{3,3} + e$. But, $V(L_S^+) \subsetneq V(L_Q^+)$ and $z \in V(L_Q^+) - V(L_S^+)$ means that L_S^+ is a subgraph of $K_5 - z$, $K_{3,3} - z$, or $K_{3,3} + e - z$. Each of these possibilities contradicts Lemma 8 which shows that L_S^+ is non-planar. □

4 The five types of obstructions and some shared properties

The connectivity-2 obstructions to the apex family are arranged into five groups. Figure 1 shows a partition of these 133 obstructions according to whether the heavy component of a 2-cut induces a planar graph and further properties of 2-cuts. This partition follows the outlines of our characterization of these graphs.

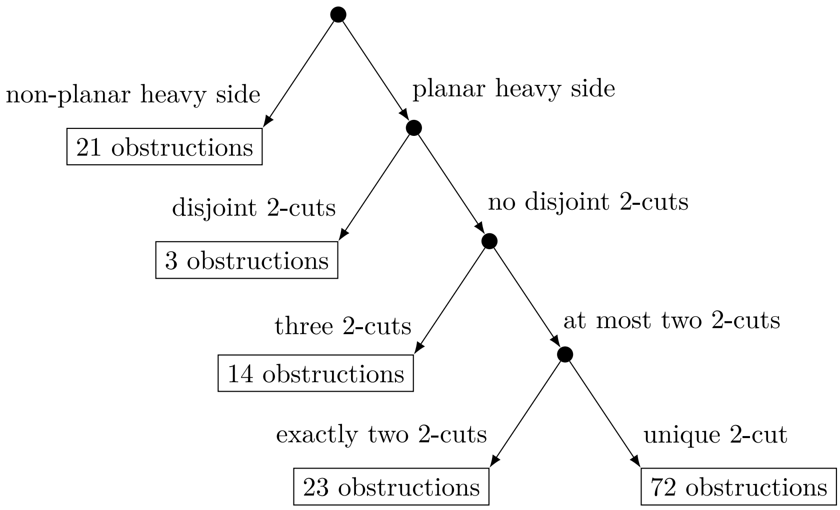

Figure 1: Partition of the 133 connectivity-2 apex obstructions into five types.

Theorem 11. *Suppose that $G \in \mathcal{F}$, $\kappa(G) = 2$, $S = \{a, b\}$ is any 2-cut of G, and C is the heavy component of $G - S$. If H_a and H_b are Kuratowski subgraphs of G avoiding a and b, respectively, then*

$$E(C) \subseteq E(H_a \cup H_b).$$

Proof. We argue by contradiction. Assume that $f \in E(C) - E(H_a \cup H_b)$. By Lemma 10, S is a basic 2-cut. Lemma 9 guarantees that the augmentation of the light component, L, of $G - S$ is isomorphic to K_5, $K_{3,3}$, or $K_{3,3} + e$. Now $G - f$ is apex with apex z, say. Observe that $z \in V(H_a \cap H_b) \subseteq V(G) - (V(L) \cup \{a, b\})$. There can therefore be no ab-path in $G[C \cup \{a, b\}] - f - z$ since otherwise the contraction of this path would, together with L, realize a Kuratowski subgraph in $G - f - z$. If the endpoints of f are in same component (we may assume the component containing a) of $G[C \cup \{a, b\}] - z$ (see Figure 2), then, because H_z must contain f and $H_a \cap H_z \neq \varnothing$, both $H_a - z$ and H_z are also in this component. Consequently the 2-cut $\{a, z\}$ has a heavy component properly contained in C_1, contradicting that S is basic.

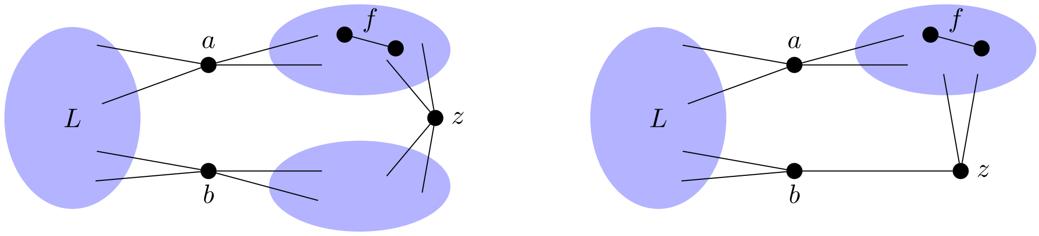

Figure 2: Cases in which endpoints of f are in same component of $G - L - S - z$.

Observe that f is not incident to z. So we may assume that the endpoints of f, u and v, are in opposite components of $G[C \cup \{a, b\}] - z$. Let us label these components A and B, where A (resp. B) denotes the component containing neighbors of a (resp. b). Without loss of generality, $u \in A$ and $v \in B$ (see Figure 3).

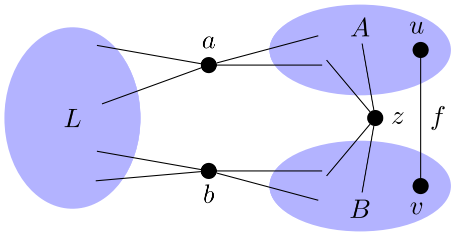

Figure 3: Endpoints of f are in different components of $G - L - S - z$.

Because $f \notin E(H_a \cup H_b)$, it follows that $|\{u, v\} \cap V(H_a)| \leqslant 1$ and $|\{u, v\} \cap V(H_b)| \leqslant 1$; consequently H_a and H_b exist in G/f, the graph obtained from G by contracting f.

Claim: There are two internally vertex-disjoint paths from a to $\{u, z\}$ in $G - b$.

Suppose, to the contrary, that there are not two internally vertex-disjoint paths from a to $\{u, z\}$ in $G - b$. Menger's theorem then guarantees that there is a vertex w in the subgraph of G induced by $A \cup \{a, z\}$ separating a from $\{u, z\}$ (see Figure 4).

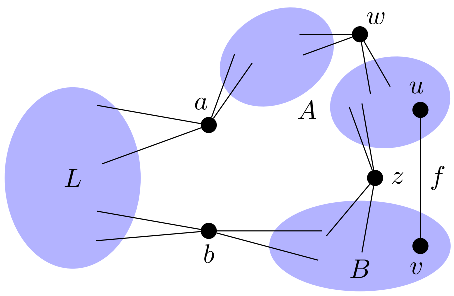

Figure 4: No two internally vertex-disjoint paths implies a vertex w.

Recall that $f \in H_z$ since z is an apex for $G - f$. Therefore H_z in on the f-side of the 2-cut $\{b, w\}$. Because $H_b \cap H_z \neq \varnothing \neq H_w \cap H_z$, it follows that H_b and H_w are also on the f-side of the 2-cut $\{b, w\}$. Consequently the 2-cut $\{b, w\}$ has a heavy component properly contained in C_1, contradicting that S is basic.

By symmetry, there are two internally vertex-disjoint paths from b to $\{v, z\}$ in $G - a$. This means that there are two internally vertex-disjoint ab-paths in $C \cup \{a, b\}$ that remain internally vertex-disjoint after contracting the edge f. Now consider G/f; it is an apex graph with an apex vertex z'. As noted earlier, H_a and H_b exist in G/f so $z' \in V(H_a \cap H_b) \subseteq V(C)$. But the deletion of the vertex z' leaves an ab-path on the C-side of the 2-cut $\{a, b\}$ in G/f. In particular, this ab-path can be contracted to the edge ab which, together with L produces a Kuratowski subgraph in $G/f - z'$, contradicting that z' is an apex vertex. □

5 The 21 connectivity-2 obstructions with a non-planar heavy component

We now turn to characterizing apex obstructions in which some 2-cut has a non-planar heavy component.

Theorem 12. *Suppose that $G \in \mathcal{F}$ and $\kappa(G) = 2$. If some 2-cut of G has a non-planar heavy component, then G is isomorphic to a graph in Figure 5.*

Proof. Let $S = \{a, b\}$ be a 2-cut of G with non-planar heavy component C of $G - S$. By Lemma 10, S is a basic 2-cut. Lemma 9 guarantees that the augmentation of the light component, L, of $G - S$ is isomorphic to K_5, $K_{3,3}$, or $K_{3,3} + e$. Observe that $ab \notin E(G)$ since otherwise G contains two disjoint Kuratowski subgraphs, one in $G[V(L) \cup S]$ and one in C, contradicting that $G \in \mathcal{F}$ and $\kappa(G) = 2$. Similar reasoning shows also that $L^+ \ncong K_{3,3} + e$, a notable characteristic of this collection of obstructions.

Let K be a minimum order Kuratowski subgraph of C. Let $H_a = K = H_b$ be the two Kuratowski subgraphs avoiding a and b. Theorem 11 implies that $E(G) \subseteq E(L) \cup E_a \cup E_b \cup E(K)$, where recall that $E_v = \{e \in E(G) : v \in e\}$ denotes the edges of G incident to v. This means that $C = K$ and, because $\delta(G) \geqslant 3$, the only possible non-branch vertices of K are neighbors of either a or b.

Step 1: For $v \in S$, $|N(v) \cap V(K)| \geqslant 2$.

We may assume $v = a$. Suppose, to the contrary, that $|N(a) \cap V(K)| \leqslant 1$. If $|N(a) \cap V(K)| = 0$, then $\kappa(G) < 2$ so we may assume that $N(a) \cap V(K) = \{w\}$, for some vertex $w \in V(K)$. Now consider the 2-cut $\{b, w\}$ of G. Because H_w can not be disjoint from $K = H_b$, it follows that H_w is a subgraph of $G[V(K) - w + b]$. Consequently the 2-cut $\{b, w\}$ has a heavy component that is properly contained in C_1, contradicting that S is basic.

Step 2: For $v \in S$, $|N(v) \cap V(K)| \leqslant 2$.

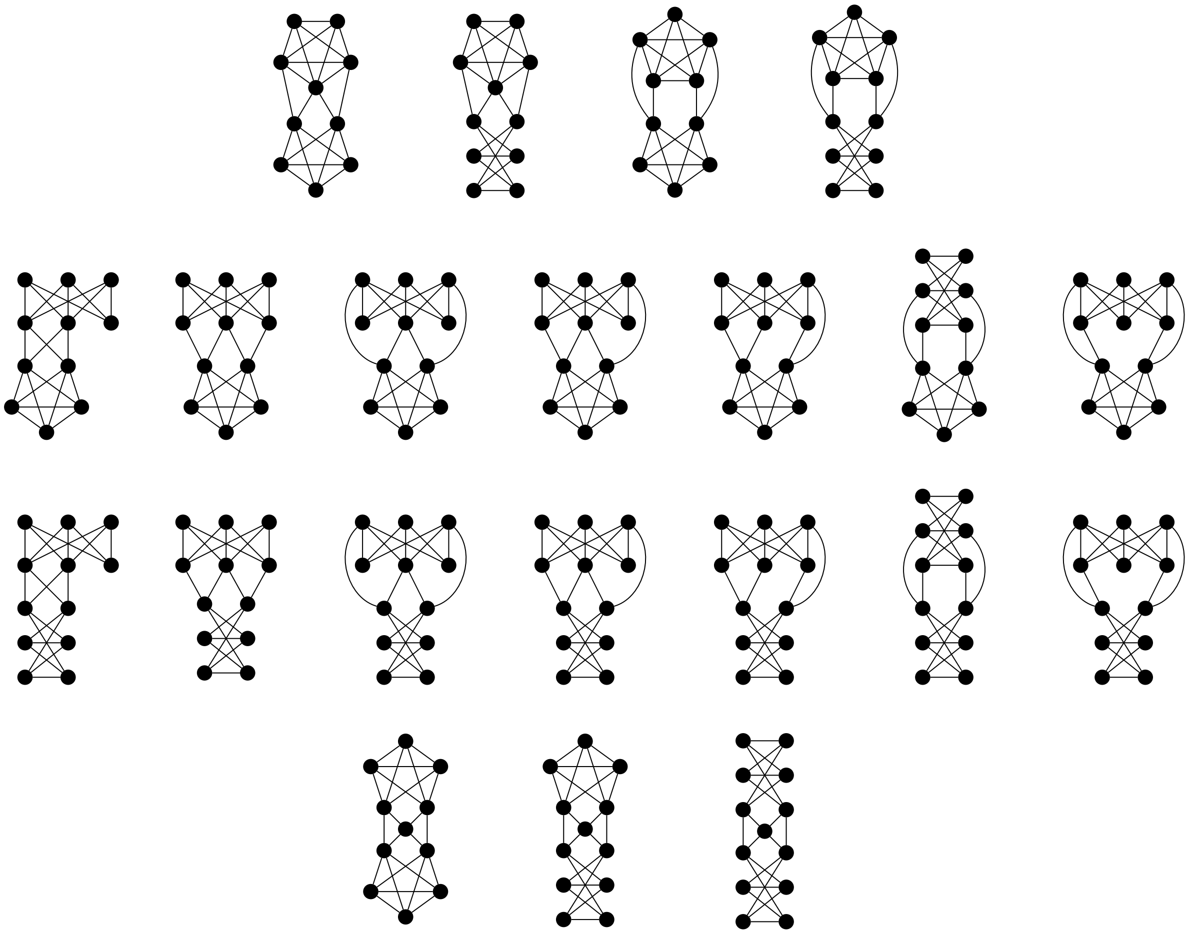

Figure 5: The 21 connectivity-2 apex obstructions with a 2-cut having a non-planar heavy component.

Without loss of generality $v = a$. Suppose, to the contrary, that $|N(a) \cap V(K)| \geqslant 3$. Choose $w \in N(a) \cap V(K)$ and set $e = aw$. Consider $G - e$; it is apex with an apex vertex z, say. Now $z \in V(K)$ since $K \subseteq G - e$. Because $|N(a) \cap V(K)| \geqslant 3$, the vertex a has at least one neighbor in $V(K) - \{w, z\}$. By Step 1, the vertex b has at least one neighbor in $G - e - z$. Since $K - z$ is connected, there is an ab-path in $G[V(K) \cup \{a, b\}] - e - z$ that produces a Kuratowski subgraph corresponding to the augmentation of L in $G - e - z$, a contradiction.

Step 3: Any subdividing vertex of K is adjacent to both a and b.

Suppose that w is a subdividing vertex of K. As noted prior to Step 1, $w \in N(a) \cup N(b)$ since otherwise the minimum degree three of w implies an edge incident to w that is not covered by K, contradicting Theorem 11. Assume now, contrary to the claim, that $|N(w) \cap \{a, b\}| = 1$. Without loss of generality, $aw \in E(G)$. Choose $w' \in (N(w) \cap K) - N(a)$; such a vertex exists since w has two neighbors in K but a only has one more neighbor in K besides w. Consider G/ww'; it is apex with apex z, say. Note that contracting ww' preserves a version of K and both a and b still have two neighbors in this version of K (because a is not adjacent to w' and b is not adjacent to w) implying that there is an ab-path in $G[V(K) \cup \{a,b\}]/ww' - z$ which determines a Kuratowski subgraph corresponding to the augmentation of L in $G/ww' - z$, a contradiction.

STEP 4: K has at most one subdividing vertex.

Observe that, by Steps 1–3, there are at most two subdividing vertices of K. Suppose, to the contrary, that x and y are subdividing vertices of K. Prior steps guarantee $N(a) \cap V(K) = \{x,y\} = N(b) \cap V(K)$. Let $w \in (V(K) \cap N(x)) - y$. Consider these three Kuratowski subgraphs of G/xw: the subgraph induced by $V(L) \cup \{a,b,x\}$, the subgraph induced by $V(L) \cup \{a,b,y\}$, and K/xw. These three Kuratowski subgraphs have no common vertex, contradicting that G/xw is apex.

STEP 5: If K has a subdividing vertex w, then $N(w) \cap V(K) \subseteq N(a) \cup N(b)$.

Let x and y be the neighbors of w in K. If $x \notin N(a) \cup N(b)$, then contracting wx preserves a version of K that must contain an apex z for G/wx. However a and b still have two neighbors to this version of K implying that there is a Kuratowski subgraph corresponding to the augmentation of L in $G/wx - z$, a contradiction. Therefore, $x \in N(a) \cup N(b)$. By symmetry, $y \in N(a) \cup N(b)$.

So, in summary, L^+ is isomorphic to K_5 or $K_{3,3}$, C is a Kuratowski subgraph, K, with at most one subdividing vertex, and $|N(a) \cap V(K)| = 2 = |N(b) \cap V(K)|$. If K has a subdividing vertex w with $N(w) \cap V(K) = \{x,y\}$, then xy is not an edge of G (by minimality of K) and, without loss of generality, $N(a) \cap V(K) = \{w,x\}$ and $N(b) \cap V(K) = \{w,y\}$. The reader can now easily verify that the graphs in Figure 5 enumerate the possible apex obstructions with these properties. □

6 The 3 connectivity-2 apex obstructions with planar heavy components and two disjoint 2-cuts

To complete the characterization of connectivity-2 apex obstructions, it suffices to consider connectivity-2 apex obstructions in which every 2-cut has a planar heavy component. We next consider such connectivity-2 apex obstructions with two disjoint 2-cuts. There are three of these (see Figure 6).

Theorem 13. *Suppose $G \in \mathcal{F}$, $\kappa(G) = 2$, and every 2-cut has a planar heavy component. If G has disjoint 2-cuts, then G is isomorphic to a graph in Figure 6.*

Proof. Suppose that S and T are disjoint 2-cuts of G. Let $S = \{a,b\}$, and let C_S be the heavy component of $G - S$. Lemma 10 yields that S and T are basic 2-cuts. Lemma 9

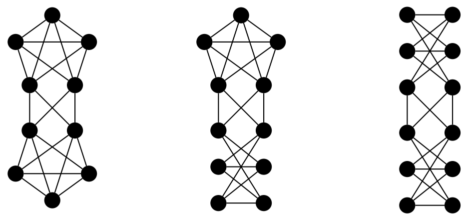

Figure 6: The 3 connectivity-2 apex obstructions in which there are disjoint 2-cuts and every 2-cut has a planar heavy component.

guarantees that the augmentation, L_S^+, of the light component, L_S, of $G-S$ is isomorphic to K_5, $K_{3,3}$, or $K_{3,3}+e$, each of which is 3-connected. Similarly L_T^+, the augmentation of the light component of T, is isomorphic to K_5, $K_{3,3}$, or $K_{3,3}+e$.

Case 1: $|T \cap V(L_S)| = 2$.

The deletion of T from L_S^+ is still connected. It follows that $G-T$ is connected, a contradiction.

Case 2: $|T \cap V(L_S)| = 1$.

The deletion of a vertex of T from L_S^+ is still connected. It follows that $G-T$ can only be disconnected if a vertex of G is a cut vertex of G, contradicting that $\kappa(G) = 2$.

Case 3: $|T \cap V(L_S)| = 0$.

Let $T = \{u, v\} \subset C_S$ with L_T (resp. C_T) the light (resp. heavy) component of $G-T$. Because $G-T$ has two components and one contains $L_S \cup S$, it follows that $L_S \cup S \subset C_T$ because T is basic. Symmetrically, $L_T \cup T \subset C_S$.

We claim that there exists an av-path in $G-\{b,u\}-V(L_S)-V(L_T)$. It suffices to find an av-path in $G-\{b,u\}$. If $G-\{b,u\}$ is connected, then there is nothing to prove; so assume that $R = \{b, u\}$ is a 2-cut of G. Because S is basic, $V(C_R)\cap(V(L_S)\cup\{a\}) \neq \varnothing$ so $a \in V(C_R)$. Symmetrically, $v \in V(C_R)$. Consequently a path in C_R connects a and v in $G-R$.

