

On the spectra of general random mixed graphs

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Abstract

A mixed graph is a graph that can be obtained from a simple undirected graph by replacing some of the edges by arcs in precisely one of the two possible directions. The Hermitian adjacency matrix of a mixed graph G of order n is the $n \times n$ matrix $H(G) = (h_{ij})$, where $h_{ij} = -h_{ji} = \mathbf{i}$ (with $\mathbf{i} = \sqrt{-1}$) if there exists an arc from v_i to v_j (but no arc from v_j to v_i), $h_{ij} = h_{ji} = 1$ if there exists an edge (and no arcs) between v_i and v_j , and $h_{ij} = 0$ otherwise (if v_i and v_j are neither joined by an edge nor by an arc). We study the spectra of the Hermitian adjacency matrix and the normalized Hermitian Laplacian matrix of general random mixed graphs, i.e., in which all arcs are chosen independently with different probabilities (and an edge is regarded as two oppositely oriented arcs joining the same pair of vertices). For our first main result, we derive a new probability inequality and apply it to obtain an upper bound on the eigenvalues of the Hermitian adjacency matrix. Our

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second main result shows that the eigenvalues of the normalized Hermitian Laplacian matrix can be approximated by the eigenvalues of a closely related weighted expectation matrix, with error bounds depending on the minimum expected degree of the underlying undirected graph.

Keywords: general random mixed graphs; random Hermitian adjacency matrix; random normalized Hermitian Laplacian matrix; spectra

Mathematics Subject Classifications: 05C50, 15A18

1 Introduction

A *Hermitian matrix* (also sometimes called *self-adjoint matrix*) is a complex square matrix that is equal to its own *conjugate transpose*, i.e., the (i, j) -th element is equal to the complex conjugate of the (j, i) -th element, for all indices i and j . The purpose of this paper is to study the spectra of the Hermitian adjacency matrix and the normalized Hermitian Laplacian matrix of general random mixed graphs. Before we give the other necessary definitions and present our results, we start off with a short overview and some background on related work.

1.1 Background and related work

Spectra of the adjacency matrix and the normalized Laplacian matrix of graphs have many applications in graph theory. For example, the spectrum of the adjacency matrix of a graph is related to its connectivity and the number of occurrences of specific subgraphs, and also to its chromatic number and its independence number. The spectrum of the normalized Laplacian matrix is related to diffusion on graphs, random walks on graphs and the Cheeger constant. For more details on these notions, and for more applications of spectra of the adjacency matrix and the normalized Laplacian matrix, we refer the interested reader to two monographs [5, 12].

Also for random graphs, spectra of their adjacency matrix and their normalized Laplacian matrix are well-studied (See, e.g., [2, 6, 7, 9, 10, 13, 14, 17, 18]). We next present a brief account of some of the results that were obtained for random graphs. We refrain from giving an exhaustive overview, and we refer the reader to the sources for more background, and for terminology and notation.

Given a graph G of order n , let μ_1, \dots, μ_n be the eigenvalues of the adjacency matrix of G in nonincreasing order. Adopting the Erdős-Rényi random graph model $G(n, p)$, Füredi and Komlós [18] showed that asymptotically almost surely $\mu_1 = (1 + o(1))np$ and $\max\{\mu_2, -\mu_{n-1}\} \leq (2 + o(1))\sqrt{np(1-p)}$ provided $np(1-p) \gg \log^6 n$. Here the notation $o(1)$ is used for a quantity (depending on n) that goes to 0 as n goes to infinity. These results were extended to sparse random graphs [14, 22] and general random matrices [13, 18]. Tropp [29] determined probability inequalities for sums of independent random self-adjoint matrices. Alon, Krivelevich, and Vu [2] studied the concentration of the s -th largest eigenvalue of a random symmetric matrix with independent random entries of absolute value at most one. Friedman et al. [15, 16, 17] proved that the second largest

eigenvalue (in absolute value) of random d -regular graphs is almost surely $(2+o(1))\sqrt{d-1}$ for any $d \geq 4$. Chung, Lu, and Vu [7] studied spectra of the adjacency matrix of random power law graphs, and spectra of the normalized Laplacian matrix of random graphs with given expected degrees. Their results on random graphs with given expected degree sequences were supplemented by Coja-Oghlan et al. [9, 10] for sparse random graphs. Lu and Peng [24, 25] studied spectra of the adjacency matrix and the normalized Laplacian matrix of edge-independent random graphs, as well as spectra of the normalized Laplacian matrix of random hypergraphs. Oliveira [26] considered the problem of approximating the spectra of the adjacency matrix and the normalized Laplacian matrix of random graphs. His results were improved by Chung and Radcliffe [8].

The purpose of our paper is to extend these studies to general random mixed graphs.

1.2 General random mixed graphs

A graph is called a *mixed graph* if it contains both directed and undirected edges. We use $G = (V(G), E(G), A(G))$ to denote a mixed graph with a set $V(G)$ of vertices, a set $E(G)$ of (undirected) edges, and a set $A(G)$ of arcs (directed edges). We define the *underlying graph* of G , denoted by $\Gamma(G)$, as the graph with vertex set $V(\Gamma(G)) = V(G)$, and edge set

$$E(\Gamma(G)) = \{v_i v_j \mid v_i v_j \in E(G) \text{ or } (v_i, v_j) \in A(G) \text{ or } (v_j, v_i) \in A(G)\}.$$

We adopt the terminology and notation of Liu and Li in [23], and define the *Hermitian adjacency matrix* of a mixed graph G of order n to be the $n \times n$ matrix $H(G) = (h_{ij})_{n \times n}$, where

$$h_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G); \\ \mathbf{i}, & \text{if } (v_i, v_j) \in A(G) \text{ and } (v_j, v_i) \notin A(G); \\ -\mathbf{i}, & \text{if } (v_i, v_j) \notin A(G) \text{ and } (v_j, v_i) \in A(G); \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\mathbf{i} = \sqrt{-1}$. This matrix, that is indeed Hermitian, as one easily sees, was also introduced independently by Guo and Mohar in [20]. We denote by $\lambda_i(H(G))$ the i -th largest eigenvalue of $H(G)$ (multiplicities counted). We use $\{\lambda_1(H(G)), \lambda_2(H(G)), \dots, \lambda_n(H(G))\}$ to denote the *spectrum* of $H(G)$ in nonincreasing order. The set of these eigenvalues is called the *Hermitian adjacency spectrum* (or briefly *H -spectrum*) of G . Let $V(G) = \{v_1, v_2, \dots, v_n\}$, and let $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be a diagonal matrix, in which d_i is the degree of the vertex v_i in the underlying graph $\Gamma(G)$. Then the matrix $L(G) = D(G) - H(G)$ is called the *Hermitian Laplacian matrix* of G , and the matrix $\mathcal{L}(G) = I - D(G)^{-\frac{1}{2}} H(G) D(G)^{-\frac{1}{2}}$ is called the *normalized Hermitian Laplacian matrix* of G . Here I is the $n \times n$ identity matrix. We denote by $\lambda_i(\mathcal{L}(G))$ the i -th largest eigenvalue of $\mathcal{L}(G)$ (multiplicities counted). We use $\{\lambda_1(\mathcal{L}(G)), \lambda_2(\mathcal{L}(G)), \dots, \lambda_n(\mathcal{L}(G))\}$ to denote the spectrum of $\mathcal{L}(G)$ in nonincreasing order. The set of these eigenvalues is called the *normalized Hermitian Laplacian spectrum* of G .

If we regard (replace) each edge $v_i v_j \in E(G)$ in $G = (V(G), E(G), A(G))$ as (by) two oppositely directed arcs (v_i, v_j) and (v_j, v_i) , then G is a directed graph. Throughout this paper, we regard mixed graphs as directed graphs, in the above sense. Next, we give the

definition of a *general random mixed graph* $\widehat{G}_n(p_{ij})$. Let K_n be a complete graph on n vertices. The *complete directed graph* DK_n is the graph obtained from K_n by replacing each edge of K_n by two oppositely directed arcs. Let p_{ij} be a function of n such that $0 < p_{ij} < 1$ ($i \neq j$). We always assume that $p_{ii} = 0$ for all indices i . The random mixed graph model $\widehat{G}_n(p_{ij})$ consists of all random mixed graphs $\widehat{G}_n(p_{ij})$ in which each arc (v_i, v_j) with $i \neq j$ is chosen randomly and independently, with probability p_{ij} from the set of arcs of DK_n , where we let the vertex set be $\{v_1, v_2, \dots, v_n\}$. Here the probabilities p_{ij} for different arcs are not assumed to be equal, that is, $\widehat{G}_n(p_{ij})$ is an arc-independent random mixed graph of order n . Then the *Hermitian adjacency matrix* of $\widehat{G}_n(p_{ij})$, denoted by $H(\widehat{G}_n(p_{ij})) = (h_{ij})$ (or H_n , for brevity), satisfies that:

- H_n is a random Hermitian matrix, with $h_{ii} = 0$ for $1 \leq i \leq n$;
- the upper-triangular entries h_{ij} , $1 \leq i < j \leq n$ are independent random variables, which take value 1 with probability $p_{ij}p_{ji}$, \mathbf{i} with probability $p_{ij}(1 - p_{ji})$, $-\mathbf{i}$ with probability $(1 - p_{ij})p_{ji}$, and 0 with probability $(1 - p_{ij})(1 - p_{ji})$.

If A is a random $n \times n$ matrix, we write $\mathbb{E}(A)$ to denote the coordinate-wise expectation of A , so $\mathbb{E}(A)_{ij} = \mathbb{E}(A_{ij})$. Similarly, $\text{Var}(A) = \mathbb{E}((A - \mathbb{E}(A))^2)$. We use $\mathbb{E}H_n$ as shorthand for $\mathbb{E}(H_n)$, and note that it is obvious that $(\mathbb{E}H_n)_{ij} = \mathbb{E}(h_{ij}) = p_{ij}p_{ji} + \mathbf{i}(p_{ij} - p_{ji})$.

