The smallest matroids with no large independent flat

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1 Introduction

Call a set S in a matroid M a *claw* of M if S is both a flat and an independent set of M. A k*-claw* is a claw of size k. These objects were introduced by Bonamy et al. [2] and studied by Nelson and Nomoto [3]; both of these papers consider the structure of 3-claw-free binary matroids.

Here we deal with general matroids, and address the simple extremal question of determining the smallest simple rank-r matroids omitting a given claw; we solve this problem and characterize the tight examples.

Theorem 1.8 of [3] shows that, for $r \geqslant 4$, the unique smallest simple rank-r binary matroid with no 3-claw is the direct sum of two binary projective geometries of ranks $\lfloor r/2 \rfloor$ and $\lceil r/2 \rceil$. We show that, perhaps surprisingly, the exact same construction is also extremal for general matroids, and that its natural generalization is still extremal for excluding larger claws. For integers $r \geqslant 1$ and $t \geqslant 1$, let $M_{r,t}$ denote the matroid that is the direct sum of t (possibly empty) binary projective geometries, whose ranks sum to r and pairwise differ by at most 1. We prove the following, which was conjectured for the special case of binary matroids in [3].

Theorem 1. *Let* $r, t \geqslant 1$ *be integers. If* M *is a simple rank-*r *matroid with no* $(t+1)$*-claw, then* $|M| \geqslant |M_{r,t}|$*. If equality holds and* $r \geqslant 2t$*, then* $M \cong M_{r,t}$*.*

Note that for $r \leqslant t$, the matroid $M_{r,t}$ is free and therefore the theorem is trivial. For $t < r < 2t$, there is a rather tame family of exceptional tight examples, which we describe

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in Theorem 8. One can also ask a similar question with 'simple' relaxed to 'loopless'. (One must still insist that M is loopless, since any matroid with a loop has no claw.) In this case the answer is much less interesting; the direct sum of $r-t$ parallel pairs and t coloops has $2r-t$ elements and is the unique smallest loopless rank-r matroid with no $(t+1)$-claw, this is a consequence of Lemma 7 below.

Graph Theory. The study of the structure of 3-claw-free binary matroids in [2, 3] was motivated by structural results in graph theory. In the context of these works, the graph-theoretic notions of induced subgraphs, cliques, chromatic number and forests have analogies in the setting of simple binary matroids: cliques are analogous to projective geometries in the sense of being maximal with a given rank, while claws correspond to induced forests. A graph-theoretic analogue of Theorem 1 using this correspondence would characterize graphs on r vertices with minimum number of edges and no induced forests of given size. From the matroidal point of view, the natural measure of the size of a forest is the number of edges, but there seem to exist no direct analogue of Theorem 1 using this measure. Defining the size of a forest as the number of its vertices works much better, as follows.

Let $G_{n,t}$ denote the graph on n vertices which is a disjoint union of t complete subgraphs, whose sizes pairwise differ by at most one. Turán's classical theorem [5] is equivalent to the statement that $|E(G)| \geqslant |E(G_{n,t})|$ for every graph G on n vertices with no stable set of size $t+1$. This observation implies that the following graph-theoretic analogue of Theorem 1 generalizes Turán's theorem for $n \geqslant 3t$.

Theorem 2. *Let $n, t \geqslant 1$ be integers such that $n \geqslant 3t$. If G is a graph on n vertices having no forest on $2t+1$ vertices as an induced subgraph, then $|E(G)| \geqslant |E(G_{n,t})|$. If $n \geqslant 4t$, then the equality holds only if G is isomorphic to $G_{n,t}$.*

We give a short proof of Theorem 2, obtained by adapting one of the standard proofs of Turan's theorem, in Section 4.

Triangle-free matroids. The extremal examples in Theorem 1 have many triangles, and our proof techniques analyze triangles closely. It seems plausible that if M is required to be triangle-free, then the sparsest examples, instead of projective geometries, come from binary affine geometries, which are triangle-free and have 2-claws but no 3-claws. (An affine geometry $\mathrm{AG}(r-1,2)$ is obtained from a projective geometry $\mathrm{PG}(r-1,2)$ by deleting a hyperplane.) This leads us to conjecture the following.