By symmetry there is an au-path in $G-\{b,v\}-V(L_S)-V(L_T)$, a bv-path in $G-\{a,u\}-V(L_S)-V(L_T)$, and a bu-path in $G-\{a,v\}-V(L_S)-V(L_T)$. All of these paths must be internally vertex disjoint since otherwise there would be an ab-path avoiding $\{u,v\}$ (which completes L_S to L_S^+ in C_T) or a uv-path avoiding $\{a,b\}$ (which completes L_T to L_T^+ in C_S), contradicting that the heavy components of S and T are planar.

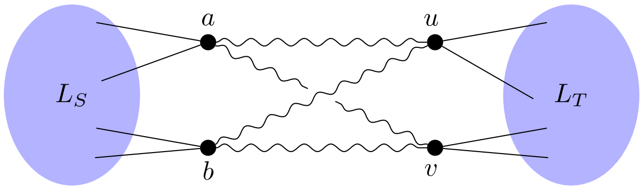

Figure 7: Four internally vertex-disjoint paths connect $\{a, b\}$ to $\{u, v\}$.

Contract all four of these paths into edges connecting $\{a, b\}$ to $\{u, v\}$ (see Figure 7). The graphs in Figure 6 are obstructions and one of them must be obtained via these contractions since $L_S^+ - ab$ and $L_T^+ - uv$ must contain $K_5 - e$ or $K_{3,3} - e$. Because the original graph is a minor-minimal apex obstruction, it must be one of these. $\square$

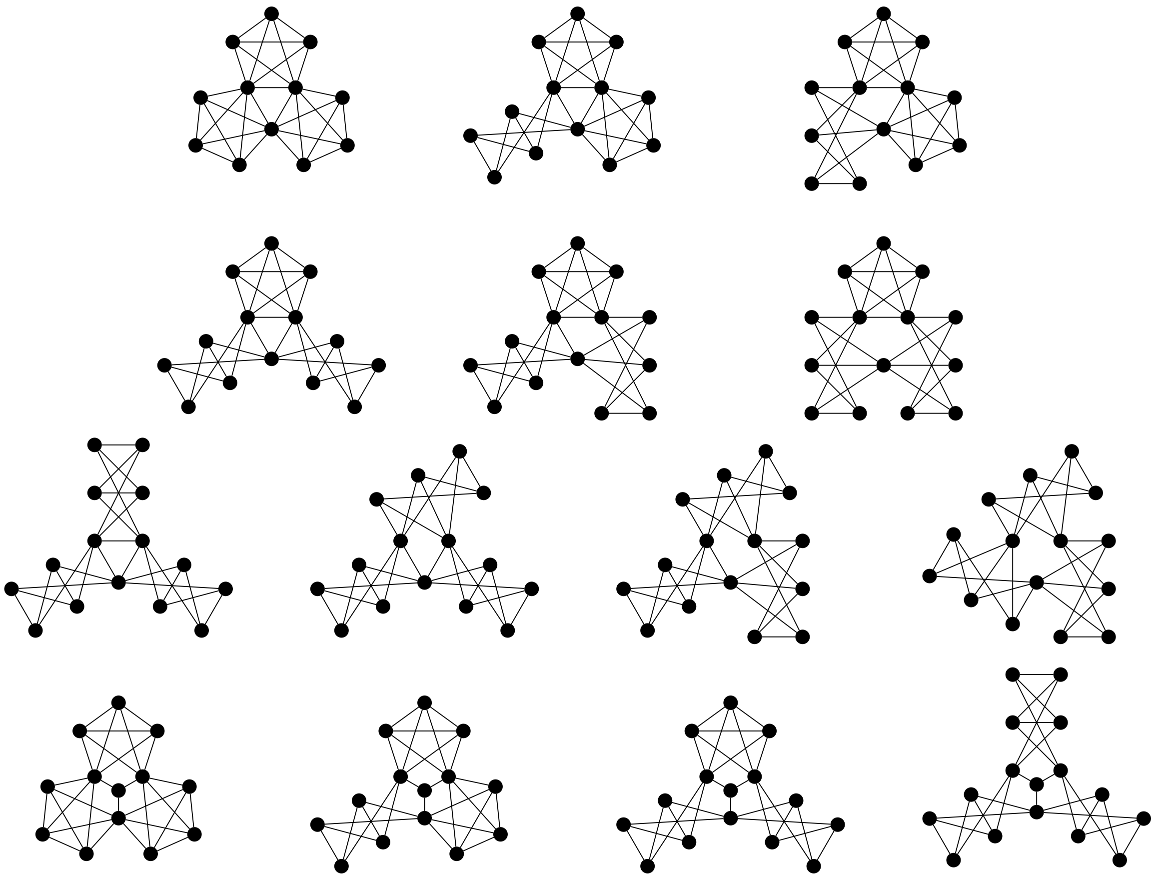

Figure 8: The 14 connectivity-2 apex obstructions with planar heavy components and at least three 2-cuts, but no disjoint 2-cuts.

7 The 37 connectivity-2 apex obstructions with planar heavy components and more than one, but no disjoint, 2-cuts

We next consider the remaining connectivity-2 apex obstructions with with planar heavy components, more than one 2-cut, and every pair of 2-cuts intersect. There are 37 of these obstructions, 14 of which have more than two 2-cuts (see Figure 8) and 23 of which have exactly two 2-cuts (see Figure 11).

Theorem 14. *Suppose $G \in \mathcal{F}$, $\kappa(G) = 2$, and every 2-cut has a planar heavy component. If G has more than one 2-cut and every pair of 2-cuts intersect, then G is isomorphic to a graph in Figure 8 or Figure 11.*

Proof. Suppose $S = \{a, b\}$ and $T = \{a, x\}$ are 2-cuts of G. By Lemma 10, all 2-cuts of G are basic. Let C_S and C_T be the heavy components of $G - S$ and $G - T$, respectively; let L_S and L_T be the corresponding light components. Lemma 9 guarantees that L_S^+ and L_T^+ are isomorphic to K_5, $K_{3,3}$, or $K_{3,3} + e$. If x were a vertex in L_S then $G - \{a, x\}$ would be connected because L_S^+ is 3-connected; so $x \notin V(L_S)$. Symmetrically $b \notin V(L_T)$. Because $G - T$ has two components and one component contains $L_S \cup \{b\}$, it follows that $L_S \cup \{b\} \subset C_T$ because T is basic. Symmetrically, $L_T \cup \{x\} \subset C_S$. Consequently $L_S \cap L_T = \varnothing$.

Case 1: $R = \{b, x\}$ is a 2-cut of G.

Let L_R denote the light component of $G - R$. Reasoning as in the paragraphs above, we find that L_R^+ is isomorphic to K_5, $K_{3,3}$, or $K_{3,3} + e$, and $L_R \cap L_S = \varnothing = L_R \cap L_T$. Consider H_a, H_b, and H_x. Note $b \in V(H_a)$ and $a \in V(H_b)$ because C_S is planar. Symmetrically $a \in V(H_x)$ and $x \in V(H_a)$ because C_T is planar; and $b \in V(H_x)$ and $x \in V(H_b)$ because C_R is planar. In this case, $H_x \cap L_T = \varnothing = H_x \cap L_R$ because $\{a, b\} \subset V(H_x)$. A symmetric statement applies to H_a and H_b.

Now let $P = V(G) - V(L_S) - V(L_T) - V(L_R) - \{a, b, x\}$.

Case 1a: $P = \varnothing$.

Because $H_x \cap L_T = \varnothing = H_x \cap L_R$ and $P = \varnothing$, we find $V(H_x) \subset V(L_S) \cup S$. This implies $G[V(L_S) \cup S] \cong L_S^+$. It follows that $G[V(L_S) \cup S]$ is isomorphic to one of K_5, $K_{3,3}$, or $K_{3,3} + e$. Symmetrically $G[V(L_T) \cup T]$ and $G[V(L_R) \cup R]$ are also isomorphic to one of these graphs. There are ten non-isomorphic graphs that result from these possibilities, and they each produce an obstruction, as shown in the top ten graphs of Figure 8.

Case 1b: $P \neq \varnothing$.

Let W be a component of P. Observe that W must have vertices adjacent to all three vertices a, b, and x since otherwise the removal from G of one of the 2-cuts R, S, or T would have more than two components, contradicting Lemma 7. Now consider the graph H obtained from G by contracting W to a vertex w. The vertex w has neighborhood $\{a, b, x\}$ in H.

If $ab, ax, bx \in E(G)$, then $H - w$ (hence $G - W$) has a minor isomorphic to one top ten graphs in Figure 8, a contradiction. Indeed if any one these three edges is in $E(G)$, then such a minor can be produced: for example, if $ab \in E(G)$, then contract H/wx. So we may assume $ab, ax, bx \notin E(G)$. We claim that $H[V(L_S) \cup S]$ is isomorphic to $K_5 - e$ or $K_{3,3} - e$. It suffices to prove that $H[V(L_S) \cup S] + ab \ncong K_{3,3} + e$. If $H[V(L_S) \cup S] + ab \cong K_{3,3} + e$, then the contraction H/wx again contains a minor isomorphic to one top ten graphs in Figure 8, a contradiction.

By symmetry we conclude that $H[L_S \cup S]$, $H[L_T \cup T]$ and $H[L_R \cup R]$ are all isomorphic to $K_5 - e$ or $K_{3,3} - e$. There are four possible graphs constructed using such components and they are shown at the bottom of Figure 8. Because these graphs are obstructions, it follows that H must be one of these graphs (and $P = W = \{w\}$); and $G \cong H$.

Case 2: $R = \{b, x\}$ is not a 2-cut of G.

As in Case 1, $b \in V(H_a)$ and $a \in V(H_b)$ since otherwise C_S is non-planar. Also $x \in V(H_a)$. Now let $P = V(G) - V(L_S) - V(L_T) - \{a, b, x\}$. Note that $V(H_a) \cap V(L_T) = \varnothing$ since $\{b, x\} \subset V(H_a)$ and $\{a, x\}$ separates b from L_T. Similarly, $V(H_a) \cap V(L_S) = \varnothing$. Consequently $V(H_a) \subseteq V(P) \cup \{b, x\}$ and $P \neq \varnothing$.

Every component of $G[P]$ has a neighbor of a; otherwise $\{b, x\}$ is a 2-cut (which is Case 1). So, there exists an edge $e = aw$ with w in P.

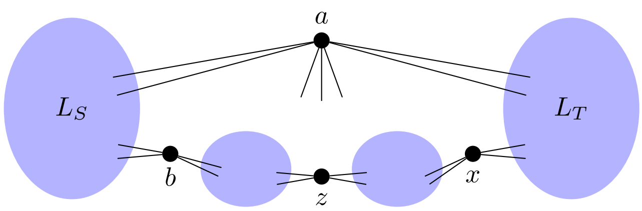

Figure 9: Augmentations of L_S and L_T to Kuratowski subgraphs.

We claim that $w \in V(H_a)$. If $w \notin V(H_a)$, then G/e must have an apex, say v, such that $v \in V(H_a) \subset P \cup \{b, x\}$. Note that $v \notin \{b, x\}$ because $G/e - b$ contains an augmentation of L_T with ax-path through $P - b$, and similarly $G/e - x$ contains an augmentation of L_S with ab-path through $P - x$. If $v \in P$ then v would separate b from x in H_a otherwise L_S can be completed to a Kuratowski subgraph in $G/e - v$ with a v-avoiding bx-path in $G[P \cup \{b, x\}]$ continuing to a through L_T. But H_a is 2-connected, so v can not separate b from x in H_a. So G/e has no apex, a contradiction.

Next we claim that $N(a) \cap (P \cup \{b, x\}) = \{w\}$. Suppose, on the contrary, there is a vertex $v \in P \cup \{b, x\}$ such that $v \neq w$ and $f = av \in E(G)$. Consider an apex u for $G - f$. Notice that $u \in V(H_a) \subseteq P \cup \{b, x\}$. Now $u \neq x$ because a Kuratowski subgraph exists by augmenting $G[V(L_S) \cup S]$ with an ab-path from a to w through P to b. Similarly, $u \neq b$; so $u \in P$. Because $u \notin \{b, x\}$, u must separate b from x in $G - f$; otherwise L_S can be completed to a Kuratowski subgraph in $G - f - u$ with a u-avoiding bx-path continuing to a through L_T. However, b and x are in H_a which is a subgraph $G[P \cup \{b, x\}]$. Because H_a is 2-connected, u can not separate b from x, a contradiction. So $N(a) \cap (P \cup \{b, x\}) = \{w\}$ (see Figure 10).

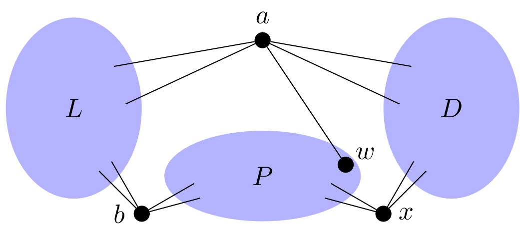

Figure 10: $N(a) \cap (P \cup \{b, x\}) = \{w\}$.

Similar reasoning as the last paragraph shows that every edge of $G[P \cup \{b, x\}]$ must be an edge of H_a; otherwise deleting this edge produces a graph with no apex.

Now suppose that there is an edge $f \in E(H_a)$ such that $|f \cap \{b, w, x\}| \leqslant 1$ and f is incident to a non-branch vertex of H_a. Consider G/f; it contains b, w, x. As before, b and x cannot be apex vertices for G/f. Furthermore, no vertex in $V(H_a)$ separates b from x, so they can not be apex vertices. Hence G/f has no apex, a contradiction. We conclude that every edge of H_a has branch vertices of H_a as endpoints or its endpoints are both in $\{b, w, x\}$.

We claim that every vertex of $P \cup \{b, x\}$ is a branch vertex of H_a. Because every edge of $G[P \cup \{b, x\}]$ is an edge of H_a and every edge of H_a has branch vertices of H_a as endpoints or its endpoints are both in $\{b, w, x\}$, it suffices to prove that b, w, and x are branch vertices of H_a.

If b is not a branch vertex of H_a, then the two neighbors of b in H_b must be w and x. Because every edge of $G[P \cup \{b, x\}]$ is an edge of H_a, it follows that b has only w and x as neighbors in $G[P \cup \{b, x\}]$. This implies $\{w, x\}$ is a 2-cut of G separating $P - \{w\}$ from b; this 2-cut is disjoint from $\{a, b\}$, contradicting that G has no disjoint 2-cuts. Hence b must be a branch vertex of H_a. By symmetry, x is also a branch vertex of H_a.

Apply similar reasoning to show w is a branch vertex. If w is not a branch vertex of H_a, then the two neighbor of w in H_a (and hence in $G[P \cup \{b, x\}]$) are b and x. This implies that $\{b, x\}$ is a 2-cut of G, (which is Case 1).

So every vertex of $P \cup \{b, x\}$ is a branch vertex of H_a. This means that there are 23 possible graphs shown in Figure 11. The three vertices in the center of these figures are, clockwise from the top of the three vertices, a, x, b. The light component, L_S, appears to the left, and L_T appears to the right. The unsubdivided Kuratowski subgraph H_a appear at the bottom of each graph drawing. $\square$

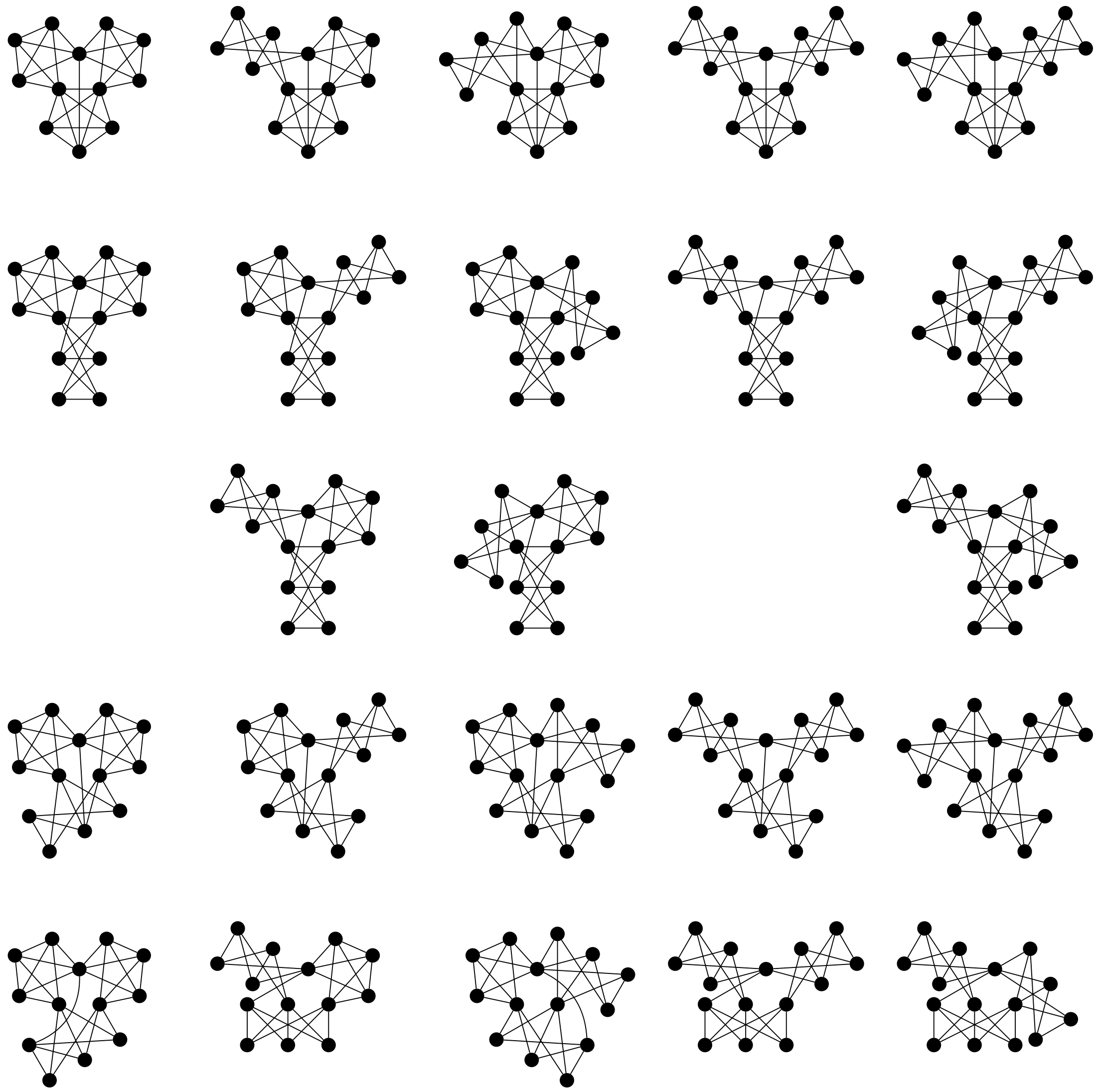

Figure 11: The 23 connectivity-2 apex obstructions with planar heavy components and exactly two 2-cuts that intersect.

8 The 72 connectivity-2 apex obstructions with a unique 2-cut having a planar heavy component

We next consider the final collection of connectivity-2 apex obstructions, the obstructions with a unique 2-cut whose removal produces a planar heavy component. These obstructions are considerably more difficult to characterize than the prior types, so we must present several structural results before tackling the final characterization.

In this section we let C denote the heavy component of an apex obstruction after deleting the unique 2-cut, with C representing either this subgraph or its vertex set, as convenient. Similar expedient recastings also apply to other subgraphs and their vertex sets, with context alleviating any ambiguities.