1.3 Our main results

In this paper we study the spectra of the Hermitian adjacency matrix and the normalized Hermitian Laplacian matrix of general random mixed graphs. Throughout adopting the above notation and terminology, our main results can be stated as follows.

For the first theorem, we use $\Delta(\Gamma(\widehat{G}_n(p_{ij})))$ to denote the maximum expected degree of the underlying graph of $\widehat{G}_n(p_{ij})$. Hence, by straightforward calculations, $\Delta(\Gamma(\widehat{G}_n(p_{ij}))) = \max_{1 \leq i \leq n} \sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij}p_{ji})$.

Theorem 1. *Let $\widehat{G}_n(p_{ij})$ and $H_n = (h_{ij})$ be defined as above, and let $\Delta = \Delta(\Gamma(\widehat{G}_n(p_{ij})))$. Let $\epsilon > 0$ be an arbitrarily small constant, chosen such that for n sufficiently large, $\Delta > \frac{4}{9} \ln(2n/\epsilon)$. Then with probability at least $1 - \epsilon$, for n sufficiently large, the eigenvalues of H_n satisfy*

$$|\lambda_i(H_n)| \leq \max_{1 \leq i \leq n} \sum_{j=1}^n \sqrt{p_{ij}^2 p_{ji}^2 + (p_{ij} - p_{ji})^2} + 2\sqrt{\Delta \ln(2n/\epsilon)}$$

for all $1 \leq i \leq n$.

Applying Theorem 1 with $\epsilon = \frac{1}{n}$, we immediately obtain the following corollaries.

Corollary 2. *Let $\widehat{G}_n(p_{ij})$ and $H_n = (h_{ij})$ be defined as in Theorem 1, and let $\Delta = \Delta(\Gamma(\widehat{G}_n(p_{ij})))$. If $\Delta > \frac{8}{9} \ln(\sqrt{2}n)$, then with probability at least $1 - \frac{1}{n} = 1 - o(1)$, the*

eigenvalues of H_n satisfy

$$|\lambda_i(H_n)| \leq \max_{1 \leq i \leq n} \sum_{j=1}^n \sqrt{p_{ij}^2 p_{ji}^2 + (p_{ij} - p_{ji})^2} + 2\sqrt{2\Delta \ln(\sqrt{2}n)}$$

for all $1 \leq i \leq n$.

In particular, consider the case that $p_{ij} = p$, with $p \in (0, 1)$ for $i \neq j$ and 0 for $i = j$. We denote this graph by $\widehat{G}_n(p)$ and define

$$\begin{aligned} \beta &:= \frac{2\sqrt{2\Delta \ln(\sqrt{2}n)}}{\max_{1 \leq i \leq n} \sum_{j=1}^n \sqrt{p_{ij}^2 p_{ji}^2 + (p_{ij} - p_{ji})^2}} \\ &= \frac{2\sqrt{2(2p - p^2)(n-1) \ln(\sqrt{2}n)}}{p^2(n-1)} \\ &= 2\sqrt{2} \cdot \sqrt{\frac{2-p}{p^3} \cdot \frac{\ln(\sqrt{2}n)}{n-1}}. \end{aligned}$$

- If p is a constant, then $\beta \rightarrow 0$ as $n \rightarrow \infty$.
- If $p \rightarrow 1$, then $\beta \rightarrow 0$ as $n \rightarrow \infty$.
- Let $p = \frac{1}{n^l}$. If $0 < l < \frac{1}{3}$, then $\beta \rightarrow 0$ as $n \rightarrow \infty$; if $l \geq \frac{1}{3}$, then $\beta \rightarrow \infty$ as $n \rightarrow \infty$.

Corollary 3. For $\widehat{G}_n(p)$, if $(n-1)(2p - p^2) > \frac{8}{9} \ln(\sqrt{2}n)$, then with probability at least $1 - \frac{1}{n} = 1 - o(1)$, we have

$$|\lambda_i(H_n)| \leq (1 + o(1))(n-1)p^2, \text{ for all } 1 \leq i \leq n,$$

when p is a constant or $p \rightarrow 1$ or $p = \frac{1}{n^l}$, $0 < l < \frac{1}{3}$.

In the next theorem, we assume that $V(\widehat{G}_n(p_{ij})) = \{v_1, v_2, \dots, v_n\}$, and we let $D_n = \text{diag}(d_1, d_2, \dots, d_n)$ denote the diagonal matrix in which d_i is the degree of the vertex v_i in the underlying graph of $\widehat{G}_n(p_{ij})$. We let $\mathbb{E}D_n$ denote the coordinate-wise expectation of D_n . Recall that $\mathcal{L}_n = I_n - D_n^{-1/2} H_n D_n^{-1/2}$ denotes the normalized Hermitian Laplacian matrix of $\widehat{G}_n(p_{ij})$, where I_n denotes the $n \times n$ identity matrix. We let $\delta(\Gamma(\widehat{G}_n(p_{ij})))$ denote the minimum expected degree of the underlying graph of $\widehat{G}_n(p_{ij})$. Hence, $\delta(\Gamma(\widehat{G}_n(p_{ij}))) = \min_{1 \leq i \leq n} \sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij}p_{ji})$.

Theorem 4. Let $\widehat{G}_n(p_{ij})$, H_n , D_n and \mathcal{L}_n be defined as above, and let $\delta = \delta(\Gamma(\widehat{G}_n(p_{ij})))$. Let $\epsilon > 0$ be an arbitrarily small constant. Then there exists a constant $k = k(\epsilon)$ such that if $\delta > k \ln n$, then with probability at least $1 - \epsilon$, the eigenvalues of \mathcal{L}_n and $\widetilde{\mathcal{L}}_n$ satisfy

$$|\lambda_i(\mathcal{L}_n) - \lambda_i(\widetilde{\mathcal{L}}_n)| \leq 7\sqrt{\frac{\ln(4n/\epsilon)}{\delta}}$$

for all $1 \leq i \leq n$, where $\widetilde{\mathcal{L}}_n = I_n - (\mathbb{E}D_n)^{-1/2}(\mathbb{E}H_n)(\mathbb{E}D_n)^{-1/2}$.

Applying Theorem 4 with $\epsilon = \frac{1}{n}$, we immediately obtain the following corollaries.

Corollary 5. Let $\widehat{G}_n(p_{ij})$, H_n , D_n and \mathcal{L}_n be defined as in Theorem 4, and let $\delta = \delta(\Gamma(\widehat{G}_n(p_{ij})))$. If $\delta \gg \ln n$, then with probability at least $1 - \frac{1}{n} = 1 - o(1)$, the eigenvalues of \mathcal{L}_n and $\widetilde{\mathcal{L}}_n$ satisfy

$$|\lambda_i(\mathcal{L}_n) - \lambda_i(\widetilde{\mathcal{L}}_n)| \leq 7\sqrt{\frac{2\ln(2n)}{\delta}} = o(1)$$

for all $1 \leq i \leq n$, where $\widetilde{\mathcal{L}}_n = I_n - (\mathbb{E}D_n)^{-1/2}(\mathbb{E}H_n)(\mathbb{E}D_n)^{-1/2}$.

If J_n is the $n \times n$ all-ones matrix, then for $\widehat{G}_n(p)$, $\mathbb{E}H_n = p^2(J_n - I_n)$, $\mathbb{E}D_n = (n-1)(2p-p^2)I_n$ and $\widetilde{\mathcal{L}}_n = I_n - \frac{p}{(n-1)(2-p)}(J_n - I_n)$. Then we have the following.

Corollary 6. For $\widehat{G}_n(p)$, if $(n-1)(2p-p^2) \gg \ln n$, then with probability at least $1 - \frac{1}{n} = 1 - o(1)$, we have

$$\left| \lambda_i(\mathcal{L}_n) - \lambda_i \left(I_n - \frac{p}{(n-1)(2-p)}(J_n - I_n) \right) \right| \leq 7\sqrt{\frac{2\ln(2n)}{(n-1)(2p-p^2)}} = o(1)$$

for all $1 \leq i \leq n$.

The spectrum of $I_n - \frac{p}{(n-1)(2-p)}(J_n - I_n)$ is $\{1 - \frac{p}{2-p}, 1 + \frac{p}{(n-1)(2-p)}\}$, where $1 + \frac{p}{(n-1)(2-p)}$ has multiplicity $n-1$. From Corollary 6, we see that, if $(n-1)(2p-p^2) \gg \ln n$ and $p \rightarrow 0$, then with high probability all eigenvalues of \mathcal{L}_n are close to 1; otherwise, if $(n-1)(2p-p^2) \gg \ln n$ and $p \not\rightarrow 0$, all eigenvalues of \mathcal{L}_n except $\lambda_{\min}(\mathcal{L}_n)$ are close to 1.

We postpone the proofs of Theorem 1 and Theorem 4 to Section 3. In the next section, we state some existing results and prove an auxiliary new result that we need in our proof of Theorem 1.

2 Preliminaries and Auxiliary Results

We start with some additional terminology and notation that we will use throughout the paper.

2.1 Additional terminology and notation

We let $\mathbb{C}_{Herm}^{n \times n}$ denote the set of $n \times n$ Hermitian matrices, which is a subset of the set $\mathbb{C}^{n \times n}$ of all $n \times n$ matrices with complex entries. For each matrix $M \in \mathbb{C}^{n \times n}$, the *spectral radius* of M is the nonnegative real number $\rho(M) = \max\{|\lambda_i(M)| : 1 \leq i \leq n\}$, where $\lambda_i(M)$ ($1 \leq i \leq n$) are all eigenvalues of M . We use $\lambda_{\max}(M)$ to denote the largest eigenvalue of M . The set $\{\lambda_i(M) : 1 \leq i \leq n\}$ is called the *spectrum* of M , and denoted by $\text{spec}(M)$. The *spectral norm* $\|M\|$ is the largest *singular value* of M , i.e., we have

$$\|M\| = \sqrt{\lambda_{\max}(M^*M)}.$$

Here M^* is the *conjugate transpose* of M . The *Spectral Theorem* for Hermitian matrices states that all $M \in \mathbb{C}_{Herm}^{n \times n}$ have n real eigenvalues (possibly with repetitions) that correspond to an orthonormal set of eigenvectors.

