Hamiltonian cycles in tough $(P_2 \cup P_3)$-free graphs

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Abstract

Let $t > 0$ be a real number and G be a graph. We say G is t-tough if for every cutset S of G, the ratio of $|S|$ to the number of components of $G - S$ is at least t. Determining toughness is an NP-hard problem for arbitrary graphs. The Toughness Conjecture of Chvátal, stating that there exists a constant t_0 such that every t_0-tough graph with at least three vertices is hamiltonian, is still open in general. A graph is called $(P_2 \cup P_3)$-free if it does not contain any induced subgraph isomorphic to $P_2 \cup P_3$, the union of two vertex-disjoint paths of order 2 and 3, respectively. In this paper, we show that every 15-tough $(P_2 \cup P_3)$-free graph with at least three vertices is hamiltonian.

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1 Introduction

Graphs considered in this paper are simple, undirected, and finite. Let G be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v in G. For $S \subseteq V(G)$ and $x \in V(G)$, define $\deg_G(x, S) = |N_G(x) \cap S|$. If $H \subseteq G$, we simply write $\deg_G(x, H)$ for $\deg_G(x, V(H))$. We skip the subscript G if the graph in consideration is clear from the context. Let $S \subseteq V(G)$. Then the subgraph induced on $V(G) \setminus S$ is denoted by $G - S$. For notational simplicity, we write $G - x$ for $G - \{x\}$. If $uv \in E(G)$ is an edge, we write $u \sim v$. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges of G with one end in V_1 and the other end in V_2.

The number of components of G is denoted by $c(G)$. Let $t \geqslant 0$ be a real number. The graph G is said to be t-*tough* if $|S| \geqslant t \cdot c(G-S)$ for each $S \subseteq V(G)$ with $c(G-S) \geqslant 2$. The *toughness* $\tau(G)$ is the largest real number t for which G is t-tough, or is ∞ if G is complete. This concept, a measure of graph connectivity and "resilience" under removal of vertices, was introduced by Chvátal [7] in 1973. It is easy to see that if G has a hamiltonian cycle then G is 1-tough. Conversely, Chvátal [7] conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian (Chvátal's toughness conjecture). Bauer, Broersma and Veldman [2] have constructed t-tough graphs that are not hamiltonian for all $t < \frac{9}{4}$, so t_0 must be at least $\frac{9}{4}$. It is not difficult to see that a non-complete t-tough graph is $2\lceil t\rceil$-connected.

There are many papers on Chvátal's toughness conjecture, and it has been verified when restricted to a number of graph classes [3], including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs. A graph G is called *$2K_2$-free* if it does not contain two independent edges as an induced subgraph. In 2014, Broersma, Patel and Pyatkin [5] proved that every 25-tough $2K_2$-free graph on at least three vertices is hamiltonian, and the author of this paper improved the required toughness in this result from 25 to 3 [13].

Let P_ℓ denote a path on ℓ-vertices. A graph is $(P_2 \cup P_3)$-free if it does not contain any induced copy of $P_2 \cup P_3$, the disjoint union of P_2 and P_3. In this paper, we confirm Chvátal's toughness conjecture for the class of $(P_2 \cup P_3)$-free graphs, a superclass of $2K_2$-free graphs.

Theorem 1. *Let G be a 15-tough $(P_2 \cup P_3)$-free graph with at least three vertices. Then G is hamiltonian.*

In [10] it was shown that every 3/2-tough split graph on at least three vertices is hamiltonian. And the authors constructed a sequence $\{G_n\}_{n=1}^{\infty}$ of *split graphs* (graphs whose vertices can be partitioned into a clique and an independent set) with no 2-factor and $\tau(G_n) \nearrow 3/2$. So 3/2 is the best possible toughness for split graphs to be hamiltonian. Since split graphs are $(P_2 \cup P_3)$-free, we cannot decrease the bound in Theorem 1 below 3/2. Although it is certain that 15-tough is not optimal, we are not sure about the best possible toughness for giving a hamiltonian cycle in a $(P_2 \cup P_3)$-free graph.

The class of $2K_2$-free graphs is well studied, for instance, see [5, 6, 8, 9, 11, 12]. It is a superclass of split graphs. One can also easily check that every *cochordal* graph (i.e., a graph that is the complement of a chordal graph) is $2K_2$-free and so the class of $2K_2$-free graphs is at least as rich as the class of chordal graphs. By the definition, the class of $(P_2 \cup P_3)$-free graphs is a superclass of $2K_2$-free graphs but with much more complicated structures than graphs that are $2K_2$-free. The proof techniques used in [5] and [13] for showing that certain tough $2K_2$-free graphs are hamiltonian do not seem to be applicable for $(P_2 \cup P_3)$-free graphs. The proof approach used in this paper for showing Theorem 1 is new and more general and reveals some structural properties of $(P_2 \cup P_3)$-free graphs.

2 Proof of Theorem 1

We start this section with some definitions. Let G be a graph and $S \subseteq V(G)$ a cutset of G, and let D be a component of $G - S$. For a vertex $x \in S$, we say that *x is adjacent to D* if x is adjacent in G to a vertex of D. We call D a *clique component* of $G - S$ if

$V(D)$ is a clique in G. We call D a *trivial component* of $G-S$ if D has only one vertex, otherwise D is *nontrivial*.

A *star-matching* is a set of vertex-disjoint copies of stars. The vertices of degree at least 2 in a star-matching are called the *centers* of the star-matching. In particular, if all the stars in a *star-matching* are isomorphic to $K_{1,t}$, where $t \geqslant 1$ is an integer, we call the star-matching a *$K_{1,t}$-matching*. For a star-matching M, we denote by $V(M)$ the set of vertices covered by M.

Let C be an oriented cycle. For $x \in V(C)$, denote the immediate successor of x on C by x^+ and the immediate predecessor of x on C by x^-. For $u, v \in V(C)$, $u\overrightarrow{C}v$ denotes the segment of C starting at u, following C in the orientation, and ending at v. Likewise, $u\overleftarrow{C}v$ is the opposite segment of C with endpoints as u and v. We assume all cycles in consideration afterwards are oriented. A path P connecting two vertices u and v is called a *(u, v)-path*, and we write uPv or vPu in specifying the two endvertices of P. Let uPv and xQy be two paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx.

Lemma 2 ([1], Theorem 2.10)**.** *Let G be a bipartite graph with partite sets X and Y, and let f be a function from X to the set of positive integers. If for every $S \subseteq X$, it holds that $|N_G(S)| \geqslant \sum_{x \in S} f(x)$, then G has a subgraph H such that $X \subseteq V(H)$, $d_H(x) = f(x)$ for every $x \in X$, and $d_H(y) = 1$ for every $y \in Y \cap V(H)$.*

We will apply the following consequences of Lemma 2 in our proof.