There are 72 apex obstructions determined in this section. Thirty-three of these obstructions (see Figure 61) have a heavy component C and a unique 2-cut $\{a, b\}$ such that $G[C \cup \{a, b\}]$ has a 2-cut separating a from b; the remaining thirty-nine (see Figure 62) have no such 2-cut.

The critical intermediate goal is to prove that there exist two Kuratowski subgraphs whose branch vertices cover every vertex of the unique 2-cut and every vertex of the heavy component (Theorem 25). Adding the light-component Kuratowski subgraph to these two makes three Kuratowski subgraphs whose branch vertices cover the entire vertex set of the apex obstruction, a property that reduces the problem of characterizing all of these remaining obstructions to a very small number of cases which we complete in the final subsection.

Many preliminary lemmas are needed for this final analysis. To simplify the presentation, we adopt the following notation and assumptions in this section.

Assumptions 15. *Standard assumptions for this section:*

- *G is a minor-minimal non-apex graph.*
- *G has connectivity two and a unique 2-cut $S = \{a, b\}$.*
- *$G - S$ has two components, the heavy component C and the light component L.*
- *C is a planar graph.*

8.1 Some preliminary lemmas for the final case

This subsection presents several lemmas establishing basic structure.

Lemma 16. *Under Assumptions 15:*

(i) If H_a and H_b are any Kuratowski subgraphs of G avoiding a and b, respectively, then $b \in V(H_a)$ and $a \in V(H_b)$.

(ii) $ab \notin E(G)$.

(iii) If H_a and H_b are any Kuratowski subgraphs of G avoiding a and b, respectively, then

$$E(G[C \cup \{a, b\}]) \subseteq E(H_a \cup H_b).$$

(iv) Any 2-cut of $G[C \cup \{a, b\}]$ that separates a from b does not induce an edge.

Proof. (i) Because C is planar, any Kuratowski subgraph of $G[C \cup \{a, b\}]$ must contain at least one vertex from $\{a, b\}$. Consequently any Kuratowski subgraph avoiding a must contain b and vice versa.

(ii) If $e = ab$ were an edge of G, then $G - e$ would have an apex $z \in V(H_a) \cap V(H_b) \subseteq C$. Indeed z would have to separate a from b in C; otherwise an ab-path in $G - e$ could complete L to L^+. Consequently, either $\{a, z\}$ or $\{b, z\}$ would be another 2-cut in G.

(iii) By Theorem 11, $E(C) \subseteq E(H_a) \cup E(H_b)$. Since (i) states that $ab \notin E(G)$, to prove (iii) it suffices (by symmetry of a and b) to prove that every edge aw with $w \in V(C)$ is also an edge in H_a or H_b. We argue by contradiction.

Assume that there is an edge $f = aw$, with $w \in V(C)$ such that f is not in $E(H_a) \cup E(H_b)$. Consider $G - f$; it has an apex vertex z. Note that H_a and H_b are still intact in $G - f$, so $z \in V(H_a) \cap V(H_b) \subset C$; in particular, $z \notin \{a, b\}$. Now z must separate a from b in $G[C \cup \{a, b\}] - f$, otherwise an ab-path avoiding z and f could be used to augment L to a Kuratowski subgraph L^+, contradicting that $G - f - z$ is planar. Define A to be the set of vertices that are connected to a via a z-avoiding path in $G[C \cup \{a, b\}] - f$. Because $a \in V(H_b)$ it has at least two neighbors in H_b; at least one of these is not z. Consequently, $A \neq \varnothing$. Now observe that $\{a, z\}$ is 2-cut of G separating b from vertices in A, contradicting that G has a unique 2-cut.

(iv) Assume, to the contrary, that a 2-cut, $\{w, x\}$ of $G[C \cup \{a, b\}]$ separates a from b and $f = wx \in E(G)$. Partition the vertices of $C - \{w, x\}$ into two sets:

$$A = \{v \in C - \{w, x\} : \text{there is a } va\text{-path avoiding } \{w, x\} \text{ in } G[C \cup \{w, x, a\}]\}$$

and

$$B = \{v \in C - \{w, x\} : \text{there is a } vb\text{-path avoiding } \{w, x\} \text{ in } G[C \cup \{w, x, b\}]\}$$

(see Figure 12).

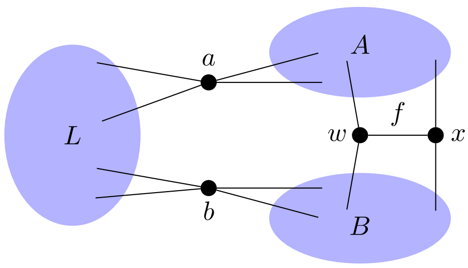

Figure 12: $V(H_b) \subseteq A \cup \{a, w, x\}$ and $V(H_a) \subseteq B \cup \{b, w, x\}$.

Let H_a and H_b be minimum-sized Kuratowski subgraphs of G avoiding a and b, respectively. Because $V(H_b) \subset C \cup \{a\}$ and the branch vertices of H_b must all be on one side of $\{w, x\}$, it follows from $wx \in E(G)$ and the minimum size of H_b that $V(H_b) \subseteq A \cup \{a, w, x\}$. A similar argument shows $V(H_a) \subseteq B \cup \{b, w, x\}$. In particular, $A \neq \varnothing \neq B$.

Observe that w must have neighbors in A and B since $\{a, x\}$ and $\{b, x\}$ are not 2-cuts of G; similarly x must have neighbors in A and B. Note too that there must be a wx-path in $G[A \cup \{w, x\}] - f$; otherwise $\{a, w\}$ or $\{a, x\}$ would be another 2-cut of G. Symmetrically there must be a wx-path in $G[B \cup \{w, x\}] - f$. Finally there must be two vertex-disjoint paths from a to $\{w, x\}$ in $G[A \cup \{a, w, x\}]$ since $\{a, b\}$ is the only 2-cut of G. Similarly, there must be two vertex-disjoint paths from b to $\{w, x\}$ in $G[B \cup \{b, w, x\}]$.

Now we claim that $G - f$ has no apex. Suppose, to the contrary, that y is an apex for $G - f$. If $y \in A \cup \{w, x\}$, then there remains an a to $\{w, x\}$-path that can be completed to an ab-path in $G[C \cup \{a, b\}] - f - y$, thus augmenting L to L^+ in $G - f - y$. A similar argument applies if $y \in B \cup \{w, x\}$. So it remains to consider $y \in \{a, b\} \cup L$. If $y \in \{a\} \cup L$, then consider H_a in $G - f - y$. Recall that $V(H_a) \subseteq B \cup \{b, w, x\}$, so H_a is only possibly missing the edge f in $G - f - y$. Now a wx-path whose internal vertices are entirely in A can substitute for the edge f in H_a. Hence no vertex in $\{a, b\} \cup L$ is an apex for $G - f$. A similar argument show that $y = b$ is not an apex in $G - f$. Hence $G - f$ has no apex, a contradiction. □

The next lemma embodies fundamental arguments to which we shall often appeal. The lemma may be interpreted as stating that $J = G[C \cup \{a, b\}]$ is a minor-minimal double apex graph with roots a and b (see Theorem 27).

Lemma 17. *Assume Assumptions 15 and $J = G[C \cup \{a, b\}]$. For any edge $e \in E(J)$,*

(i) $J - e - a$ is planar or $J - e - b$ is planar (or both), and

(ii) $J/e - a$ is planar or $J/e - b$ is planar (or both).

In part (ii) the notation means that if e is incident to vertex a (resp. b) in J, then e is contracted in J/e to form a new vertex also labeled a (resp. b).

Proof. (i) Assume, to the contrary, that $J - e - a$ and $J - e - b$ are both non-planar. Consider Kuratowski subgraphs H_a and H_b in $J - e$ avoiding a and b, respectively. Now observe that $e \notin E(H_a) \cup E(H_b)$, contradicting Lemma 16 part (iii).

(ii) Assume, to the contrary, that $J/e - a$ and $J/e - b$ are both non-planar. Consider G/e; it has an apex z. Because $J/e - a$ and $J/e - b$ are both non-planar, $z \notin \{a, b\}$. Now z must separate a from b in J/e, otherwise a z-avoiding path in G/e could complete L to L^+, contradicting that z is an apex for G/e. Because $\{a, z\}$ and $\{b, z\}$ are not 2-cuts of G, it follows that z is the contracted vertex created by contracting the edge e. Consequently the vertices in e form a 2-cut in J separating a from b, contradicting Lemma 16 part (iv). □

The next lemma gives information about edges shared by Kuratowski subgraphs missing a and b.

Lemma 18. *Assume Assumptions 15. Suppose H_a and H_b are any Kuratowski subgraphs avoiding a and b respectively. If $uv = e \in E(H_a) \cap E(H_b)$, then*

(i) u and v are both branch vertices of H_a, or

(ii) u and v are both branch vertices of H_b,

or both.

Proof. Let $J = G[C \cup \{a, b\}]$. If u is a branch vertex of neither H_a nor H_b, then H_a and H_b remain non-planar after contracting e. Consequently $J/e - a$ and $J/e - b$ are both non-planar, contradicting Lemma 17 part (ii). So u is a branch vertex of H_a or H_b; similarly v is a branch vertex of H_a or H_b. If u or v is a branch vertex of both, then

(i) or (ii) follows. So we may assume, to the contrary, that u is a branch vertex of H_a but not a branch vertex of H_b and, symmetrically, v is a branch vertex of H_b but not a branch vertex of H_a. In this case, again H_a and H_b remain non-planar after contracting e so $J/e - a$ and $J/e - b$ are both non-planar, again contradicting Lemma 17 part (ii). $\square$

The next lemma give a powerful tool to prove that vertices are branch vertices of Kuratowski subgraphs avoiding a and b.

Lemma 19. *Assume Assumptions 15. Suppose H_a and H_b are any Kuratowski subgraphs avoiding a and b respectively.*

(i) If $v \in V(H_a) - (V(H_b) \cup \{a, b\})$, then v and all of its neighbors are branch vertices of H_a.

(ii) If $v \in V(H_b) - (V(H_a) \cup \{a, b\})$, then v and all of its neighbors are branch vertices of H_b.

Proof. By symmetry it suffices to prove (i). Let $v \in V(H_a) - (V(H_b) \cup \{a, b\})$. Assume, to the contrary, that w is v or a neighbor of v but w is not a branch vertex of H_a. If $w = v$, then because w has only degree two in H_a, there is an edge incident to w (since $\delta(G) \geqslant 3$, Lemma 1) that is not in $E(H_a) \cup E(H_b)$, contradicting Lemma 16 part (iii). So $w \neq v$. Let $e = wv$ and $J = G[C \cup \{a, b\}]$.

Because w is not branch of H_a, the Kuratowski H_a survives in J/e. Also, H_b survives in J/e because $v \notin V(H_b)$. Consequently $J/e - a$ and $J/e - b$ are non-planar, contradicting Lemma 17 part (ii). $\square$

8.2 Vertices in the unique 2-cut must be branch vertices

The main goal in this subsection is to prove that vertex a must be a branch vertex of any Kuratowski subgraph avoiding b and, symmetrically, b must be a branch vertex of any Kuratowski subgraph avoiding a (statement of Corollary 22). To simplify statements in this subsection we focus just on proving the latter claim since the former claim follows from symmetry. So the reader should keep in mind the symmetrical consequences of the results that follow.

First we focus on the connectivity of C.

Lemma 20. *Assume Assumptions 15 and H_a is any Kuratowski subgraph avoiding a. If b is not a branch vertex of H_a, then C is 2-connected.*

Proof. Let H_b be any Kuratowski subgraph missing b. If b is not a branch vertex of H_a, then it has exactly two neighbors in H_a. Since all edges of $G[C \cup \{a, b\}]$ must be covered by $E(H_a) \cup E(H_b)$ (Lemma 16 part (iii)), it follows that $N(b) \cap C = \{x, y\}$, for some $x, y \in C$. Suppose, to the contrary, that c is a cut vertex of C. If $c = x$, $c = y$, or x and y are in the same component of $C - c$, then $\{a, c\}$ is another 2-cut of G, a contradiction. Therefore c separates x from y in C. In $G[C \cup \{b\}]$, the set $\{b, c\}$ is a 2-cut so all branch vertices of H_a must be on one side of this 2-cut. If one side of this 2-cut contains no branch vertices of either H_a or H_b, then (since each side is non-empty) there is an edge e incident to c that can be contracted that preserves H_a and H_b. Consequently, $G[C \cup \{a, b\}]/e - a$ and $G[C \cup \{a, b\}]/e - b$ are non-planar, contradicting Lemma 17 part (ii). So, each side of $G[C \cup \{b\}] - c$ has branch vertices. That is, the branch of H_a are all in one side, and the branch vertices of H_b, except possibly a, are on the other side. Consequently, the vertex a can have at most one neighbor, call it z, in one of the sides, which without loss of generality, contains y. If there is a cz-path avoiding y, then by is an edge that can be contracted preserving H_a and H_b; that is, $G[C \cup \{a, b\}]/by - a$ and $G[C \cup \{a, b\}]/by - b$ are non-planar, contradicting Lemma 17 part (ii). So every cz-path contains y. If $y = z$, then $\{c, y\}$ or $\{b, y\}$ is another 2-cut of G. If $y \neq z$, then $\{a, y\}$ is a 2-cut separating c from z. □

Now we are ready to prove a claim that plays a major role in the final characterization of the connectivity-2 apex obstructions. The claim states that, under the Assumptions 15, the vertices in the unique 2-cut, a and b, must be branch vertices of all of their Kuratowski witnesses. This is the statement of Corollary 22, which will be a consequence of Proposition 21. Though Proposition 21 proves a seemingly weaker existential claim, its significance is indicated by its long proof in which several 3-connected apex obstructions play a vital role, including the Petersen-family graphs M, Y^-, P_7 and the Petersen graph itself.

Proposition 21. *If Assumptions 15 are satisfied, then there exists a Kuratowski subgraph avoiding vertex a in which b is a branch vertex.*

Proof. Suppose, to the contrary, that b is not a branch vertex of any Kuratowski subgraph avoiding a. Let H_a be a Kuratowski subgraph of G avoiding a. Necessarily $b \in V(H_a)$ (Lemma 16 part (i)), so b has at least two neighbors in H_a. If $|N_G(b) \cap C| > 2$, then some edge in $G[C \cup \{b\}]$ incident to b does not appear in H_a or any Kuratowski subgraph of G avoiding b, contradicting Lemma 16 part (iii). Consequently we may assume $|N_G(b) \cap C| = 2$; say $N_G(b) \cap C = \{x, y\}$.

Fix a plane embedding of C. Let $K = V(H_a) - b$. The plane embedding of C includes a plane embedding of $G[K]$. Because $H_a - b$ is a subdivision of $K_5 - e$ or $K_{3,3} - e$, a plane embedding of $H_a - b$ is unique; consequently, a plane embedding of $G[K]$ is unique. Define A to be the component of $G[C \cup \{a\} - K]$ that contains the vertex a, and let $\overline{a}$ be the vertex obtained by contracting A in this graph to a single vertex.

Clearly $G[C \cup \{a\}]$ is not planar. It is possible that this graph becomes planar after contracting A to $\overline{a}$. We focus on $H_a - b$, a subgraph of $G[C \cup \{a\}]$. If it is possible to append $\overline{a}$ (and its incident edges to neighbors in $H_a - b$) to the plane embedding of $H_a - b$ without introducing a crossing in the plane (i.e. extend the plane embedding of $H_a - b$ to a plane embedding of $H_a - b + \overline{a}$), then we say that $\overline{a}$ *hits only one face*; otherwise $\overline{a}$ *hits multiple faces*.

The following steps provide contradictions that complete the proof of the proposition.

$\boxed{\text{Step 1: For any } H_a,\ \overline{a} \text{ hits only one face of the plane embedding of } H_a - b.}$

Suppose, on the contrary, there is a choice of H_a such that $\overline{a}$ hits multiple faces of the plane embedding of $H_a - b$. There are two cases according to whether $H_a - b$ is a subdivision of $K_5 - e$ or $K_{3,3} - e$.

Consider first the case in which $H_a - b$ is a subdivision of $K_5 - e$. The planar embedding of this subdivision is unique; it is shown in Figure 13.

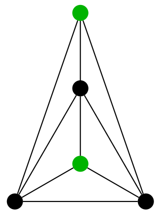

Figure 13: The case when $H_a - b$ is a subdivision of $K_5 - e$; neighbors of b are green.

Lemma 20 implies that $\overline{a}$ must have at least two neighbors in $H_a - b$. Suppose that $\overline{a}$ has only two such neighbors. There are seven non-isomorphic ways (see Figure 14) that $\overline{a}$ has exactly two neighbors in this subdivision of $K_5 - e$ so that the resulting graph is not planar.

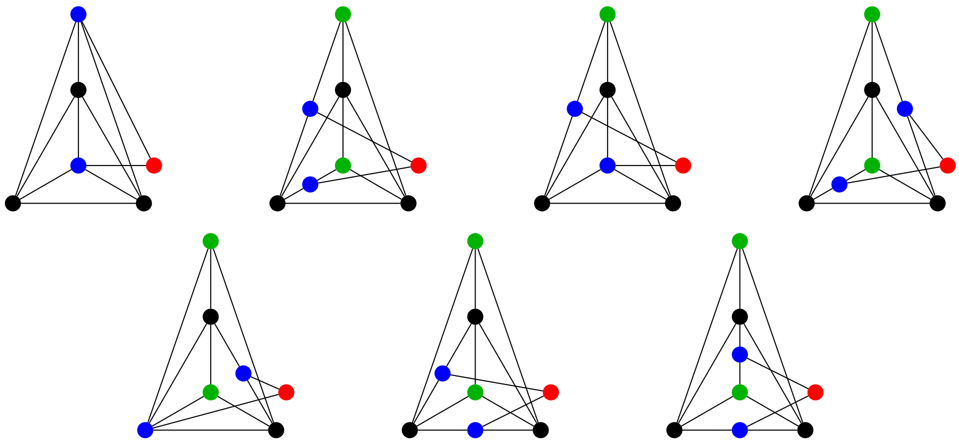

Figure 14: In the case $H_a - b$ is a subdivision of $K_5 - e$, there are seven ways that $\overline{a}$ has two neighbors in $H_a - b$ not both of which are in the same face.

The three right-most cases on the top row of Figure 14 all include neighbors of $\overline{a}$ (shown in blue) that can be contracted along edges of $H_a - b$ so that one is contracted to x and the other is contracted to y. These contractions produce a proper minor of $J = G[C \cup \{a, b\}]$ in which the contracted $H_a - b$, still a subdivision of $K_5 - e$, extends to a subdivision of K_5 by adding either b or $\overline{a}$. This contradicts Lemma 17 part (ii), so these cases cannot occur.

The left-most case on the top row of Figure 14 implies that $\{x, y\}$ is another 2-cut of G, a contradiction. As an aside, this last configuration actually has disjoint 2-cuts, $\{x, y\}$ and $\{a, b\}$, showing that this case would produce obstructions shown in Figure 6.

The cases shown on the bottom row of Figure 14 can be dismissed as follows. Consider adding the vertex b to each graph along with the edge $\overline{a}b$ (which corresponds to contracting L in G to the edge ab). Figure 15 shows the resulting graphs (where now vertex b is shown in red). The leftmost graph of Figure 15 is a 3-connected apex obstruction from the Petersen family; it is usually called M. Because M cannot appear as a proper minor of G, this case does not occur. The other two graphs of Figure 15 contain M as a proper minor, so those cases too cannot occur.