When $M \in \mathbb{C}_{Herm}^{n \times n}$, we have $\|M\| = \max\{|\lambda_i(M)| : 1 \leq i \leq n\}$. Then $\rho(M) = \|M\| = \max\{\lambda_{\max}(M), \lambda_{\max}(-M)\}$. We use $\text{Tr}(M)$ (the *trace* of M) to denote the sum of the eigenvalues of M .

We will use the notation $A \succeq \mathbf{0}$ to indicate that A is *positive semidefinite*, i.e., $A \in \mathbb{C}_{Herm}^{n \times n}$ and its eigenvalues are nonnegative, and use the notation $A \succ \mathbf{0}$ to indicate that A is *positive definite*, i.e., $A \in \mathbb{C}_{Herm}^{n \times n}$ and its eigenvalues are positive, where $\mathbf{0}$ is the zero matrix of the same size as A . With \preceq we denote the *positive semidefinite order* on Hermitian matrices, as follows. Given two Hermitian matrices A and B , we use $A \preceq B$ or $B \succeq A$ to indicate that $B - A \succeq \mathbf{0}$.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire analytic function with a power-series representation $f(x) \equiv \sum_{n=0}^{\infty} a_n x^n$ ($x \in \mathbb{C}$). If all a_n are real, the expression:

$$f(A) \equiv \sum_{n=0}^{\infty} a_n A^n \quad (A \in \mathbb{C}_{Herm}^{d \times d})$$

corresponds to a mapping from $\mathbb{C}_{Herm}^{d \times d}$ to itself. We note that notions of convergence are as in [21]. The *Spectral Mapping Theorem* states that each eigenvalue of $f(A)$ is equal to $f(\lambda)$ with $\lambda \in \text{spec}(A)$, i.e.,

$$\text{spec}(f(A)) = f(\text{spec}(A)). \quad (2.1)$$

In the sequel, we use the following lemma applied to the matrix exponential, to be defined shortly.

Lemma 7 (Lieb [29]). *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$, and suppose there is a subset $S \subseteq \mathbb{R}$ with $f(a) \leq g(a)$ for all $a \in S$. If A is a Hermitian matrix with all eigenvalues contained in S , then $f(A) \preceq g(A)$.*

In our proofs, we make use of the *matrix exponential*, defined as $\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$. From the Spectral Mapping Theorem we know that $\exp(A)$ is always positive definite when A is Hermitian, and that $\exp(A)$ converges for all choices of A . By Lemma 7, we have:

$$\text{for any } A \in \mathbb{C}_{Herm}^{d \times d}, \quad I + A \preceq e^A. \quad (2.2)$$

Moreover, we shall require brief use of the matrix logarithm. The *matrix logarithm* is defined as the functional inverse of the matrix exponential:

$$\text{for any } A \in \mathbb{C}_{Herm}^{d \times d}, \quad \log(e^A) := A. \quad (2.3)$$

This formula defines the logarithm of a positive definite matrix. In general, if $B = \exp(A)$, we say that A is the logarithm of B . As our matrices will be Hermitian, it is sufficient for uniqueness of this function to require that the logarithm also be Hermitian. The matrix logarithm is monotone with respect to the positive semidefinite order (See [4]):

$$\text{for any } A, B \in \mathbb{C}_{Herm}^{d \times d}, \text{ if } A \succ \mathbf{0}, B \succ \mathbf{0} \text{ and } A \preceq B, \text{ then } \log(A) \preceq \log(B). \quad (2.4)$$

Any notation not mentioned here pertaining to matrices is as in [21].

2.2 Auxiliary concentration results

We shall require the following concentration inequalities in order to prove our main theorems. Various matrix concentration inequalities have been derived by many authors, including Ahlswede and Winter [1], Cristofides and Markström [11], Oliveira [26], Gross [19], Recht [28], Tropp [29], and Chung and Radcliffe [8]. In [8], Chung and Radcliffe give a short proof for the following relatively simple version that is particularly suitable for random graphs.

Theorem 8 ([8]). *Let X_1, X_2, \dots, X_m be independent random $n \times n$ Hermitian matrices. Moreover, assume that $\|X_i - \mathbb{E}(X_i)\| \leq c$ for all i . Let $X = \sum_{i=1}^m X_i$. Then for any $a > 0$,*

$$\Pr(\|X - \mathbb{E}(X)\| \geq a) \leq 2n \exp\left(-\frac{a^2}{2\|\sum_{i=1}^m \text{Var}(X_i)\| + 2ac/3}\right).$$

A strengthened version of Theorem 8 that we need for one of our proofs in Section 3, is as follows.

Theorem 9. *Let X_1, X_2, \dots, X_m be independent random $n \times n$ Hermitian matrices. Moreover, assume that $\|X_i\| \leq c$ for all i . Let $X = \sum_{i=1}^m X_i$. Then*

$$\Pr(\lambda_{\max}(X) \geq a) \leq n \exp\left(-\frac{(a - \|\mathbb{E}(X)\|)^2}{2\|\sum_{i=1}^m \mathbb{E}(X_i^2)\| + \frac{2c}{3}(a - \|\mathbb{E}(X)\|)}\right) \text{ for } a > \|\mathbb{E}(X)\|.$$

In particular,

$$\Pr(\|X\| \geq a) \leq 2n \exp\left(-\frac{(a - \|\mathbb{E}(X)\|)^2}{2\|\sum_{i=1}^m \mathbb{E}(X_i^2)\| + \frac{2c}{3}(a - \|\mathbb{E}(X)\|)}\right) \text{ for } a > \|\mathbb{E}(X)\|. \quad (2.5)$$

Before we present our proof of Theorem 9, we will first show that Theorem 9 implies Theorem 8. For this purpose, let X_i ($1 \leq i \leq m$) be as in Theorem 8. Let $X'_i = X_i - \mathbb{E}(X_i)$ and $X' = \sum_{i=1}^m X'_i = X - \mathbb{E}(X)$. Then $\mathbb{E}(X') = \mathbf{0}$. From the hypothesis of Theorem 8, we see that

$$\|X'_i\| \leq c \text{ for all } i \in \{1, \dots, m\}.$$

We also have

$$\begin{aligned} \left\| \sum_{i=1}^m \mathbb{E}(X_i'^2) \right\| &= \left\| \sum_{i=1}^m \mathbb{E}(X_i - \mathbb{E}(X_i))^2 \right\| \\ &= \left\| \sum_{i=1}^m \text{Var}(X_i) \right\|. \end{aligned}$$

Applying Theorem 9, we get that for $a > 0 = \|\mathbb{E}(X')\|$,

$$\Pr(\|X - \mathbb{E}(X)\| \geq a) = \Pr(\|X'\| \geq a)$$

$$\begin{aligned} &\leq 2n \exp \left(- \frac{(a - \|\mathbb{E}(X')\|)^2}{2\|\sum_{i=1}^m \mathbb{E}(X_i'^2)\| + \frac{2c}{3}(a - \|\mathbb{E}(X')\|)} \right) \\ &= 2n \exp \left(- \frac{a^2}{2\|\sum_{i=1}^m \text{Var}(X_i)\| + 2ac/3} \right). \end{aligned}$$

This shows that Theorem 9 implies Theorem 8.

2.3 The proof of Theorem 9

We are now going to prove Theorem 9. For our proof, we will rely on Lemma 7 and the following known result.

Lemma 10 ([29]). *Consider a finite sequence $\{X_i\}_{i=1}^m$ of independent, random, Hermitian matrices. Then*

$$\mathbb{E} \left[\text{Tr} \left(\exp \left(\sum_{i=1}^m \theta X_i \right) \right) \right] \leq \text{Tr} \left(\exp \left(\sum_{i=1}^m \log \mathbb{E}(\exp(\theta X_i)) \right) \right) \text{ for any constant } \theta \in \mathbb{R}.$$

Proof of Theorem 9. We define

$$g(x) = 2 \sum_{k=2}^{\infty} \frac{x^{k-2}}{k!} = \frac{2(e^x - 1 - x)}{x^2},$$

and first show the following facts about g , followed by short justifications for the statements.

- $g(0) = 1$. In fact,

$$g(0) = \lim_{x \rightarrow 0} \frac{2(e^x - 1 - x)}{x^2} = \lim_{x \rightarrow 0} \frac{2(e^x - 1)}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{1} = 1.$$

- $g(x)$ is monotone increasing for $x \geq 0$. Note that for $x \neq 0$, $g'(x) = 2x^{-3}((x-2)e^x + x + 2)$, and so it suffices to show that $h(x) = (x-2)e^x + x + 2$ satisfies $h(x) \geq 0$ for all $x \in \mathbb{R}$. Clearly, $h(0) = 0$ and $h'(x) = (x-1)e^x + 1$. Hence, $h'(0) = 0$ and $h''(x) = xe^x$, so $h''(x) < 0$ for $x < 0$ and $h''(x) > 0$ for $x > 0$. Therefore, $h'(x)$ is monotone decreasing in $x \in (-\infty, 0]$ and $h'(x)$ is monotone increasing in $x \in (0, +\infty)$. So, $h'(x) \geq h'(0) = 0$ for all $x \in \mathbb{R}$. Thus, $h(x)$ is monotone increasing for all $x \in \mathbb{R}$. Indeed, $h(x) \geq h(0) = 0$ for all $x \in \mathbb{R}$, as required.
- $g(x) \leq 1$ for $x < 0$. In fact, $g'(x) = 2x^{-3}h(x) \leq 0$ if $x < 0$. So, the function g is decreasing for $x < 0$. Thus, $g(x) \leq g(0) = 1$ for $x < 0$.

