Cycle decompositions of complete digraphs

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Abstract

In this paper, we consider the problem of decomposing the complete directed graph K_n^* into cycles of given lengths. We consider general necessary conditions for a directed cycle decomposition of K_n^* into t cycles of lengths $m_1, m_2, \ldots, m_t$ to exist and provide a powerful construction for creating such decompositions in the case where there is one 'large' cycle. Finally, we give a complete solution in the case when there are exactly three cycles of lengths $\alpha, \beta, \gamma \neq 2$. Somewhat surprisingly, the general necessary conditions turn out not to be sufficient in this case. In particular, when $\gamma = n$, $\alpha + \beta > n + 2$ and $\alpha + \beta \equiv n \pmod 4$, K_n^* is not decomposable.

Mathematics Subject Classifications: 05C51, 05B30, 05C20, 05C38

1 Introduction

Let G be a graph and $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ be a collection of subgraphs of G. We say that $\mathcal{H}$ *decomposes* G if the edges of the graphs in $\mathcal{H}$ partition the edges of G. In this case, we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_r$. If $H_1 \cong H_2 \cong \cdots \cong H_r \cong H$, we refer to an

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H-*decomposition* of G; in this case, we call the decomposition *uniform*. An oft-studied case is that G is the complete graph K_n and each element of $\mathcal{H}$ is a cycle. It is easy to see that if n is even, then there can be no cycle decomposition of K_n. In this case, it is common instead to consider $K_n - I$ the complete graph with the edges of a 1-factor I removed. We will use C_m to denote a cycle of length m.

The existence question for cycle decompositions of complete graphs has a history dating to the mid-1800s; among the first landmark results were Kirkman's 1847 proof that K_n has a C_3-decomposition if and only if $n \equiv 1$ or $3 \pmod 6$ [6] and Walecki's construction of Hamilton cycle decompositions of the complete graph [8]. The existence problem for uniform cycle decomposition of K_n and $K_n - I$ was only settled over a century later [2, 9].

Theorem 1 ([2, 9])**.** *Let $n, k \geqslant 3$ be integers. There exists a C_m-decomposition of K_n if and only if n is odd and $k \mid \frac{n(n-1)}{2}$. There exists a C_m-decomposition of $K_n - I$ if and only if n is even and $m \mid \frac{n(n-2)}{2}$.*

A more general question is the existence of possibly non-uniform cycle decompositions of K_n or $K_n - I$. It was conjectured by Alspach [1] in 1981 that the obvious necessary conditions for the existence of such a decomposition were sufficient. Alspach's conjecture was finally verified in a 2014 paper by Bryant, Horsley and Pettersson [5].

Theorem 2 ([5])**.** *Let $n \geqslant 3$ be an integer, and let G_n denote K_n if n is odd, and $K_n - I$ if n is even. Let $\mathcal{H} = \{C_{m_1}, C_{m_2}, \ldots, C_{m_r}\}$. Then $\mathcal{H}$ decomposes G_n if and only if $3 \leqslant m_i \leqslant n$ for each $i = 1, 2, \ldots, r$ and $m_1 + m_2 + \cdots + m_r = n\lfloor \frac{n-1}{2} \rfloor$.*

More recently, Bryant, Horsley, Maenhaut and Smith [4] have extended this result, finding necessary and sufficient conditions for the existence of a cycle decomposition of the complete multigraph λK_n.

Theorem 3 ([4])**.** *Let $\mathcal{C} = \{C_{m_1}, C_{m_2}, \ldots, C_{m_r}\}$. Then $\mathcal{C}$ decomposes λK_n if and only if the following conditions all hold:*

1. $\lambda(n-1)$ *is even;*
2. $2 \leqslant m_1, m_2, \ldots, m_r \leqslant n$*;*
3. $m_1 + m_2 + \cdots + m_r = \lambda\binom{n}{2}$*;*
4. $\max(m_1, m_2, \ldots, m_r) + r - 2 \leqslant \frac{\lambda}{2}\binom{n}{2}$ *when λ is even; and*
5. $\sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{n}{2}$ *when λ is odd.*

Also, $\mathcal{C}$ decomposes $\lambda K_n - I$ if and only if the following conditions all hold:

1. $\lambda(n-1)$ *is odd;*
2. $2 \leqslant m_1, m_2, \ldots, m_r \leqslant n$*;*

3. $m_1 + m_2 + \cdots + m_r = \lambda\binom{n}{2} - \frac{n}{2}$; *and*

4. $\sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{n}{2}$.

In this paper, we consider an alternative generalization of Alspach's conjecture, to cycle decompositions of the complete symmetric digraph.

For a graph G, we let G^* denote the digraph formed from G by replacing each edge $\{x, y\}$ with two arcs xy and yx. In particular, K_n^* denotes the complete symmetric digraph on n vertices. We will use $\overrightarrow{P}_k$ to denote a directed path on k vertices, and $\overrightarrow{C}_k$ a directed cycle of length k. We use $[u_1, u_2, \ldots, u_k]$ to denote a directed path with arcs $u_1u_2, u_2u_3, \ldots, u_{k-1}u_k$ and $(u_1, u_2, \ldots, u_k)$ to denote a directed cycle with arcs $u_1u_2, u_2u_3, \ldots, u_{k-1}u_k$ together with u_ku_1.

The case of uniform cycle decomposition of K_n^* was settled by Alspach, Gavlas, Šajna and Verrall [3].

Theorem 4 ([3])**.** *Let $n, m \geqslant 2$ be integers. There is a $\overrightarrow{C}_m$-decomposition of K_n^* if and only if $m \mid n(n-1)$ and $(n, m) \notin \{(4, 4), (6, 3), (6, 6)\}$.*

Let $\mathcal{C}$ denote a collection of r directed cycles of lengths $m_1, m_2, \ldots, m_r$. It is clear that by ignoring the direction on the arcs, any $\mathcal{C}$-decomposition of K_n^* yields a $\mathcal{C}$-decomposition of $2K_n$. Thus, any such decomposition must satisfy the conditions of Theorem 3 with $\lambda = 2$, giving the following necessary conditions:

Lemma 5. *Let $\mathcal{C} = \{\overrightarrow{C}_{m_1}, \overrightarrow{C}_{m_2}, \ldots, \overrightarrow{C}_{m_r}\}$. If $\mathcal{C}$ decomposes K_n^*, then:*

1. $2 \leqslant m_1, m_2, \ldots, m_r \leqslant n$;

2. $m_1 + m_2 + \cdots + m_r = n(n-1)$; *and*

3. $\max(m_1, m_2, \ldots, m_r) + r - 2 \leqslant \binom{n}{2}$

In light of Theorem 3, one possible method of constructing directed cycle decompositions of K_n^* would be to orient the edges in a cycle decomposition of $2K_n$. However, it is worth noting that the methods of [4] for $2K_n$ do not in general produce decompositions whose cycles can be oriented to give a directed cycle decomposition of K_n^*. In fact, as we shall see in Theorem 24, the necessary conditions stated in Lemma 5 are not also sufficient in all cases.