Corollary 3. *Let G be a graph and $X \subseteq V(G)$ be an independent set in G. If G does not have a subgraph H such that $X \subseteq V(H)$, $d_H(x) = 2$ for every $x \in X$, and $d_H(y) = 1$ for every $y \in Y \cap V(H)$, where $Y \subseteq V(G) \setminus X$, then there exists $X_1 \subseteq X$ such that $|N_G(X_1) \cap Y| < 2|X_1|$.*

Proof. Let $R[X, Y]$ be the bipartite graph with bipartition X and Y and with $E(R)$ being the set of edges in G between X and Y. Let f be a function on X such that $f(x) = 2$ for each $x \in X$. The assumption that G does not have a subgraph H with the requirements implies that R does not have such a subgraph also. Applying Lemma 2, we find $X_1 \subseteq X$ such that $|N_R(X_1)| < 2|X_1|$. Since X is an independent set in G, $N_R(X_1) = N_G(X_1) \cap Y$. Therefore there exists $X_1 \subseteq X$ such that $|N_G(X_1) \cap Y| < 2|X_1|$, as desired. □

Corollary 4. *Let G be a 2-tough graph with at least three vertices and $X \subseteq V(G)$ be an independent set in G. Then G has a subgraph H such that $X \subseteq V(H)$, $d_H(x) = 2$ for every $x \in X$, and $d_H(y) = 1$ for every $y \in (V(G) \setminus X) \cap V(H)$.*

Proof. Let $Y = V(G) \setminus X$, and $R[X, Y]$ be the bipartite graph with bipartition X and Y and with $E(R)$ being the set of edges in G between X and Y. Let f be a function on X such that $f(x) = 2$ for each $x \in X$. Let $S \subseteq X$. If $|S| \leqslant 1$, then since G is 4-connected, $|N_R(S)| = |N_G(S)| \geqslant 2|S|$. Thus, $|S| \geqslant 2$. Note that $c(G - N_G(S)) \geqslant |S| \geqslant 2$. By the toughness of G, $|N_R(S)| = |N_G(S)| \geqslant 2|S|$. Therefore, by Lemma 2, R and so G has a desired subgraph H such that $X \subseteq V(H)$, $d_H(x) = 2$ for every $x \in X$, and $d_H(y) = 1$ for every $y \in (V(G) \setminus X) \cap V(H)$. □

Lemma 5 (Bauer et al. [4])**.** *Let $t > 0$ be real and G be a t-tough n-vertex graph ($n \geqslant 3$) with $\delta(G) > \frac{n}{t+1} - 1$. Then G is hamiltonian.*

Lemmas 6 and 7 below are consequences of $(P_2 \cup P_3)$-freeness.

Lemma 6. *Let G be a $(P_2 \cup P_3)$-free graph and $S \subseteq V(G)$ a cutset of G. If $G - S$ has a component that is not a clique component, then all other components of $G - S$ are trivial. Consequently, if $G - S$ has at least two nontrivial components, then all components of $G - S$ are clique components.*

Lemma 7. *Let G be a $(P_2 \cup P_3)$-free graph and $S \subseteq V(G)$ a cutset of G, and let $x \in S$. Suppose that x is adjacent to exactly one component D of $G - S$, and $G - S$ has a nontrivial component to which x is not adjacent, then x is adjacent in G to all vertices of D.*

Lemma 8. *Let G be a connected $(P_2 \cup P_3)$-free graph and $S \subseteq V(G)$ a cutset of G such that each vertex in S is adjacent to at least two components of $G - S$. Then each of the following statement holds.*

(i) For every nontrivial clique component $D \subseteq G - S$ and for every vertex $x \in S$, x is adjacent to D.

(ii) For every nontrivial clique component $D \subseteq G - S$ and for every vertex $x \in S$, if x is adjacent in G to at least three components of $G - S$, then x is adjacent in G to at least $|V(D)| - 1$ vertices of D.

(iii) Let D_1 and D_2 be two nontrivial clique components of $G - S$. Then for every vertex $x \in S$, either x is adjacent in G to at least $|V(D_i)| - 1$ vertices of each D_i, or x is adjacent in G to all vertices of one of D_i, $i = 1, 2$.

Proof. Let w_1 and w_2 be two neighbors of x in G respectively from two distinct components of $G - S$. Then w_1xw_2 is an induced P_3. Now for every nontrivial component D, if $V(D) \cap \{w_1, w_2\} \neq \varnothing$, then x is already adjacent to D in G. So $V(D) \cap \{w_1, w_2\} = \varnothing$. For every edge $uv \in E(D)$, x is adjacent to u or v by the assumption of G being $(P_2 \cup P_3)$-free. This proves (i). For (ii), let $x \in S$ and D be a nontrivial clique component of $G - S$. Since x is adjacent in G to at least three components of $G - S$, there exists u, w, respectively from two components of $G - S$ that are distinct from D such that $x \sim u$ and $x \sim w$ in G. Thus, uxw is an induced P_3 in G. Furthermore, since $u, w \in V(G) \setminus (S \cup V(D))$, $E_G(\{u, w\}, V(D)) = \varnothing$. Thus, by the $(P_2 \cup P_3)$-freeness assumption, for every edge in D, x is adjacent to at least one endvertex of that edge. This, together with the fact that D is a clique component of $G - S$, we know that x is adjacent in G to at least $|V(D)| - 1$ vertices of D. For (iii), assume to the contrary that the statement does not hold. By symmetry, we assume that there exists $uv \in E(D_1)$ such that $x \not\sim u, v$ in G, and there exists $w \in V(D_2)$ such that $x \not\sim w$ in G. Let $y \in V(D_2) \cap N_G(x)$ that exists by Lemma 8 (i). Then $uv \cup xyw$ is an induced $P_2 \cup P_3$, giving a contradiction. □

Lemma 9. *Let $t > 0$ and G be a non-complete n-vertex t-tough graph. Then $|W| \leqslant \frac{1}{t+1}n$ holds for every independent set W in G.*

Proof. Since G is $2\lceil t \rceil$-connected, $n \geqslant 2\lceil t \rceil + 1 \geqslant 2t + 1 \geqslant t + 1$. Therefore, if $|W| = 1$, then $|W| \leqslant \frac{1}{t+1}n$. Suppose $|W| \geqslant 2$. Let $S = V(G) \setminus W$ and $\alpha = \frac{|W|}{n}$. Clearly $|S| = (1 - \alpha)n$. Since $c(G - S) = |W| \geqslant 2$ and G is t-tough, we get

$$(1 - \alpha)n = |S| \geqslant t \cdot c(G - S) = t|W| = t\alpha n.$$

Therefore, we get $(1 - \alpha)n \geqslant t\alpha n$, which yields $\alpha \leqslant \frac{1}{t+1}$ and $|W| \leqslant \frac{1}{t+1}n$. □