Conjecture 3. Let t, r be integers with $t \geqslant 1$ and $t|r$. If M is a simple triangle-free matroid with no $(2t+1)$-claw, then $|M| \geqslant t2^{r/t-1}$.

This conjectured bound holds with equality when M is the direct sum of t copies of a rank-(r/t) binary affine geometry; these should be the only cases where equality holds. We prove this in the easy case where $t=1$; see Lemma 16.

In what follows, we use the notation of Oxley [4]; flats of a matroid of rank 1 and 2 are *points* and *lines* respectively. We additionally write $|M|$ for $E(M)$. A *simplification* of M is any matroid obtained from M by deleting all loops and all but one element from each parallel class. All such matroids are clearly isomorphic; we write $\mathrm{si}(M)$ for a generic matroid isomorphic to a simplification of M, and write $\varepsilon(M)$ for $|\mathrm{si}(M)|$, the number of points of M. A family $\mathcal{X}$ of sets are *skew* in a matroid M if $r_M(\cup\mathcal{X}) = \sum_{X\in\mathcal{X}} r_M(X)$, and we say that a set X is *skew* to a set Y if $\{X, Y\}$ is a skew family: i.e. $r_M(X \cup Y) = r_M(X) + r_M(Y)$.

2 The Bound

In this section we give the easy proof of the lower bound in Theorem 1. Our first lemma shows that the property of being $(t+1)$-claw-free is essentially closed under contraction; if F is a k-claw of some simplification of M, call F a *k-pseudoclaw* of M.

Lemma 4. *Let $k \geqslant 1$. If M is a simple matroid and $X \subseteq E(M)$, then every k-pseudoclaw of M/X is a k-claw of M.*

Proof. Let $M' = (M/X)\backslash P$ be a simplification of M/X, and suppose that M' has a k-claw F. Since F is independent in M/X, it is independent in M, and is skew to X in M. Since F is a flat of M/X, we have $\varnothing = \mathrm{cl}_{M'}(F) - F = \mathrm{cl}_M(X \cup F) - (F \cup X \cup P) \supseteq (\mathrm{cl}_M(F) - F) - (X \cup P)$, giving $\mathrm{cl}_M(F) - F \subseteq X \cup P$.

The sets $\mathrm{cl}_M(F)$ and X are skew in M, so $\mathrm{cl}_M(F) - F \subseteq P$. Suppose that $e \in (\mathrm{cl}_M(F) - F) \cap P$. Then there exists $e' \in E(M')$ parallel to e in M/X; since $e' \in \mathrm{cl}_M(F) - X \subseteq \mathrm{cl}_{M/X}(F)$ we also have $e \in \mathrm{cl}_{M/X}(F)$ and so $e \in \mathrm{cl}_{M'}(F) = F$, contrary to the choice of e. It follows that $\mathrm{cl}_M(F) - F$ intersects neither X nor P so is empty; therefore F is a k-claw of M. □

Let $f(r, t) = |M_{r,t}|$. Since a rank-n projective geometry has $2^n - 1$ elements, we clearly have $f(r, t) = (t - a)2^{\lfloor r/t \rfloor} + a2^{\lceil r/t \rceil} - t$, where $a \in \{0, \dots, t - 1\}$ is the integer with $a \equiv r \pmod{t}$. More importantly for our purposes, we can define f recursively; it is easy to check that $f(r, t) = r$ for all $0 \leqslant r \leqslant t$ and $f(r, t) = 2f(r - t, t) + t$ for $r > t$. We use this recurrence and the previous lemma to prove the lower bound in our main theorem.

Theorem 5. *If $t \geqslant 1$ is an integer and M is a simple rank-r matroid with no $(t+1)$-claw, then $|M| \geqslant f(r, t)$.*

Proof. Let M be a counterexample for which $r + |M|$ is minimized. If M is a free matroid then clearly $r \leqslant t$, in which case $f(r, t) = r \leqslant |M|$ so M is not a counterexample. Therefore M has a non-coloop e.