Definition 6. A nondecreasing list $M = (m_1, m_2, \ldots, m_r)$ of nonnegative integers will be called *n-admissible* if it satisfies conditions 1–3 of Lemma 5. We refer to a directed cycle decomposition of K_n^* corresponding to the admissible list M as an *M-decomposition of K_n^**.

Using the notation of [4] and [5], for a given n-admissible list M, we let $\nu_i(M)$ denote the number of instances of i in M. It is worth noting that one of the major differences in decomposing K_n^* (or λK_n) rather than K_n is the possibility of cycles of length 2. Further, if there are cycles of length 2 in a decomposition of K_n^*, removing them corresponds to removing edges from the underlying undirected graph, K_n. Thus directed cycle decomposition of K_n^* can be seen as a directed cycle decomposition of G^* into cycles of lengths greater than 2, where G is an arbitrary graph of order n. This observation leads to a strategy for decomposing K_n^*.

Definition 7. We say that a directed cycle $\overrightarrow{C}$ *decouples* the arc xy if $\overrightarrow{C}$ contains yx but not xy. If $\mathcal{C} = \{\overrightarrow{C}_{m_1}, \overrightarrow{C}_{m_2}, \ldots, \overrightarrow{C}_{m_r}\}$ is a collection of pairwise edge-disjoint cycles, we say that an arc xy is *decoupled by* $\mathcal{C}$ if yx is contained in a cycle of $\mathcal{C}$ but xy is not.

The following is an obvious consequence of Definition 7.

Lemma 8. *Let $\mathcal{C} = \{\overrightarrow{C}_{m_1}, \overrightarrow{C}_{m_2}, \ldots, \overrightarrow{C}_{m_r}\}$ be a set of pairwise edge-disjoint directed cycles in K_n^*. There is a $\overrightarrow{C}_2$-decomposition of $K_n^* - \mathcal{C}$ if and only if $\mathcal{C}$ leaves no arc decoupled.*

For convenience of notation, we will henceforth let $\nu = \nu_2(M)$ denote the number of cycles of length 2 in the list M. In light of Lemma 8, it will sometimes be convenient to list only the cycle lengths which are greater than 2, particularly when a decomposition is to contain many 2-cycles. Therefore, we make the following definition.

Definition 9. Let M be an n-admissible list. The *associated canonical list* is the non-decreasing list $\hat{M}$ formed from M by removing all instances of 2 from M. Note that the sum of the entries in $\hat{M}$ is $n(n-1) - 2\nu$.

In this paper, we give further results on cycle decompositions of K_n^*. In Section 2, we give general constructions and show sufficiency of the necessary conditions in certain cases. In Section 3, we determine necessary and sufficient conditions for the existence of an M-decomposition of K_n^* when $\nu \geqslant |M| - 3$, summarized in Theorem 24. As we will see, the necessary conditions of Lemma 5 turn out not to be sufficient in general. In particular, in Lemma 23 we exhibit a family of n-admissible lists for which K_n^* admits no M-decomposition.

2 General constructions

We begin by noting some easy consequences of conditions 2 and 3 of Lemma 5.

Lemma 10. *Let M be an n-admissible list with associated canonical list $\hat{M} = (m_1, \ldots, m_r)$. Then*

1. $m_1 + \cdots + m_{r-1} \equiv m_r \pmod 2$
2. $m_1 + \cdots + m_{r-1} \geqslant m_r + 2(r-2)$. *Hence when $r \geqslant 3$, $m_1 + \cdots + m_{r-1} > m_r$.*

Conversely, given a list $\hat{M} = (m_1, \ldots, m_r)$ with $3 \leqslant m_i \leqslant n$ for each $i \in \{1, \ldots, r\}$ and $m_1 + \cdots + m_r \leqslant n(n-1)$, if $\hat{M}$ satisfies conditions 1 and 2 above, then $\hat{M}$ is the associated canonical list of some n-admissible list M.

Note that the case where equality holds in condition 2 corresponds with the case where equality holds in condition 3 of Lemma 5.

We now give some constructions of cycle decompositions of K_n^* from known decompositions of complete graphs or complete symmetric digraphs.

Theorem 11. *Let $M = (m_1, m_2, \ldots, m_r)$ be an n-admissible list, where $\nu_k(M)$ is even for each $3 \leqslant k \leqslant n$. If either*

1. *n is odd and either $\nu = 0$ or $\nu \geqslant 3$; or*
2. *n is even and either $\nu = \frac{n}{2}$ or $\nu \geqslant \frac{n}{2} + 3$,*

then K_n^* is M-decomposable.

Proof. We form a new list M' which decomposes K_n. We then form the corresponding directed graph K_n^*, using some of the cycles from K_n to form 2-cycles in K_n^*. We consider two cases according to the parity of n.

First, suppose that n is odd. Let $s \in \{0, 4, 5\}$ be such that $\nu \equiv s \pmod 3$. Let M' be the nondecreasing list satisfying that, for $i \in \{3, 4, \ldots, n\}$,

$$\nu_i(M') = \begin{cases} \frac{1}{2}\nu_3(M) + \frac{1}{3}(\nu - s), & \text{if } i = 3 \\ \frac{1}{2}\nu_s(M) + 1, & \text{if } i = s \neq 0 \\ \frac{1}{2}\nu_i(M), & \text{otherwise} \end{cases}$$

Since $m_1 + m_2 + \cdots m_r = n(n-1)$, it follows that the sum of the entries of M' is $\binom{n}{2}$, and so by Theorem 2, K_n is M'-decomposable. From such a decomposition, take $\frac{\nu_i(M)}{2}$ cycles of length i for each $3 \leqslant i \leqslant n$, and orient them both ways to obtain $\nu_i(M)$ directed cycles of length i. Each edge in the remaining cycles can be directed both ways to form a directed 2-cycle, giving $\nu_2(M)$ in total. It is easy to see that we obtain an M-decomposition of K_n^*.