- for $x < 3$, using $k! \geq 2 \cdot 3^{k-2}$, we obtain

$$g(x) = 2 \sum_{k=2}^{\infty} \frac{x^{k-2}}{k!} \leq \sum_{k=2}^{\infty} \frac{x^{k-2}}{3^{k-2}} = \frac{1}{1-x/3}. \quad (2.6)$$

Recalling that $g(x)$ is monotone increasing for $x \geq 0$, for $0 < x \leq c$, we get $g(x) \leq g(c)$. Now let X_i ($1 \leq i \leq m$) be as in the hypothesis of Theorem 9.

Given a real constant $\theta > 0$, we have $\|\theta X_i\| \leq \theta c$. Applying Lemma 7, we obtain that $g(\theta X_i) \preceq g(\theta c)I$. Therefore, noting that $e^x = 1 + x + \frac{1}{2}x^2g(x)$, we have

$$\begin{aligned} e^{\theta X_i} &= I + \theta X_i + \frac{1}{2}\theta^2 g(\theta X_i)X_i^2 \\ &\preceq I + \theta X_i + \frac{1}{2}\theta^2 g(\theta c)X_i^2. \end{aligned} \quad (2.7)$$

We now use that the expectation respects the positive semidefinite order (See [29]), i.e.,

$$\text{for any } A, B \in \mathbb{C}_{Herm}^{d \times d}, \quad A \preceq B \text{ almost surely implies } \mathbb{E}A \preceq \mathbb{E}B. \quad (2.8)$$

Using (2.2), (2.7), and (2.8), we obtain

$$\begin{aligned} \mathbb{E}(e^{\theta X_i}) &\preceq \mathbb{E}(I + \theta X_i + \frac{1}{2}\theta^2 g(\theta c)X_i^2) \\ &= I + \theta \mathbb{E}(X_i) + \frac{1}{2}\theta^2 g(\theta c)\mathbb{E}(X_i^2) \\ &\preceq e^{\theta \mathbb{E}(X_i) + \frac{1}{2}\theta^2 g(\theta c)\mathbb{E}(X_i^2)}. \end{aligned} \quad (2.9)$$

Next, we prove the following claim related to the trace of the matrix exponential.

Claim 1. For the given matrices X_i ,

$$\mathbb{E}\left[\text{Tr}\left(\exp\left(\sum_{i=1}^m \theta X_i\right)\right)\right] \leq \text{Tr}\left(\exp\left(\theta \mathbb{E}(X) + \frac{1}{2}\theta^2 g(\theta c) \sum_{i=1}^m \mathbb{E}(X_i^2)\right)\right). \quad (2.10)$$

Proof of Claim 1. Here we work with the trace of the matrix exponential, $\text{Tr}(\exp) : A \mapsto \text{Tr}(\exp(A))$. This trace exponential function is monotone with respect to the positive semidefinite order, i.e.,

$$\forall A, B \in \mathbb{C}_{Herm}^{d \times d}, \quad A \preceq B \text{ implies } \text{Tr}(\exp(A)) \leq \text{Tr}(\exp(B)). \quad (2.11)$$

See, e.g., [27], Section 2 for a short proof of this fact. Now, using Lemma 10, (2.3), (2.4), (2.9) and (2.11), we obtain

$$\mathbb{E}\left[\text{Tr}\left(\exp\left(\sum_{i=1}^m \theta X_i\right)\right)\right] \leq \text{Tr}\left(\exp\left(\sum_{i=1}^m \log \mathbb{E}(\exp(\theta X_i))\right)\right)$$

$$\begin{aligned}
&\leq \text{Tr} \left(\exp \left(\sum_{i=1}^m \log e^{\theta \mathbb{E}(X_i) + \frac{1}{2} \theta^2 g(\theta c) \mathbb{E}(X_i^2)} \right) \right) \\
&= \text{Tr} \left(\exp \left(\sum_{i=1}^m \left(\theta \mathbb{E}(X_i) + \frac{1}{2} \theta^2 g(\theta c) \mathbb{E}(X_i^2) \right) \right) \right) \\
&= \text{Tr} \left(\exp \left(\theta \mathbb{E}(X) + \frac{1}{2} \theta^2 g(\theta c) \sum_{i=1}^m \mathbb{E}(X_i^2) \right) \right),
\end{aligned}$$

as required. \square

Note that $\exp(\theta \lambda_{\max}(X))$ is a random variable. Then, by Markov's inequality, for any real constant $a > 0$,

$$\Pr(\lambda_{\max}(X) \geq a) = \Pr(e^{\theta \lambda_{\max}(X)} \geq e^{\theta a}) \leq e^{-\theta a} \mathbb{E}(e^{\theta \lambda_{\max}(X)}) \quad (2.12)$$

By (2.1), for any $s \geq 0$, and for any $A \in \mathbb{C}_{Herm}^{d \times d}$, the largest eigenvalue of e^{sA} is $e^{s \lambda_{\max}(A)}$ and all eigenvalues of e^{sA} are nonnegative. Hence,

$$\exp(s \lambda_{\max}(A)) = \lambda_{\max}(\exp(sA)) \leq \text{Tr}(\exp(sA)). \quad (2.13)$$

We need two more inequalities from matrix analysis, where the first one is usually referred to as the *Golden-Thompson inequality* (See, e.g., [3]), and the second one can be found, e.g., in [31].

$$\forall d \in \{1, 2, 3, \dots\}, \text{ and any } A, B \in \mathbb{C}_{Herm}^{d \times d}, \text{Tr}(e^{A+B}) \leq \text{Tr}(e^A e^B). \quad (2.14)$$

If A and B are $n \times n$ positive semidefinite Hermitian matrices, then

$$0 \leq \text{Tr}(A \cdot B) \leq \text{Tr}(A) \cdot \lambda_{\max}(B) \leq \text{Tr}(A) \cdot \text{Tr}(B). \quad (2.15)$$

Now, given a real constant $a > \|\mathbb{E}(X)\|$, for every real constant $\theta > 0$, using (2.10), (2.12), (2.13), (2.14), and (2.15), we obtain

$$\begin{aligned}
\Pr(\lambda_{\max}(X) \geq a) &\leq e^{-\theta a} \mathbb{E}(e^{\theta \lambda_{\max}(X)}) \\
&\leq e^{-\theta a} \mathbb{E}(\text{Tr}(\exp(\theta X))) \\
&= e^{-\theta a} \mathbb{E}\left[\text{Tr}(\exp(\sum_{i=1}^m \theta X_i))\right] \\
&\leq e^{-\theta a} \text{Tr}\left[\exp\left(\theta \mathbb{E}(X) + \frac{1}{2} \theta^2 g(\theta c) \sum_{i=1}^m \mathbb{E}(X_i^2)\right)\right] \\
&\leq e^{-\theta a} \text{Tr}\left[\exp\left(\theta \mathbb{E}(X)\right) \cdot \exp\left(\frac{1}{2} \theta^2 g(\theta c) \sum_{i=1}^m \mathbb{E}(X_i^2)\right)\right] \\
&\leq e^{-\theta a} \lambda_{\max}\left(\exp\left(\theta \mathbb{E}(X)\right)\right) \cdot \text{Tr}\left[\exp\left(\frac{1}{2} \theta^2 g(\theta c) \sum_{i=1}^m \mathbb{E}(X_i^2)\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq e^{-\theta a} \lambda_{\max} \left(\exp \left(\theta \mathbb{E}(X) \right) \right) \cdot n \lambda_{\max} \left[\exp \left(\frac{1}{2} \theta^2 g(\theta c) \sum_{i=1}^m \mathbb{E}(X_i^2) \right) \right] \\
&= n e^{-\theta a} \left(\exp \left(\theta \lambda_{\max}(\mathbb{E}(X)) \right) \right) \cdot \exp \left(\frac{1}{2} \theta^2 g(\theta c) \lambda_{\max} \left(\sum_{i=1}^m \mathbb{E}(X_i^2) \right) \right) \\
&= n \exp \left(-\theta a + \theta \lambda_{\max}(\mathbb{E}(X)) + \frac{1}{2} \theta^2 g(\theta c) \lambda_{\max} \left(\sum_{i=1}^m \mathbb{E}(X_i^2) \right) \right) \\
&\leq n \exp \left(-\theta a + \theta \|\mathbb{E}(X)\| + \frac{1}{2} \theta^2 g(\theta c) \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| \right).
\end{aligned}$$

The final inequality holds as $\|\mathbb{E}(X)\| \geq \lambda_{\max}(\mathbb{E}(X))$ and $\left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| \geq \lambda_{\max} \left(\sum_{i=1}^m \mathbb{E}(X_i^2) \right)$.