Since $|M\backslash e| < |M| \leqslant f(r, t)$ but $M\backslash e$ is not a counterexample, there must be a $(t + 1)$-claw S' in $M\backslash e$. Now the matroid $M|\mathrm{cl}_M(S')$ has rank $t + 1$ and has at most $t + 2$ elements, so has at most one circuit. There is thus a t-element subset S of $\mathrm{cl}_M(S')$ containing at most $|C| - 2$ elements of each C circuit of M; this set S is a t-claw.

If there is some rank-$(t + 1)$ flat F containing S for which $|F - S| = 1$, then F is a $(t + 1)$-claw. Therefore every such flat satisfies $|F - S| \geqslant 2$; since S is a flat, it follows that every parallel class of M/S has size at least 2. Moreover, $\mathrm{si}(M/S)$ is a rank-$(r - t)$

matroid that by Lemma 4 has no $(t+1)$-claw; inductively we have $|\operatorname{si}(M/S)| \geqslant f(r-t,t)$. Now

$$|M| = |S| + |M/S| \geqslant t + 2|\operatorname{si}(M/S)| \geqslant t + 2f(r-t,t) = f(r,t),$$

as required. □

3 Equality

We now characterize matroids for which the bound in Theorem 5 holds with equality. This requires two lemmas; the first (which uses Tutte's characterization of binary matroids as those with no $U_{2,4}$-minor [6]) corresponds to the case $t = 1$ of Theorem 5.

Lemma 6. *If M is a simple rank-r matroid with no 2-claw, then $|M| \geqslant 2^r - 1$. If equality holds then $M \cong \mathrm{PG}(r-1,2)$.*

Proof. Let M be a minor-minimal counterexample. Clearly $r(M) \geqslant 3$. Let $e \in E(M)$ and let H be a hyperplane of M not containing e. Since $M|H$ has no 2-claw, we have $|H| \geqslant 2^{r-1} - 1$ by the minimality of M. For each $x \in H$, the line spanned by x and e contains an element of $E(M) - \{e,x\}$, and these lines pairwise intersect only in e, so we see that $|M| \geqslant 2|H| + 1 \geqslant 2^r - 1$ as required.

If $|M| = 2^r - 1$, then equality holds above, so $|H| = 2^{r-1} - 1$ and thus $M|H \cong \mathrm{PG}(r-2,2)$. Moreover, for each $x \in H$ we have $|\operatorname{cl}_M(\{e,x\})| = 3$ and $E(M) = \cup_{x \in H}(\operatorname{cl}_M(\{e,x\}))$, which implies that $\operatorname{si}(M/e) \cong M|H \cong \mathrm{PG}(r-2,2)$ so M/e is binary. The choice of e was arbitrary, so M/e is binary for all e; since $r \geqslant 3$ this gives that M has no $U_{2,4}$-minor so is binary. Since M is simple with $2^r - 1$ elements, this implies $M \cong \mathrm{PG}(r-1,2)$, a contradiction. □

Note that if $t < r < 2t$ then $|M_{r,t}| = 2r - t$. In this range, the matroids $M_{r,t}$ are not the only ones satisfying the bound in Theorem 5 with equality. The other examples include direct sums of circuits and coloops, and the matroid $M_{r,t}$ is the special case where all these circuits are triangles. The following lemma shows that these are the only examples. It also implies the characterization of the smallest $(t+1)$-claw-free matroids that are not required to be simple that was claimed in the introduction.

Lemma 7. *Let $r \geqslant t \geqslant 1$ be integers. If M is a loopless rank-r matroid with no $(t+1)$-claw, then $|M| \geqslant 2r - t$. If equality holds, then M is the direct sum of $r - t$ circuits and some number of coloops.*

Proof. Suppose first that every circuit of M^* has at most two elements. Then, since M^* is coloopless, it is the direct sum of loops and parallel classes of size at least two; let $\mathcal{P}$ be its set of parallel classes, so $r(M^*) = |\mathcal{P}|$ and $r = |M| - |\mathcal{P}|$. Let U be a set comprising exactly two elements from each $P \in \mathcal{P}$. Since $r(M^*|U) = r(M^*)$, the set $(E - U)$ is independent in M, and since $M^*|U$ is coloopless, the set $(E - U)$ is also a flat of M, so is a claw of M. By hypothesis, it follows that $t \geqslant |E - U| = |M| - 2|\mathcal{P}|$. Therefore $2r - t \leqslant 2(|M| - |\mathcal{P}|) - (|M| - 2|\mathcal{P}|) = |M|$, as required. If equality holds, then

$t = |M| - 2|\mathcal{P}| = r - |\mathcal{P}|$, so $|\mathcal{P}| = r - t$. By the definition of $\mathcal{P}$, each $P \in \mathcal{P}$ is a circuit of M, and each other element of M is a coloop; this gives the required structure.