The case where n is even is similar. Again, let $s \in \{0, 4, 5\}$ such that $\nu \equiv s + \frac{n}{2} \pmod 3$. Let M' be the nondecreasing list satisfying that, for $i \in \{3, 4, \ldots, n\}$

$$\nu_i(M') = \begin{cases} \frac{1}{2}\nu_3(M) + \frac{1}{3}(\nu - s - \frac{n}{2}), & \text{if } i = 3 \\ \frac{1}{2}\nu_s(M) + 1, & \text{if } i = s \neq 0 \\ \frac{1}{2}\nu_i(M), & \text{otherwise.} \end{cases}$$

In this case, we note that $K_n - I$ is M'-decomposable, and proceed as before, except that the edges of the 1-factor I are also directed both ways to form $\frac{n}{2}$ directed 2-cycles. $\square$

Lemma 12. *Let M be an n-admissible list and M' an n'-admissible list, where $n < n'$. If $\nu_k(M) = \nu_k(M')$ for each $k \geqslant 3$ and there exists an M-decomposition of K_n^*, then there exists an M'-decomposition of $K_{n'}^*$.*

Proof. Decompose $K^*_{n'} = K^*_n \oplus K^*_{n'-n} \oplus K^*_{n,n'-n}$. Decomposing K^*_n into directed cycles of lengths given in M, it is easy to see that no arc of $K^*_{n'}$ is left decoupled. Since all remaining directed cycles to be formed are of length 2, the result follows by Lemma 8. $\square$

In the remainder of this section, we will give constructions which give directed cycle decomposition in a wide array of cases. Before proceeding, we note the following result which settles existence in the case that the canonical list has size at most 2.

Theorem 13. *Let $M = (m_1, m_2, \ldots, m_r)$. Suppose M has at most two cycle lengths other than 2, i.e. $\nu \geqslant r-2$. There is an M-decomposition of K^*_n if and only if M is n-admissible.*

Proof. It is easy to see that if $\nu = r$ (i.e. $M = (2, 2, \ldots, 2)$), an M-decomposition of K^*_n exists. Also, note that conditions 2 and 3 of Lemma 5 imply that no n-admissible list has $\nu = r-1$.

Finally, if M is an n-admissible list with $\nu = r-2$, its canonical list has the form $\hat{M} = (m_{r-1}, m_r)$. By condition 3 of Lemma 5 and since M is non-decreasing, $m_{r-1}+r-2 \leqslant m_r + r - 2 \leqslant \frac{n(n-1)}{2}$. But condition 2 implies that $2(r-2) + m_{r-1} + m_r = n(n-1)$, i.e. $(m_{r-1}+r-2)+(m_r+r-2) = n(n-1)$, so it must be that $m_{r-1}+r-2 = m_r+r-2 = \frac{n(n-1)}{2}$, and hence $m_{r-1} = m_r$. In this case, it is easy to see that an M-decomposition of K^*_n exists. $\square$

Recall that condition 3 of Lemma 5 states that in an n-admissible list M of size t with maximum entry m, we have that $m + t - 2 \leqslant \binom{n}{2}$. Written in terms of the canonical list $\hat{M} = (m_1, m_2, \ldots, m_r)$, the total number of cycles is $t = r + \nu$, so this condition becomes $m_r + r + \nu - 2 \leqslant \binom{n}{2}$. The next result shows that if equality holds, then an M-decomposition does indeed exist.

Lemma 14. *Let M be an n-admissible list with associated canonical list $\hat{M} = (m_1, m_2, \ldots, m_r)$. If*

$$m_r + r + \nu - 2 = \binom{n}{2},$$

then K^*_n is M-decomposable.

Proof. First, suppose that $m_{r-1} = m_r$. Since M is n-admissible and contains ν entries equal to 2, counting edges we have that

$$2\nu+m_1+m_2+\cdots+m_r = 2\nu+m_1+m_2+\cdots+m_{r-2}+2m_r = n(n-1) = 2(m_r+r+\nu-2).$$

Rearranging gives $m_1 + m_2 + \cdots + m_{r-2} = 2(r-2)$, and since $m_i \geqslant 3$ for each $i \in \{1, 2, \ldots, r-2\}$, it follows that $r = 2$, which was dealt with in Theorem 13.

Now suppose $m_{r-1} < m_r$. Let the vertex set of K^*_n be $\mathbb{Z}_n$. We form the cycles of lengths $m_1, m_2, \ldots, m_r$. An illustration of this construction can be found in Figure 1.

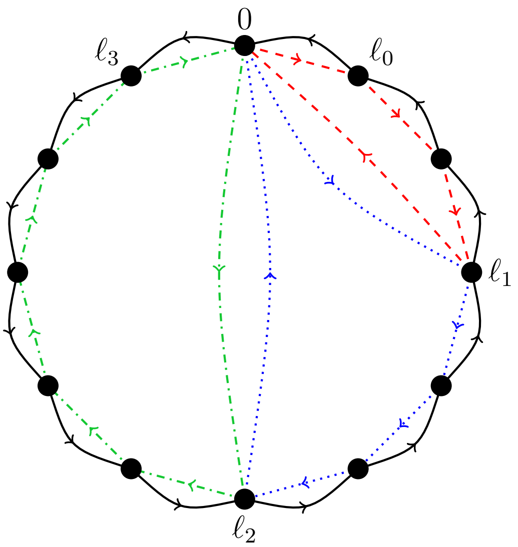

Figure 1: A $(2,\ldots,2,4,5,7,12)$-decomposition of K_{12}^*

Let $\overrightarrow{C}_{m_r} = (m_r-1, m_r-2, \ldots, 2, 1, 0)$ be the directed cycle of length m_r. Define $\ell_0 = 1$, $\ell_{r-1} = m_r - 1$, and for $i = 1, 2, \ldots, r-2$, let $\ell_i = \left(\sum_{j=1}^{i} m_j\right) - (2i-1)$. Note that, since $n(n-1) = 2(m_r + r + \nu - 2) = 2\nu + m_1 + m_2 + \cdots + m_r$,

$$\begin{aligned} \ell_{r-2} &= (m_1 + m_2 + \cdots + m_{r-2}) - (2(r-2)-1) \\ &= (m_r - m_{r-1} + 2r - 4) - (2r-5) \\ &= m_r - m_{r-1} + 1. \end{aligned}$$

Hence, $\ell_i - \ell_{i-1} = m_i - 2$ for each $i = 1, \ldots, r-1$. For $i = 1, 2, \ldots, r-1$, define the m_i-cycle

$$\overrightarrow{C}_{m_i} = (0, \ell_{i-1}, \ell_{i-1}+1, \ell_{i-1}+2, \ldots, \ell_i).$$

It is easy to verify that $\overrightarrow{C}_{m_i}$ has length $\ell_i - \ell_{i-1} + 2 = m_i$.

Note that the cycles $\overrightarrow{C}_{m_1}, \ldots, \overrightarrow{C}_{m_{r-1}}, \overrightarrow{C}_{m_r}$ leave no decoupled arc, and the result follows by Lemma 8. $\square$

Theorem 15. *Let M be an n-admissible list with associated canonical list $\hat{M} = (m_1, m_2, \ldots, m_r)$. Let $S = (s_0, s_1, \ldots, s_{r-2}, s_{r-1}, s_r)$ be a sequence of non-negative integers with $s_0 = s_{r-1} = s_r = 0$, and let s be the sum of the entries of S. If M is an n-admissible list such that:*

1. *$m_r + r + \nu - 2 = \binom{n}{2}$; and*

2. *$m_r + s \leqslant n$,*

then K_n^* is M'-decomposable, where $\hat{M}' = (m'_{\sigma(1)}, m'_{\sigma(2)}, \ldots, m'_{\sigma(r)})$ such that $m'_i = m_i + s_{i-1} + s_i$ for each $i \in \{1, \ldots, r\}$ and σ is a permutation which ensures that $\hat{M}'$ is in nondecreasing order.