Recall that we assume $a > \|\mathbb{E}(X)\|$. Now take $\theta = \frac{a - \|\mathbb{E}(X)\|}{\left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| + \frac{c}{3}(a - \|\mathbb{E}(X)\|)}$. Then, clearly $\theta c < 3$. Using (2.6), we obtain

$$\begin{aligned}
\Pr(\lambda_{\max}(X) \geq a) &\leq n \exp \left(-\theta a + \theta \|\mathbb{E}(X)\| + \frac{1}{2} \theta^2 g(\theta c) \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| \right) \\
&\leq n \exp \left(-\theta a + \theta \|\mathbb{E}(X)\| + \frac{\theta^2 \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\|}{2(1 - \frac{\theta c}{3})} \right) \\
&= n \exp \left(-\theta \left[a - \|\mathbb{E}(X)\| - \frac{\theta \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\|}{2(1 - \frac{\theta c}{3})} \right] \right) \\
&= n \exp \left(-\theta \left[a - \|\mathbb{E}(X)\| - \frac{\theta \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\|}{2 \left(1 - \frac{a - \|\mathbb{E}(X)\|}{\left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| + \frac{c}{3}(a - \|\mathbb{E}(X)\|)} \cdot \frac{c}{3} \right)} \right] \right) \\
&= n \exp \left(-\theta \left[a - \|\mathbb{E}(X)\| - \frac{\theta \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\|}{\frac{2 \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\|}{\left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| + \frac{c}{3}(a - \|\mathbb{E}(X)\|)}} \right] \right) \\
&= n \exp \left(-\theta \left[a - \|\mathbb{E}(X)\| - \frac{\theta}{2} \left(\left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| + \frac{c}{3}(a - \|\mathbb{E}(X)\|) \right) \right] \right) \\
&= n \exp \left(-\theta \left[a - \|\mathbb{E}(X)\| - \frac{1}{2}(a - \|\mathbb{E}(X)\|) \right] \right) \\
&= n \exp \left(-\frac{\theta}{2}(a - \|\mathbb{E}(X)\|) \right) \\
&= n \exp \left(-\frac{(a - \|\mathbb{E}(X)\|)^2}{2 \left\| \sum_{i=1}^m \mathbb{E}(X_i^2) \right\| + \frac{2c}{3}(a - \|\mathbb{E}(X)\|)} \right). \tag{2.16}
\end{aligned}$$

This proves the first statement of Theorem 9. To obtain the norm bound (2.5) in the

second statement of Theorem 9, recall that

$$\text{for any } Y \in \mathbb{C}_{Herm}^{n \times n}, \|Y\| = \max\{\lambda_{\max}(Y), -\lambda_{\min}(Y)\} = \max\{\lambda_{\max}(Y), \lambda_{\max}(-Y)\}.$$

Using this, we next apply the inequality (2.16) to the sequence $\{-X_i\}$, i.e., we replace the sequence $\{X_i\}$ by the sequence $\{-X_i\}$ in the above inequality (2.16). We obtain

$$\begin{aligned} \Pr(\lambda_{\max}(-X) \geq a) &\leq n \exp \left(- \frac{(a - \|\mathbb{E}(-X)\|)^2}{2\|\sum_{i=1}^m \mathbb{E}((-X_i)^2)\| + \frac{2c}{3}(a - \|\mathbb{E}(-X)\|)} \right) \\ &= n \exp \left(- \frac{(a - \|\mathbb{E}(X)\|)^2}{2\|\sum_{i=1}^m \mathbb{E}(X_i^2)\| + \frac{2c}{3}(a - \|\mathbb{E}(X)\|)} \right). \end{aligned}$$

Applying the union bound to the estimates for $\lambda_{\max}(X)$ and $-\lambda_{\min}(X)$, we obtain

$$\Pr(\|X\| \geq a) \leq 2n \exp \left(- \frac{(a - \|\mathbb{E}(X)\|)^2}{2\|\sum_{i=1}^m \mathbb{E}(X_i^2)\| + \frac{2c}{3}(a - \|\mathbb{E}(X)\|)} \right).$$

This completes the proof of Theorem 9. \square

3 Proofs of Theorem 1 and Theorem 4

In this section, we provide proofs for the two main theorems of this paper, using Theorem 8, Theorem 9, and a number of other known results.

3.1 The proof of Theorem 1

Before presenting our proof of Theorem 1, we recall one more known result that will be used in the sequel of the paper.

Lemma 11 ([21]). *Let $M = (m_{ij})$ be an $n \times n$ matrix. Then*

$$\rho(M) \leq \min \left\{ \max_{1 \leq i \leq n} \sum_{j=1}^n |m_{ij}|, \max_{1 \leq j \leq n} \sum_{i=1}^n |m_{ij}| \right\}.$$

Proof of Theorem 1. Let $\widehat{G}_n(p_{ij})$ and $H_n = (h_{ij})$ be defined as before, and let $\Delta = \Delta(\Gamma(\widehat{G}_n(p_{ij})))$. Recall that this implies that $\Delta = \max_{1 \leq i \leq n} \sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij}p_{ji})$.

For the indices i and j with $1 \leq i, j \leq n$, let H^{ij} be the $n \times n$ matrix with a 1 in the (i, j) -th position and a 0 everywhere else. Recall that h_{ij} takes value 1 with probability $p_{ij}p_{ji}$, value \mathbf{i} with probability $p_{ij}(1 - p_{ji})$, value $-\mathbf{i}$ with probability $(1 - p_{ij})p_{ji}$, and value 0 with probability $(1 - p_{ij})(1 - p_{ji})$. So, $\overline{h_{ji}} = \overline{h_{ij}}$, i.e., h_{ji} is the complex conjugate of h_{ij} . Take $X_{ij} = h_{ij}H^{ij} + h_{ji}H^{ji} = h_{ij}H^{ij} + \overline{h_{ij}}H^{ji}$. Then, $H_n = \sum_{1 \leq i < j \leq n} X_{ij}$. Now, we can apply Theorem 9 to H_n if we derive a suitable upper bound c on $\|X_{ij}\|$. Note that X_{ij}

$(1 \leq i < j \leq n)$ are independent random $n \times n$ Hermitian matrices, and that, with the choice $c = 1$,

$$\|X_{ij}\| = \|h_{ij}H^{ij} + \overline{h_{ij}}H^{ji}\| = |h_{ij}| < 1 = c.$$

Before applying Theorem 9, we first perform some additional calculations in order to obtain upper bounds for $\|\sum_{1 \leq i < j \leq n} \mathbb{E}(X_{ij}^2)\|$ and $\|\mathbb{E}H_n\|$.

For all $1 \leq i < j \leq n$, we have

$$\begin{aligned} \mathbb{E}(X_{ij}^2) &= \mathbb{E}(h_{ij}H^{ij} + \overline{h_{ij}}H^{ji})^2 \\ &= \mathbb{E}[h_{ij} \cdot \overline{h_{ij}}](H^{ii} + H^{jj}) \\ &= \mathbb{E}[|h_{ij}|^2](H^{ii} + H^{jj}) \\ &= (p_{ij} + p_{ji} - p_{ij}p_{ji})(H^{ii} + H^{jj}). \end{aligned}$$

We set $p_{ii} = 0$. Then,

$$\begin{aligned} \left\| \sum_{1 \leq i < j \leq n} \mathbb{E}(X_{ij}^2) \right\| &= \left\| \sum_{i=1}^n \left(\sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij}p_{ji}) \right) H^{ii} \right\| \\ &= \max_{i=1, \dots, n} \sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij}p_{ji}) \\ &= \Delta. \end{aligned}$$

Recall that $(\mathbb{E}H_n)_{ij} = \mathbb{E}h_{ij} = p_{ij}p_{ji} + \mathbf{i}(p_{ij} - p_{ji})$, and in particular, $\mathbb{E}H_n$ is a Hermitian matrix. So, $\|\mathbb{E}H_n\| = \rho(\mathbb{E}H_n)$. By Lemma 11, we have

$$\begin{aligned} \|\mathbb{E}H_n\| &= \rho(\mathbb{E}H_n) \\ &\leq \min \left\{ \max_{1 \leq i \leq n} \sum_{j=1}^n |\mathbb{E}h_{ij}|, \max_{1 \leq j \leq n} \sum_{i=1}^n |\mathbb{E}h_{ij}| \right\} \\ &\leq \max_{1 \leq i \leq n} \sum_{j=1}^n \sqrt{p_{ij}^2 p_{ji}^2 + (p_{ij} - p_{ji})^2}. \end{aligned}$$

Now, we take $a = \|\mathbb{E}H_n\| + \sqrt{4\Delta \ln(2n/\epsilon)}$. By the assumption that $\Delta > \frac{4}{9} \ln(2n/\epsilon)$, we obtain that $a - \|\mathbb{E}H_n\| < 3\Delta$. Applying Theorem 9, and using $c = 1$, we get

$$\begin{aligned} \Pr(\|H_n\| \geq a) &\leq 2n \exp \left(- \frac{(a - \|\mathbb{E}H_n\|)^2}{2\|\sum_{1 \leq i < j \leq n} \mathbb{E}(X_{ij}^2)\| + \frac{2c}{3}(a - \|\mathbb{E}H_n\|)} \right) \\ &\leq 2n \exp \left(- \frac{4\Delta \ln(2n/\epsilon)}{4\Delta} \right) \\ &= \epsilon. \end{aligned}$$

Thus, with probability at least $1 - \epsilon$, we have that for all $1 \leq i \leq n$,

$$|\lambda_i(H_n)| \leq \|H_n\|$$

$$\begin{aligned} &\leq a \\ &= \|\mathbb{E}H_n\| + \sqrt{4\Delta \ln(2n/\epsilon)}. \end{aligned}$$

This completes the proof of Theorem 1. \square

3.2 The proof of Theorem 4

Before presenting the proof of Theorem 4, we recall some known results that will be used in the sequel of the paper.

Lemma 12 (Weyl [30]). *Let X , Y and Z be $n \times n$ Hermitian matrices such that $X = Y + Z$. Suppose that X, Y, Z have eigenvalues $\lambda_1(X) \geq \dots \geq \lambda_n(X)$, $\lambda_1(Y) \geq \dots \geq \lambda_n(Y)$, $\lambda_1(Z) \geq \dots \geq \lambda_n(Z)$, respectively. Then for $i = 1, 2, \dots, n$ the following inequalities hold:*

$$\lambda_i(Y) + \lambda_n(Z) \leq \lambda_i(X) \leq \lambda_i(Y) + \lambda_1(Z).$$

We also use the following concentration result. It involves a variation on the Chernoff bound, and can, e.g., be found as Lemma A in [6].

Lemma 13. *Let X_1, X_2, \dots, X_m be independent random variables satisfying $|X_i| \leq c$ for all i . Let $X = \sum_{i=1}^m X_i$. Then for any $a > 0$,*

$$\Pr(|X - \mathbb{E}(X)| \geq a) \leq \exp\left(-\frac{a^2}{2\sum_{i=1}^m \text{Var}(X_i) + 2ac/3}\right).$$

Finally, we will use the following min-max result due to Courant-Fischer (Theorem 4.2.11 in [21]).