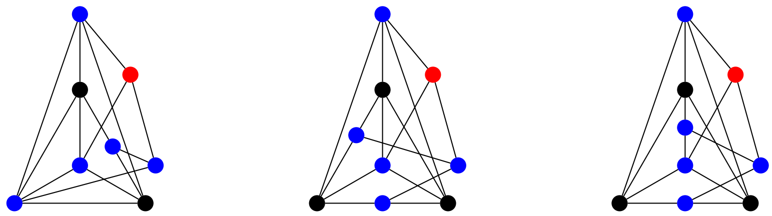

Figure 15: The leftmost graph is M, a Petersen family apex obstruction. The other two graphs properly contain M as a minor. The vertex b is shown in red.

To complete the analysis of the case in which $H_a - b$ is a subdivision of $K_5 - e$, suppose now that $\overline{a}$ has at least three neighbors in $H_a - b$. Observe that $\overline{a}$ cannot have two neighbors that attach as in Figure 14 since each of these cases produces a contradiction to Lemma 17 (parts (i) and (ii)) or a minor of M. Consequently, we may assume that the three neighbors of $\overline{a}$ are not all on one face, but any two of them are on a single face. There are four non-isomorphic ways to select such neighbors for $\overline{a}$ (see Figure 16).

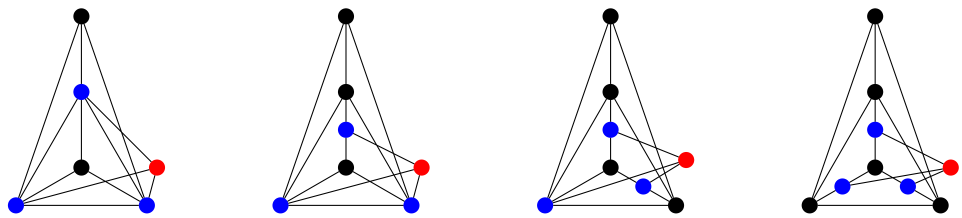

Figure 16: The four non-isomorphic ways that $\overline{a}$ (shown in red) has three neighbors (shown in blue) in $H_a - b$ (which is a subdivision of $K_5 - e$) so that all three neighbors are not on the same face, but any two are on a single face.

One way is to choose three neighbors of $\overline{a}$ to be the three degree four vertices of $H_a - b$. The remaining three cases can be contracted to this case. Now adding to this graph the vertex b and the edge $\overline{a}b$, as before, produces the graph shown in Figure 17. It is an apex obstruction from the Petersen family; it is called Y^-. It cannot occur as a minor of G, so this completes the analysis of Step 1 when $H_a - b$ is a subdivision of $K_5 - e$

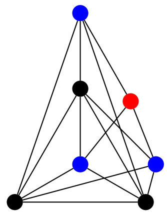

Figure 17: The graph Y^{-}, a Petersen family apex obstruction. Vertex b is shown in red.

Consider next the case in which $H_a - b$ is a subdivision of $K_{3,3} - e$. The planar embedding of this subdivision is unique; it is shown in Figure 18.

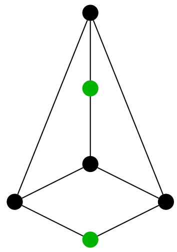

Figure 18: The case when $H_a - b$ is a subdivision of $K_{3,3} - e$; neighbors of b are green.

The analysis now follows along similar reasoning as the case in which $H_a - b$ is a subdivision of $K_5 - e$. Suppose that $\overline{a}$ has only two neighbors in $H_a - b$. There are four non-isomorphic ways (see Figure 19) that $\overline{a}$ has exactly two neighbors in this subdivision of $K_{3,3} - e$ so that the resulting graph is not planar.

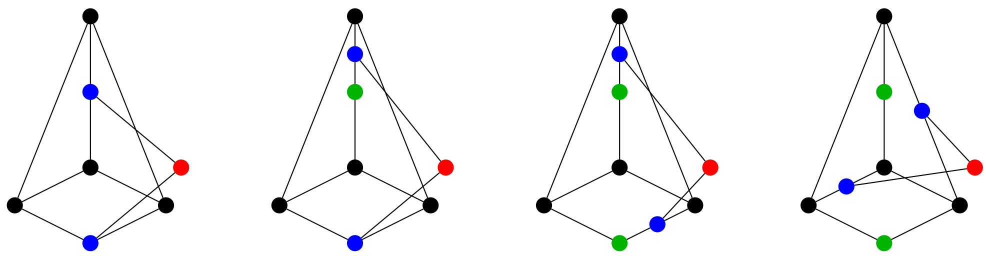

Figure 19: In the case $H_a - b$ is a subdivision of $K_{3,3} - e$, there are four ways that $\overline{a}$ has two neighbors in $H_a - b$ not both of which are in the same face.

In the left three graphs of Figure 19 one can reason, as before in the $K_5 - e$ case, that either $\{x, y\}$ is another 2-cut of G or one neighbor of $\overline{a}$ can be contracted to x and the other to y so that a contradiction to Lemma 17 part (ii) occurs. So $\overline{a}$ has no two neighbors of this type, leaving only the exclusion of the graph at the right of Figure 19. This graph extends with b and the edge $\overline{a}b$ to the graph shown in Figure 20. This graph is isomorphic to the Petersen graph, an apex obstruction, so cannot be a proper minor of G.

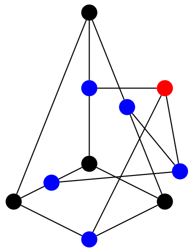

Figure 20: In the case $H_a - b$ is a subdivision of $K_{3,3} - e$, a graph isomorphic to the Petersen graph emerges. Vertex b is red.

So it remains to consider the case in which $\overline{a}$ has three (or more) neighbors in $H_a - b$ such that, not all are in a single face, but any two of them are on a single face. There are ten non-isomorphic ways $\overline{a}$ has three such neighbors (see Figure 21).

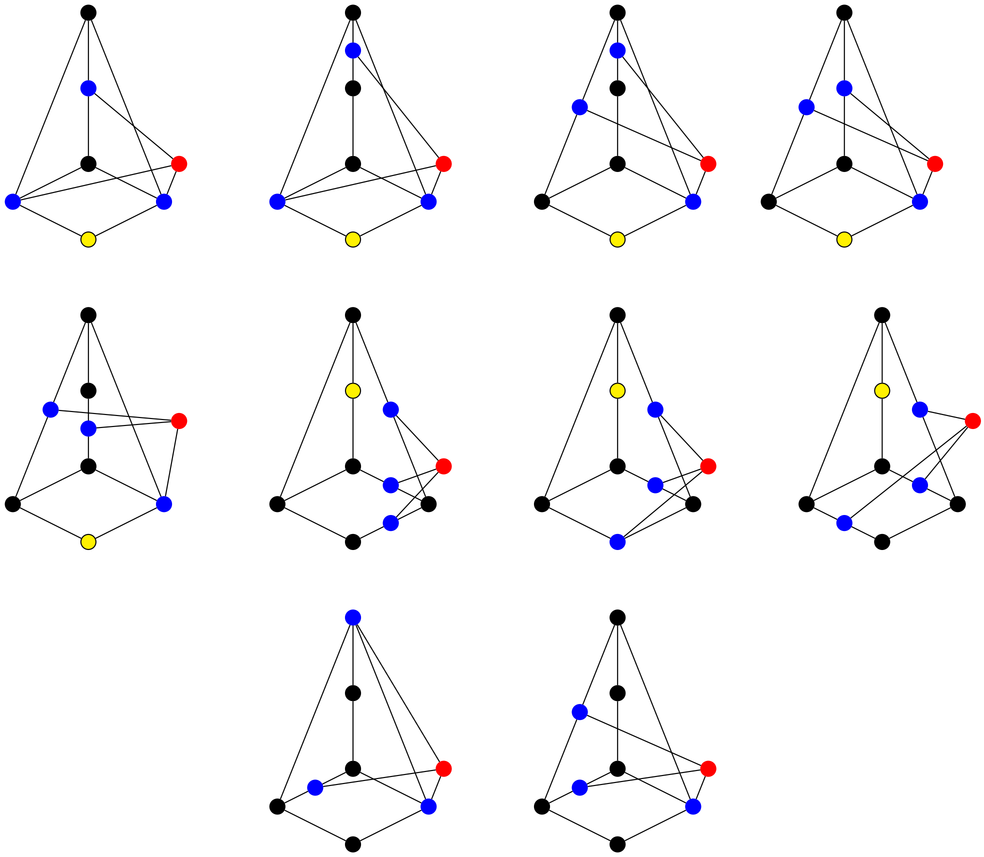

Figure 21: The ten non-isomorphic ways $\overline{a}$ (shown in red) has three neighbors in $H_a - b$ (which is a subdivision of $K_{3,3} - e$) so that all three neighbors (shown in blue) are not on the same face, but any two are on a single face.

In the eight graphs in the top two rows of Figure 21 a vertex, either x or y, can be deleted and the remaining graph remain non-planar; in each case the deletable vertex is shown in yellow. Consequently in these graphs either bx or by, according to whether x or y is deletable, is an edge that contradicts Lemma 17 (part (ii)). So these eight graphs cannot occur.

The bottom right graph of Figure 21 can be contracted to the bottom left graph, so it remains to dismiss the bottom left graph. As before, add vertex b and the edge $\overline{a}b$ to this graph produces the graph shown in Figure 22. This graph is a graph isomorphic to the graph P_7, a Petersen-family minor. Because it is an apex obstruction it cannot occur as a proper minor of G. This completes the proof of Step 1.

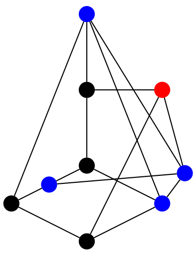

Figure 22: In the case $H_a - b$ is a subdivision of $K_{3,3} - e$, a graph isomorphic to the graph P_7, a Petersen family apex obstruction, emerges. Vertex b is red.

By Step 1, we may assume that $\overline{a}$ has neighbors in only one face of the plane embedding of $H_a - b$. Let F be a face of the plane embedding of $H_a - b$ such that $\overline{a}$ has only neighbors in it. There could be two such faces, a technicality addressed later (see Step 5). Without loss of generality, the face F is a face not incident to y (see Figure 23). Recall that $K = V(H_a) - b$ and A is the component of $G[C \cup \{a\} - K]$ that contains the vertex a. Let $int(F)$ denote all of the vertices of $C - K$ that appear in the interior of F. More precisely, $int(F)$ consists of the vertices in the interior of the region of the plane avoiding y that is bounded by F in our fixed plane embedding of C.

Choose H_a that minimizes the number of vertices that are on the boundary or the interior of F. We may also assume that embedding has been fixed to minimize the number of crossings produced when a is reinserted into the face F.

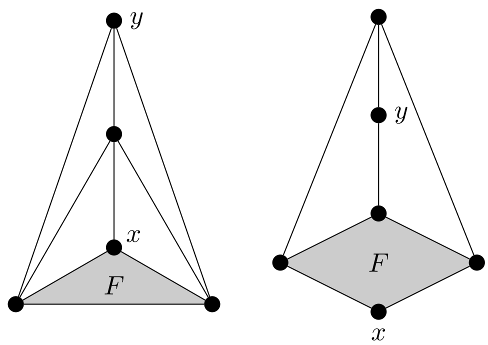

Figure 23: Whether $H_a - b$ is a subdivision of $K_5 - e$ or $K_{3,3} - e$, we may assume F is a face that is not incident to y in the unique plane embedding of $H_a - b$.

Define $C_1, \ldots, C_t$ to be the components of $G[C \cup \{a\} - K]$ that intersect $int(F) \cup \{a\}$. Note that A is one of these components; we assume $C_1 = A$. Observe that $|N(C_i) \cap F| \geqslant 3$, for $i > 1$; otherwise there would be another 2-cut in G. This argument does not apply to the component C_1 because it contains the vertex a which has neighbors in L. However, Lemma 20 implies that $\overline{a}$ must have at least two neighbors in $H_a - b$; hence C_1 must have at least two neighbors on F.

Define S to be the set of vertices on F with at least one neighbor in $int(F) \cup \{a\}$. Note that $|S| \geqslant 2$ since if $S = \{v\}$, then $\{b, v\}$ is another 2-cut of G. An important observation is that, by Lemma 19, all the vertices in $int(F)$ and their neighbors on F are all branch vertices of H_b. So all vertices in S are branch vertices of H_b, except possibly vertices whose only neighbor in $int(F) \cup \{a\}$ is a.

Step 2: $|C_i| = 1$, for all $i = 1, \ldots, t$.

Assume, to the contrary, that $|C_i| > 1$ for some $i = 1, \ldots, t$. Let $u \in C_i - \{a\}$ and $v \in C_i$ be chosen so that uv is an edge. Let $e = uv$. Lemma 19 implies that all of the vertices in C_i and the neighbors of u must be branch vertices of H_b.

Recall that $J = G[C \cup \{a, b\}]$. It suffices to show that there exists a Kuratowski minor in $G[C \cup \{a\}]$ avoiding y, since then yb can be contracted so that $J/yb - a$ and $J/yb - b$ are both non-planar, contradicting Lemma 17 part (ii). Consider J/e and let w be the composite vertex in J/e that results from identifying u and v. Because H_a is untouched by the contraction of e, Lemma 17 part (ii) implies that $J/e - b$ is planar. Because $H_a - b$ has a unique planar embedding and it is a subgraph of $J/e - b$, the cycle F must separate y from w in any plane embedding of $J/e - b$. In particular, the faces containing w do not contain y. Let W be the plane graph formed by the union of the faces containing w in a plane embedding of $J/e - b$. Now it suffices to find a Kuratowski minor in the graph obtained from W by splitting w back into u and v.

Case 1: u and v have a common neighbor.

Let w_1 be the common neighbor of u and v. Because w_1 is also a branch vertex of H_b and all edges of C are covered by $E(H_a) \cup E(H_b)$, it follows that all edges of the triangle $G[\{u, v, w_1\}]$ are in H_b; so H_b is a subdivision of K_5. In this case, u and v must have degree exactly four in $G[C \cup \{a\}]$. Suppose $N(u) = \{v, w_1, w_2, w_3\}$. Note that u and its four neighbors, are the five branch vertices of H_b.

Suppose that $\{w_1, w_2, w_3\} \subset N_J(v)$. This implies $N_J(u) - \{v\} = N_J(v) - \{u\} = \{w_1, w_2, w_3\}$. Splitting w back into u and v in W produces a K_5 subdivision in $G[C \cup \{a\}]$ avoiding y (see Figure 24.

If $\{w_2, w_3\} \not\subset N(v)$, then $v = a$ (because $v \neq a$ implies that all of v's neighbors are branch vertices). Without loss of generality, w_3 is not a neighbor of v.

The vertices w_1, w_2, and w_3 partition the cycle $W - w$ into into three segments. Splitting w back into u and v must produce a crossing since $G[C \cup \{a\}]$ is non-planar. In particular, splitting w back into u and v produces a Kuratowski subdivision or minor in $G[C \cup \{a\}]$ avoiding y in each of the remaining cases:

(i) v has a neighbor in the segment between w_2 and w_3 ($K_{3,3}$ subdivision),

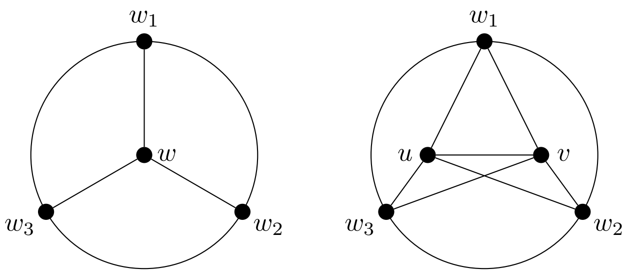

Figure 24: The faces of the plane embedding of $J/e - b$ containing w on left. On the right, a subdivision of K_5 is found in $J - \{b, y\}$ if $\{w_1, w_2, w_3\} \subset N(u) \cap N(v)$.

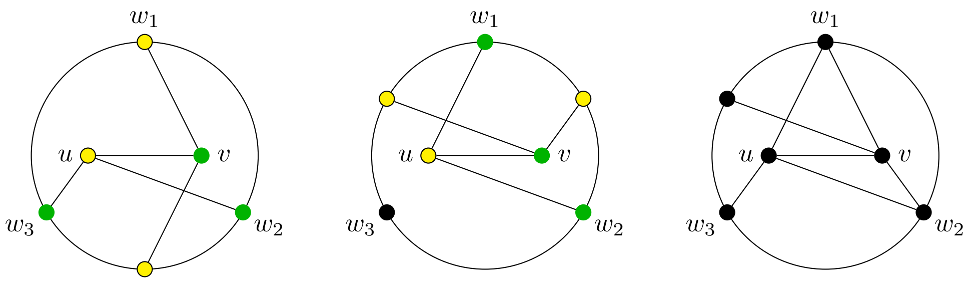

Figure 25: Splitting w back into u and v produces a Kuratowski minor in $G[C \cup \{a\}]$ avoiding y. Cases (i), (ii), and (iii) in this order from left to right.

(ii) v has neighbors in the segments between w_1 and w_3 and between w_2 and w_3 ($K_{3,3}$ subdivision), and

(iii) v is adjacent to w_1 and w_2 so has a neighbor between w_1 and w_3 (K_5 minor).

See Figure 25 which gives drawings for cases 1), 2), and 3) in this order from left to right.

Case 2: u and v have no common neighbor.

Suppose first that $v \neq a$. Under this supposition, all of the neighbors of v are also branch vertices of H_b (Lemma 19). Also the degree of u and v are at least three, so u and v have four different neighbors implying that H_b is a subdivision of $K_{3,3}$. It follows that u and v both have degree three in $G[C \cup \{a\}]$ and have no common neighbors. Set $N(u) - \{v\} = \{\alpha, \beta\}$ and $N(v) - \{u\} = \{\gamma, \delta\}$. In W the neighbors of w (the four vertices $\alpha, \beta, \gamma, \delta$) must appear along the face created by deleting w so that the α and β are not consecutive (see left of Figure 26), otherwise the plane embedding of $J/e - b$ could be extended to $J - b$, contradicting that $G[C \cup \{a\}]$ is non-planar. This produces a subdivision of $K_{3,3}$ that does not contain y (see right of Figure 26).

So $v = a$. We may assume that, except for v, all of the neighbors of u are on F (otherwise we could substitute one of them for v instead).

If C_1 contains three vertices, say $C_1 = \{u, v, z\}$ then, because $u \neq a \neq z$, the closed neighborhoods of u and z are all branch vertices of H_b (Lemma 19). Because all of the neighbors of u are on F, u and z are not adjacent so H_b is a subdivision of $K_{3,3}$. All edges of C are covered by $E(H_a) \cup E(H_b)$, so u and z must have exactly three neighbors. Let w_1 and w_2 be the two neighbors of u on F. This implies $N_J(u) = \{v, w_1, w_2\} = N_J(z)$. Let

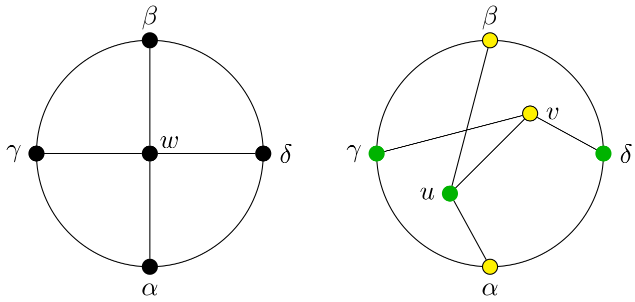

Figure 26: The faces of the plane embedding of $J/e - b$ containing w on left. On the right, a subdivision of $K_{3,3}$ is found in $J - \{b, y\}$.

α be a neighbor of v on F. A subdivision of $K_{3,3}$ exists in $G[F \cup \{u, v, z\}]$ (with branch vertices $u, v, z, w_1, w_2, \alpha$). Because this subdivision of $K_{3,3}$ avoids y, we may assume that $C_1 = \{u, v\}$.