Lemma 10. *Let $t \geqslant 1$ and G be an n-vertex t-tough graph, and let C be a non-hamiltonian cycle of G. If $x \in V(G) \setminus V(C)$ satisfies that $\deg(x, C) > \frac{n}{t+1}$, then G has a cycle C' such that $V(C') = V(C) \cup \{x\}$.*

Proof. It is clear that if x is adjacent to two consecutive vertices u, w on C, then

$$C' = (C - \{uw\}) \cup \{ux, xw\}$$

is a cycle with the desired property. So we assume that for any $u, w \in N_G(x) \cap V(C)$, $uw \notin E(C)$. Let $W = \{u^+ \,|\, u \in N_G(x) \cap V(C)\}$ be the set of the successors of the neighbors of x on C. Because there is a one-to-one correspondence between W and $N_G(x) \cap V(C)$, by the assumption that $\deg(x, C) > \frac{n}{t+1}$, we know that

$$|W| > \frac{n}{t+1}. \tag{1}$$

Thus, W is not an independent set in G by Lemma 9, and there exist $u^+, w^+ \in W$ with $u, w \in N_G(x) \cap V(C)$ such that $u^+ \sim w^+$ in G. Then

$$C' = u^+ \overrightarrow{C} w x u \overleftarrow{C} w^+ u^+$$

is a desired cycle. □

Lemma 11. *Let G be an n-vertex 15-tough $(P_2 \cup P_3)$-free graph, and let C be a non-hamiltonian cycle of G. Let $P \subseteq G - V(C)$ be an (x, z)-path. If both x and z are adjacent in G to more than $\frac{4.5n}{16}$ vertices from $V(C)$, then G has a cycle C' such that $V(C') = V(C) \cup V(P)$.*

Proof. It is clear that if x is adjacent to a vertex u on C and z is adjacent to a vertex w on C such that $uw \in E(C)$, then

$$C' = (C - \{uw\}) \cup \{ux, zw\} \cup P$$

is a cycle with the desired property. So we assume that

$$\text{for any } u \in N_G(x) \cap V(C) \text{ and any } w \in N_G(z) \cap V(C),\ uw \notin E(C). \tag{2}$$

Let

$$\begin{aligned} W_x &= \{u^+ \mid u \in N_G(x) \cap V(C)\}, \\ W_z &= \{u^+ \mid u \in N_G(z) \cap V(C)\}. \end{aligned}$$

Clearly,

$$|W_x| = |N_G(x) \cap V(C)| > \frac{4.5n}{16}, \quad \text{and} \quad |W_z| = |N_G(z) \cap V(C)| > \frac{4.5n}{16}. \tag{3}$$

If there exist $u^+ \in W_x$ and $w^+ \in W_z$ with $u \in N_G(x) \cap V(C)$ and $w \in N_G(z) \cap V(C)$ such that $u^+ \sim w^+$ in G, then

$$C' = u^+ \overrightarrow{C} wzPxu \overleftarrow{C} w^+ u^+$$

is a desired cycle. Therefore, we assume

$$E_G(W_x, W_z) = \varnothing. \tag{4}$$

We further claim that

$$\text{no two vertices in } N_G(x) \cap V(C) \text{ or } N_G(z) \cap V(C) \text{ are consecutive on } C. \tag{5}$$

By symmetry, we only show that no two vertices in $N_G(x) \cap V(C)$ are consecutive on C.

Assume to the contrary that there exists a path $v_1 v_2 \cdots v_\ell \subseteq C$ with $\ell \geqslant 2$ such that for each i with $1 \leqslant i \leqslant \ell$, $v_i \in N_G(x) \cap V(C)$, $v_1^- \notin N_G(x) \cap V(C)$, and $v_\ell^+ \notin N_G(x) \cap V(C)$. Note that such vertices v_1 and v_ℓ exist by the assumption in (**2**) and the fact that $N_G(z) \cap V(C) \neq \varnothing$. By (**3**) and Lemma 9, W_z is not an independent set in G and so there exist $w_1, w_2 \in W_z$ such that $w_1 \sim w_2$ in G.

Then $x v_\ell v_\ell^+$ is an induced P_3 in G. Consider the edge $w_1 w_2$. By the assumption in (**2**), $x \not\sim w_1, w_2$ in G (otherwise, $w_1^- w_1 \in E(C)$ or $w_2^- w_2 \in E(C)$ with $w_1^-, w_2^- \in N_G(z) \cap V(C)$), and by the assumption in (**4**), $v_\ell^+ \not\sim w_1, w_2$ in G. Thus, $v_\ell \sim w_1$ or $v_\ell \sim w_2$ in G by the $(P_2 \cup P_3)$-freeness assumption. However, $v_\ell = v_{\ell-1}^+ \in W_x$, showing a contradiction to (**4**).

Therefore, by (**5**),

$$(N_G(x) \cap V(C)) \cap W_x = \varnothing, \quad \text{and} \quad (N_G(z) \cap V(C)) \cap W_z = \varnothing. \tag{6}$$

Also, by (**2**),

$$(N_G(x) \cap V(C)) \cap W_z = \varnothing, \quad \text{and} \quad (N_G(z) \cap V(C)) \cap W_x = \varnothing. \tag{7}$$

Let

$$W_{xz} = W_x \cap W_z.$$

By the assumption in (**4**), W_{xz} is an independent set in G. By Lemma 9, $|W_{xz}| \leqslant \frac{n}{16}$. Therefore, $|N_G(x) \cap N_G(z) \cap V(C)| \leqslant \frac{n}{16}$. These, together with (**3**), (**6**) and (**7**), imply

$$\begin{aligned} n &\geqslant |(N_G(x) \cap V(C)) \cup (N_G(z) \cap V(C)) \cup W_x \cup W_z| \\ &> \frac{9n}{16} + \frac{9n}{16} - |N_G(x) \cap N_G(z) \cap V(C)| - |W_{xz}| \\ &\geqslant \frac{16n}{16} = n, \end{aligned}$$

showing a contradiction. □

Lemma 12. *Let G be an n-vertex 15-tough $(P_2 \cup P_3)$-free graph, and let $S \subseteq V(G)$ be a cutset of G with $|S| \leqslant \frac{3n}{4}$. Assume that $G-S$ has at least two nontrivial clique components, and that for every edge $uv \in E(G)$, $d(u) + d(v) \geqslant |S|$. Then G has a hamiltonian cycle.*

Proof. By Lemma 6, every component of $G - S$ is a clique component. If there exists $x \in S$ such that x is adjacent to exactly one component, say D of $G - S$, then we move x from S into D. By Lemma 7, every component of $G-(S \setminus \{x\})$ is still a clique component. We move out all such vertex x from S iteratively and denote the remaining vertices in S by S_1. Note that $S_1 \neq \varnothing$, since G is a connected graph and S is a cutset of G. Also, $c(G - S) = c(G - S_1)$ and $G - S_1$ has at least two nontrivial components. By Lemma 6, every component of $G - S_1$ is a clique component. Let

$$\begin{aligned} S_0 &= \{x \in S_1 \,|\, x \text{ is not adjacent to any component of } G - S_1\}, \\ S_2 &= \{x \in S_1 \,|\, x \text{ is adjacent to at least two components of } G - S_1\}. \end{aligned}$$

Note that $S_2 = S_1 - S_0$.