Proof. We proceed in a similar manner to the proof of Lemma 14, except that we add s vertices $x_1, \ldots, x_s$ to the cycles in $\hat{M}$, and modify the cycles $\overrightarrow{C}_{m_i}$, $1 \leqslant i \leqslant r$, from that lemma as follows.

Let ℓ_i and $\overrightarrow{C}_{m_r}$ be as defined as in the proof of Lemma 14. For $i \in \{0, 1, \ldots, r\}$, let $s'_i = s_0 + s_1 + \cdots + s_i$. We define directed paths

$$P_i = [0, x_{s'_{i-1}+1}, x_{s'_{i-1}+2}, \ldots, x_{s'_i}, \ell_i].$$

Note that if $s_i = 0$, we take $P_i = [0, \ell_i]$ of length 1. We also define Q_i to be the reversal of P_i, i.e. if $s_i \neq 0$ then

$$Q_i = [\ell_i, x_{s'_i}, x_{s'_i-1}, \ldots, x_{s'_{i-1}+1}, 0],$$

and $Q_i = [\ell_i, 0]$ otherwise. Now, define $\overrightarrow{C}_{m_1}$ as the concatenation of the path $[0, 1, \ldots, \ell_1]$ and Q_1. For $i \in \{2, \ldots, r-2\}$, define $\overrightarrow{C}_{m_i}$ as the concatenation of the paths P_{i-1}, $[\ell_{i-1}, \ell_{i-1}+1, \ldots, \ell_i]$ and Q_i. Finally, define $\overrightarrow{C}_{m_{r-1}}$ as the concatenation of P_{r-2} and $[\ell_{r-2}, \ell_{r-2}+1, \ldots, \ell_{r-1}, 0]$ (recall that $\ell_{r-1} = m_r - 1$). See Figure 2.

It is clear that these cycles leave no decoupled arcs, so the result follows by Lemma 8. □

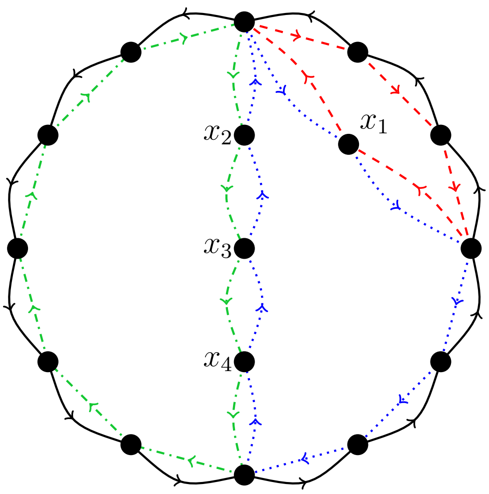

Figure 2: A $(2, \ldots, 2, 5, 9, 10, 12)$-decomposition of K^*_{16} formed using Theorem 15 with $\hat{M} = (4, 5, 7, 12)$, $S = (0, 1, 3, 0, 0)$.

Note that although the statements of Lemma 14 and Theorem 15 are written so that the cycle lengths $m_1, \ldots, m_r$ are in non-decreasing order, this is not actually required in the proof, as long as $m_r \geqslant \max_{1 \leqslant i \leqslant r-1}\{m_i\}$.

Theorem 16. *Let H be an n-admissible list with associated canonical list $\hat{H} = (h_1, \ldots, h_r)$ such that $r \geqslant 3$, $h_r \geqslant 2(r-1)$, $h_1 + \cdots + h_r \leqslant 2(n + r - 2)$, and, if r is even, $r \neq 4$ and $h_1 \leqslant r$. Then K^*_n is H-decomposable.*

Proof. We begin by defining a list $K = (k_1, k_2, \ldots, k_r)$ by permuting the elements of $\hat{H}$ with the purpose of ensuring that K satisfies the inequalities (1) and (2) below, while retaining the property that $k_r \geqslant \max_{1\leqslant i\leqslant r-1}\{k_i\}$.

(1) If r is odd, then $k_1 + k_3 + \cdots + k_{r-2} + k_r \geqslant k_2 + k_4 + \cdots + k_{r-1} + (r+1)$.

(2) If r is even, then $k_2 + k_4 + \cdots + k_{r-2} + k_r \geqslant k_1 + k_3 + \cdots + k_{r-1} + (r-2)$.

If r is odd, take $k_r = h_r$, and define $k_1, \ldots, k_{r-1}$ by

$$(k_2, k_4, \ldots, k_{r-1}) = (h_1, h_2, \ldots, h_{(r-1)/2})$$

and

$$(k_1, k_3, \ldots, k_{r-2}) = (h_{(r+1)/2}, h_{(r+3)/2}, \ldots, h_{r-1}).$$

This ensures that the elements of $(k_1, \ldots, k_r)$ with odd index are greater than or equal to those with even index, i.e. $k_{2i-1} \geqslant k_{2i}$ for $i \in \{1, \ldots, (r-1)/2\}$. Since $k_r \geqslant 2(r-1)$, it is now easy to see that (1) is satisfied.

If r is even, take $k_r = h_r$, $k_{r-1} = h_4$, $k_{r-2} = h_3$, $k_{r-3} = h_2$ and $k_1 = h_1$, and define $k_2, \ldots, k_{r-4}$ so that

$$(k_3, k_5, \ldots, k_{r-5}) = (h_5, h_6 \ldots, h_{(r+2)/2})$$

and

$$(k_2, k_4, \ldots, k_{r-4}) = (h_{(r+4)/2}, h_{(r+6)/2}, \ldots, h_{r-1}).$$

Thus, $k_r = \max\{k_1, \ldots, k_r\}$, $k_{r-2} \geqslant k_{r-3}$, $k_{r-4} \geqslant k_{r-1}$, and $k_{2i} \geqslant k_{2i+1}$ for $i \in \{1, \ldots, (r-6)/2\}$. Since the elements of H are written in nondecreasing order, using the assumptions that $k_r = h_r \geqslant 2(r-1)$ and $k_1 = h_1 \leqslant r$, it is now easy to check that (2) is satisfied.

We now define lists $(m_1, \ldots, m_r)$ and $(s_0, \ldots, s_r)$ of integers with $m_r \geqslant \max\limits_{1\leqslant i\leqslant r-1}\{m_i\}$ satisfying the conditions of Theorem 15, such that for each $i \in \{1, \ldots, r\}$, $k_i = m_i + s_{i-1} + s_i$. The inequalities (1) and (2) will ensure that $m_{r-2} \geqslant 3$.