If H_b is a subdivision of K_5, then u has three neighbors w_1, w_2,and w_3 on F. Because u and v have no common neighbors and the neighbors of v must be on F, at least two of the neighbors of v must be in F in different interior faces, otherwise this planar embedding of C could be extended to a planar embedding of $G[C \cup \{a\}]$. This implies a $K_{3,3}$ subdivision avoiding y, a contradiction (see left and center panel of Figure 25).

If H_b is a subdivision of $K_{3,3}$, then u and v have degree three but no common neighbors in $G[C \cup \{a\}]$. In this case u has two neighbors w_1, w_2 on F. The two neighbors of v on F must appear in different interior faces, otherwise this planar embedding of C could be extended to a planar embedding of $G[C \cup \{a\}]$. This implies a $K_{3,3}$ subdivision avoiding y, a contradiction (see center panel of Figure 25).

$\boxed{\text{Step 3: } t \leqslant 2.}$

We shall prove Step 3 by contradiction: assume that $t > 2$. Step 2 implies C_2 and C_3 each contain exactly one vertex. These two vertices are nonadjacent branch vertex of H_b. The neighbors C_2 and C_3 on F must be branch vertices of H_b too. So H_b is necessarily a subdivision of $K_{3,3}$; C_1 and C_2 have the same three neighbors in S. Consequently, the graph $G[F \cup C_2 \cup C_3]$ contains a subdivision of $K_5 - e$ that has an embedding in the plane (inherited from the planar embedding of C) in which F contains the degree-four branch vertices of the subdivided $K_5 - e$ and the other branch vertices are in $int(F)$. This is a contradiction because $K_5 - e$ has a unique planar embedding in which the triangle connecting the degree-four vertices is a Jordan curve separating the degree-three vertices. So this proves Step 3 and we may assume that $t \leqslant 2$.

$\boxed{\text{Step 4: } t \leqslant 1.}$

For the rest of the proof we may assume, by Step 2, that $C_1 = \{a\}$. Assume, to the contrary, that $t = 2$. Let w denote the one vertex in C_2. Note that all of the vertices in the closed neighborhood of w are branch vertices of H_b (by Lemma 19) and, $d_G(w) = 3$ or $d_G(w) = 4$ (because $E(C) \subset E(H_a) \cup E(H_b)$ by Lemma 16 part (iii)).

Recall that $K = V(H_a) - b$. An edge in $G[K]$ that is not used by the subdivision H_a is called a *chord*. To the given fixed, plane embedding of C add the vertex a (and its incident edges), placing a inside F in general position so as to minimize the number of crossings. The next definition of crossing chords and related arguments are now with respect to this embedding of $G[C \cup \{a\}]$. A chord $e = uv$ is *crossing* if it is a chord in the face F (that produces a crossing involving a in the plane embedding of $H_a - b$ inherited from the plane embedding of C) and a has neighbors in different components of $F - \{u, v\}$. Recall the strong Tutte-Hannani theorem: a graph is planar if and only if it has an embedding so that no two vertex-disjoint edges cross an odd number of times. Applying the Tutte-Hannani to the fixed embedding of $G[C \cup \{a\}]$, it follows that there must be two disjoint edges that cross exactly once.

Lemma 1 yields $d_G(w) \geqslant 3$. Let w_1, w_2 be the two neighbors of w in S chosen consecutively in the circular ordering of the neighbors of w determined by the plane embedding of C so that a has been placed in the face, F_1, using the edges w_1w and ww_2. By the crossing-minimizing placement of a, there must be a neighbor of a in F_1, call it a_1 (possibly $a_1 \in \{w_1, w_2\}$). Because there is a crossing, there must be a neighbor of a outside of F_1, call it a_2. We may choose a_2 to be the first neighbor of a clockwise from w_2 (away from w_1) along F that is outside of F_1. Let w_3 be the next neighbor (clockwise) of w that is at least as far as a_2. If a has a neighbor in $F_1 - \{w_1, w_2\}$, then delete all edges incident to w except w_1w and ww_2 and then contract these two edges to produce a crossing chord in this minor of G. Otherwise we may assume all neighbors of a in F_1 are in $\{w_1, w_2\}$, in particular, we may assume $a_1 = w_1$. If $a_2 \neq w_3$, then contracting w_1w and ww_3 produces a crossing chord in a minor of G. So assume $\{a_1, a_2\} = \{w_1, w_3\}$. If $N_J(w) = \{w_1, w_2, w_3\}$ and $N_J(a) = \{w_1, w_3\}$, then a crossing-free drawing of $G[C \cup \{a\}]$ could be made by placing a in the face defined by the edges w_1w and ww_3. If w has another neighbor, w_4, then contracting w_2w and ww_4 produces a crossing chord in a minor of G. Otherwise $N_J(w) = \{w_1, w_2, w_3\}$, and all of a's neighbors are in the face determined by w_1w and ww_3.

So the only case in which we cannot create a minor with a crossing chord is when a and w are both branch of H_b, which is a subdivided $K_{3,3}$, and $N(a) \cap F = \{w_1, w_2, w_3\} = N(w) \cap F$.

If H_a is a subdivided K_5, then F has three branch vertices of H_a which partition F into three paths. If w_i, w_j with $1 \leqslant i < j \leqslant 3$ are on the same path of H_a, then rerouting this path through w_i, w_j and w produces another H_a that has smaller $|int(F)|$. So the only remaining case is if w_1, w_2, and w_3 are interior vertices on the three different paths of H_a on F. This is shown on the left panel of Figure 27.

Adding a (and its incident edges to the neighbors of w) produces graph (shown in right panel of Figure 27) which remains non-planar after deleting y, a contradiction.

Consider now the case in which H_a is a subdivided $K_{3,3}$. If w_i, w_j with $1 \leqslant i < j \leqslant 3$ are on the same path of H_a, then rerouting this path through w_i, w_j and w produces another H_a with smaller $|int(F)|$. So the only remaining case is if w_1, w_2, and w_3 are interior vertices on the three different paths of H_a on F. The non-isomorphic cases are shown on the left panel of Figure 28.

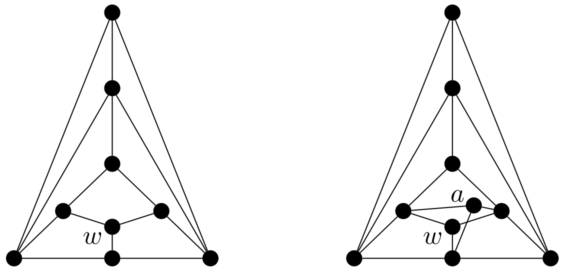

Figure 27: H_a is a subdivided K_5 and the neighbors of w are interior vertices on the three paths of H_a on F (left). Adding a, produces graph shown in center panel which remains non-planar after deleting y (top vertex).

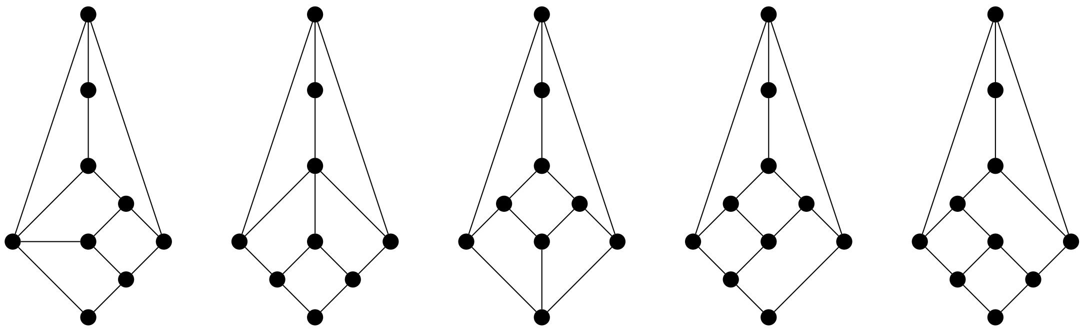

Figure 28: H_a is a subdivided $K_{3,3}$ and the neighbors of w are interior vertices on the three different paths of H_a on F. The non-isomorphic cases are shown here.

In each case it is possible to find a new subdivision of $K_{3,3}$ for H_a (in which x and y are branch vertices connected by a path through b) with smaller $|int(F)|$ as shown in Figure 29.

If a has neighbors in C that are all in a single path of $H_a - b$, then a technical difficulty arises: $\overline{a}$ indeed hits only one face (as Step 1 guarantees), but there are two choices for this one face — either of the two faces incident to this path (in the unique plane embedding of $H_a - b$). The next step excludes this possibility, resolving this difficulty.

A path in C whose internal vertices are disjoint from $H_a - b$ is called an *external* path.

STEP 5: The neighbors of a in C are not all on one path of $H_a - b$.

Assume, to the contrary, that the neighbors of a in C are all on one path of $H_a - b$. Because $t = 1$, every crossing pair of edges involves a crossing chord and an edge incident to a. Keep in mind that, by the minimality of H_a, the endpoints of any crossing chord cannot both be on the same path connecting branch vertices of H_a.

Let F_1 and F_2 be the two faces (in the unique plane embedding of $H_a - b$) incident to the path P (connecting branch vertices $H_a - b$) that contains all of the neighbors of a in C. It is very important to note that Steps 2-4 apply to both F_1 and F_2. In particular, this means that no vertices of C are in the interior of F_1 or F_2; so these faces of the plane embedding of $H_a - b$ are actually faces of the plane embedding of C. As in prior steps,

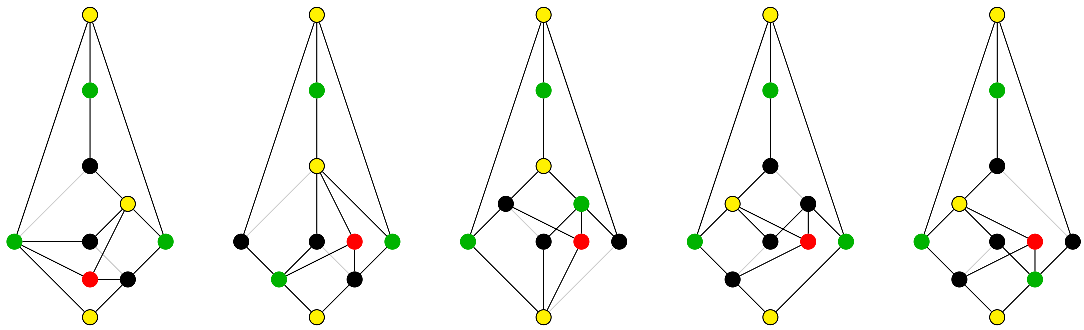

Figure 29: In each case a new subdivision of $K_{3,3}$ can be chosen for H_a with smaller $|int(F)|$. The branch vertices of the new subdivision of $K_{3,3}$ are shown in green and yellow. Vertex a is shown in red. The non-branch vertex b of this $K_{3,3}$ subdivision, which connects x and y, is not shown.

we now consider cases depending on whether $H_a - b$ is a subdivision of $K_5 - e$ or $K_{3,3} - e$.

First consider the easier case in which $H_a - b$ is a subdivision of $K_5 - e$. Suppose that P is a path ending at x. Let a_1 and a_2 be the extreme neighbors that a has on P; that is, a_1 is the furthest from x and a_2 is the closest to x. There must be a chord in F_1 that produces a crossing with a. This chord must have one endpoint between a_1 and a_2. The other endpoint cannot be on P by the minimality of H_a. There must be a chord in F_2 with similar properties. These chords are drawn in color in Figure 30. Regardless of where the other endpoints of these chords occur, contractions produce the graph on the right of Figure 30, a K_5 minor that avoids y, a contradiction.

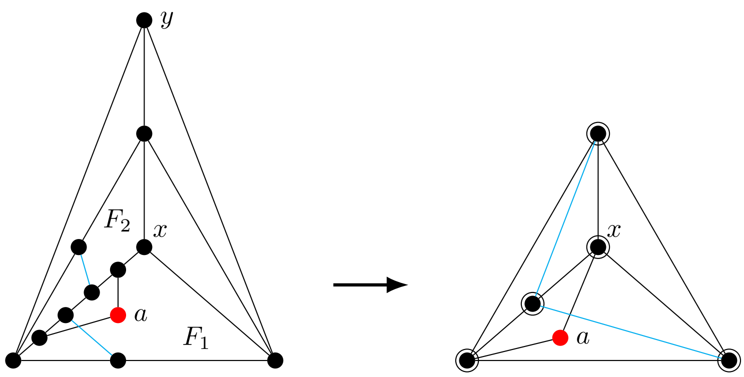

Figure 30: $H_a - b$ is a subdivision of $K_5 - e$ and a has neighbors in C only along a path of $H_a - b$ ending at x. A K_5 minor avoiding y appears.

So suppose that P is a path of $H_a - b$ that does not end at x. Again F_1 and F_2 have chords that produce crossing with a (see left of Figure 31). Regardless of where the other endpoints of these chords occur, contractions produce the graph shown in the center of Figure 31. Adding the vertex b and the edge ab (which corresponds to contracting the light component) produces the apex obstruction M as a minor of G, a contradiction.

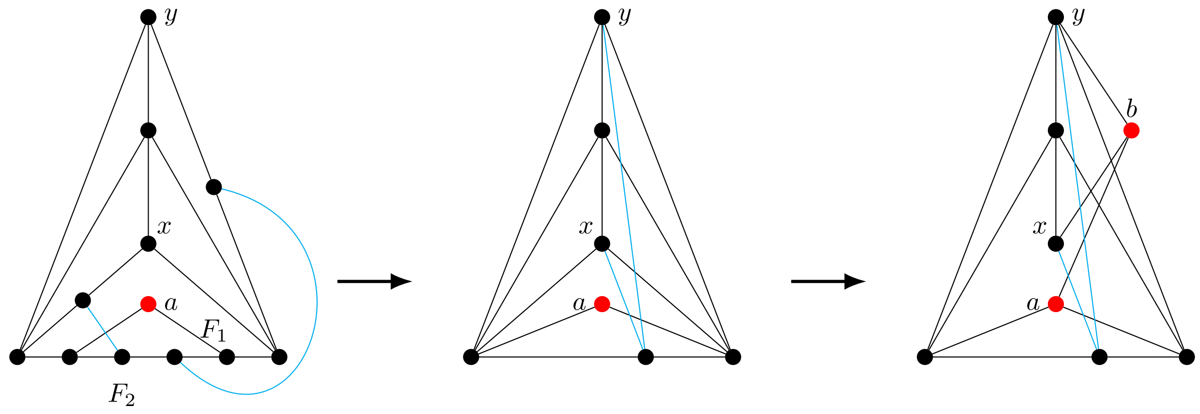

Figure 31: $H_a - b$ is a subdivision of $K_5 - e$ and a has neighbors in C only along a path of $H_a - b$ that does not end at x. An M minor appears after adding b and the edge ab (which corresponds to contracting the light component).

Next consider the case in which $H_a - b$ is a subdivision of $K_{3,3} - e$. Two subcases occur depending on whether the path P (containing all neighbors of a in C) is incident to x or y, or neither x nor y.

First consider the subcase in which P is incident to neither x nor y. Again define a_1 and a_2 to be the extreme neighbors of a along P; that is, the neighbors of a closest to the branch vertices of $H_a - b$ at the ends of P. As before, F_1 and F_2 must contain chords producing crossings with edges incident to a (see Figure 32) These chords must have ends in the interval of P between a_1 and a_2; call these ends α and β. The vertices at the other ends of these chords are of three possible types,

type (I): can be contracted to x or y along paths in $H_a - b$ (without passing through a branch vertex of $H_a - b$),

type (II): appear in the interior of the path in $H_a - b$ opposite x or y (along the face F_1 or F_2),

type (III): branch vertex of $H_a - b$ (adjacent to x or y in $H_a - b$).

First we argue that $\alpha = \beta$. If $\alpha \neq \beta$, then we claim that the edges along P can be contracted, preserving H_a, so that a Kuratowski avoiding b exists. This is a contradiction because then neither a nor b is an apex in $G[C \cup \{a, b\}]$ after contracting α to β. All nine cases (two end vertices each have independently any of three types) can be dismissed by examining graphs shown in Figure 33. All graphs in this figure have a_1 and a_2 contracted (if necessary) to the ends of the path P and α contracted to β. If both ends of the crossing chords are type (I), the leftmost graph shows that there is a minor of $K_{3,3}$ avoiding b. If the ends of the crossing chords are both type (III), the right graph shows that there is a minor of K_5 avoiding b. The right-most graph also works if one chord is of type (I) and the other is of type (III). If one end of a crossing chords is type (I) and other is type (II),

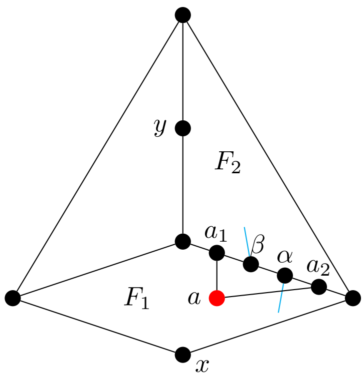

Figure 32: The case in which $H_a - b$ is a subdivision of $K_{3,3} - e$ and a has neighbors in C only along a path of $H_a - b$ that ends neither at x nor y.

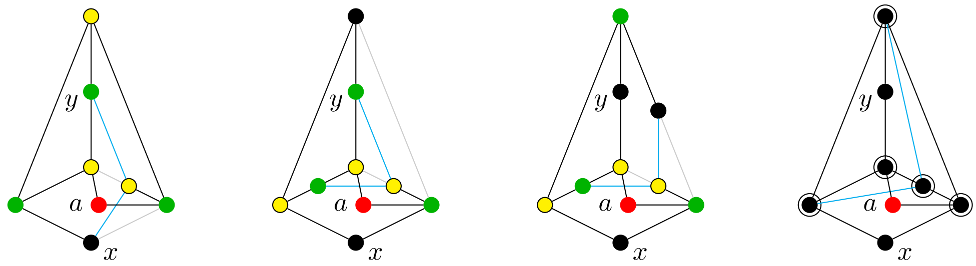

Figure 33: If $\alpha \neq \beta$, then edges along P can be contracted, preserving H_a, so that a Kuratowski subgraph avoiding b emerges.

the left-center graph shows that there is a minor of $K_{3,3}$ avoiding b. If one end is of type (II) and other is type (II) or (III), then the right-center graph of Figure 33 shows a minor of $K_{3,3}$ avoiding b.

So we may assume that $\alpha = \beta$.

If both chords have ends that are of type (I), then an edge e exists (shown dotted in Figure 34) such that $J - e - a$ and $J - e - b$ are non-planar, contradicting Lemma 17 part (i). This can be seen in Figure 34 where a subdivision of $K_{3,3}$ is found in both $J - e - a$ (new H_a), and $J - e - b$ (new H_b).

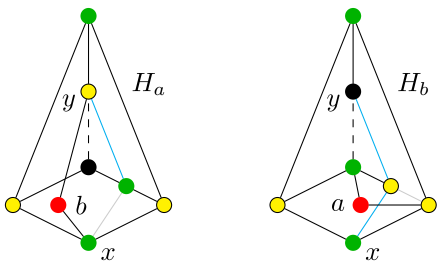

Figure 34: If $\alpha = \beta$ and both crossing chords have ends that are of type (I), then a choice of H_a and H_b are shown that avoid an edge of C (shown dotted).

If both chords have ends that are of type (III), then an edge e exists (shown in pink in Figure 35) such that $J/e - a$ and $J/e - b$ are non-planar, contradicting Lemma 17 part (ii). This figure also shows a contradiction if one end of a crossing chord has type (III).