Since $G - S_1$ has a nontrivial component that has no edge going to S_0, the $(P_2 \cup P_3)$-freeness of G implies that $G[S_0]$ consists of vertex-disjoint complete subgraphs of G. Thus S_2 is a cutset of G with components consisting those from $G - S_1$ and $G[S_0]$. Also, all components of $G - S_2$ are clique components in which at least two of them are nontrivial. By the toughness of G, $|S_2| \geqslant 15c(G - S_2)$.

We will construct a hamiltonian cycle in G through two steps: (1) combing spanning cycles from every clique component of $G - S_2$ that has at least three vertices into a single cycle C, and (2) inserting remaining vertices in $V(G) \setminus V(C)$ into C to obtain a hamiltonian cycle of G.

Suppose that $G - S_2$ has exactly h clique components $D_1, D_2, \cdots, D_h$ with $|V(D_1)| \geqslant |V(D_2)| \geqslant \cdots \geqslant |V(D_h)| \geqslant 1$, and that the first t $(0 \leqslant t \leqslant h)$ of them are components that contain at least three vertices. Since $G - S_2$ has at least two nontrivial components, both D_1 and D_2 are nontrivial.

Claim 1. The component D_1 contains at least 5 vertices.

Proof: Since $|S_2| \leqslant |S| \leqslant \frac{3n}{4}$, $n \geqslant \frac{4|S_2|}{3}$. Also, $c(G - S_2) \leqslant \frac{|S_2|}{15}$ by $\tau(G) \geqslant 15$. Therefore, a largest component of $G - S_2$ contains at least

$$\frac{n - |S_2|}{c(G - S_2)} \geqslant \frac{\frac{4|S_2|}{3} - |S_2|}{\frac{|S_2|}{15}} = 5$$

vertices. ■

Let

$$\begin{aligned} Q_1 &= \{x \in S_2 \,|\, x \text{ is adjacent to a component distinct from } D_1 \text{ and } D_2\}, \\ Q_2 &= \{x \in S_2 \,|\, x \text{ is adjacent to less than } \tfrac{|V(D_1)|-1}{2} \text{ vertices of } D_1\}, \\ Q_3 &= \{x \in S_2 \,|\, x \text{ is adjacent to less than } \tfrac{|V(D_2)|-1}{2} \text{ vertices of } D_2\}. \end{aligned}$$

By Lemma 8 (i) and the definition of Q_1, we know that if $Q_1 \neq \varnothing$, then every vertex in Q_1 is adjacent to at least three components of $G - S_2$. By Lemma 8 (ii), we get the following claim.

Claim 2. Suppose that $Q_1 \neq \varnothing$. Then for every $x \in Q_1$ and for every nontrivial component D of $G - S_2$, x is adjacent to at least $|V(D)| - 1$ vertices of D.

Claim 3. Suppose that $Q_2 \neq \varnothing$. Then for every $x \in Q_2$, x is adjacent to all vertices of D_2 and Q_2 is a clique in G.

Proof: Note that both D_1 and D_2 are nontrivial components of $G - S_2$. Since D_1 is a nontrivial component, $\frac{|V(D_1)|+1}{2} > 1$. Hence, by the definition of Q_2, D_1 contains at least two vertices that are not adjacent to x in G. Therefore, x is adjacent in G to all vertices of D_2 by Lemma 8 (iii). For the second part, suppose to the contrary that there exist $x, y \in Q_2$ such that $x \not\sim y$ in G. Let $w \in V(D_2)$. Then $w \sim x$ and $w \sim y$ in G by the first part of this claim. Thus, we find an induced $P_3 = xwy$. Since $E_G(\{w\}, V(D_1)) = \varnothing$, the $(P_2 \cup P_3)$-freeness implies that for every edge in D_1, at least one of x and y is adjacent to at least one endpoint of the edge. Since D_1 is complete, by Pigeonhole Principle, one of x and y is adjacent to at least $\frac{|V(D_1)|-1}{2}$ vertices of D_1. This gives a contradiction to the assumption that $x, y \in Q_2$. ■

Similarly, we have the following result.

Claim 4. Suppose that $Q_3 \neq \varnothing$. Then for every $x \in Q_3$, x is adjacent to all vertices of D_1 and Q_3 is a clique in G.

By Claims 2 to 4, we have that

$$Q_i \cap Q_j = \varnothing, i \neq j, i, j = 1, 2, 3. \tag{8}$$

Define

$$W = \bigcup_{\max\{t+1,3\} \leqslant i \leqslant h} V(D_i).$$

Since $|V(D_i)| \leqslant 2$ for each i with $t + 1 \leqslant i \leqslant h$, we have $\sum_{i=t+1}^{h} |V(D_i)| \leqslant 2(h - t)$. Moreover, since S_2 is a cutset of G, the toughness of G yields $|S_2| \geqslant 15c(G - S_2) = 15h$. Therefore, we have

$$|W| \leqslant \sum_{i=t+1}^{h} |V(D_i)| \leqslant 2(h - t) \leqslant \frac{2|S_2|}{15} - 2t. \tag{9}$$

If $W \neq \varnothing$, we claim that there is a $K_{1,2}$-matching M between W and S_2 such that every vertex in W is the center of a $K_{1,2}$-star. This is clearly true if $|W| \leqslant 2$, as G is non-complete and 15-tough and so is 30-connected. Thus, we assume that $|W| \geqslant 3$, and suppose to the contrary that there is no $K_{1,2}$-matching between W and S_2. Let G^* be obtained from G by deleting all edges within W. Applying Corollary 3 on G^* with W and S_2, there exists $W_1 \subseteq W$ such that $2|W_1| > |N_{G^*}(W_1) \cap S_2|$. Note that $|W_1| \geqslant 3$ by the argument in the beginning of this paragraph. Let $W_1' \subseteq W \setminus W_1$ be the set of all vertices that is adjacent in G to a vertex in W_1. As each component in $G[W]$ is either K_1 or K_2, $|W_1'| \leqslant |W_1|$, and $G - ((N_G(W_1) \cap S_2) \cup W_1')$ has at least $\lceil |W_1|/2 \rceil \geqslant 2$ components. Therefore,

$$\frac{|(N_G(W_1) \cap S_2) \cup W_1'|}{c(G - ((N_G(W_1) \cap S_2) \cup W_1'))} < \frac{3|W_1|}{|W_1|/2} < 15.$$

This gives a contradiction to the toughness.

$$\text{Let } M \text{ be a } K_{1,2}\text{-matching between } W \text{ and } S_2. \tag{10}$$

Claim 5. It holds that $N_G(W) \cap S_2 \subseteq Q_1$ and $N_G(W) \cap S_2 = Q_1$ if $t = 2$. Consequently, for every $x \in N_G(W) \cap S_2$ and every nontrivial component D of $G - S_2$, x is adjacent in G to at least $|V(D)| - 1$ vertices of D.