Lemma 14. *Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, and let k be an integer with $1 \leq k \leq n$. Then*

$$\lambda_k = \min_{w_1, w_2, \dots, w_{n-k} \in \mathbb{C}^n} \max_{\substack{x \neq 0, x \in \mathbb{C}^n \\ x \perp w_1, w_2, \dots, w_{n-k}}} \frac{x^* A x}{x^* x}.$$

Let $G = (V(G), E(G), A(G))$ be a mixed graph of order n . For brevity, we write D for $D(G)$, L for $L(G)$ and \mathcal{L} for $\mathcal{L}(G)$. Hence, $\mathcal{L} = I - D^{-\frac{1}{2}} H D^{-\frac{1}{2}} = D^{-1/2} L D^{-1/2}$. We are first going to show that \mathcal{L} is positive semidefinite, by deriving an alternative expression for $\frac{x^* \mathcal{L} x}{x^* x}$ for an arbitrary nonzero complex $n \times 1$ column vector x . After that, we are going to expand the alternative expression in order to obtain an upper bound for the eigenvalues of \mathcal{L} , using Lemma 14.

In the following expansion, $y = D^{-1/2}x$, $N(v_i)$ denotes the neighborhood of v_i in the underlying graph $\Gamma(G)$, and $\sum_{e=v_i v_j}$ denotes the sum over all unordered pairs $\{v_i, v_j\}$ for which v_i and v_j are adjacent in $\Gamma(G)$.

$$\frac{x^* \mathcal{L} x}{x^* x} = \frac{x^* D^{-1/2} L D^{-1/2} x}{x^* x}$$

$$\begin{aligned}
&= \frac{y^* Ly}{(D^{1/2}y)^*(D^{1/2}y)} \\
&= \frac{y^* Ly}{y^* Dy} \\
&= \frac{(y_1^*, y_2^*, \dots, y_n^*) \begin{pmatrix} d_1 & -h_{12} & \cdots & -h_{1n} \\ -h_{21} & d_2 & \cdots & -h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -h_{n1} & -h_{n2} & \cdots & d_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}{(y_1^*, y_2^*, \dots, y_n^*) \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} \\
&= \frac{\sum_{v_i \in V(G)} d_i |y_i|^2 - \sum_{v_i \neq v_j} h_{ij} y_i^* y_j}{\sum_{v_i \in V(G)} d_i |y_i|^2} \\
&= \frac{\sum_{v_i} \sum_{v_j \in N(v_i)} |y_j|^2 - \sum_{v_i \neq v_j} h_{ij} y_i^* y_j}{\sum_{v_i \in V(G)} d_i |y_i|^2} \\
&= \frac{\sum_{e=v_i v_j} (|y_i|^2 + |y_j|^2) - \sum_{e=v_i v_j} (h_{ij} y_i^* y_j + h_{ji} y_i y_j^*)}{\sum_{v_i \in V(G)} d_i |y_i|^2} \\
&= \frac{\sum_{e=v_i v_j} (|y_i|^2 + |y_j|^2) - \sum_{e=v_i v_j} (h_{ij} y_i^* y_j + h_{ij}^* y_i y_j^*)}{\sum_{v_i \in V(G)} d_i |y_i|^2} \\
&= \frac{\sum_{e=v_i v_j} (y_i - h_{ij} y_j)(y_i^* - h_{ij}^* y_j^*)}{\sum_{v_i \in V(G)} d_i |y_i|^2} \\
&= \frac{\sum_{e=v_i v_j} (y_i - h_{ij} y_j)(y_i - h_{ij} y_j)^*}{\sum_{v_i \in V(G)} d_i |y_i|^2} \\
&= \frac{\sum_{e=v_i v_j} |y_i - h_{ij} y_j|^2}{\sum_{v_i \in V(G)} d_i |y_i|^2}.
\end{aligned}$$

Before we continue our calculations, we note that the derived expression for $\frac{x^* \mathcal{L} x}{x^* x}$ implies that \mathcal{L} is positive semidefinite. Next, we are going to expand the obtained expression for $\frac{x^* \mathcal{L} x}{x^* x}$, using the known fact that

$$|f(x) - f(y)|^2 \leq 2(|f(x)|^2 + |f(y)|^2), \quad (3.1)$$

where equality holds if and only if $f(x) = -f(y)$.

We split $\sum_{e=v_i v_j}$ in the above expression by distinguishing undirected edges (or pairs of oppositely oriented arcs), denoted as $v_i \leftrightarrow v_j$, and arcs, denoted as $v_i \rightarrow v_j$ if the orientation is from v_i to v_j , and as $v_i \leftarrow v_j$ if the orientation is from v_j to v_i . Adopting

this notation, and using (3.1), we obtain

$$\begin{aligned}
\sum_{e=v_i v_j} |y_i - h_{ij} y_j|^2 &= \sum_{v_i \leftrightarrow v_j} |y_i - y_j|^2 + \sum_{v_i \rightarrow v_j \text{ or } v_i \leftarrow v_j} |y_i - h_{ij} y_j|^2 \\
&\leq \sum_{v_i \leftrightarrow v_j} 2(|y_i|^2 + |y_j|^2) + \sum_{v_i \rightarrow v_j \text{ or } v_i \leftarrow v_j} 2(|y_i|^2 + |h_{ij} y_j|^2) \\
&= \sum_{v_i \leftrightarrow v_j} 2(|y_i|^2 + |y_j|^2) + \sum_{v_i \rightarrow v_j \text{ or } v_i \leftarrow v_j} 2(|y_i|^2 + |y_j|^2) \\
&= 2 \left(\sum_{v_i \leftrightarrow v_j} (|y_i|^2 + |y_j|^2) + \sum_{v_i \rightarrow v_j \text{ or } v_i \leftarrow v_j} (|y_i|^2 + |y_j|^2) \right).
\end{aligned}$$

We also obtain

$$\begin{aligned}
\sum_{v_i \in V(G)} d_i |y_i|^2 &= \sum_{v_i} \sum_{v_j \in N(v_i)} |y_j|^2 \\
&= \sum_{e=v_i v_j} (|y_i|^2 + |y_j|^2) \\
&= \sum_{v_i \leftrightarrow v_j} (|y_i|^2 + |y_j|^2) + \sum_{v_i \rightarrow v_j \text{ or } v_i \leftarrow v_j} (|y_i|^2 + |y_j|^2).
\end{aligned}$$

Therefore, using the latter two expressions and applying Lemma 14, we get the following upper bound on the eigenvalues of \mathcal{L} .

$$\lambda_i(\mathcal{L}) \leq \sup_x \frac{\sum_{e=v_i v_j} |y_i - h_{ij} y_j|^2}{\sum_{v_i \in V(G)} d_i |y_i|^2} \leq 2.$$

This shows that the normalized Hermitian Laplacian spectrum is in $[0, 2]$, and hence that $\|I - \mathcal{L}\| \leq 1$. We will use this conclusion near the end of the proof of Theorem 4. We now have all the ingredients to present our proof of Theorem 4.

Proof of Theorem 4. Let $\widehat{G}_n(p_{ij})$ and $H_n = (h_{ij})$ be defined as before, and let $\delta = \delta(\Gamma(\widehat{G}_n(p_{ij})))$. Recall that this implies that $\delta = \min_{1 \leq i \leq n} \sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij} p_{ji})$.

For each vertex v_i of $\widehat{G}_n(p_{ij})$, we let d_i denote the degree of v_i in the underlying graph $\Gamma(\widehat{G}_n(p_{ij}))$, and we use $t_i = \mathbb{E}(d_i)$ to denote the expected degree of v_i , so $\mathbb{E}D_n = \text{diag}(\mathbb{E}(d_1), \mathbb{E}(d_2), \dots, \mathbb{E}(d_n)) = \text{diag}(t_1, t_2, \dots, t_n)$. This means that the matrix $\widetilde{\mathcal{L}}_n = I_n - (\mathbb{E}D_n)^{-1/2}(\mathbb{E}H_n)(\mathbb{E}D_n)^{-1/2}$ can be considered as the “expected Laplacian matrix” of $\widehat{G}_n(p_{ij})$. Let $C_n = I_n - (\mathbb{E}D_n)^{-1/2}H_n(\mathbb{E}D_n)^{-1/2}$. Then, clearly

$$\|\mathcal{L}_n - \widetilde{\mathcal{L}}_n\| \leq \|C_n - \widetilde{\mathcal{L}}_n\| + \|\mathcal{L}_n - C_n\| = \|\widetilde{\mathcal{L}}_n - C_n\| + \|C_n - \mathcal{L}_n\|.$$

In the next stages, we derive bounds for each of the last two terms separately.

We first consider $\widetilde{\mathcal{L}}_n - C_n = (\mathbb{E}D_n)^{-1/2}(H_n - \mathbb{E}H_n)(\mathbb{E}D_n)^{-1/2}$. Let

$$Y_{ij} = (\mathbb{E}D_n)^{-1/2}[(h_{ij} - \mathbb{E}h_{ij})H^{ij} + (h_{ji} - \mathbb{E}h_{ji})H^{ji}](\mathbb{E}D_n)^{-1/2}$$

$$= \frac{(h_{ij} - \mathbb{E}h_{ij})H^{ij} + (\overline{h_{ij}} - \mathbb{E}\overline{h_{ij}})H^{ji}}{\sqrt{t_i t_j}}.$$

Then, $\widetilde{\mathcal{L}}_n - C_n = \sum_{1 \leq i < j \leq n} Y_{ij}$. We are going to apply Theorem 8 to obtain an upper bound for $\|\widetilde{\mathcal{L}}_n - C_n\|$. Before we can do so, we have to perform some preliminary calculations in order to obtain an upper bound c_0 for $\|Y_{ij} - \mathbb{E}(Y_{ij})\|$, and a suitable upper bound for $\|\sum_{1 \leq i < j \leq n} \text{Var}(Y_{ij})\|$. First of all, note that for all $1 \leq i < j \leq n$,

$$\mathbb{E}(Y_{ij}) = \mathbb{E} \left[\frac{(h_{ij} - \mathbb{E}h_{ij})H^{ij} + (\overline{h_{ij}} - \mathbb{E}\overline{h_{ij}})H^{ji}}{\sqrt{t_i t_j}} \right] = \mathbf{0}.$$