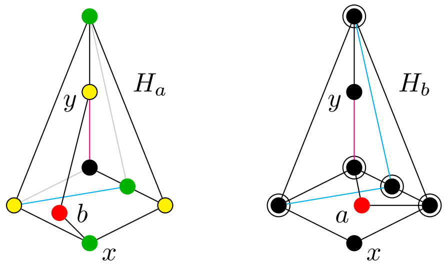

Figure 35: If $\alpha = \beta$ and one crossing chord has type (III), then a choice of H_a and H_b are shown that a share a contractible edge (shown in pink).

If the end of one crossing chord has type (I) and the other is of type (II), then an edge e exists (shown dotted in Figure 36) such that $J - e - a$ and $J - e - b$ are non-planar, contradicting Lemma 17 part (i).

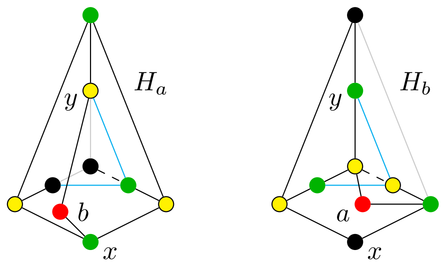

Figure 36: If $\alpha = \beta$ and one crossing chord has type (I) and the other has type (II), then a choice of H_a and H_b are shown that a share a deletable edge (shown dotted).

If both ends of the crossing chords have type (II), then an edge e exists (shown in pink in Figure 37) such that $J/e - a$ and $J/e - b$ are non-planar, contradicting Lemma 17 part (i).

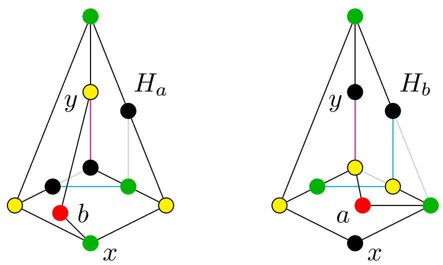

Figure 37: If $\alpha = \beta$ and both ends of the crossing chords have type (II), then a choice of H_a and H_b are shown that a share a contractible edge (shown in pink).

Finally consider the subcase in which a has neighbors only along the path P of $H_a - b$ and P is incident to x or y. Without loss of generality, the path P is incident to x. There are two possibilities, shown in Figure 38, according to whether a has neighbors on one side of x (along P) or both sides of x. Figure 38 introduces the labeling of the remaining branch vertices of H_a as p, q, r, s as well as the extreme neighbors, a_1 and a_2, of a along P, as shown.

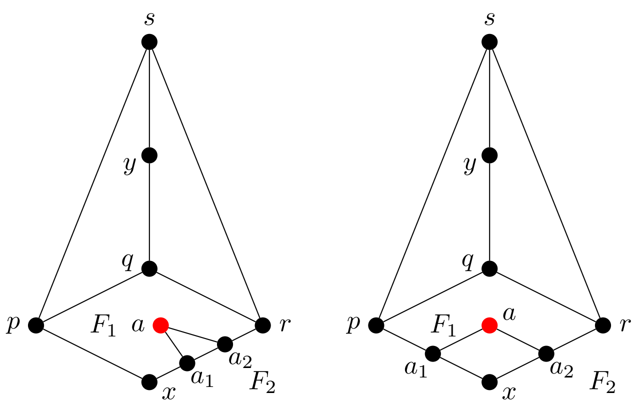

Figure 38: The two possibilities when $H_a - b$ is a subdivision of $K_{3,3} - e$, the path P is incident to x, and a has neighbors on one side of x (left) or both sides of x (right).

Recall that there must be a crossing chord in both faces F_1 and F_2 preventing a from being placed into each face. By the minimality of H_a, these chords must have ends on distinct paths of H_a.

Regarding the leftmost graph in Figure 38, we may assume that $a_1 = x$ and $a_2 = r$, since contracting edges along the $[x, a_1]$ and $[a_2, r]$ segments of P cannot destroy crossing chords. The crossing chords must have one endpoint between a_1 and a_2 along P and another endpoint outside this interval. For the F_1 face, the other endpoint is contractible to p or to q. For the F_2 face, the other endpoint is contractible to p or to s. The four resulting cases are depicted in Figure 39.

In all cases, except the bottom right case of Figure 39, the ends of the crossing chords must be distinct because the chords must cross each other as well cross edges incident to a. If the chords did not cross each other, then they could be brought into a single face. In the bottom right case, we may contract the ends of the crossing chords in the (x, r)-interval of P to the same vertex. As the figure shows, all four cases result in a $K_{3,3}$ subdivision avoiding y, a contradiction.

In the final case of Step 5, consider the rightmost graph in Figure 38. This graph can be redrawn as shown in Figure 40; the ends of the crossing chords are left undecided in this drawing.

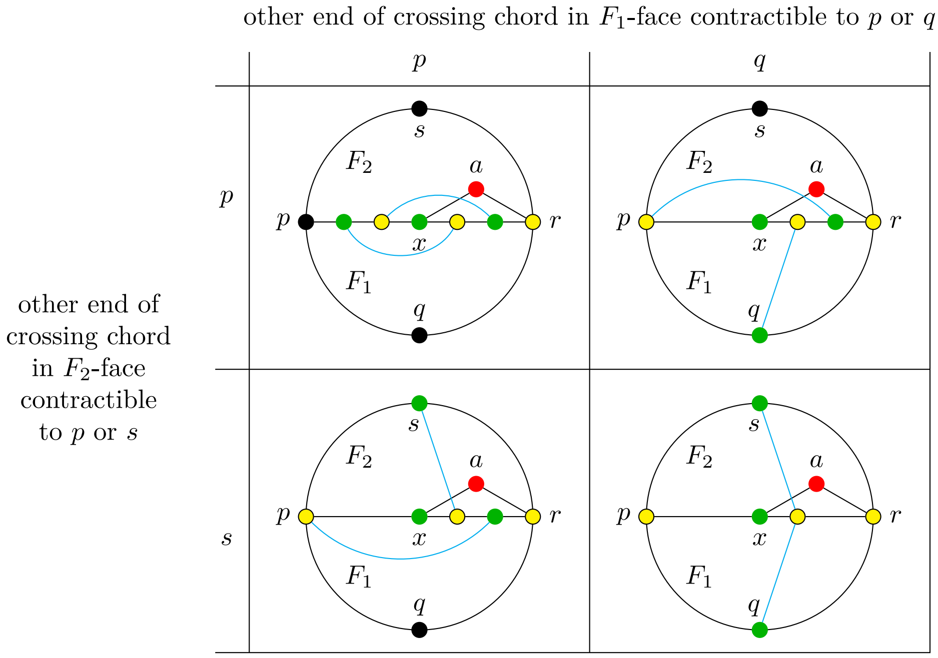

Figure 39: The four possible combinations for ends of crossing chords (shown in light blue).

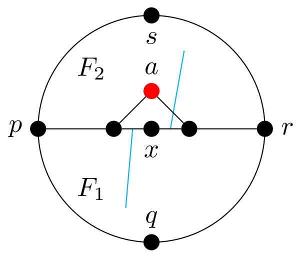

Figure 40: A redrawing of a portion of the rightmost graph from Figure 38 with crossing chords shown in light blue.

If the ends of the chords can be contracted to s and q, then a minor isomorphic to M (a Petersen-family graph) in G emerges (Figure 41) after adding b and the edge ab, the latter of which corresponds to contracting the light component of G.

In the remaining possible placements of the ends of the crossing chords, each possibility can be contracted to one of the two shown in Figure 42. Observe that the crossing chords must again cross each other (as well as edges incident to a) to avoid being placed into a single face. In both cases, a $K_{3,3}$ avoiding y is shown, a contradiction.

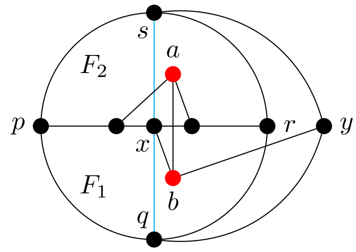

Figure 41: If the crossing chords have ends that can be contracted to s and q, then a minor of M emerges.

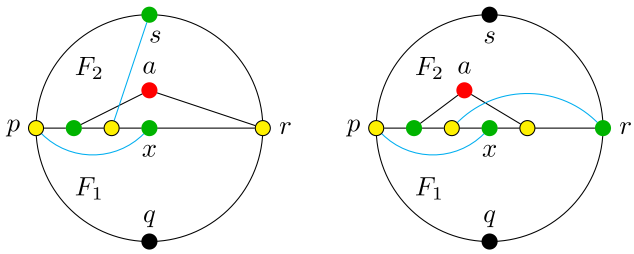

Figure 42: Crossing chords must cross each other in the final two possible placements of the ends of these crossing chords (shown in blue). The cases can each be contracted to the graphs depicted, or similar graphs. A $K_{3,3}$ avoiding y arises in both cases.

STEP 6: The final contradiction to complete the proof.

In this final step, we may assume that a has neighbors in only one face of the plane embedding of $H_a - b$ (Step 1). Furthermore there are no other vertices of C in this face (Steps 2-4). Finally we may assume that all of a's neighbors in C are not along a single path of $H_a - b$ (Step 5). Consequently, there is exactly one face, F, containing all of the neighbors of a in C and there must be a crossing chord in F.

If H_a is a subdivision of K_5, then there are four remaining non-isomorphic positions for a crossing chord. These cases are shown in Figure 43.

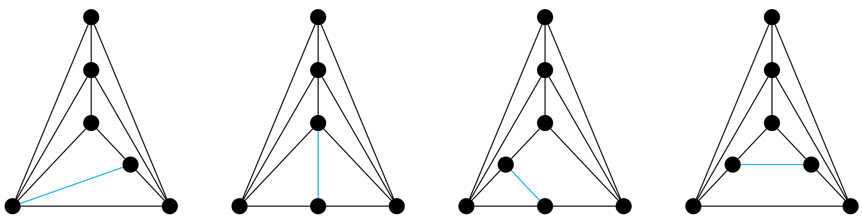

Figure 43: H_a is a subdivision of K_5. The four non-isomorphic positions for the crossing chord (shown in light blue) when endpoints are not along the same path of H_a.

In each of these cases (see Figure 43), a new $K_{3,3}$ subdivision for H_a exists with x and y branch vertices connected by b (see Figure 44). One can argue in each case that this new

H_a has a unique embedding of $H_a - b$ such that a now hits multiple faces, contradicting Step 1. This argument requires some care. For example, in the left-most graph, clearly a must have neighbors that cross the blue chord. These could simply be a vertex along the (u, x)-path and another along the (u, v)-path. However all of the neighbors of a could not be along these paths by Step 5.

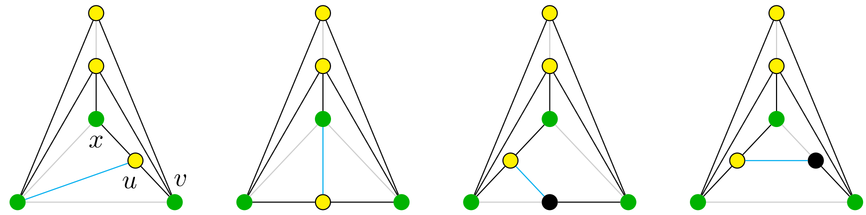

Figure 44: In each case of Figure 43 a $K_{3,3}$ subdivision for H_a exists with x and y branch vertices connected by b.

Now consider the ten non-isomorphic ways a crossing chord can appear in F when H_a is a subdivision of $K_{3,3}$. Six of these cases (see Figure 45) can be dismissed using an argument similar to the one given in the prior paragraphs.

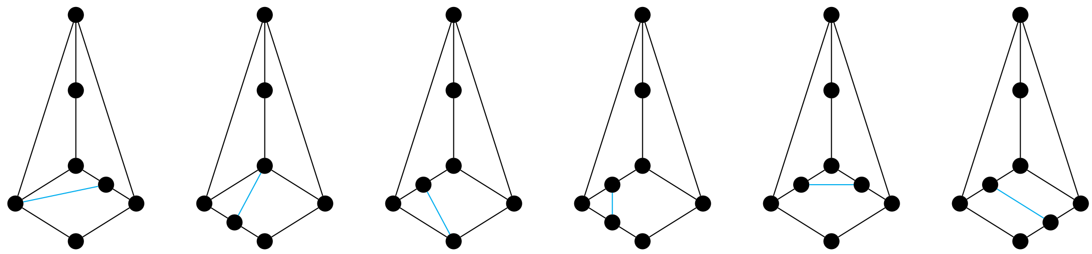

Figure 45: Six of the ten non-isomorphic ways a crossing chord (shown in light blue) can appear in F when H_a is a subdivision of $K_{3,3}$.

Each case of Figure 45 has a new $K_{3,3}$ subdivision with x and y branch vertices connected by b. This new $H_a - b$ has a that now hits multiple faces (by Step 5), contradicting Step 1 (see Figure 46);

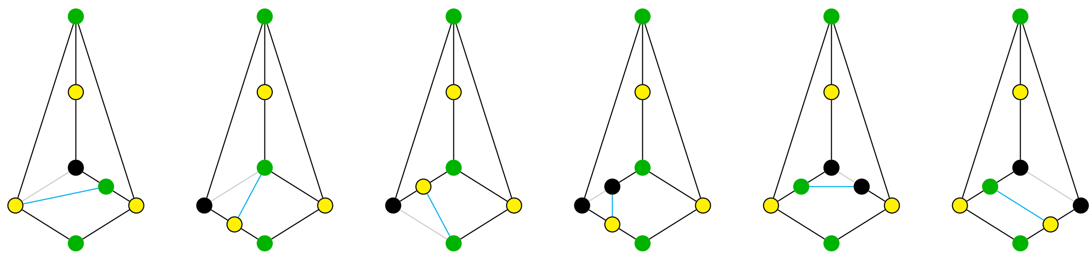

Figure 46: Each case of Figure 45 has a new $K_{3,3}$ subdivision with x and y branch vertices connected by b. This new $H_a - b$ has a that now hits multiple faces, contradicting Step 1.

The last four remaining cases are shown in Figure 47.

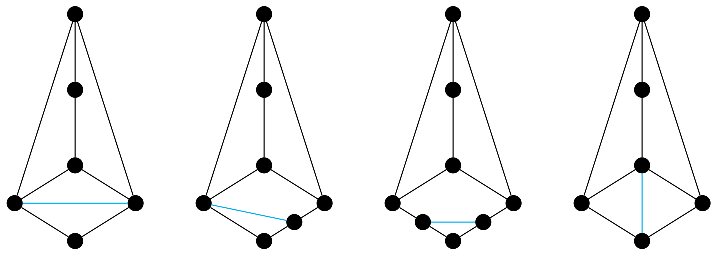

Figure 47: The four remaining chord placements in $H_a - b$.

Consider left-most graph in Figure 47 in which uv is the crossing chord. This means that a has a neighbor on the upper half (above the chord uv) of face F and a neighbor on the lower half (below the chord uv) of face F. Observe that there must be an external path P that connects the lower open interval (v, u) of the exterior face (shown in red in Figure 48) to the upper open interval (u, v) of the exterior face (shown in green in Figure 48); otherwise the uv chord could be drawn on the exterior face, reducing the number of crossings produced when reinserting a into the planar embedding of C; that is, the external path P *blocks* the chord uv. The resulting graph can be contracted to the one shown on the right of Figure 48. Adding vertex a to this graph produces a K_5 minor of G which implies that $H_a - b = H_b - a$ in the original graph G, so $\{x, y\}$ is another 2-cut of G, a contradiction.

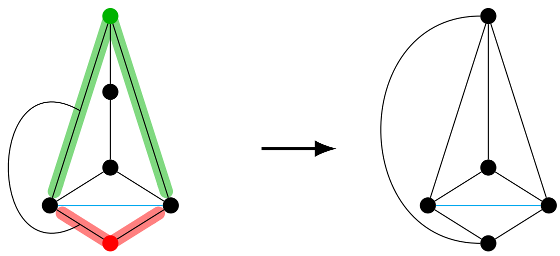

Figure 48: An external path connecting the lower exterior face (red) to the upper exterior face (green) implies a K_5 minor of G.

Consider next the center-left graph in Figure 47. Again let uv be the crossing chord; let w be the branch vertex of $H_a - b$ opposite u on face F (as shown in Figure 49).

The vertex a has neighbors on F above and below the chord uv because uv is a crossing chord. By Step 5, a has neighbors between u and w (on the upper part of F — Figure 49 shows one case); otherwise all of a's neighbors would occur on the path of $H_a - b$ containing x. Now the same analysis as given in the prior paragraph (Figure 48) applies. We omit further details for this case. Similar reasoning applies to the center-right case shown in Figure 47.

The final analysis regards the case in which the crossing chord is a vertical through F connecting branch vertices of $H_a - b$ (shown as the rightmost graph of Figure 47). By

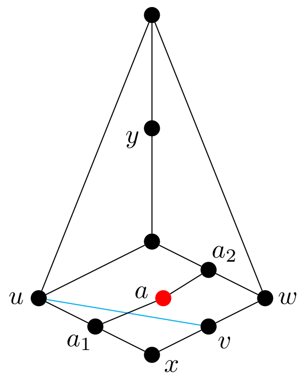

Figure 49: By Step 5, the vertex a must have neighbors above uw and below the blue crossing chord.

Step 5 not all of a neighbors can occur on the path of $H_a - b$ containing x. Thus there are three cases that remain; these are shown in Figure 50.

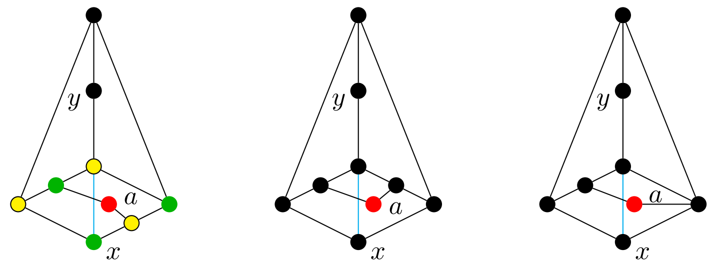

Figure 50: The left graph contains a subdivision of $K_{3,3}$ avoiding b and y. The other two contract to P_7 after adding b and the edge ab.

Observe that the left most graph of Figure 50 contains a subdivision of $K_{3,3}$ avoiding b and y. The other two graphs contain a minor of P_7 (see Figure 22) after adding b and the edge ab. $\square$

The next corollary strengthens the statement of Proposition 21 to show that vertices in the unique 2-cut must be branch vertices of all of their Kuratowski witnesses.

Corollary 22. *If Assumptions 15 are satisfied, then b is a branch vertex for any Kuratowski subgraph of G avoiding a (and vice versa, a is a branch vertex for any Kuratowski subgraph of G avoiding b).*

Proof. Note that b must have at least three neighbors in C since Proposition 21 guarantees a Kuratowski subgraph avoiding a (with vertices all in $C \cup \{a\}$) in which b must appear as a branch vertex. Suppose now that H_a is an arbitrary Kuratowski subgraph of G avoiding a. Clearly $V(H_a) \subseteq C \cup \{b\}$. If one of the edges in $G[C \cup \{b\}]$ incident to b does not appear in H_a, then it is not in $E(H_a) \cup E(H_b)$, contradicting Lemma 16 part (iii). $\square$

8.3 Two Kuratowski subgraphs have branch vertices that cover $C \cup \{a, b\}$

With Corollary 22, establishing that a and b must be branch vertices of all of their Kuratowski witnesses, we are almost ready to prove the main objective of this subsection: the existence of two Kuratowski subgraphs whose branch vertices cover $C \cup \{a, b\}$ (Theorem 25). First we focus on properties of vertices in C that are not branch vertices of Kuratowski witnesses.