Proof: For the first part of the Claim, we may assume that $N_G(W) \cap S_2 \neq \varnothing$. If $G - S_2$ has at least three nontrivial components, then every vertex of S_2 is adjacent to all those nontrivial components by Lemma 8 (i). Therefore, $S_2 = Q_1$ by the definition of Q_1. In particular, $N_G(W) \cap S_2 \subseteq Q_1$. Hence, we assume that $G - S_2$ has exactly two nontrivial components, which are D_1 and D_2. This assumption implies that $|V(D_3)| \leqslant 1$. Consequently, $|V(D_3)| = 1$ since $N_G(W) \cap S_2 \neq \varnothing$. Then for every $x \in N_G(W) \cap S_2$, x is adjacent to both D_1 and D_2 by Lemma 8 (i), and also x is adjacent to a trivial component of $G - S_2$. Thus $N_G(W) \cap S_2 \subseteq Q_1$. When $t = 2$, if $Q_1 \neq \varnothing$, then for every $x \in Q_1$, $x \in N_G(W)$. Therefore, $Q_1 \subseteq N_G(W) \cap S_2$. Hence, $N_G(W) \cap S_2 = Q_1$ when $t = 2$. The second part of Claim 5 is a consequence of Claim 2. ■

Claim 6. There is a cycle C in $G - V(M)$ with at least $\frac{3n}{20}$ vertices such that C contains all vertices from every D_i, $i = 1, 2, \cdots, \max\{2, t\}$, and $Q_2 \cup Q_3 \subseteq V(C)$.

Proof: Suppose first that $G - S_2$ has at least three nontrivial components, that is, $t \geqslant 3$. Then by Lemma 8 (i), every vertex of S_2 is adjacent to all those nontrivial components of $G - S_2$. Consequently, $S_2 = Q_1$ and $Q_2 = Q_3 = \varnothing$. Therefore, for every $x \in S_2$ and every D_i, x is adjacent to at least $|V(D_i)| - 1$ vertices of D_i by Claim 2.

Let $x_1, \cdots, x_t$ be t distinct vertices in $S_2 \setminus V(M)$. (By the toughness of G, $|S_2| \geqslant 15c(G - S_2)$. Since $|V(M) \cap S_2| \leqslant 4c(G - S_2)$, we have enough vertices in $S_2 \setminus V(M)$ to pick.) Let C_i be a hamiltonian cycle of D_i, and let $u_i, v_i \in V(C_i)$ with $u_iv_i \in E(C_i)$ such that for $i = 1, 2, \cdots, t-1$, $x_i \sim v_i, u_{i+1}$, and $x_t \sim u_1, v_t$ in G. Then

$$C = u_1\overrightarrow{C}_1v_1x_1u_2\overrightarrow{C}_2v_2 \cdots u_{t-1}\overrightarrow{C}_{t-1}v_{t-1}x_{t-1}u_t\overrightarrow{C}_tv_tx_tu_1$$

is a cycle that contains all vertices from each D_i and the vertices $x_1, \cdots, x_t$ from $S_2 \setminus V(M)$. Also $Q_2 \cup Q_3 \subseteq V(C)$ trivially as $Q_2 = Q_3 = \varnothing$.

So we assume that $G - S_2$ has exactly two nontrivial clique components, which are D_1 and D_2, call this **assumption (∗)**. Let

$$G_1 = G[V(D_1) \cup Q_3] \quad \text{and} \quad G_2 = G[V(D_2) \cup Q_2].$$

By Claims 3 and 4, we know that both G_1 and G_2 are complete subgraphs of G.

Suppose firstly that $t = 1$ and $Q_2 = \varnothing$. Then D_2 has exactly two vertices. Consequently, $Q_3 = \varnothing$ by Lemma 8 (i) and $G_1 = D_1$. Let $V(D_2) = \{u, v\}$. By Lemma 8 (i), every vertex from S_2 is adjacent in G to a vertex of D_2. Since $Q_2 = \varnothing$, every vertex from S_2 is adjacent in G to at least $\frac{|V(D_1)-1|}{2} \geqslant 2$ vertices of D_1. Suppose there exist distinct $u_1, v_1 \in S_2 \setminus V(M)$ such that $u \sim u_1$ and $v \sim v_1$. Let $u_2, v_2 \in V(D_1)$ be distinct such that $u_1 \sim u_2$ and $v_1 \sim v_2$. Let P be a hamiltonian (u_2, v_2)-path of D_1. Then $C = u_1u_2Pv_2v_1vuu_1$ is a desired cycle. Thus, we assume, without loss of generality, that for every $x \in S_2 \setminus V(M)$, $x \sim u$ and $x \not\sim v$. Since G is 15-tough, there exists $v_1' \in V(M) \cap S_2$ such that $v \sim v_1'$. Let $v_1'wv_1 \in M$ be the $K_{1,2}$-star that contains the vertex v_1', where $v_1, v_1' \in S_2$ and $w \in W$. Let $u_1 \in S_2 \setminus V(M)$, and $u_2, v_2 \in V(D_1)$ be distinct such that $u_1 \sim u_2$ and $v_1 \sim v_2$. Let P be a hamiltonian (u_2, v_2)-path of D_1. Then $C = u_1u_2Pv_2v_1wv_1'vuu_1$ is a desired cycle. (For the latter case, we still denote the $K_{1,2}$-matching $M \setminus \{v_1'wv_1\}$ by M.)

Thus, we assume that $t \geqslant 2$ or $Q_2 \neq \varnothing$. By assumption $(*)$, we have either $t = 2$ or $t = 1$ and $Q_2 = \varnothing$. Since D_1 has at least three vertices by Claim 1, G_1 contains at least three vertices. Note that $|V(D_2)| \geqslant 2$ by the assumption that $G - S_2$ has at least two nontrivial components and D_2 is one of them. Thus, G_2 contains at least three vertices either by $t = 2$ or $Q_2 \neq \varnothing$.

If there are two disjoint edges between G_1 and G_2, then $G[V(G_1) \cup V(G_2)]$ has a hamiltonian cycle C. Thus, we may assume, without loss of generality, that there is either no edge between G_1 and G_2 or all edges between G_1 and G_2 are incident to only a single vertex, say in G_1.