Note that by Lemma 10 and the assumptions that $k_r = h_r \geqslant 2(r-1)$ and $r \geqslant 3$, we have

$$3(r-1) \leqslant k_r + 2(r-2) \leqslant k_1 + \cdots + k_{r-1}.$$

Thus, we can find integers $m_1, \ldots, m_r$ such that $3 \leqslant m_i \leqslant k_i$ for each $i \in \{1, \ldots, r\}$ and $m_1 + \cdots + m_{r-1} = k_r + 2(r-2)$. Specifically, define

$$t = \frac{k_1 + k_2 + \cdots + k_{r-1} - k_r - 2(r-2)}{2},$$

and note that t is a non-negative integer by Lemma 10.

We define a sequence of non-negative integers $s_0, s_1, s_2, \ldots, s_r$ by $s_0 = s_{r-1} = s_r = 0$, for $i \in \{1, 2, \ldots, r-3\}$, set

$$s_i = \min\{k_i - s_{i-1} - 3, t - (s_1 + \cdots + s_{i-1})\},$$

and define $s_{r-2} = t - (s_1 + \cdots + s_{r-3})$. It is evident from the definition that $s_{i-1} + s_i \leqslant k_i - 3$ for each $i \in \{1, \ldots, r-3\}$ and $s_1 + s_2 + \cdots + s_r = t$.

We now show by induction that $s_i \geqslant 0$ for each $0 \leqslant i \leqslant r$. First note that $s_0 = 0 \geqslant 0$ and $s_1 = \min\{k_1 - 3, t\} \geqslant 0$. Now suppose that $i > 1$ and assume as the inductive hypothesis that $s_j \geqslant 0$ for each $0 \leqslant j < i$. Firstly, if $s_i = k_i - s_{i-1} - 3$ then $s_{i-1} + s_{i-2} \leqslant k_{i-1} - 3$, so $s_{i-1} \leqslant k_{i-1} - 3$, which implies that $k_i - s_{i-1} - 3 \geqslant 0$, since $k_i \geqslant k_{i-1}$. Alternatively if $s_i = t - (s_1 + \cdots + s_{i-1})$, then by the definition of s_{i-1}, $s_{i-1} \leqslant t - (s_1 + \cdots + s_{i-2})$ which implies $t - (s_1 + \cdots + s_{i-1}) \geqslant 0$. Hence, noting that $s_{r-1} = s_r = 0$, each $s_i \geqslant 0$.

For each $i \in \{1, \ldots, r\}$, set $m_i = k_i - s_{i-1} - s_i$. By choice of s_{r-2}, it is easy to see that

$$s = \sum_{i=1}^{r-1} s_i = t,$$

and hence

$$\sum_{i=1}^{r-1} m_i = \sum_{i=1}^{r-1} k_i - 2\sum_{i=1}^{r-1} s_i = \sum_{i=1}^{r-1} k_i - 2t = k_r + 2(r-2) = m_r + 2(r-2).$$

It is now easy to check that $(m_1, \ldots, m_r)$ satisfy conditions 1 and 2 of Lemma 10. Also, since $m_i \leqslant k_i$ for each $i \in \{1, \ldots, r\}$, we have that each $m_i \leqslant k_i \leqslant n$, and $m_1 + \cdots + m_r \leqslant k_1 + \cdots + k_r \leqslant n(n-1)$. Thus, to show that $(m_1, \ldots, m_r)$ can be viewed as the (suitably ordered) canonical list of an n-admissible list M, we need only to show that $m_i \geqslant 3$ for each $i \in \{1, \ldots, r\}$. This is clear if $1 \leqslant i \leqslant r-3$ since $s_i \leqslant k_i - s_{i-1} - 3$, and if $i = r$ since $m_r = k_r$.

For $i = r-2$, we have

$$m_{r-2} = k_{r-2} - s_{r-3} - s_{r-2}.$$

If $s_{r-2} = 0$, then $m_{r-2} = k_{r-2} - s_{r-3} \geqslant k_{r-3} - s_{r-3} \geqslant m_{r-3} \geqslant 3$. Otherwise,

$$\begin{aligned} m_{r-2} &= k_{r-2} - s_{r-3} - s_{r-2} \\ &= k_{r-2} - s_{r-3} - (t - (s_1 + s_2 + \cdots + s_{r-3})) \\ &= k_{r-2} - t + s_1 + s_2 + \cdots + s_{r-4}. \end{aligned}$$

To bound the value of m_{r-2}, we rewrite this quantity in terms of $k_1, \ldots, k_r$. Note that for each $j \in \{1, \ldots, r-3\}$, if $s_j = t - (s_1 + \cdots + s_{j-1})$, then $s_{j+1} = 0$. Hence the condition $s_{r-2} \neq 0$ implies that $s_j = k_j - s_{j-1} - 3$ for each $j \in \{1, \ldots, r-3\}$. Thus

$$\begin{aligned} s_1 + \cdots + s_{r-4} &= \sum_{i=1}^{r-4} (k_i - s_{i-1} - 3) \\ &= (k_1 - 3) + [(k_2 - 3) - (k_1 - 3)] + [(k_3 - 3) - (k_2 - 3) + (k_1 - 3)] + \cdots \\ &\qquad + [(k_{r-4} - 3) + \cdots + (-1)^{r-5}(k_1 - 3)] \\ &= \begin{cases} k_1 + k_3 + \cdots + k_{r-4} - 3(r-3)/2 & \text{if } r \text{ is odd} \\ k_2 + k_4 + \cdots + k_{r-4} - 3(r-4)/2 & \text{if } r \text{ is even.} \end{cases} \end{aligned}$$

If r is odd, it follows that

$$\begin{aligned}
m_{r-2} &= k_{r-2} - t + s_1 + \cdots + s_{r-4} \\
&= k_1 + k_3 + \cdots + k_{r-4} + k_{r-2} - \frac{3(r-3)}{2} - \frac{k_1 + k_2 + \cdots + k_{r-1} - k_r - 2(r-2)}{2} \\
&= \frac{k_1 + k_3 + \cdots + k_{r-2} + k_r}{2} - \frac{k_2 + k_4 + \cdots + k_{r-1}}{2} - \frac{r-5}{2} \\
&\geqslant 3,
\end{aligned}$$

since $k_1 + k_3 + \cdots + k_{r-2} + k_r \geqslant k_2 + k_4 + \cdots + k_{r-1} + r + 1$.