We may therefore assume M^* has a circuit C of size at least 3. Let B be a basis of M^* containing all but one element of C. Since M^* has no coloops, for each $x \in B$, there is a circuit C_x of M^* for which $x \in C_x$ and $|C_x \cap B| = |C_x| - 1$; choose the C_x so that $C_x = C$ for each $x \in C$. Let $X = \cup_{x \in X} C_x$. Since each C_x contains only one element outside B and the element of $C - B$ is chosen at least twice, we have $|X| < 2r(M^*)$.

By construction, the set X contains a basis and, since X is a union of circuits of M^*, the matroid $M^*|X$ has no coloops. Let $Y = E(M) - X$; by construction the set Y is independent in M, and M/Y has no loops, so Y is a flat, and thus a claw, of M. Hence $|Y| \leqslant t$ and so $|X| \geqslant |M| - t$. By our upper bound on $|X|$, this gives $|M| - t < 2r(M^*) = 2(|M| - r)$ and so $|M| > 2r - t$, as required. □

We are now ready to strengthen Theorem 5 with an equality characterisation. Note that both outcomes in the equality case imply that M has a t-claw.

Theorem 8. *Let $t \geqslant 1$. If M is a simple rank-r matroid with no $(t+1)$-claw, then $|M| \geqslant f(r,t)$. If equality holds, then either*

- *$M \cong M_{r,t}$, or*
- *$t < r < 2t$ and M is the direct sum of coloops and exactly $r - t$ circuits, not all of which are triangles.*

Proof. Consider a counterexample M for which $|M| + r$ is minimized. Clearly $r > t$, as otherwise $|M| \geqslant r = f(r,t)$ and there is nothing to prove. Therefore M is not a free matroid, since otherwise any $(t+1)$-element subset of $E(M)$ is a claw.

Claim 9. *M has a t-claw.*

Subproof: Let e be a non-coloop of M; by the minimality of M, the matroid $M \backslash e$ has a $(t+1)$-claw F; now $M|\operatorname{cl}_M(F)$ has rank at least $t+1$ and has at most $t+2$ elements, so has at most one circuit. There is thus a t-element subset of $\operatorname{cl}_M(F)$ that contains at most $|C| - 2$ elements of each circuit C of $M|\operatorname{cl}_M(F)$; this set is a t-claw of M. ■

Call a t-claw S of M *generic* if no four-point line of M intersects S, and exactly $f(r-t,t)$ triangles of M intersect S. Let S be a t-claw of M, chosen *not* to be generic if such a choice is possible.

Claim 10. *$|M| = f(r,t)$, each parallel class of M/S has size 2, and $\varepsilon(M/S) = f(r-t,t)$.*

Subproof: The matroid M/S has rank $r - t$ and, by Lemma 4, has no $(t+1)$-pseudoclaw. Therefore $\operatorname{si}(M/S)$ has no $(t+1)$-claw, so $\varepsilon(M/S) \geqslant f(r-t,t)$. Moreover, if some parallel class Y of M/S has size 1, then $S \cup Y$ is a $(t+1)$-claw of M, so every parallel class of M/S has size at least 2, giving $|M/S| \geqslant 2\varepsilon(M/S)$. Therefore

$$f(r,t) \geqslant |M| \geqslant 2\varepsilon(M/S) + |S| \geqslant 2f(r-t,t) + t = f(r,t).$$

Equality holds throughout, which gives the claim. ■

The matroid $\mathrm{si}(M/S)$ has no $(t+1)$-claw and has $f(r-t,t)$ elements, so inductively satisfies one of the conclusions of the theorem. For each component N of M/S, the matroid $\mathrm{si}(N)$ is either a circuit or a binary projective geometry.