If $c(G - S_2) = 2$, then $M = \varnothing$ by the definitions of W and M. Since G is 15-tough and thus is 2-connected, there are vertex-disjoint paths P_1 and P_2 connecting G_1 and G_2 in G such that each P_i only has exactly one of its endvertices in G_1 and G_2. Let $V(P_i) \cap V(G_1) = \{x_i\}$ and $V(P_i) \cap V(G_2) = \{y_i\}$, $i = 1, 2$. Let C_1 be a hamiltonian cycle in G_1 such that $x_1x_2 \in E(C_1)$, and C_2 be a hamiltonian cycle in G_2 such that $y_1y_2 \in E(C_2)$. Then

$$C = x_1P_1y_1\overrightarrow{C}_2y_2P_2x_2\overleftarrow{C}_1x_1$$

is a cycle that contains all vertices in clique components of $G - S_2$ that contain at least three vertices and the vertices from P_1 and P_2. Also $Q_2 \cup Q_3 \subseteq V(C)$ by the construction of C.

So we assume that $c(G - S_2) \geqslant 3$. By Claims 2 to 4, Q_1, Q_2 and Q_3 are pairwise disjoint. Now, by the definition of Q_1, $D_3, \ldots, D_h$ are all components of $G - Q_1$. Moreover, there exits a component of $G - Q_1$ which contains D_1. This together with $h = c(G - S_2) \geqslant 3$ yields $c(G - Q_1) \geqslant h - 1 \geqslant 2$. Hence we have $|Q_1| \geqslant 15(h - 1)$ and $|Q_1 \setminus V(M)| \geqslant 15(h - 1) - 2(h - 2) = 13h - 11 \geqslant 28$ since each component D_i with $i \in \{t + 1, \ldots, h\}$ is a trivial component and so uses exactly two vertices from $S_2 \cap V(M)$. Hence, we can find two vertices $x, y \in Q_1 \setminus V(M) \subseteq S_2 \setminus (Q_2 \cup Q_3 \cup V(M))$ such that both x and y are adjacent to at least $|V(D_1)| - 1$ vertices of D_1, and at least $|V(D_2)| - 1$ vertices of D_2 by Claim 5. We claim that x is adjacent to at least two vertices of G_2. This is clear if x is adjacent to at least two vertices of D_2. So we assume $|N_G(x) \cap V(D_2)| \leqslant 1$. Then since $|N_G(x) \cap V(D_2)| \geqslant |V(D_2)| - 1$ and D_2 is a nontrivial component, $|V(D_2)| = 2$ and $|N_G(x) \cap V(D_2)| = 1$. This means $t = 1$ by Claim 1 and hence $Q_2 \neq \varnothing$. Let $V(D_2) = \{w, w_1\}$ and $N_G(x) \cap V(D_2) = \{w\}$. Also, since $Q_2 \neq \varnothing$, we can take $w_2 \in Q_2$. If $x \sim w_2$, we get $|N_G(x) \cap V(G_2)| \geqslant 2$. Thus, we may assume $x \not\sim w_2$. Therefore, $x \not\sim w_1, w_2$ in G. Note that w_1 is not adjacent to any vertex of D_1, and w_2 is adjacent to less than

$\frac{|V(D_1)|-1}{2}$ vertices of D_1. Therefore, we can find a vertex $w^* \in V(D_1)$ such that $w_1, w_2 \not\sim w^*$ in G and $x \sim w^*$ in G. By the choice of x, there is a vertex $w' \in V(G)\setminus(S_2 \cup V(D_1) \cup V(D_2))$ such that $x \sim w'$ in G. However, $w_1w_2 \cup w^*xw'$ is an induced $P_2 \cup P_3$. This gives a contradiction. Since D_1 has at least 5 vertices, both x and y have at least four neighbors in D_1. Thus we can select distinct vertices $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$ such that $x \sim x_1, x_2$ and $y \sim y_1, y_2$ in G.

Let C_1 be a hamiltonian cycle of G_1 such that $x_1y_1 \in E(C_1)$, and let C_2 be a hamiltonian cycle of G_2 such that $x_2y_2 \in E(C_2)$. Then

$$C = x_1xx_2\overrightarrow{C}_2y_2yy_1\overleftarrow{C}_1x_1$$

is a cycle that contains all vertices in nontrivial clique components D_1 and D_2 of $G - S_2$ and the vertices x and y. Furthermore, $Q_2 \cup Q_3 \subseteq V(C)$.

Since for each i, $1 \leqslant i \leqslant \max\{2, t\}$, $V(D_i) \subseteq V(C)$ and $\bigcup_{1 \leqslant i \leqslant t} V(D_i) \subseteq V(G) \setminus (S_2 \cup W)$, we have

$$\begin{aligned} |V(C)| &\geqslant n - |S_2| - |W| \geqslant n - |S_2| - 2c(G - S_2) \\ &\geqslant n - |S_2| - \frac{2|S_2|}{15} \geqslant n - \frac{17}{15} \cdot \frac{3n}{4} = \frac{3n}{20}. \end{aligned}$$

■

Claim 7. Let C be the cycle defined in Claim 6. For any $x \in S_2 \setminus V(C)$, x has more than $\frac{n}{16}$ neighbors on C.

Proof: Note that every vertex in S_2 is adjacent to at least two components of $G - S_2$. If $G - S_2$ has at least three nontrivial clique components, then Lemma 8 (ii) implies that for every $x \in S_2$, and for every nontrivial clique component D of $G - S_2$, x is adjacent to at least $|V(D)| - 1$ vertices of D. By (**9**) that $\sum_{i=t+1}^{h} |V(D_i)| \leqslant \frac{2|S_2|}{15} - 2t$, we get

$$\begin{aligned} \left| N_G(x) \cap \left(\bigcup_{1 \leqslant i \leqslant t} V(D_i) \right) \right| &\geqslant \sum_{i=1}^{t} (|V(D_i)| - 1) = \sum_{i=1}^{t} |V(D_i)| - t \\ &= (n - |S_2| - \sum_{i=t+1}^{h} |V(D_i)|) - t \\ &\geqslant n - |S_2| - \frac{2|S_2|}{15} + 2t - t \\ &\geqslant n - \frac{17|S_2|}{15} \geqslant \frac{3n}{20} > \frac{n}{16}, \end{aligned}$$

since $|S_2| \leqslant |S| \leqslant \frac{3n}{4}$. Therefore, x has more than $\frac{n}{16}$ neighbors on C.