We set $\mathbb{E}(Y_{ii}) = \mathbf{0}$. Then,

$$\begin{aligned} \|Y_{ij} - \mathbb{E}(Y_{ij})\| &= \|Y_{ij}\| \\ &= \frac{\|(h_{ij} - \mathbb{E}h_{ij})H^{ij} + (\overline{h_{ij}} - \mathbb{E}\overline{h_{ij}})H^{ji}\|}{\sqrt{t_i t_j}} \\ &= \frac{\|(h_{ij} - \mathbb{E}h_{ij})H^{ij} + (\overline{h_{ij}} - \mathbb{E}\overline{h_{ij}})H^{ji}\|}{\sqrt{t_i t_j}} \\ &= \frac{|h_{ij} - \mathbb{E}h_{ij}|}{\sqrt{t_i t_j}} \\ &= \frac{|h_{ij} - [p_{ij}p_{ji} + \mathbf{i}(p_{ij} - p_{ji})]|}{\sqrt{t_i t_j}} \\ &= \begin{cases} \frac{\sqrt{(1-p_{ij}p_{ji})^2 + (p_{ij}-p_{ji})^2}}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = 1, \\ \frac{\sqrt{(p_{ij}p_{ji})^2 + (1-(p_{ij}-p_{ji}))^2}}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = \mathbf{i}, \\ \frac{\sqrt{(p_{ij}p_{ji})^2 + (1+(p_{ij}-p_{ji}))^2}}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = -\mathbf{i}, \\ \frac{\sqrt{(p_{ij}p_{ji})^2 + (p_{ij}-p_{ji})^2}}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = 0. \end{cases} \\ &\leq \begin{cases} \frac{\sqrt{2}}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = 1, \\ \frac{\sqrt{4}}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = \mathbf{i}, \\ \frac{\sqrt{4}}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = -\mathbf{i}, \\ \frac{1}{\sqrt{t_i t_j}}, & \text{for } h_{ij} = 0. \end{cases} \\ &\leq \frac{2}{\sqrt{t_i t_j}} \\ &\leq \frac{2}{\delta}. \end{aligned}$$

So, we are going to use $c_0 = \frac{2}{\delta}$. Next, we consider $\text{Var}(Y_{ij})$ for all $1 \leq i < j \leq n$, and obtain

$$\text{Var}(Y_{ij}) = \mathbb{E}((Y_{ij} - \mathbb{E}(Y_{ij}))^2)$$

$$\begin{aligned}
&= \mathbb{E}Y_{ij}^2 \\
&= \frac{\mathbb{E}[(h_{ij} - \mathbb{E}h_{ij})H^{ij} + (\overline{h_{ij}} - \mathbb{E}\overline{h_{ij}})H^{ji}]^2}{t_i t_j} \\
&= \frac{\mathbb{E}[(h_{ij} - \mathbb{E}h_{ij})(\overline{h_{ij}} - \mathbb{E}\overline{h_{ij}})](H^{ii} + H^{jj})}{t_i t_j} \\
&= \frac{\mathbb{E}[(h_{ij} - \mathbb{E}h_{ij})(\overline{h_{ij}} - \mathbb{E}\overline{h_{ij}})](H^{ii} + H^{jj})}{t_i t_j} \\
&= \frac{\text{Var}(h_{ij})(H^{ii} + H^{jj})}{t_i t_j} \\
&= \frac{(p_{ij} + p_{ji} + p_{ij}p_{ji} - p_{ij}^2 - p_{ji}^2 - p_{ij}^2 p_{ji}^2)(H^{ii} + H^{jj})}{t_i t_j}.
\end{aligned}$$

We also have $\text{Var}(Y_{ii}) = \mathbb{E}Y_{ii}^2 = \mathbf{0}$ as $p_{ii} = 0$. Therefore,

$$\begin{aligned}
\left\| \sum_{1 \leq i < j \leq n} \text{Var}(Y_{ij}) \right\| &= \left\| \sum_{1 \leq i < j \leq n} \mathbb{E}Y_{ij}^2 \right\| \\
&= \left\| \sum_{i=1}^n \sum_{j=1}^n \frac{(p_{ij} + p_{ji} + p_{ij}p_{ji} - p_{ij}^2 - p_{ji}^2 - p_{ij}^2 p_{ji}^2)H^{ii}}{t_i t_j} \right\| \\
&= \max_{i=1, \dots, n} \sum_{j=1}^n \frac{p_{ij} + p_{ji} + p_{ij}p_{ji} - p_{ij}^2 - p_{ji}^2 - p_{ij}^2 p_{ji}^2}{t_i t_j} \\
&= \max_{i=1, \dots, n} \sum_{j=1}^n \frac{p_{ij} + p_{ji} - p_{ij}p_{ji} + 2p_{ij}p_{ji} - p_{ij}^2 - p_{ji}^2 - p_{ij}^2 p_{ji}^2}{t_i t_j} \\
&= \max_{i=1, \dots, n} \sum_{j=1}^n \frac{p_{ij} + p_{ji} - p_{ij}p_{ji} - (p_{ij} - p_{ji})^2 - p_{ij}^2 p_{ji}^2}{t_i t_j} \\
&\leq \max_{i=1, \dots, n} \sum_{j=1}^n \frac{p_{ij} + p_{ji} - p_{ij}p_{ji}}{t_i t_j} \\
&\leq \max_{i=1, \dots, n} \frac{1}{\delta} \sum_{j=1}^n \frac{p_{ij} + p_{ji} - p_{ij}p_{ji}}{t_i} \\
&= \frac{1}{\delta}.
\end{aligned}$$

For the final equality, we used that $d_i = \sum_{j=1}^n |h_{ij}|$, so $t_i = \mathbb{E}(d_i) = \sum_{j=1}^n \mathbb{E}|h_{ij}| = \sum_{j=1}^n [p_{ij}p_{ji} + p_{ij}(1 - p_{ji}) + (1 - p_{ij})p_{ji}] = \sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij}p_{ji})$. Now, in order to apply Theorem 8, we take $a = \sqrt{\frac{4 \ln(4n/\epsilon)}{\delta}}$, and we let k be large enough so that $\delta > k \ln n$ implies $a < 1$ (in particular, choosing $k > 4(1 + \ln(4/\epsilon))$ is sufficient). Now, noting that

$\mathbb{E}(\widetilde{\mathcal{L}}_n - C_n) = \mathbf{0}$, applying Theorem 8, we obtain

$$\begin{aligned}
\Pr(\|\widetilde{\mathcal{L}}_n - C_n\| \geq a) &\leq 2n \exp\left(-\frac{a^2}{2\|\sum_{1 \leq i < j \leq n} \text{Var}(Y_{ij})\| + 2c_0 a/3}\right) \\
&\leq 2n \exp\left(-\frac{\frac{4 \ln(4n/\epsilon)}{\delta}}{2/\delta + 4a/3\delta}\right) \\
&= 2n \exp\left(-\frac{4 \ln(4n/\epsilon)}{2 + 4a/3}\right) \\
&\leq 2n \exp\left(-\frac{4 \ln(4n/\epsilon)}{4}\right) \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

So, with probability at least $1 - \frac{\epsilon}{2}$, $\|\widetilde{\mathcal{L}}_n - C_n\| \leq a$. For the second term, we first rewrite $C_n - \mathcal{L}_n$, as follows.

$$\begin{aligned}
C_n - \mathcal{L}_n &= I_n - (\mathbb{E}D_n)^{-1/2} H_n (\mathbb{E}D_n)^{-1/2} - I_n + D_n^{-1/2} H_n D_n^{-1/2} \\
&= D_n^{-1/2} H_n D_n^{-1/2} - (\mathbb{E}D_n)^{-1/2} D_n^{1/2} D_n^{-1/2} H_n D_n^{-1/2} D_n^{1/2} (\mathbb{E}D_n)^{-1/2} \\
&= I_n - \mathcal{L}_n - (\mathbb{E}D_n)^{-1/2} D_n^{1/2} (I_n - \mathcal{L}_n) D_n^{1/2} (\mathbb{E}D_n)^{-1/2} \\
&= (I_n - \mathcal{L}_n) - (I_n - \mathcal{L}_n) D_n^{1/2} (\mathbb{E}D_n)^{-1/2} \\
&\quad - (\mathbb{E}D_n)^{-1/2} D_n^{1/2} (I_n - \mathcal{L}_n) D_n^{1/2} (\mathbb{E}D_n)^{-1/2} + (I_n - \mathcal{L}_n) D_n^{1/2} (\mathbb{E}D_n)^{-1/2} \\
&= (I_n - \mathcal{L}_n) [I_n - D_n^{1/2} (\mathbb{E}D_n)^{-1/2}] \\
&\quad + [I_n - (\mathbb{E}D_n)^{-1/2} D_n^{1/2}] (I_n - \mathcal{L}_n) D_n^{1/2} (\mathbb{E}D_n)^{-1/2}.
\end{aligned}$$

Recalling that $\|I_n - \mathcal{L}_n\| \leq 1$, we obtain the following expression for $\|C_n - \mathcal{L}_n\|$.

$$\begin{aligned}
\|C_n - \mathcal{L}_n\| &\leq \|I_n - \mathcal{L}_n\| \|I_n - D_n^{1/2} (\mathbb{E}D_n)^{-1/2}\| \\
&\quad + \|I_n - (\mathbb{E}D_n)^{-1/2} D_n^{1/2}\| \|I_n - \mathcal{L}_n\| \|D_n^{1/2} (\mathbb{E}D_n)^{-1/2}\| \\
&\leq \|I_n - D_n^{1/2} (\mathbb{E}D_n)^{-1/2}\| + \|I_n - (\mathbb{E}D_n)^{-1/2} D_n^{1/2}\| \|D_n^{1/2} (\mathbb{E}D_n)^{-1/2}\|.
\end{aligned}$$

Next, we are going to obtain an upper bound for $\|I_n - D_n^{1/2} (\mathbb{E}D_n)^{-1/2}\|$. For this, we will apply Lemma 13 to the random variables $|h_{ij}|$ (in the role of X_i), and using the observations that $d_i = \sum_{j=1}^n |h_{ij}|$, and $t_i = \mathbb{E}(d_i) = \sum_{j=1}^n \mathbb{E}|h_{ij}|$. We first need some preparation in order to obtain upper bounds for $|h_{ij}|$ and $\sum_{j=1}^n \text{Var}(|h_{ij}|)$.