Claim 11. *Let $e_1, e_2 \in E(M/S)$. If e_1 and e_2 are in different components, then M/S has a t-pseudoclaw containing e_1 and e_2. If e_1 and e_2 are in the same component, then there is a $(t-1)$-element set U such that $U \cup \{e_1\}$ and $U \cup \{e_2\}$ are both t-pseudoclaws of M/S.*

Subproof: We first argue that M/S has a t-pseudoclaw. Since $\mathrm{si}(M/S)$ satisfies one of the outcomes of the theorem, this only fails if $\mathrm{si}(M/S) \cong M_{r-t,t}$ and $r-t<t$. If this holds then $|M| = f(r,t) = 2r-t$ and M satisfies the hypothesis of Lemma 7, so is the direct sum of coloops and $r-t$ circuits. If these circuits are all triangles then $M \cong M_{r,t}$, and otherwise M satisfies the second outcome of the theorem; both are contrary to the choice of M as a counterexample. Therefore M/S has a t-pseudoclaw.

Since every component of $\mathrm{si}(M/S)$ is a circuit or projective geometry, given any pseudoclaw K of M and any e, e' in the same component of M/S for which $e \in K$ and $e' \notin K$, the set $(K-e) \cup \{e'\}$ is also a pseudoclaw. Since M/S has at least one t-pseudoclaw, both conclusions of the claim easily follow. ■

The above claim implies in particular that every element of M/S is in a t-pseudoclaw.

Claim 12. *For each t-pseudoclaw U of M/S, there is a bijection ψ_U from U to S so that for each $e \in U$, the flat $T_e = \mathrm{cl}_M(e, \psi_U(e))$ is a triangle of M, and so that $M|\,\mathrm{cl}_M(S \cup U) = \oplus_{e \in U}(M|T_e)$.*

Subproof: Since the closure of U in M/S is obtained from U by extending each element of U once in parallel, we have $|\,\mathrm{cl}_M(S \cup U)| = |S| + 2|U| = 3t$. By Lemma 7, it follows that the simple rank-$2t$ matroid $M' = M|\,\mathrm{cl}_M(S \cup U)$ is the direct sum of t circuits and some set of coloops, and therefore that is precisely the direct sum of t triangles. Since S is a t-claw of M' and U is a t-pseudoclaw of M'/S, both S and U must be transversals of this set of triangles. The claim follows. ■

Every element e of M/S is contained in a t-pseudoclaw, so the above claim implies that each such e is in exactly one triangle that intersects S. Write $\psi(e)$ for the unique element of S for which e and $\psi(e)$ are contained in a triangle; we have $\psi_U(e) = \psi(e)$ for each t-pseudoclaw U of M/S containing e.

Since S is a claw and no rank-1 flat of M/S has more than two elements, no line of M that intersects S has more than three elements. Moreover, each $e \in E(M/S)$ is in exactly one triangle of M that intersects S, so the number of triangles of M that intersect S is exactly $\frac{1}{2}|M/S| = \frac{1}{2}(2\varepsilon(M/S)) = f(r-t,t)$. Therefore S is generic. It follows from the choice of S that every t-claw of M is generic.

Claim 13. *For all $e_1, e_2 \in E(M/S)$, we have $\psi(e_1) = \psi(e_2)$ if and only if e_1 and e_2 are in the same component of M/S. Moreover, M/S has exactly t components.*

Subproof: Suppose that e_1 and e_2 are in the same component of M/S. By Claim 11, there is a set $U \subseteq E(M/S)$ such that $U \cup \{e_1\}$ and $U \cup \{e_2\}$ are both t-pseudoclaws of M/S. For each $i \in \{1,2\}$, there is a bijection $\psi_i = \psi_{U\cup\{e_i\}}$ from $U \cup \{e_i\}$ to S, and moreover for each $e \in U$ we have $\psi_1(e) = \psi(e) = \psi_2(e)$. Therefore ψ_1 and ψ_2 agree on all $t-1$ elements of U; thus $\psi_1(e_1) = \psi_1(e_2)$ and so $\psi(e_1) = \psi(e_2)$.