If r is even, then

$$\begin{aligned}
m_{r-2} &= k_{r-2} - t + s_1 + \cdots + s_{r-4} \\
&= k_2 + k_4 + \cdots + k_{r-4} + k_{r-2} - \frac{3(r-4)}{2} - \frac{k_1 + k_2 + \cdots + k_{r-1} - k_r - 2(r-2)}{2} \\
&= \frac{k_2 + k_4 + \cdots + k_{r-2} + k_r}{2} - \frac{k_1 + k_3 + \cdots + k_{r-1}}{2} - \frac{r-8}{2} \\
&\geqslant 3
\end{aligned}$$

since $k_2 + k_4 + \cdots + k_{r-2} + k_r \geqslant k_1 + k_3 + \cdots + k_{r-1} + r - 2$.

Finally, we show that $m_{r-1} \geqslant 3$. Since $s_{r-1} = 0$ and by construction $k_{r-1} \geqslant k_{r-2}$, we have that

$$m_{r-1} = k_{r-1} - s_{r-2} \geqslant k_{r-2} - s_{r-2} - s_{r-3} = m_{r-2} \geqslant 3.$$

We now show that the list satisfies Condition 2 of Theorem 15, i.e. that $m_r + s \leqslant n$. Since

$$s = \sum_{i=1}^{r-1} s_i = \sum_{i=1}^{r-1} (k_i - m_i - s_{i-1}) = \left(\sum_{i=1}^{r-1} k_i \right) - (k_r + 2(r-2)) - s,$$

we have

$$2s = \left(\sum_{i=1}^{r} k_i \right) - 2k_r - 2(r-2),$$

giving

$$s = \frac{1}{2} \left(\sum_{i=1}^{r} k_i \right) - k_r - r + 2 \leqslant (n + r - 2) - k_r - r + 2 = n - k_r,$$

so that $m_r + s = k_r + s \leqslant n$.

The result now follows by applying Theorem 15, taking $\hat{M}' = K$. □

We note that the conclusion of Theorem 16 holds whenever $r \geqslant 3$, $h_r \geqslant 2(r-1)$, $h_1 + \cdots + h_r \leqslant 2(n + r - 2)$ and there is a reordering of the h_i, $1 \leqslant i < r$, so that inequalities (1) and (2) are satisfied. In particular, when r is even, the conditions $r \neq 4$ and $h_1 \leqslant r$ may be dropped if (2) holds in the reordering of the h_i.

Recall that a directed cycle decomposition of K_n^* can be considered as a decomposition of G^* into directed cycles of lengths greater than 2 for some graph G of order n. Theorem 16 applies when the underlying (undirected) graph G formed by cycles not of length two is sparse. In particular, it solves all cases where G has one vertex of degree $r \geqslant 5$ ($r \geqslant 3$ if r is odd) and all the rest of degree at most three, and which is a subdivision of a Hamiltonian graph. It thus gives a solution in all cases where G can be decomposed into subgraphs which either have this form or are 2-regular.

3 Admissible lists with three cycles of length greater than 2

Recall that Theorem 13 states that K_n^* is M-decomposable whenever M is an admissible list containing at most two cycles of length greater than 2. In this section we give necessary and sufficient conditions for the existence of an M-decomposition of K_n^* in the case that the associated canonical list $\hat{M}$ has size 3. We begin by noting the following special case of Lemma 10.

Lemma 17. *If M is an n-admissible list with associated canonical list $\hat{M} = (\alpha, \beta, \gamma)$, then $\alpha + \beta \equiv \gamma \pmod 2$ and $\alpha + \beta > \gamma$.*

For the case that $\alpha + \beta + \gamma \leqslant 2(n + 1)$, sufficiency follows directly from Theorem 16. Specifically, we have the following lemma.

Lemma 18. *Let M be an n-admissible list with associated canonical list $\hat{M} = (\alpha, \beta, \gamma)$. If $\alpha + \beta + \gamma \leqslant 2(n + 1)$, then K_n^* is M-decomposable.*

Proof. Note that the conditions of Theorem 16 when $r = 3$ are that $\alpha + \beta + \gamma \leqslant 2(n + 1)$ (as in the assumption) and $\gamma \geqslant 4$. However, if $\gamma = 3$, then $\alpha = \beta = \gamma = 3$, so that $\alpha + \beta \not\equiv \gamma \pmod 2$, in contradiction to Lemma 17. □

If $\hat{M} = (\alpha, \beta, \gamma)$ and $\alpha + \beta + \gamma > 2(n + 1)$, the existence of an M-decomposition of K_n^* depends on the value of γ as well as the congruence classes of $\alpha + \beta$ and γ modulo 4.

Lemma 19. *Let M be an n-admissible list with associated canonical list $\hat{M} = (\alpha, \beta, \gamma)$. If $\alpha + \beta + \gamma > 2(n + 1)$ and $\alpha + \beta \not\equiv \gamma \pmod 4$, then K_n^* is M-decomposable.*

Proof. We construct directed cycles of lengths α, β and γ, leaving no decoupled arcs, so that the remaining arcs can be used to form directed 2-cycles by Lemma 8. By Lemma 12, it is sufficient to consider the case $\gamma = n$.

Let $\overrightarrow{C}_n = (0, n-1, n-2, \ldots, 1)$. To construct $\overrightarrow{C}_\alpha$ and $\overrightarrow{C}_\beta$, we must use the decoupled arcs along the directed cycle $\overrightarrow{C} = (0, 1, \ldots, n-1)$, together with $t = (\alpha+\beta-n)/2$ further pairs of arcs. Note that t is odd and $t > 1$.

We first form the cycle of length β. Let $\ell = n - \beta$. If $t = 3$, let

$$\overrightarrow{C}_\beta = (0, \ell + 3, \ell + 4, \ell + 1, \ell + 2, \ell + 5, \ell + 6, \ell + 7, \ldots, n - 1).$$

Otherwise, we build $\overrightarrow{C}_\beta$ by concatenating directed paths. Let

$$P = [0, \ell + 3, \ell + 4, \ell + 1, \ell + 2].$$

For each $i = 0, 1, \ldots, (t-5)/2$, let

$$P_i = [\ell + 4i + 2, \ell + 4i + 7, \ell + 4i + 8, \ell + 4i + 5, \ell + 4i + 6].$$

Note that:

- The initial vertex of P_0 is $\ell + 2$, the terminal vertex of P.
- For each $i = 1, 2, \ldots, (t-5)/2$, the terminal vertex of P_{i-1} and the initial vertex of P_i coincide. Moreover, the paths $P, P_0, P_1, \ldots, P_{(t-5)/2}$ are pairwise internally vertex-disjoint, so that the concatenation $PP_0P_1 \cdots P_{(t-5)/2}$ is a path.
- Each P_i and P uses two arcs of $\overrightarrow{C}$ and two other arcs. In total, these paths contain $(t-1)$ arcs of $\overrightarrow{C}$ and decouple $(t-1)$ further arcs.