So we assume that $G - S_2$ has exactly two nontrivial clique components. Since $S_2 \setminus V(C) \subseteq S_2 \setminus (Q_2 \cup Q_3)$ (recall that $Q_2 \cup Q_3 \subseteq V(C)$), we know that x is adjacent to at least $\frac{|V(D_1)|-1}{2}$ vertices of D_1, and is adjacent to at least $\frac{|V(D_2)|-1}{2}$ vertices of D_2. We show that $|V(D_1)| + |V(D_2)|$ is large. Since G is 15-tough, $|S_2| \geqslant 15c(G - S_2) = 15h$. On the other hand, since D_1 and D_2 are the only nontrivial components of $G - S_2$, we have $n - |S_2| = |V(D_1)| + |V(D_2)| + h - 2$. Combining these inequalities, we have

$$\begin{aligned} |V(D_1) + |V(D_2)|| &= n - |S_2| - h + 2 \geqslant n - |S_2| - \frac{|S_2|}{15} + 2 \\ &> n - \frac{16|S_2|}{15} \geqslant n - \frac{16}{15} \cdot \frac{3n}{4} = \frac{n}{5}. \end{aligned}$$

Since C contains all vertices from $D_1 \cup D_2$, we conclude that x is adjacent to at least $\frac{n}{10} - 1 > \frac{n}{16}$ (by $n \geqslant 31$) neighbors on C. ■

By Claim 7, and by applying Lemma 10 for C and vertices in $S_2 \setminus (V(C) \cup V(M))$ iteratively, we get a longer cycle C' such that $V(C') = V(C) \cup \big(S_2 \setminus (V(C) \cup V(M))\big)$. Note also that

$$S_2 \setminus V(C') = V(M) \cap S_2 \subseteq N_G(W) \cap S_2 \text{ and } V(G) \setminus (S_2 \cup V(C')) = V(M) \cap (V(G) \setminus S_2) = W.$$

Recall that for every $x \in S_2 \setminus V(C') = S_2 \cap V(M)$, x is adjacent to at least $|V(D_i)| - 1$ vertices in each D_i, $i = 1, 2, \cdots, t$ by Claim 5. Assume $|S_2| \leqslant \frac{7n}{12}$. Then by the same argument as in the first case of proving Claim 7, we have

$$\begin{aligned} |N_G(x) \cap V(C')| &\geqslant \left| N_G(x) \cap \left(\bigcup_{1 \leqslant i \leqslant t} V(D_i) \right) \right| \geqslant n - |S_2| - \frac{2|S_2|}{15} + t \\ &\geqslant n - \frac{17}{15} \cdot \frac{7}{12} n = \frac{61}{180} n > \frac{4.5}{16} n. \end{aligned}$$

Applying Lemma 11 for C' and every path in M iteratively, we obtain a hamiltonian cycle in G. Hence we assume

$$|S_2| > \frac{7n}{12}. \tag{11}$$

Claim 8. *For any two $K_{1,2}$-stars $x_1u_1y_1, x_2u_2y_2 \in M$, if u_1u_2 is a 2-vertex component of $G - S_2$ and $|S_2| > \frac{7n}{12}$, then at least one of u_1 and u_2 has more than $\frac{n}{16}$ neighbors on C'.*

Proof: For otherwise, since u_i is adjacent to exactly one vertex in $V(M) \cap (V(G) \setminus S_2)$, and $|V(M) \cap S_2| \leqslant 2|V(M) \cap (V(G) \setminus S_2)| = 2|W| \leqslant \frac{4|S_2|}{15}$,

$$\begin{aligned} d_G(u_1) + d_G(u_2) &\leqslant 2\left(\frac{n}{16} + 1 + |V(M) \cap S_2|\right) \leqslant 2\left(\frac{n}{16} + 1 + \frac{4|S_2|}{15}\right) \\ &< 2(\frac{1}{16} \cdot \frac{12}{7}|S_2| + 1 + \frac{4|S_2|}{15}) = \frac{157}{210}|S_2| + 2. \end{aligned}$$

Since $n \geqslant 31$, we have $|S_2| > \frac{7}{12} \cdot 31 > 18$. Therefore, $\frac{157}{210}|S_2| + 2 < |S_2| \leqslant |S|$. This contradicts the assumption that for every edge $uv \in E(G)$, $d_G(u) + d_G(v) \geqslant |S|$. ■

Let

$$M_1 = \{uwv \in M \mid \deg_G(w, C') > \frac{n}{16}\}, \qquad M_2 = M \setminus M_1.$$

Take $uwv \in M_1$, note that $u, v \in S_2$ and $w \in V(G) \setminus S_2$. By the definition of M_1, $\deg(w, C') > \frac{n}{16}$. By Claim 7, $\deg(u, C') > \frac{n}{16}$ and $\deg(v, C') > \frac{n}{16}$. Now applying Lemma 10 for C' and every path in M_1 iteratively, we get a longer cycle C^* such that $V(C^*) = V(C') \cup V(M_1)$.

By the toughness of G, $G - S_2$ has at most $\frac{|S_2|}{15}$ components in total. Particularly, $G - S_2$ has at most $\frac{|S_2|}{15}$ components that have at most two vertices in total. By Claim 8, we know that for every 2-vertex component uv of $G - S_2$, at least one of u or v has more than $\frac{n}{16}$ neighbors on C'. Therefore, at least one of the two $K_{1,2}$-stars centered, respectively, at u and v is contained in M_1. In other words, there is at most one $K_{1,2}$-star from M_2 that centers at a vertex from a same component of $G - S_2$. Therefore,

$$|V(M_2)| \leqslant \frac{|S_2|}{15} + \frac{2|S_2|}{15} = \frac{|S_2|}{5}.$$

By the definition of M_2 and by the assumption that for any $uv \in E(G)$, $d_G(u)+d_G(v) \geqslant |S|$, we know that for any path $xwy \in M_2$, where $x, y \in S_2$ and $w \in V(G) \setminus S_2$, we have $d_G(x) + d_G(w) \geqslant |S| \geqslant |S_2|$. Therefore, the number of neighbors that x has in G on C^* is at least

$$\begin{aligned}
& |S_2| - \deg_G(x, G - V(C^*)) - d_G(w) \\
\geqslant\ & |S_2| - \deg_G(x, V(M_2)) - \big(\deg_G(w, C^*) + \deg_G(w, S_2 \cap V(M_2))\big) \\
\geqslant\ & |S_2| - \frac{|S_2|}{5} - \left(\frac{n}{16} + \frac{2|S_2|}{15}\right) = \frac{2|S_2|}{3} - \frac{n}{16} \\
>\ & \frac{2 \cdot 7n}{3 \cdot 12} - \frac{n}{16} = \frac{47n}{144} > \frac{4.5n}{16}.
\end{aligned}$$

Similarly, the vertex y has in G at least $\frac{4.5n}{16}$ neighbors on C^*. Now applying Lemma 11 for C^* and every path in M_2 iteratively gives a hamiltonian cycle in G. $\square$

Proof of Theorem 1. We may assume that G is not a complete graph. Since G is 15-tough, it is 30-connected, and consequently, $\delta(G) \geqslant 30$. By Lemma 5, we may assume that

$$n \geqslant (\delta(G) + 1) \cdot (\tau(G) + 1) \geqslant 31 \cdot 16, \quad \text{and} \quad \delta(G) \leqslant \frac{n}{16} - 1. \qquad \textbf{(12)}$$

We consider two cases to finish the proof.