Suppose now that e_1 and e_2 are in different components of M/S. By Claim 11 there is a t-pseudoclaw U containing e_1 and e_2. Since ψ_U is a bijection we have $\psi(e_1) = \psi_U(e_1) \neq \psi_U(e_2) = \psi(e_2)$, as required.

It follows from the first part that the image of ψ has size equal to the number of components of M/S. But clearly the image of ψ contains the image of ψ_U, which is equal to S, for each t-pseudoclaw U. Therefore ψ has image S, so M/S has exactly $|S| = t$ components. ■

Let $\mathcal{N}$ be the set of components of M/S. By Claim 13, for each $N \in \mathcal{N}$ there is some $\psi(N)$ for which $\psi(e) = \psi(N)$ for each $e \in E(N)$. Since $|S| = |\mathcal{N}| = t$, the t-tuple $\psi(N) : N \in \mathcal{N}$ is a permutation of S. For each $N \in \mathcal{N}$, let $\widehat{N} = M|(E(N) \cup \psi(N))$.

Claim 14. *If $N \in \mathcal{N}$ and L is a line of M intersecting $E(N)$, then either $|L| = 2$, or $|L| = 3$ and $L \subseteq E(\widehat{N})$.*

Subproof: Let U be a t-pseudoclaw of M/S containing an element $e \in E(N) \cap L$. Note that U is a generic t-claw in M, which gives $|L| \leqslant 3$.

Suppose that $|L| = 3$. Note that $1 = r_{M/S}(e) \leqslant r_{M/S}(L - S) \leqslant r_M(L) = 2$. If $r_{M/S}(L - S) = 2$ then L is a triangle of M/S that intersects the component N of M/S, so obviously $L \subseteq E(N)$. If $r_{M/S}(L - S) = 1$ then $L \subseteq \mathrm{cl}_M(S \cup \{e\}) \subseteq \mathrm{cl}_M(S \cup U)$, so 12 gives $L = \mathrm{cl}_M(\{e, \psi(e)\})$. Since $L - \psi(e)$ is a two-element rank-1 set in M/S, the third element of L is the element of N parallel to e, so $L \subseteq E(\widehat{N})$ as required. ■

For each $N \in \mathcal{N}$ and $e \in E(N)$, let $\tau(e)$ be the number of 3-element lines of M containing e, and let $\sigma(e)$ be the number of elements of $E(\widehat{N}\backslash e)$ that are *not* in a 3-element line of M with e. By the previous claim we have $2\tau(e) + \sigma(e) = |\widehat{N}\backslash e| = |N|$.

Claim 15. *For each $N \in \mathcal{N}$, every line of $E(\widehat{N})$ has size* 3.

Subproof: Suppose not, so there is some $e \in E(N)$ for which $\sigma(e) > 0$. Let U be a t-pseudoclaw of M/S containing e. Since U is a generic t-claw of M, we have $\sum_{u\in U} \tau(u) = f(r-t, t)$, so

$$\begin{aligned} |M| &= |S| + \sum_{N\in\mathcal{N}} |N| \\ &= t + \sum_{u\in U} (2\tau(u) + \sigma(u)) \\ &= t + 2f(r-t,t) + \sum_{u\in U} \sigma(u) \\ &\geqslant f(r,t) + \sigma(e). \end{aligned}$$

Since $|M| = f(r,t)$ and $\sigma(e) > 0$, this is a contradiction. ■

Let $N \in \mathcal{N}$. It is clear, since N is obtained from $\widehat{N}/\psi(N)$ by $t-1$ successive extension-contraction operations, that $r(N) \leqslant r(\widehat{N}) - 1$. The matroid $\mathrm{si}(N)$ is a circuit or a binary projective geometry, so

$$|\widehat{N}| = |N| + 1 \leqslant 2(2^{r(N)} - 1) + 1 = 2^{r(N)+1} - 1 \leqslant 2^{r(\widehat{N})} - 1.$$