Next let $Q = [\ell+2t-4, \ell+2t-1]$. Note that the initial vertex of Q is the terminal vertex of $P_{(t-5)/2}$, and the terminal vertex of Q is $\ell+2t-1 = \alpha-1 \leqslant n-1$, which has not occurred in any of the paths so far. Finally, let $Q' = [\ell+2t-1, \ell+2t, \ldots, \ell+2t-1+(n-2t-\ell+1)]$. Since the terminal vertex of Q' is $\ell + 2t - 1 + n - 2t - \ell + 1 = n = 0$, it is now easy to see that the concatenation

$$PP_0P_1 \cdots P_{(t-5)/2}QQ'$$

forms a directed cycle of length

$$4 + 4(t-3)/2 + 1 + (n - 2t - \ell + 1) = n - \ell = \beta.$$

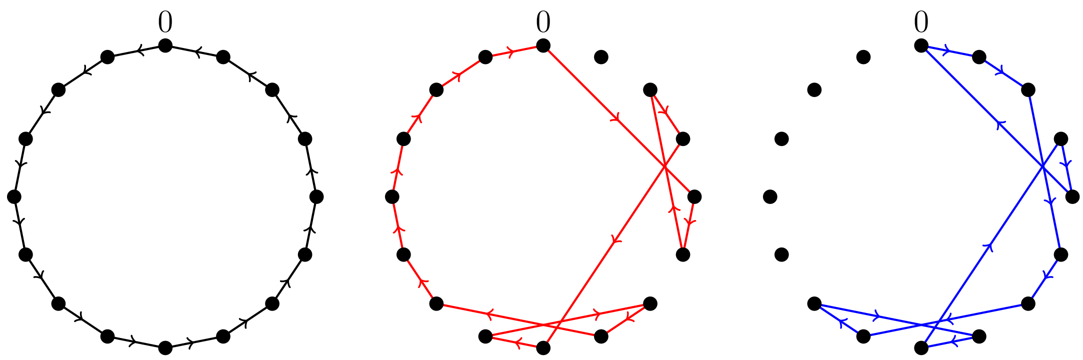

Figure 3: An illustration of the construction of Lemma 19. Here $\alpha = 11$, $\beta = 15$, $\gamma = 16 = n$.

It is not difficult to show that the remaining decoupled arcs form a cycle of length α; see Figure 3. □

Lemma 20. *Let M be an n-admissible list with associated canonical list $\hat{M} = (\alpha, \beta, \gamma)$. If $\alpha + \beta + \gamma > 2(n+1)$, $\alpha + \beta \equiv \gamma \pmod 4$ and $\gamma < n$, then K_n^* is M-decomposable.*

Proof. First, note that $\alpha \geqslant 4$, as otherwise $2(n+1) < \alpha+\beta+\gamma \leqslant 3+\beta+(n-1) = \beta+n+2$, implying $\beta > n$. Let M' be the $(n-1)$-admissible list with associated canonical list $\hat{M}' = (\alpha - 1, \beta - 1, \gamma)$.

Since $\alpha + \beta + \gamma > 2(n + 1)$, we have that $(\alpha - 1) + (\beta - 1) + \gamma > 2n$, and so by Lemma 19, K^*_{n-1} is M'-decomposable. Consider the cycles $\overrightarrow{C}_{\alpha-1}$, $\overrightarrow{C}_{\beta-1}$ and $\overrightarrow{C}_{\gamma}$ in the decomposition constructed by Lemma 19. It is easy to see that there exist arcs xy and yx with $xy \in \overrightarrow{C}_{\alpha-1}$ and $yx \in \overrightarrow{C}_{\beta-1}$. Add a new vertex z, and replace arcs xy and yx with directed paths xzy and yzx. We thus obtain three cycles of lengths α, β and γ in K^*_n which between them leave no arc decoupled. The result follows by Lemma 12. $\square$

Note that the conditions of Lemma 20 require that the largest cycle be non-Hamiltonian. Indeed, if $\gamma = n$ but all other conditions remain the same as those of Lemma 20, no M-decomposition exists. To prove this result, we will exploit a connection between an M-decomposition with $\hat{M} = (\alpha, \beta, n)$ and perfect 1-factorizations, which we now define.

Definition 21. A 1-factorization $\mathcal{F}$ of a graph G is a *perfect 1-factorization* if, for any two 1-factors F_1 and F_2 in $\mathcal{F}$, $F_1 \cup F_2$ is a Hamiltonian cycle.

We will need the following result on perfect 1-factorizations of cubic bipartite graphs, due to Kotzig and Labelle [7].

Lemma 22 ([7])**.** *Let G be a cubic bipartite graph with bipartition (X, Y), where $|X| = |Y| = t$. If G admits a perfect 1-factorization, then t is odd.*

Lemma 23. *Let M be an n-admissible list with associated canonical list $\hat{M} = (\alpha, \beta, n)$. If $\alpha + \beta + n > 2(n + 1)$ (i.e. $\alpha + \beta > n + 2$) and $\alpha + \beta \equiv n \pmod 4$, then K^*_n is not M-decomposable.*

Proof. Suppose that such a decomposition exists, and let $\overrightarrow{C} = (n - 1, n - 2, \ldots, 1, 0)$ be the directed n-cycle in the decomposition. Colour the edges of the directed α-cycle green and those of the directed β-cycle red, and without loss of generality, suppose that the arc 01 is green. Note that each arc of $\overleftarrow{C} = (0, 1, \ldots, n - 1)$, the reversal of $\overrightarrow{C}$, must be coloured green or red; in fact, $\overleftarrow{C}$ is partitioned into an equal number of green and red directed paths, say $G_0, G_1, \ldots, G_{t-1}$ and $R_0, R_1, \ldots, R_{t-1}$, where the green path G_i has initial and terminal vertices x_i and y_i, and the red path R_i has initial and terminal vertices y_i and x_{i+1} (where $x_{t+1} = x_0$), and $t = \frac{1}{2}(\alpha + \beta - n)$ is the number of edges of K^*_n which are in the cycles of length α and β but not in $\overleftarrow{C}$.

We form a bipartite graph B with partite sets $\{x_0, x_1, \ldots, x_{t-1}\}$ and $\{y_0, y_1, \ldots, y_{t-1}\}$ as follows. For $i = 0, 1, \ldots, t - 1$, we join x_i to y_i, forming a 1-factor F_1 of B, and y_i to x_{i+1} (computing subscripts modulo t), forming a second 1-factor F_2 of B. Also, join y_i to x_j, where $y_i x_j$ is an arc of the green cycle; these edges form a third 1-factor F_3 of B. Note that $j \neq i+1$, as otherwise the α-cycle in the decomposition must be $(y_i, y_i + 1, \ldots, y_{i+1})$, implying $t = 1$ and hence $\alpha + \beta - n = 2 \equiv 2 \pmod 4$. An example of the α- and β-cycles together with the associated cubic bipartite graph can be found in Figure 4.