Case 1: For every edge $e = uv \in E(G)$, $d_G(u) + d_G(v) > \frac{3n}{4}$.

Denote by

$$V_1 = \{v \in V(G) \mid d_G(v) \leqslant \frac{3n}{8}\}. \qquad \textbf{(13)}$$

By the assumption of Case 1, we know that V_1 is an independent set in G. Therefore,

$$|V_1| \leqslant \frac{n}{16}, \qquad \textbf{(14)}$$

by Lemma 9.

Since G is 15-tough, Corollary 4 implies that G has a $K_{1,2}$-matching M with all vertices in V_1 as the centers of the $K_{1,2}$-matching. Let V_2 be the set of the vertices contained in M. By (**14**), we have that

$$|V_2| \leqslant \frac{3n}{16}. \tag{15}$$

Denote by $G_1 = G - V_2$. Then by the definitions of V_1, V_2 and (**15**), we get that

$$\delta(G_1) > \frac{3n}{8} - |V_2| \geqslant \frac{3n}{16}, \tag{16}$$

$$\deg_G(x, G_1) > \frac{3n}{8} - |V_2| \geqslant \frac{3n}{16}, \quad \text{for any } x \in V_2 \setminus V_1. \tag{17}$$

We first assume that G_1 has a hamiltonian cycle C. For every copy of $K_{1,2}$, say $xyz \in M$, by (**17**),

$$\begin{aligned} \deg_G(x, G_1) > \frac{3n}{16} > \frac{n}{16}, \\ \deg_G(z, G_1) > \frac{3n}{16} > \frac{n}{16}. \end{aligned} \tag{18}$$

Let

$$M_1 = \{uwv \in M \mid \deg_G(w, C) > \frac{n}{16}\}, \qquad M_2 = M \setminus M_1.$$

By (**18**), applying Lemma 10 with respect to C and every vertex in M_1 iteratively, we get a longer cycle C^* such that $V(C^*) = V(C) \cup V(M_1)$.

By the definition of M_2 and by the assumption that for any $uv \in E(G)$, $d_G(u)+d_G(v) > \frac{3n}{4}$, we know that for any path $xwy \in M_2$, where $x, y \in V_2 \setminus V_1$ and $w \in V_1$, we have $d_G(x) + d_G(w) > \frac{3n}{4}$. Therefore, the number of neighbors that x has in G on C^* is at least

$$\begin{aligned} & \frac{3n}{4} - \deg_G(x, G - V(C^*)) - d_G(w) \\ \geqslant\ & \frac{3n}{4} - \deg_G(x, V(M_2)) - \big(\deg_G(w, C^*) + \deg_G(w, V_2)\big) \\ \geqslant\ & \frac{3n}{4} - |V_2| - \Big(\frac{n}{16} + |V_2 \setminus V_1|\Big) \\ \geqslant\ & \frac{3n}{4} - \frac{3n}{16} - \frac{n}{16} - \frac{2n}{16} \\ =\ & \frac{6n}{16} > \frac{4.5n}{16}. \end{aligned}$$

Similarly, the vertex y has in G at least $\frac{4.5n}{16}$ neighbors on C^*. Now applying Lemma 11 for C^* and every path in M_2 iteratively gives a hamiltonian cycle in G.

Hence we assume that G_1 does not have a hamiltonian cycle. By Lemma 5, we have $\delta(G_1) \leqslant \frac{|V(G_1)|}{\tau(G_1)+1} \leqslant \frac{n}{\tau(G_1)+1}$. On the other hand, (**16**) yields $\delta(G_1) > \frac{3n}{16}$. Combining these inequalities, we have $\frac{3n}{16} < \frac{n}{\tau(G_1)+1}$, which implies $\tau(G_1) < \frac{13}{3} < 7$. Therefore, there exists $S_1 \subseteq V(G_1)$ such that $c(G_1 - S_1) \geqslant 2$ and

$$|S_1|/c(G_1 - S_1) < 7. \tag{19}$$

Note $c(G_1 - S_1) = c(G - (S_1 \cup V_2))$. If $|S_1| \geqslant \frac{3n}{16}$, then we have $c(G_1 - S_1) > \frac{|S_1|}{7} \geqslant \frac{3n}{16\cdot 7}$, and thus by (**15**),

$$\frac{|S_1 \cup V_2|}{c(G - (S_1 \cup V_2))} = \frac{|S_1|}{c(G_1 - S_1)} + \frac{|V_2|}{c(G_1 - S_1)} < 7 + \frac{3n/16}{3n/(16\cdot 7)} = 14.$$

This contradicts $\tau(G) \geqslant 15$. So we assume $|S_1| < \frac{3n}{16}$. Thus $|S_1| \leqslant \lfloor \frac{3n}{16} \rfloor$. As $\delta(G_1) \geqslant \lfloor \frac{3n}{16} \rfloor + 1$ by (**16**), we know that each component of G_1 contains at least

$$\delta(G_1) - |S_1| \geqslant \lfloor \frac{3n}{16} \rfloor + 1 - \lfloor \frac{3n}{16} \rfloor + 1 = 2$$

vertices. By Lemma 6, we know that every component of $G_1 - S_1$ is a clique component. Let $S = S_1 \cup V_2$. We then see that all components of $G - S$ are nontrivial. Also, $|S| < \frac{6n}{16} < \frac{3n}{4}$ since $|S_1| < \frac{3n}{16}$ and $|V_2| \leqslant \frac{3n}{16}$ by (**15**). Furthermore, by the assumption of Case 1, for every edge $uv \in E(G)$, $d_G(u) + d_G(v) > \frac{3n}{4} > |S|$. Now we can apply Lemma 12 on G and S to find a hamiltonian cycle in G.

Case 2: There exists an edge $e = uv \in E(G)$ such that $d_G(u) + d_G(v) \leqslant \frac{3n}{4}$.

Let

$$S = (N_G(u) \cup N_G(v)) \setminus \{u, v\},$$

such that $d_G(u) + d_G(v)$ is smallest among all the degree sums of two adjacent vertices in G.

By the assumption of this case and the choice of S, we know that

$$|S| \leqslant \frac{3n}{4} - 2, \quad \text{and} \quad \text{for any } u'v' \in E(G),\ d(u') + d(v') \geqslant |S|. \tag{20}$$

By the definition of S, $c(G - S) \geqslant 2$ and uv is one of the components of $G - S$. Since $\tau(G) \geqslant 15$, and $|V(G) \setminus (S \cup \{u, v\})| = n - |S| - 2 \geqslant \frac{|S|}{3} = \frac{5|S|}{15}$, $G - S - \{u, v\}$ has a component with at least 5 vertices. This, together with the fact that uv is one of the components of $G - S$, Lemma 6 implies that every component of $G - S$ is a clique component, and $G - S$ has at least two nontrivial components. Again Lemma 12 implies that G has a hamiltonian cycle. □

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