By Claim 15 and Lemma 6 we have $|\widehat{N}| \geqslant 2^{r(\widehat{N})} - 1$, so equality holds and therefore each matroid $\widehat{N}$ is a binary projective geometry of rank $r(N) + 1$. The sets $E(\widehat{N}) : N \in \mathcal{N}$ partition $E(M)$, so

$$r \leqslant \sum_{N \in \mathcal{N}} r(\widehat{N}) = \sum_{N \in \mathcal{N}} (r(N) + 1) = r(M/S) + |\mathcal{N}| = t + r(M/S) = r,$$

so equality holds throughout, and the sets $\{E(\widehat{N}) : N \in \mathcal{N}\}$ are skew in M. Thus M is the direct sum of t nonempty binary projective geometries. If M has components of ranks r_1, r_2 with $r_2 \geqslant r_1 + 2$, then deleting both and replacing them with projective geometries of rank $r_2 - 1$ and $r_1 + 1$ respectively gives a matroid M' with no $(t+1)$-claw satisfying

$$|M| - |M'| = 2^{r_2} + 2^{r_1} - 2^{r_2 - 1} - 2^{r_1 + 1} = 2^{r_2 - 1} - 2^{r_1 + 1} > 0,$$

which contradicts the minimality of $|M|$. It follows that no two components of M have ranks differing by more than 1, so $M \cong M_{r,t}$, contrary to the choice of M as a counterexample. □

Finally, we prove the $t = 1$ case of Conjecture 3 as promised.

Lemma 16. *If M is a simple rank-r triangle-free matroid with no 3-claw, then $|M| \geqslant 2^{r-1}$. If equality holds, then $M \cong AG(r-1, 2)$.*

Proof. We may assume that $r \geqslant 3$. We first show that every triple of distinct elements of M is contained in a four-element circuit; indeed, given such a triple I, since I is not a triangle or a 3-claw, we have $r_M(I) = 3$ and $\mathrm{cl}_M(I) \neq I$. Thus there is some $x \in \mathrm{cl}_M(I) - I$. Since M is triangle-free, no pair of elements of I spans x, so $I \cup \{x\}$ is a 4-element circuit.

Let $e \in E(M)$. Since M is triangle-free, the matroid M/e is simple. If M/e has a 2-claw I, then $I \cup \{e\}$ is clearly a 3-claw of M; therefore M/e is 2-claw-free and so $|M/e| \geqslant 2^{r-1} - 1$ by Lemma 6. It follows that $|M| \geqslant 2^{r-1}$ as required.

If equality holds, then $M/e \cong \mathrm{PG}(r-2, 2)$ so M/e is binary. This holds for arbitrary $e \in E(M)$; it follows (since $r \geqslant 3$) that M has no $U_{2,4}$-minor so is also binary. A simple rank-r triangle-free binary matroid has at most 2^{r-1} elements and equality holds only for binary affine geometries (see [1], for example); therefore $M \cong \mathrm{AG}(r-1, 2)$. □

4 Graphs

Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined recursively by

- $g(n,t) = 0$ for $n < 2t$,
- $g(n,t) = 3(n-2t)$ for $2t \leqslant n \leqslant 4t$,
- $g(n,t) = g(n-1,t) + \lceil \frac{n}{t} \rceil - 1$, for $n > 4t$.

It is easy to check that $|E(G_{n,t})| = g(n,t)$ for $n \geqslant 3t$ (although not for smaller n). The recursion for $n > 4t$ in fact also holds when $3t < n \leqslant 4t$. Thus the next theorem implies Theorem 2.

Theorem 17. *Let $n, t \geqslant 1$ be integers. Let G be a simple graph on n vertices such that no forest on $2t+1$ vertices is an induced subgraph of G. Then*

$$|E(G)| \geqslant g(n,t).$$

If equality holds and $n < 4t$, then every component of G is a complete graph on $1, 3$ or 4 vertices. If equality holds and $n \geqslant 4t$, then G is isomorphic to $G_{n,t}$.

Proof. We prove the theorem by induction on $|V(G)|$. We may clearly assume that $n \geqslant 2t+1$, as otherwise the result is easy. Let v be a vertex of G of maximum degree.