Lemma 23 (Non-Branch Vertex Lemma)**.** *Assume Assumptions 15 and H_a and H_b are Kuratowski subgraphs of G avoiding a and b, respectively. If $w \in C$ is a branch vertex of neither H_a nor H_b:*

(i) $d_G(w) = 4$, with w incident to two edges in $E(H_a)$ and two in $E(H_b)$.

(ii) If $e = wx \in E(H_a)$ then $x \in V(H_b) \cup \{b\}$, $H_b \not\subset G/e$, and b is the only apex for G/e.

(iii) If $e = wx \in E(H_b)$ then $x \in V(H_a) \cup \{a\}$, $H_a \not\subset G/e$, and a is the only apex for G/e.

Proof. (i) Assume that $c \in C$ is not a branch vertex of H_a or H_a. So the degree of c in H_a and H_b is two. Lemma 1 implies $d_G(w) \geqslant 3$. If $d_G(w) > 4$, then there is an edge of $G[C \cup \{a, b\}]$ that is not covered by H_a or H_b contradicting Lemma 16 part (iii). To prove claim (i), it suffices to prove that $d_G(w) \neq 3$. Assume, to the contrary, that $d_G(w) = 3$. The pigeon-hole principle guarantees an edge $wx \in E(H_a) \cap E(H_b)$ and so $x \notin \{a, b\}$. Note that G/wx must have an apex z in $H_a/wx \cap H_b/wx$, so $z \notin \{a, b\}$. However, z must also separate a from b in $G[C \cup \{a, b\}]$ since otherwise L^+ would still be a minor of G/wx. But then either $\{a, z\}$ or $\{b, z\}$ is another 2-cut of G (contradicting that S is the only 2-cut) or z is the vertex resulting from the contraction of the edge wx, contradicting Lemma 16 part (iv).

(ii-iii) By symmetry, it suffices to prove (ii). Consider $e = wx \in E(H_a)$. If $x \notin V(H_b) \cup \{b\}$, then H_a and H_b remain Kuratowski subgraphs in G/e implying that any apex z for G/e must be in $H_a \cap H_b \subset C$. However, this means that z must also separate a from b in $G[C \cup \{a, b\}]$ since otherwise L^+ would still exist in G/e. But then either $\{a, z\}$ or $\{b, z\}$ is another 2-cut of G (contradicting that S is the only 2-cut) or z is the vertex resulting from the contraction of the edge wx, contradicting Lemma 16 part (iv). Therefore $x \in V(H_b) \cup \{b\}$. The reader can check that similar reasoning applies if H_b remains a Kuratowski subgraph of G/e or if b is not the only apex for G/e. □

Theorem 24. *Assume Assumptions 15. Choose any Kuratowski subgraphs H_a and H_b avoiding a and b, respectively, that also minimize $|E(H_a)|+|E(H_b)|$. If H_a is a subdivision of K_5 or H_b is a subdivision of K_5, then any vertex in C is a branch vertex of H_a or a branch vertex of H_b.*

Proof. Without loss of generality, H_a is a subdivision of K_5. Assume, to the contrary, there is a vertex $w \in C$ that is a not a branch vertex either H_a or H_b. By Theorem 23 part (i), $deg_G(w) = 4$ and $w \in V(H_a) \cap V(H_b)$. Now w has two neighbors in H_b, at least one of which is not vertex a. Consider a neighbor x of w such that $wx \in E(H_b)$ and $x \neq a$. By Theorem 23 part (iii), $x \in V(H_a)$. There are three cases shown in Figure 51.

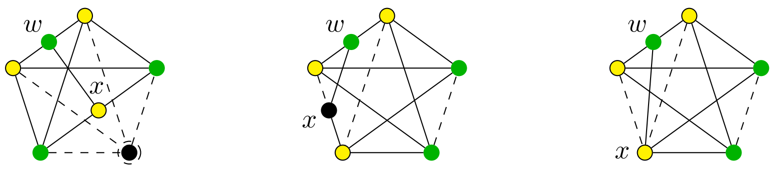

Figure 51: Deleting dotted lines/vertex and adding the edge wx reveals a subdivision of $K_{3,3}$ with fewer edges than the original subdivision of K_5.

The figure shows a subdivision of K_5 that represents H_a; w is a non-branch vertex with a neighbor x such that $wx \in E(H_b) - E(H_a)$. The vertex x could be in any of three positions. The dotted lines indicates paths of H_a that can be deleted leaving a new subdivision of $K_{3,3}$ with fewer edges than H_a, in each case. This new Kuratowski subgraph avoids vertex a also, so contradicts the choice of H_a. So these cases essentially follow from a commonly rediscovered fact that a vertex minimal non-planar graph that is not just a subdivision of K_5 has a spanning $K_{3,3}$ subdivision.

In each case, a new choice of H_a as a subdivision of $K_{3,3}$ has fewer edges than the current H_a; this contradicts that the original choice of H_a and H_b minimized $|E(H_a)| + |E(H_b)|$. $\square$

The next theorem is the main result of this subsection.

Theorem 25. *Assume Assumptions 15. There are Kuratowski subgraphs H_a and H_b avoiding a and b respectively, such that any vertex in $C \cup \{a, b\}$ is a branch vertex of H_a or a branch vertex of H_b.*

Proof. Choose Kuratowski subgraphs H_a and H_b as follows:

(i) $a \notin V(H_a)$ and $b \notin V(H_b)$,

(ii) maintaining (i), minimize $|E(H_a)| + |E(H_b)|$,

(iii) maintaining (i) and (ii), minimize $|W|$, where

$$W = (C \cup \{a, b\}) - \{v : v \text{ is branch vertex of } H_a \text{ or } H_b\}.$$

It suffices to prove that this choice produces $W = \varnothing$. Corollary 22 implies $a, b \notin W$.

If H_a is a subdivision of K_5 or H_b is a subdivision of K_5, then Theorem 24 yields $W = \varnothing$. So, we may assume H_a and H_b are subdivisions of $K_{3,3}$.

Assume, to the contrary, that $W \neq \varnothing$. Let w be an arbitrary vertex in W. If possible, choose H_a and H_b, subdivisions of $K_{3,3}$, satisfying (i) - (iii), minimum $|W|$ and $w \in W$ so that $w \notin N(a)$. By Theorem 23, $d_G(w) = 4$, $w \in V(H_a) \cap V(H_b)$, and all of the neighbors

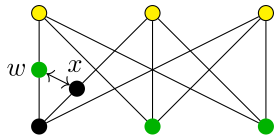

Figure 52: A depiction of H_a and the extra edge $wx \in E(H_b)$. Contracting wx preserves H_b and non-planarity of H_a.

of w are also vertices in $(V(H_a) \cap V(H_b)) \cup \{a, b\}$. Let x, y be the neighbors of w such that $wx, wy \in E(H_b)$. We may assume that $x \notin \{a, b\}$ because w has two neighbors in H_b and $b \notin V(H_b)$. Note also the minimality of H_a implies that x, y are not on the same path of H_a that contains w.

Now consider H_a. If w and x are internal vertices of paths of H_a that intersect at a common branch vertex, then contracting wx preserves H_b and also the non-planarity of H_a (see Figure 52). This contradicts Lemma 19 part (ii).

Therefore x must be a branch vertex of H_a or it is an internal vertex on a path of H_a that does not intersect at a common branch vertex with w.

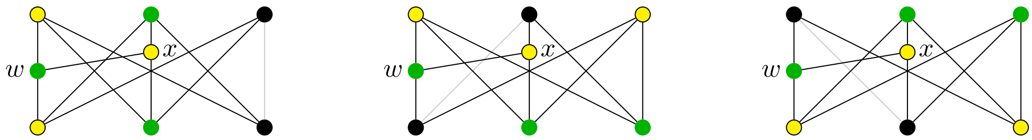

Figure 53: For any branch vertex t of H_a, there is a subdivision of $K_{3,3}$ in $H_a + wx$ in which t is not a branch vertex.

Assume that x is not a branch vertex of H_a. In this case, Figure 53 shows that for any branch vertex t of H_a, there is a subdivision of $K_{3,3}$ in $H_a + wx$ in which t is not a branch vertex. Recall that b is a branch vertex of H_a. So, if x is not a branch vertex of H_a, then there exists a subdivision of $K_{3,3}$ missing a that does not have b as a branch vertex, contradicting Corollary 22.

Consequently we may assume that x is a branch vertex of H_a different from the ones at the end of the path of H_a containing w. Without loss of generality H_a appears as in see Figure 54, where the label v has been introduced on the branch vertex of H_a along the path of H_a containing w that has opposite color as x.

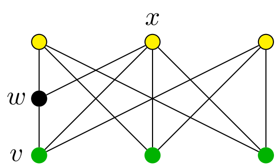

Figure 54: x is a branch vertex of H_a (different from the ones at the end of the path of H_a containing w).

Now consider the subdivision of $K_{3,3}$, call it J_v, in $H_a + wx$ that remains after deleting the edges along the vx-path (see Figure 55).

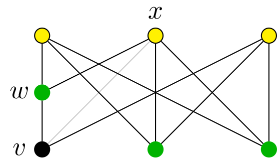

Figure 55: The graph J_v that is a subdivision of $K_{3,3}$ in $H_a + wx$.

Note that J_v covers all of the branch vertices of H_a and does not use any vertices outside of H_a. By choice of H_a, there cannot be fewer edges in J_v than in H_a. Consequently J_v and H_a must have the same number of edges. In particular the vx-path in H_a is just the edge joining v and x. Because H_a minimizes $|W|$ and the branch vertices of J_v and H_a only differ at v and w, the vertex v is a branch vertex of neither H_b nor J_v. In particular the edge vx must be an edge of H_a and H_b.

If w and v are both neighbors of a, a 4-cycle formed by a, w, x, and v appears in H_b (see Figure 56). However, since w and v are not branch vertices of H_b, this 4-cycle is an impossible configuration in H_b.

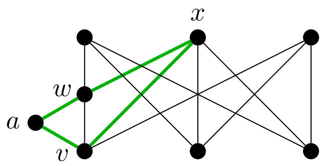

Figure 56: If both w and v are neighbors of a, then an impossible 4-cycle (green) appears in H_b.

By the choice of w we conclude that $w \notin N_G(a)$ (if $w \in N_G(a)$ then replace w with v). It follows from earlier reasoning that, like x, the other neighbor of w in H_b, namely y, must also be a branch vertex of H_a.

If x and y were branch vertices of H_a with the same color, then there would be a subdivision of $K_{3,3}$ with fewer edges than H_a that could have been chosen (as shown in Figure 57) contradicting the choice of H_a.

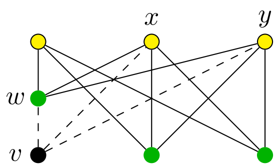

Figure 57: Deleting edges along the three dotted paths and adding the two edges wx, wy produces a subdivision of $K_{3,3}$ with fewer edges than H_a.

So, without loss of generality, H_a appears as shown in Figure 58. This figure introduces labels for all of the branch vertices of H_a. Observe that, like v, the vertex u is not a branch vertex of H_b. Also, like the edge vx, the edge uy must be an edge in H_a and H_b.

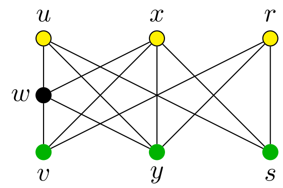

Figure 58: In H_a, the vertex w must have two neighbors, x and y, that are branch vertices of opposite color in H_a and not on the path of H_a containing w.

If $u, v \in N_G(a)$, then the edges au,uy,yw,wx,xv,va form a 6-cycle in H_b, as shown in Figure 59. Recall a is a branch vertex of H_b but u, v, and w are not branch vertices of H_b. If x or y are not branch vertices of H_b, then a cycle with at most two branch vertices of H_b exists in H_b, an impossibility. So, a, x and y are three branch vertices of H_b. However the 6-cycle induced by the edges au,uy,yw,wx,xv,va from H_b implies that a, x and y cannot be 2-colored as the branch vertices of a subdivision of $K_{3,3}$, contradicting that $H_b \cong TK_{3,3}$.

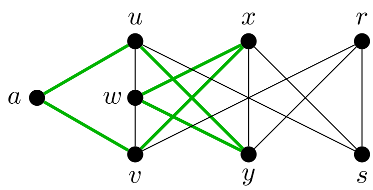

Figure 59: If $u, v \in N_G(a)$, then an impossible 6-cycle (shown in green) emerges in H_b.

Consequently $u \notin N_G(a)$ or $v \notin N_G(a)$. Without loss of generality, $v \notin N_G(a)$. Now $d_G(v) = 4$ so v has another neighbor in $V(H_a) \cap V(H_b)$. Applying the same reasoning to v as we have applied previously to w, we conclude that the remaining unknown neighbor of v must be a branch vertex of H_a. Further applying this reasoning to J_v (see Figure 55) reveals that this neighbor of v must be either y or s. However, if y is a neighbor of v in H_b, then the four edges vy, yw, wx, xv form an impossible 4-cycle in H_b with at most two branch vertices (since w and v cannot be branch vertices of H_b). Therefore, s must be the final neighbor of v in H_b and vs must be an edge of H_b (see leftmost graph in Figure 60).

Because $s \in N_G(v)$ there is a subdivision of $K_{3,3}$, call it J_r, that covers the branch vertices of H_a, contains only vertices from H_a, but does not have r as a branch vertex (see middle of Figure 60). Applying the same reasoning to J_r as we applied before to J_v, we conclude that r is a branch vertex of neither J_r nor H_b. Moreover the edge rs is in H_b. Consequently the path $uywxvsr$ has all its edges in H_b and covers all of the branch vertices of H_a. In particular the vertices of this path must be in H_b. This is

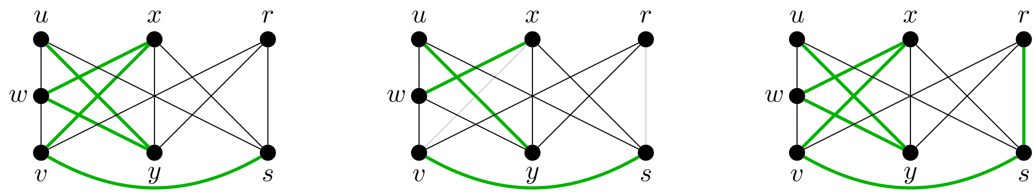

Figure 60: $s \in N_G(v)$ (left) implies a subdivision of $K_{3,3}$ (middle) that implies a path in H_b that covers the branch vertices of H_a (path shown in green edges at right).

a contradiction because b is in $V(H_a) - V(H_b)$ and, by Corollary 22, it must also be a branch vertex of H_a. This contradiction proves that $W = \varnothing$, as desired. □

8.4 The final computation

In this subsection we outline how to show the list of 72 connectivity-2 apex obstructions satisfying Assumptions 15 is complete. While much of the case-analysis can be reduced by hand, ultimately we confirmed the final list using computers. We omit many details. Much of the case analysis applies to small (order $\leqslant 10$) graphs and is routine, but it is sufficiently tedious that it precludes comprehensive presentation.

Theorem 26. *Suppose that $G \in \mathcal{F}$, $\kappa(G) = 2$, $S = \{a, b\}$ is the unique 2-cut of G, C is the heavy component of $G - S$ and C is non-planar.*

- *If $G[C \cup \{a, b\}]$ has a 2-cut separating a from b, then G is isomorphic to a graph in Figure 61.*
- *If $G[C \cup \{a, b\}]$ has no 2-cut separating a from b, then G is isomorphic to a graph in Figure 62.*

Proof. By Theorem 25, there are Kuratowski witnesses H_a and H_b in G avoiding a and b respectively, such that any vertex in $C \cup \{a, b\}$ is a branch vertex of H_a or a branch vertex of H_b. Consequently, every vertex of G is a branch vertex of H_a, H_b, or the Kuratowski witness in L^+.

Because a and b are branch vertices of H_b and H_a respectively, it follows that $|C| \leqslant 10$. Indeed, the only way that $|C| = 10$ is if H_a and H_b are subdivisions of $K_{3,3}$ with disjoint branch sets. This case can be shown never to occur by examining possible subdivided edges of H_a and H_b that must involve branch vertices from the other Kuratowski witness. We omit the details.

Clearly $|C| \geqslant 4$ because H_a, for one, has at least 5 branch vertices. If $|C| = 4$, then it is easy to show K_6 is a minor of G, an impossibility. Thus it suffices to consider cases in which $5 \leqslant |C| \leqslant 9$, C is connected and planar. There are 87,816 non-isomorphic, connected planar graphs with order between 5 and 9 inclusive. A computer check of all of these graphs (together with adding a light component) reveals the 72 obstructions indicated. □

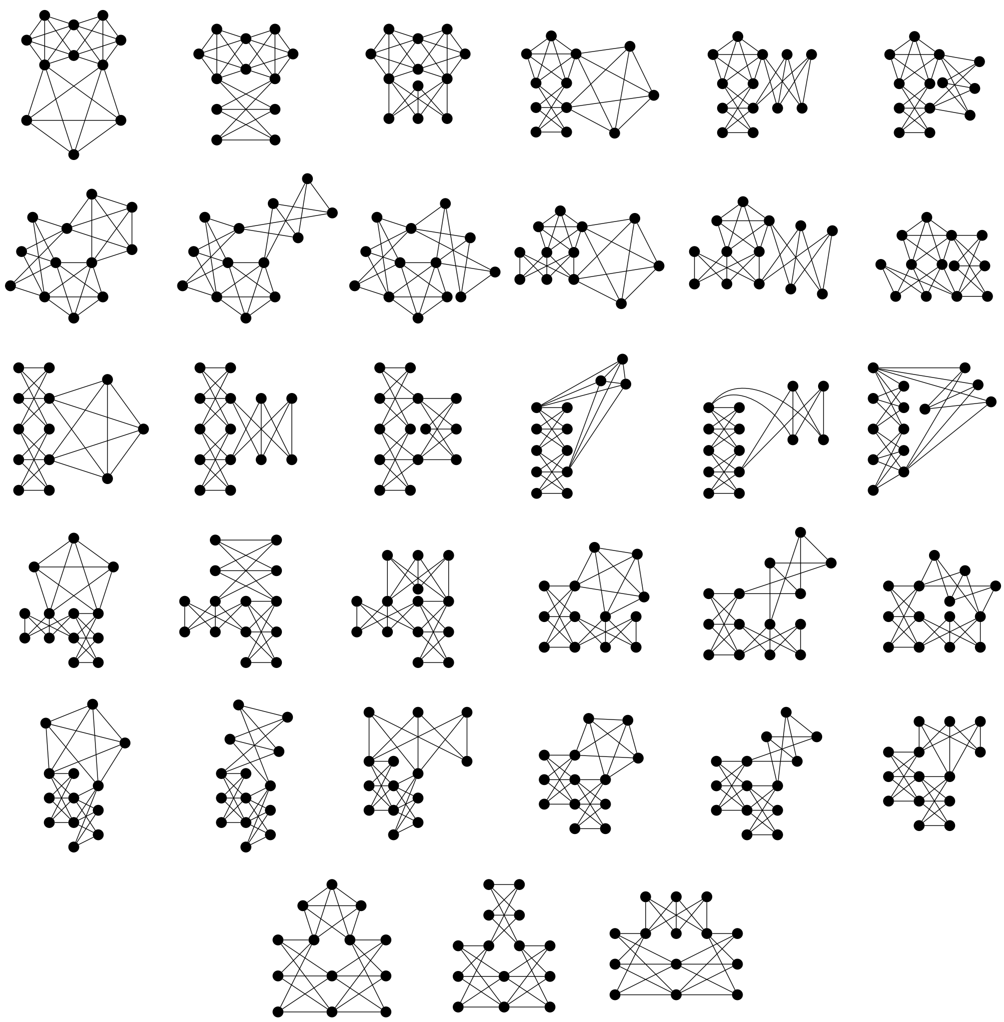

Figure 61: The 33 connectivity-2 apex obstructions that have a unique 2-cut $\{a, b\}$, a planar heavy component C, and a 2-cut separating a from b in $G[C \cup \{a, b\}]$.