Obviously, $|h_{ij}| \leq 1$, so we can take $c = 1$, and

$$\sum_{j=1}^n \text{Var}(|h_{ij}|) = \sum_{j=1}^n [\mathbb{E}(|h_{ij}|^2) - (\mathbb{E}(|h_{ij}|))^2]$$

$$\begin{aligned}
&= \sum_{j=1}^n [p_{ij} + p_{ji} - p_{ij}p_{ji} - (p_{ij} + p_{ji} - p_{ij}p_{ji})^2] \\
&\leq \sum_{j=1}^n (p_{ij} + p_{ji} - p_{ij}p_{ji}) \\
&= \sum_{j=1}^n \mathbb{E}|h_{ij}| \\
&= t_i.
\end{aligned}$$

Since we already used a above, with $a = \sqrt{\frac{4\ln(4n/\epsilon)}{\delta}} < 1$, we are going to use a b instead of an a when applying Lemma 13. We choose $b = \sqrt{3t_i \ln(4n/\epsilon)}$. Then, since $a = \sqrt{\frac{4\ln(4n/\epsilon)}{\delta}} < 1$, $t_i \geq \delta > 4\ln(4n/\epsilon)$, implying that $b = \sqrt{3t_i \ln(4n/\epsilon)} < t_i$. Applying Lemma 13, we have for all i ,

$$\Pr(|d_i - t_i| \geq b) \leq e^{-\frac{b^2}{2(t_i+b/3)}} < e^{-\frac{3t_i \ln(4n/\epsilon)}{3t_i}} = \frac{\epsilon}{4n}.$$

This implies that with probability at least $1 - o(1)$, for all $1 \leq i \leq n$, $|d_i - t_i| \leq b = \sqrt{3t_i \ln(4n/\epsilon)}$.

Next, we use the known fact that $|\sqrt{x} - 1| \leq |x - 1|$ for any real number $x > 0$. Taking $x = \frac{d_i}{t_i} > 0$, we obtain that with probability at least $1 - \frac{\epsilon}{2}$,

$$\left| \sqrt{\frac{d_i}{t_i}} - 1 \right| \leq \left| \frac{d_i}{t_i} - 1 \right| = \frac{|d_i - t_i|}{t_i} \leq \frac{\sqrt{3t_i \ln(4n/\epsilon)}}{t_i} \leq \sqrt{\frac{3 \ln(4n/\epsilon)}{\delta}} = \frac{\sqrt{3}}{2}a.$$

Thus, we obtain

$$\|I_n - D_n^{1/2}(\mathbb{E}D_n)^{-1/2}\| = \max_{i=1,2,\dots,n} \left| \sqrt{\frac{d_i}{t_i}} - 1 \right| \leq \frac{\sqrt{3}}{2}a$$

with probability at least $1 - \frac{\epsilon}{2}$. So, with probability at least $1 - \frac{\epsilon}{2}$,

$$\begin{aligned}
\|C_n - \mathcal{L}_n\| &\leq \|I_n - D_n^{1/2}(\mathbb{E}D_n)^{-1/2}\| + \|I_n - (\mathbb{E}D_n)^{-1/2}D_n^{1/2}\| \|D_n^{1/2}(\mathbb{E}D_n)^{-1/2}\| \\
&\leq \frac{\sqrt{3}}{2}a + \frac{\sqrt{3}}{2}a \left(\frac{\sqrt{3}}{2}a + 1 \right) \\
&= \frac{3}{4}a^2 + \sqrt{3}a.
\end{aligned}$$

Combining the above bound with the bound we obtained for $\|C_n - \widetilde{\mathcal{L}}_n\|$, and using that $a < 1$, we conclude that with probability at least $1 - \epsilon$,

$$\begin{aligned}
\|\mathcal{L}_n - \widetilde{\mathcal{L}}_n\| &\leq \|C_n - \widetilde{\mathcal{L}}_n\| + \|C_n - \mathcal{L}_n\| \\
&\leq a + \frac{3}{4}a^2 + \sqrt{3}a
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{7}{2}a \\
&= \frac{7}{2}\sqrt{\frac{4\ln(4n/\epsilon)}{\delta}} \\
&= 7\sqrt{\frac{\ln(4n/\epsilon)}{\delta}}.
\end{aligned}$$

For the final step in our proof, we use Lemma 12, which states that for Hermitian matrices M and N , $\max_k |\lambda_k(M) - \lambda_k(N)| \leq \|M - N\|$. Thus, with probability at least $1 - \epsilon$, we have that for all $1 \leq i \leq n$,

$$|\lambda_i(\mathcal{L}_n) - \lambda_i(\widetilde{\mathcal{L}}_n)| \leq \|\mathcal{L}_n - \widetilde{\mathcal{L}}_n\| \leq 7\sqrt{\frac{\ln(4n/\epsilon)}{\delta}}.$$

This completes the proof of Theorem 4. □

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References

- [1] R. Ahlswede and A. Winter. Strong converse for identification via quantum channels. *IEEE Trans. Inform. Theory*, **48**(3) (2002), 569–579.
- [2] N. Alon, M. Krivelevich, and V. H. Vu. Concentration of eigenvalues of random matrices. *Israel Math. J.*, **131** (2002), 259–267.
- [3] R. Bhatia. *Matrix Analysis*, Graduate Texts in Mathematics, vol. 169, Springer, Berlin, 1997, p.10.
- [4] R. Bhatia. *Positive Definite Matrices*, Princeton Univ. Press, Princeton, 2007.
- [5] F. Chung. *Spectral graph theory*, AMS publications, 1997.
- [6] F. Chung, L. Lu, and V. H. Vu. Eigenvalues of random power law graphs. *Ann. Combin.*, **7** (2003), 21–33.
- [7] F. Chung, L. Lu, and V. H. Vu. Spectra of random graphs with given expected degrees. *Proc. Nat. Acad. Sci. USA*, **100**(11) (2003), 6313–6318.
- [8] F. Chung and M. Radcliffe. On the spectra of general random graphs. *Electron. J. Combin.*, **18** (2011), #P215.
- [9] A. Coja-Oghlan. On the Laplacian eigenvalues of $G(n, p)$. *Combin. Probab. Comput.*, **16** (2007), 923–946.
- [10] A. Coja-Oghlan and A. Lanka. The spectral gap of random graphs with given expected degrees. *Electron. J. Combin.*, **16** (2009), #R138.
- [11] D. Cristofides and K. Markström. Expansion properties of random Cayley graphs and vertex transitive graphs via matrix martingales. *Random Struct. Alg.*, **32** (2008), 88–100.

- [12] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs, Theory and Applications*, Academic Press, 1980.
- [13] X. Ding and T. Jiang. Spectral distributions of adjacency and Laplacian matrices of random graphs. *Ann. Appl. Probab.*, **20** (2010), 2086–2117.
- [14] U. Feige and E. Ofek. Spectral techniques applied to sparse random graphs. *Random Struct. Alg.*, **27**(2) (2005), 251–275.
- [15] J. Friedman. *A Proof of Alon’s Second Eigenvalue Conjecture and Related Problem*, Memoirs of the American Mathematical Society 2008, 100 pp.
- [16] J. Friedman. On the second eigenvalue and random walks in random d -regular graphs. *Combinatorica*, **11**(4) (1991), 331–362.
- [17] J. Friedman, J. Kahn, and E. Szemerédi. On the second eigenvalue in random regular graphs, in *Proc. 21st ACM Symp. Theory of Computing*, 1989, 587–598.
- [18] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, **1**(3) 1981, 233–241.
- [19] D. Gross. Recovering low-rank matrices from few coefficients in any basis. *IEEE Trans. Inform. Theory*, **57** (2011), 1548–1566.
- [20] K. Guo and B. Mohar. Hermitian adjacency matrix of digraphs and mixed graphs. *J. Graph Theory*, **85** (2017), no. 1, 217–248.
- [21] R. A. Horn and C. R. Johnson. *Matrix Analysis*, 2nd, Cambridge University Press, 2012.
- [22] M. Krivelevich and B. Sudakov. The largest eigenvalue of sparse random graphs. *Combin. Probab. Comput.*, **12** (2003), 61–72.
- [23] J. Liu and X. Li. Hermitian-adjacency matrices and Hermitian energies of mixed graphs. *Linear. Algebra. Appl.*, **466** (2015) 182–207.
- [24] L. Lu and X. Peng. Loose Laplacian spectra of random hypergraphs. *Random Struct. Alg.*, **41** (2012), no. 4, 521–545.
- [25] L. Lu and X. Peng. Spectra of edge-independent random graphs. *Electron. J. Combin.*, **20**(4) (2013), #P27.
- [26] R. Oliveira. Concentration of the adjacency matrix and of the Laplacian in random graphs with independent edges. [arXiv:0911.0600](https://arxiv.org/abs/0911.0600), 2009.
- [27] D. Petz. A survey of certain trace inequalities, in *Functional Analysis and Operator Theory*, Banach Center Publications, vol. 30(Polish Acad. Sci., Warsaw, 1994), pp. 287–298.
- [28] B. Recht. Simpler approach to matrix completion. *J. Mach. Learn. Res.*, **12** (2011), 3413–3430.
- [29] J. Tropp. User-friendly tail bounds for sums of random matrices. *Found. Comput. Math.*, **12** (2012), 389–434.
- [30] H. Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen. *Math. Ann.*, **71** (2010), 441–479.
- [31] F. Zhang. *Matrix theory*, Springer-Verlag, New York, 1999.