If $\deg(v) \leqslant 2$, every component of G is a path or a cycle. Let S be the set of vertices of cycles of G, and b be the number of cycles of G; note that $b \leqslant \frac{1}{3}|S| \leqslant \frac{1}{3}n$. Clearly G contains an induced forest on $n-b$ vertices, so $n - b \leqslant 2t$. This gives $n \leqslant 2t + \frac{n}{3}$, so $n \leqslant 3t$, which in turn implies that $g(n,t) = 3(n-2t) \leqslant 3b$. On the other hand, we have $|E(G[S])| = |S|$, so

$$|E(G)| \geqslant |E(G[S])| = |S| \geqslant 3b \geqslant g(n,t),$$

giving the bound. If equality holds, then $E(G) = E(G[S])$ and $b = \frac{1}{3}|S|$, so every component of G is an isolated vertex or triangle. We have argued that $n \leqslant 3t$; thus G has the claimed structure. We may therefore assume that $\deg(v) \geqslant 3$.

Let $X \subseteq V(G)$ be maximal so that $G[X]$ is a forest, so $|X| \leqslant 2t$. Let Z be the set of non-isolated vertices of $G[X]$. As $G[X \cup \{w\}]$ contains a cycle for every $w \in V(G) \setminus X$, every such w has at least two neighbors in Z. Thus

$$\sum_{z \in Z} \deg(z) \geqslant |Z| + 2|V(G) - X| \geqslant |Z| + 2(n-2t).$$

Hence there exists $z_0 \in Z$ such that $\deg(z_0) \geqslant 2(n-2t)/|Z|+1 \geqslant (n-2t)/t+1 = n/t-1$; thus $\deg(v) \geqslant \lceil \frac{n}{t} \rceil - 1$ by the choice of v.

By the above, we can assume that $\deg(v) \geqslant \max(3, \lceil \frac{n}{t} \rceil - 1)$. Let $H = G - v$. It follows that

$$|E(G)| = |E(H)| + \deg(v) \geqslant g(n-1,t) + \max(3, \lceil \tfrac{n}{t} \rceil - 1) = g(n,t);$$

the last equality is easy to check. This gives the desired bound.

Suppose now that $|E(G)| = g(n,t)$. Then $\deg(v) = \max(3, \lceil \frac{n}{t} \rceil - 1)$, and $|E(H)| = g(n-1,t)$. Call a component of H with at least two vertices *big*. By the induction hypothesis, every component of H is a complete graph. Therefore, $H[X]$ is a maximal induced forest in H if and only if X contains at least one vertex from every component of H and exactly two vertices from every big component. It follows that $|X| = 2t$ for every such X. (Otherwise we could remove any edge from H to get a graph containing no $(2t+1)$-vertex induced forest and with fewer than $g(n-1,t)$ edges.)

If each big component of H contains a non-neighbour of v, then we can choose a set X as above so that $X \cup \{v\}$ induces a forest on $2t+1$ vertices, a contradiction. Therefore H has a big component C such that v is complete to C. By the induction hypothesis, each big component of H has at least $\max(3, \lceil \frac{n}{t} \rceil - 1) = \deg(v)$ vertices; it follows that $|V(C)| = \deg(v) = \max(3, \lceil \frac{n}{t} \rceil - 1)$, and that G is obtained from H by adding a new vertex with neighbourhood $V(C)$.

If $2t+1 \leqslant n \leqslant 4t$, then $|V(C)| = \deg(v) = \max(3, \lceil \frac{n}{t} \rceil - 1) = 3$, so G is obtained from H by adding a vertex complete to a component on three vertices; thus, every component of G is complete with 1, 3 or 4 vertices. If $n < 4t$ then this implies that G has the claimed structure. If $n = 4t$ then $|E(G)| = g(4t,t) = 6t = \frac{3}{2}|V(G)|$ and G has maximum degree 3. This implies that every vertex of G has degree three, and so G is isomorphic to $G_{4t,t}$, as required. If $n > 4t$, then H is isomorphic to $G_{n-1,t}$, so $|V(C)| = \deg(v) = \lceil \frac{n}{t} \rceil - 1 = \lfloor \frac{n-1}{t} \rfloor$. Thus C is a smallest component of H, so G is isomorphic to $G_{n,t}$, as required. □

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