9 A double apex graph interpretation

In this section we discuss another interpretation of our characterization of connectivity-2 apex obstructions because it may be of independent interest.

A graph H with vertices a and b is *double apex* (with respect to a and b) if $H - a$ and $H - b$ are non-planar, but $H - a - b$ is planar. Vertices a and b are the *roots* of H. For example, consider the rooted graphs obtained from K_5 and $K_{3,3}$ by replacing an edge with two subdivided edges and making the degree-two vertices the roots; denote these two graphs K_5^* and $K_{3,3}^*$, respectively. These graphs (shown in Figure 63 with the roots colored red) are double apex.

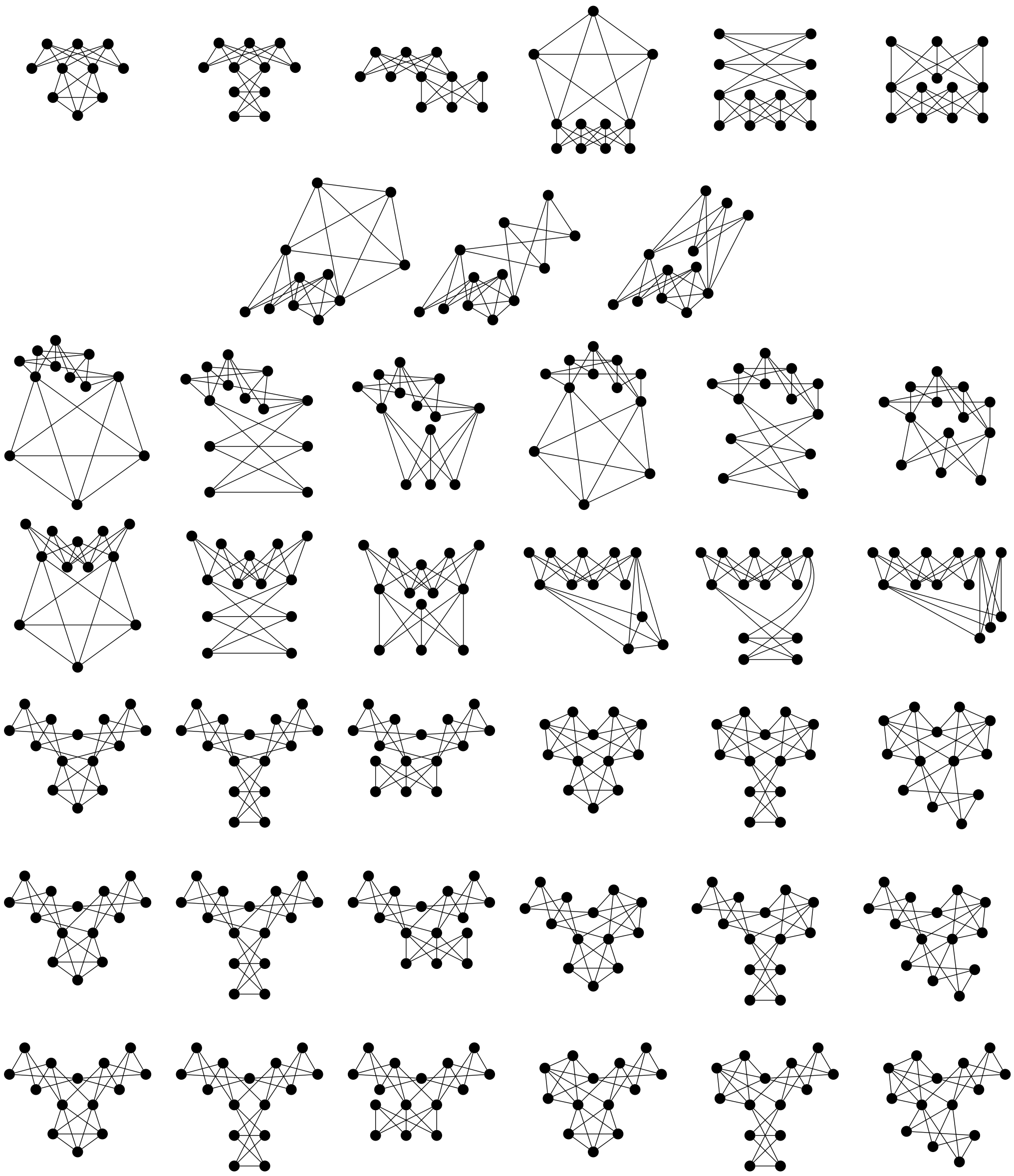

Figure 62: The 39 connectivity-2 apex obstructions that have a unique 2-cut $\{a, b\}$, a planar heavy component C, and no 2-cut separating a from b in $G[C \cup \{a, b\}]$.

Figure 63: The two rooted double apex graphs K_5^* and $K_{3,3}^*$.

Characterizing minor-minimal double apex graphs is a special type of 'intertwine' problem in which the goal is to determine minimal graphs that contain two Kuratowski subgraphs, each containing one root but avoiding the other. Note that K_5^* and $K_{3,3}^*$ are minor-minimal double apex graphs. Recall that the Petersen family of graphs consists of the seven graphs: $K_6, K_{1,3,3}, Y^-, K_{4,4} - e, M, P_7$, P. Significant minor-minimal double apex graphs can be obtained from one of the seven Petersen family graphs by removing an edge and making its endpoints the roots. Note that not all such edge deletions produce a minor-minimal double apex graph ($K_{1,3,3}, Y^-, K_{4,4} - e$, and M have problematic edges).

One could restate Corollary 22 in the language of double apex graphs this way:

Theorem 27. *Suppose that H is a minor-minimal double apex graph with roots a and b. If $H + \{ab\} \ncong Y^-, M, P_7, P$ and $H \ncong K_5^*, K_{3,3}^*$, then a (resp. b) is a branch vertex of every Kuratowski subgraph in $H - b$ (resp. $H - a$).*

Using Theorem 27 one can prove (see the proof of Theorem 25) that in any minor-minimal double apex graph H satisfying $H + \{ab\} \ncong Y^-, M, P_7, P$ and $H \ncong K_5^*, K_{3,3}^*$, two Kuratowski subgraphs exist, one avoiding a and the other avoiding b, whose branch vertices cover the entire vertex set. In this way all minor-minimal double apex graphs can be enumerated. Indeed it follows from our characterization of connectivity-2 apex obstructions, that there are 57 non-isomorphic (as rooted graphs!) minor-minimal double apex graphs. Three are disconnected graphs; these are rooted versions of $2K_5$, $2K_{3,3}$ and $K_5 + K_{3,3}$ in which a root appears in each component. Twelve can be obtained from Petersen family graphs by removing an edge. For example, there are four non-isomorphic rooted minor-minimal double apex graphs that can be obtained from M by removing an edge. The remaining 42 non-isomorphic rooted minor-minimal double apex graphs (including K_5^* and $K_{3,3}^*$) can be found by inspecting the augmented heavy components of the connectivity-2 apex obstructions: see Appendix B and consider the 42 'cards' shown there in which all three 2-sums (with K_5, $K_{3,3}$, or $K_{3,3} + e$) produce an apex obstruction.

Assuming Theorem 27, it is easy to reason that if H is a minor-minimal double apex graph with roots a and b and $H + \{ab\}$ is not isomorphic to a Petersen family graph, then a 2-sum at the roots a and b of H with K_5 or $K_{3,3}$ will produce a connectivity-2 apex obstruction. Indeed Lemma 9 states that, under the right circumstances, a connectivity-2 apex obstruction must be generated in this way. So it is tempting to believe that knowing the 57 non-isomorphic rooted minor-minimal double apex graphs suffices to generate all connectivity-2 apex obstructions via such 2-sums with K_5 and $K_{3,3}$. But these 2-sums may generate the same graph more than one way. More significantly, because of non-planar heavy components or vertices that are not branch of any Kuratowski subgraph, there are 23 apex obstructions that can not be generated as a 2-sum in this way: nineteen of the 21 graphs in Figure 5 and four graphs at the bottom of Figure 8. The two left-most graphs in the middle rows of Figure 5 can be generated from a 2-sum applied to K_5^* and $K_{3,3}^*$, respectively. Incidentally it is worth mentioning that, among the 23 apex obstructions that can not be generated as a 2-sum but excepting the seven graphs that contain vertices that are not branch of any Kuratowski subgraph, the remaining 16 graphs require only the branch vertices of two Kuratowski subgraphs to cover the entire vertex set.

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A graph6 codes for connectivity-2 apex obstructions

Computer-readable files of the graphs for Figures 5, 6, 8, 11, 61 and 62 can be found in the supplementary data file published with this article.

B Augmented heavy components

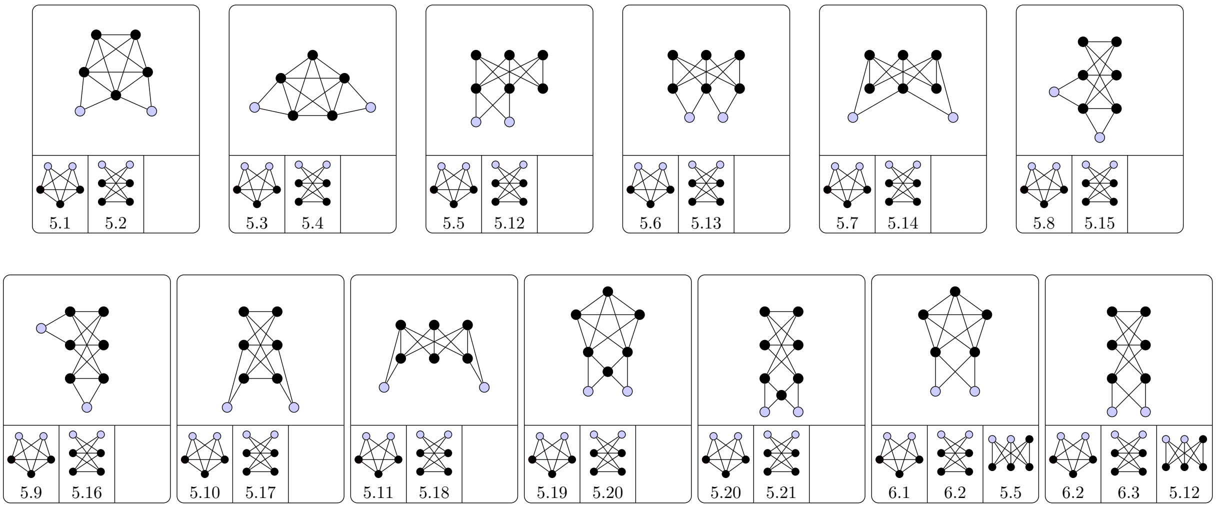

Figure 64: Augmented heavy components appearing in Figure 5 or Figure 6

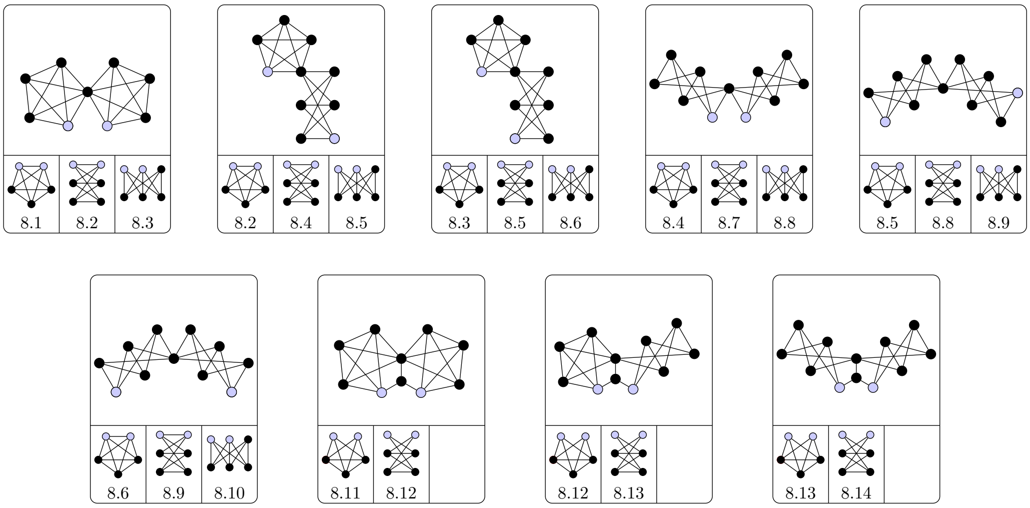

Figure 65: Augmented heavy components appearing in Figure 8

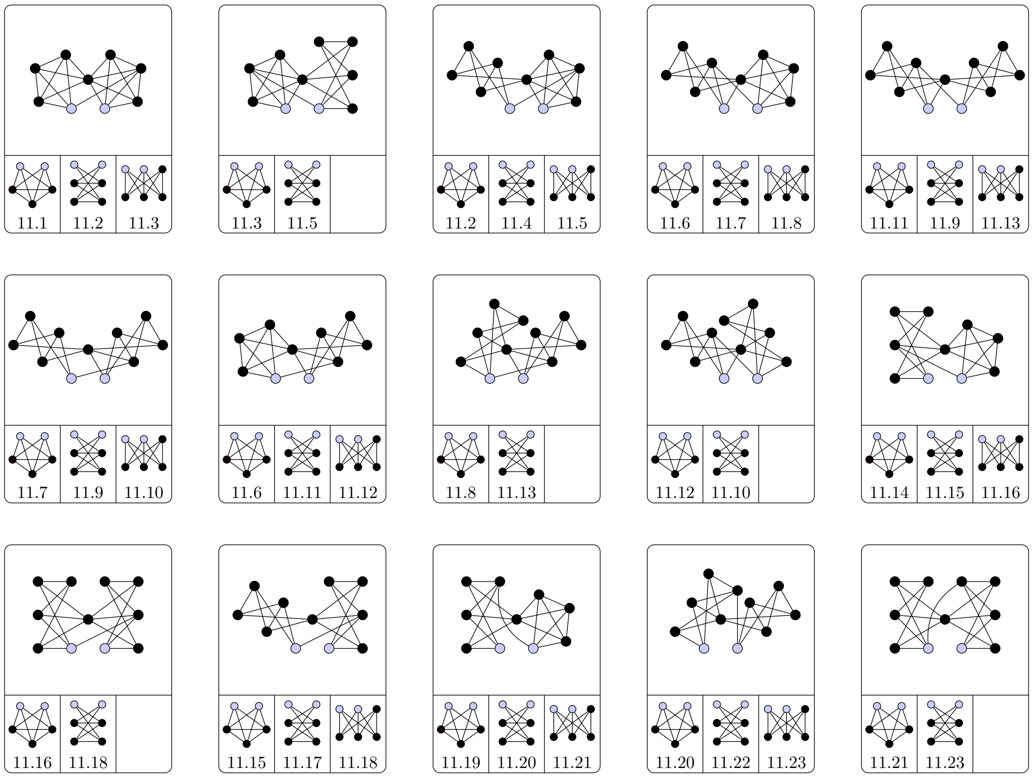

Figure 66: Augmented heavy components appearing in Figure 11

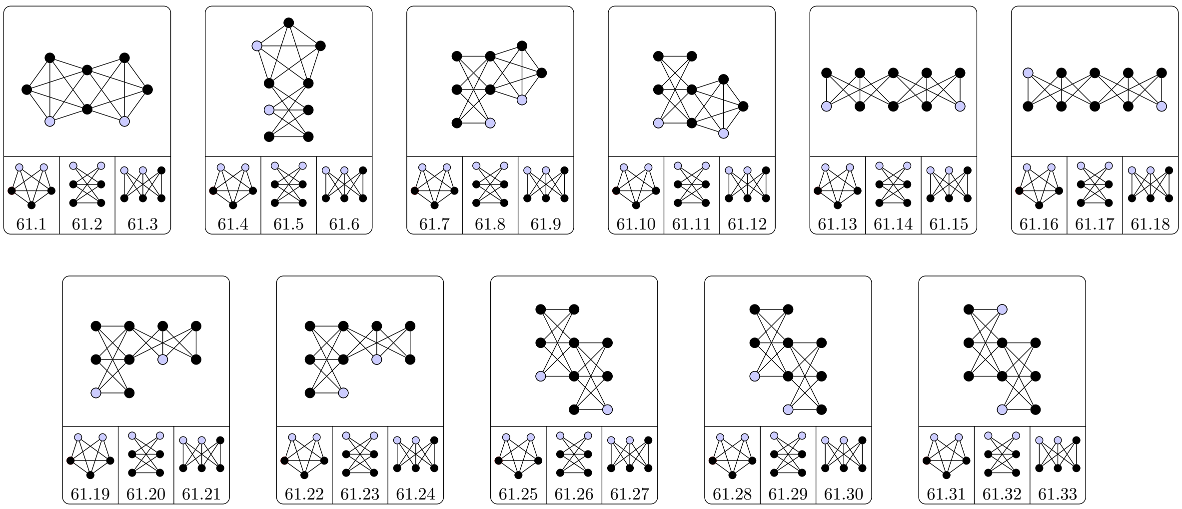

Figure 67: Augmented heavy components appearing in Figure 61

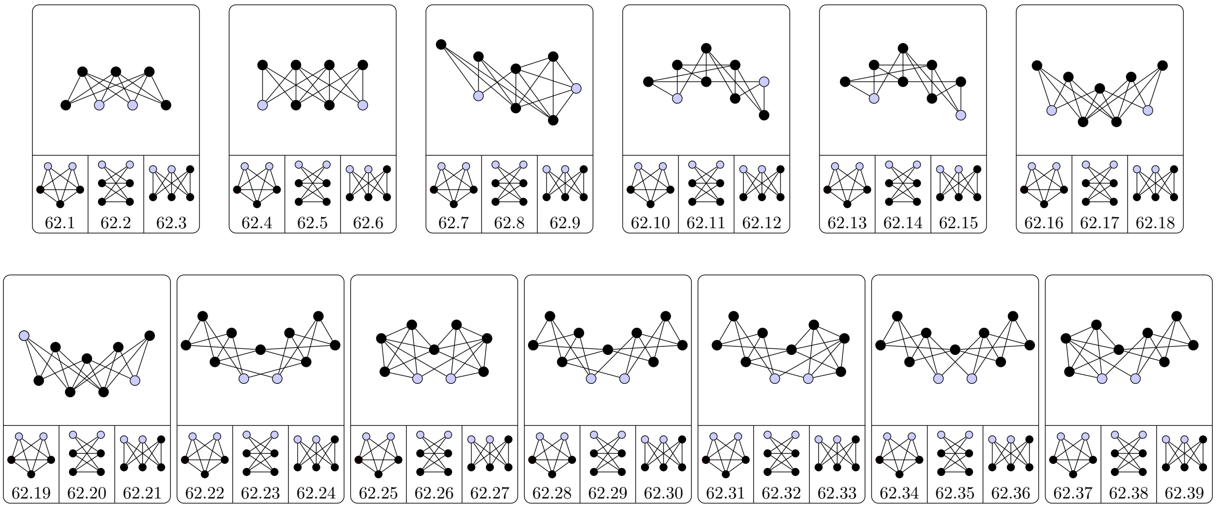

Figure 68: Augmented heavy components appearing in Figure 62