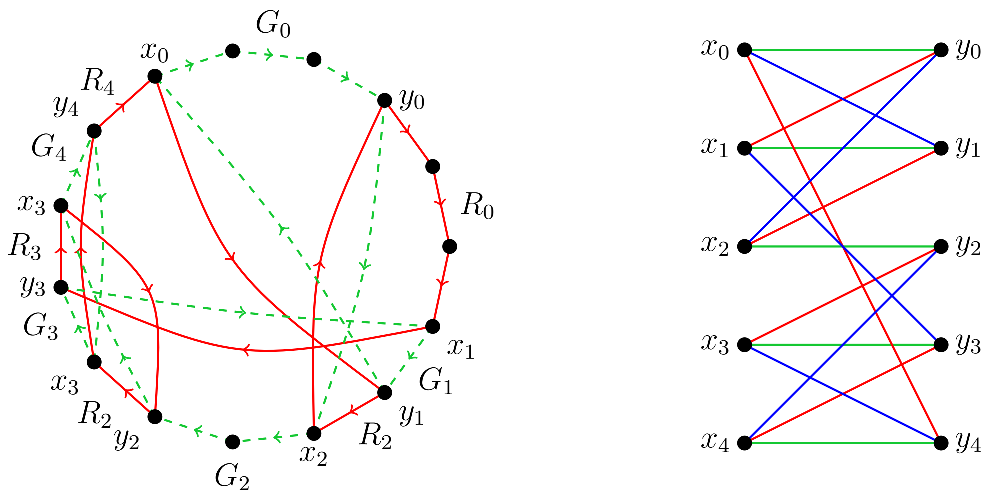

Figure 4: Directed α- and β-cycles together with the associated cubic bipartite graph

It is clear that B is a cubic bipartite graph. We show that $\{F_1, F_2, F_3\}$ is a perfect 1-factorization of B. First, notice that $F_1 \cup F_2$ induces the Hamiltonian cycle $(x_0, y_0, x_1, y_1, \ldots, x_{t-1}, y_{t-1})$. Next, consider the green α-cycle in the decomposition. Replacing the arcs of each G_i with a single arc $x_i y_i$ and disregarding direction, we obtain a cycle whose edges correspond to those of F_1 and F_3, and so $F_1 \cup F_3$ induces a Hamiltonian cycle in B. In a similar way, looking at the red β-cycle shows that $F_2 \cup F_3$ induces a Hamiltonian cycle in B.

By Lemma 22, the existence of a perfect 1-factorization of B implies that t is odd, contradicting the fact that $\alpha + \beta - n \equiv 0 \pmod 4$. □

Summarizing the results of this section, we have the following complete result for canonical lists of size at most three.

Theorem 24. *Let $M = (2, 2, \ldots, 2, \alpha, \beta, \gamma)$, where $2 < \alpha \leqslant \beta \leqslant \gamma \leqslant n$. The complete symmetric digraph K_n^* is M-decomposable if and only if M is n-admissible and it is not the case that $\alpha + \beta > n + 2$, $\gamma = n$ and $\alpha + \beta \equiv n \pmod 4$.*

One immediate consequence is the following result regarding cycle decomposition of certain 3-regular digraphs.

Corollary 25. *Let α, β, $\gamma \geqslant 3$ be integers with $\alpha + \beta > \gamma + 2$ and $\alpha + \beta \equiv \gamma \pmod 4$. If G is a 3-regular graph with γ vertices and $(\alpha + \beta + \gamma)/2$ edges, then G^* is not (α, β, γ)-decomposable.*

4 Conclusion

In this paper, we have made progress on the problem of decomposing a complete symmetric digraph into cycles of given lengths. Theorem 16 shows that if the greatest cycle length and the number of 2-cycles in an n-admissible list M are both large enough, then K_n^* admits an M-decomposition.

In the case that there are at most three cycles of length greater than 2, we have given a complete solution (Theorem 24). Notably, there is a family of n-admissible lists M, namely those with $\hat{M} = (\alpha, \beta, n)$ where $\alpha + \beta > n + 2$ and $\alpha + \beta \equiv n \pmod 4$, for which $2K_n$ is decomposable but K_n^* is not. This result implies that no cubic graph G of order n exists such that G^* is decomposable into three cycles of lengths α, β and n, where $\alpha + \beta > n + 2$ and $\alpha + \beta \equiv n \pmod 4$. In particular, G^* has no Hamiltonian cycle decomposition if $n \equiv 0 \pmod 4$.

The method used in this paper to show non-existence by constructing an auxiliary bipartite graph with a perfect 1-factorization does not apply in general to canonical lists of size greater than three. It remains an interesting open question to determine if there are other infinite families of n-admissible lists M for which K_n^* is not M-decomposable. We have checked all n-admissible lists with $n \leqslant 13$ and have verified that the corresponding decompostions exist, except for those given by Lemma 23 and the decompositions corresponding to the canonical lists:

$$\begin{aligned} n = 4: &\quad (4,4,4); \\ n = 5: &\quad (4,4,5,5), (3,4,4,4,5), (3,3,3,3,3,5); \\ n = 6: &\quad (3,3,3,3,3,3,3,3,3,3), (3,3,6,6,6,6), (4,6,6,6), (6,6,6,6,6), \end{aligned}$$

which do not exist. We conjecture that these are the only exceptions.

References

[1] B. Alspach. Research Problem 3. *Discrete Math.* **36** (1981), 333.

[2] B. Alspach and H. Gavlas. Cycle decompositions of K_n and $K_n - I$. *J. Combin. Theory Ser. B* **81** (2001), 77–99.

[3] B. Alspach, H. Gavlas, M. Šajna and H. Verrall. Cycle decompositions IV: complete directed graphs and fixed length directed cycles. *J. Combin. Theory Ser. A* **103** (2003), 165–208.

[4] D. Bryant, D. Horsley, B. Maenhaut and B.R. Smith. Decompositions of complete multigraphs into cycles of varying lengths. *J. Combin. Theory Ser. B* **129** (2018), 79–106.

[5] D. Bryant, D. Horsley and W. Pettersson. Cycle decompositions V: Complete graphs into cycles of arbitrary lengths. *Proc. London Math. Soc.* **108** (2014), 1153–1192.

[6] T.P. Kirkman. On a problem in combinations. *Cambridge and Dublin Math J.* **2** (1847), 191–204.

[7] A. Kotzig and J. Labelle. Strongly Hamiltonian graphs. *Utilitas Math.* **14** (1978), 99–116.

[8] E. Lucas. *Récréations Mathématiques*, Vol. II. Gauthier-Villars, Paris, 1892.

[9] M. Šajna. Cycle decompositions III: Complete graphs and fixed length cycles. *J. Combin. Des.* **10** (2002), 27–78.