Classification of toric log del Pezzo surfaces with few singular points

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Abstract

We give a classification of toric log del Pezzo surfaces with two or three singular points. Our proofs are purely combinatorial, relying on the bijection between toric log del Pezzo surfaces and the so-called LDP-polygons introduced by Dais and Nill.

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1 Introduction

A normal projective surface is called *log del Pezzo surface* if it has at worst log-terminal singularities (that is, quotient singularities) and its anticanonical divisor is a $\mathbb{Q}$-Cartier ample divisor. Log del Pezzo surfaces have been extensively studied and many results are known (for example [13, 14, 15, 1, 12, 9]).

When they are toric, log del Pezzo surfaces are known to be in bijection to the so-called LDP-polygons, introduced by Dais and Nill [8]. This makes their classification be a purely combinatorial problem. Let us explain this connection from the point of view of the polygon, skipping all technical algebraic-geometric details.

An LDP-polygon is a convex lattice polygon with the origin in its interior and with the property that all vertices are primitive lattice points; that is, they are of the form $(p, q) \in \mathbb{Z}^2$ with $\gcd(p, q) = 1$. The bijection sends isomorphism classes of toric log del Pezzo surfaces to equivalence classes of LDP-polygons, where two LDP-polygons Q and Q' are called *equivalent* if there is an automorphism of the lattice $\mathbb{Z}^2$ sending Q to Q'. Remember that an automorphism of $\mathbb{Z}^2$ is (the restriction to $\mathbb{Z}^2$ of) a linear map

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of determinant ±1 and with integer coefficients. See Section 2, in particular Corollary 8, for more details. Moreover, several algebraic-geometric invariants of a toric log del Pezzo surface X_Q admit an easy description in terms of the corresponding LDP-polygon Q. Among them:

Picard number: The Picard number of X_Q equals the number of edges (equivalently, of vertices) of Q minus two. That is, triangles, quadrilaterals, and pentagons correspond respectively to Picard number 1, 2, 3, etc.

Singular points: Each torus-invariant point of X_Q corresponds to an edge of Q, and those points are the only possible singularities in a toric surface. Since the number and distribution of such singular points around the polygon plays an important role in this paper, we call an edge of Q *singular* or *nonsingular* accordingly.

Proposition 1 ([4, Theorem 1.3.12])**.** *An edge of an LDP-polygon Q is nonsingular (that is, the corresponding torus-fixed point of the associated toric surface* X_Q *is nonsingular) if and only if its two vertices form a basis of* $\mathbb{Z}^2$*. That is, if and only if* $|ps - rq| = 1$*, where* (p, q) *and* (r, s) *are the vertex coordinates.*

It is well known that there are exactly five nonsingular LDP-polygons (that is, five nonsingular toric del Pezzo surfaces). They are the polars of the five *smooth reflexive polygons*, and they are depicted in Figure 1.

Blow-ups: Suppose that $v_i v_{i+1}$ is a nonsingular edge in an LDP-polygon Q, and let Q' be the convex hull of $Q \cup \{v_i + v_{i+1}\}$. If v_i and v_{i+1} are still vertices of Q', then $X_{Q'}$ is the blow-up of X_Q at the corresponding nonsingular point. In this situation we say that the LDP-polygon Q' is a *blow-up* of Q at the nonsingular edge $v_i v_{i+1}$. Blow-ups make sense algebraically even if v_i or v_{i+1} are not vertices of Q' anymore but in this case the resulting toric surface is no longer log del Pezzo. Thus, we do not consider it.

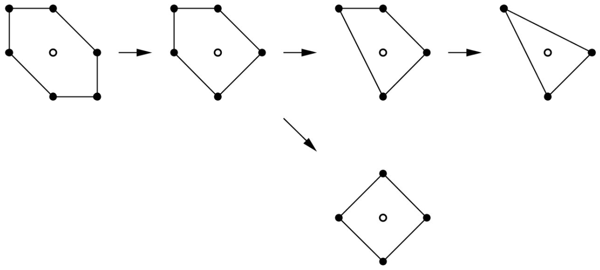

Figure 1: The five nonsingular LDP-polygons, polars of the smooth reflexive polygons. The interior dot in each polygon is the origin and arrows represent blow-ups. As customary in algebraic geometry, blow-ups are shown as reverse arrows, since there is a morphism from the blown-up variety to the original one.

Index: The index of X_Q is the minimum $k \in \mathbb{N}$ such that Q can be described by linear inequalities of the form $a_1x \leqslant k, \dots, a_dx \leqslant k$, with $a_1, \dots, a_d \in \mathbb{Z}^2$. Put differently, it is the minimum $k \in \mathbb{N}$ such that $kQ^\vee$ is a lattice polygon, where $Q^\vee$ is the polar polygon of Q. For example, an LDP-polygon has index 1 if and only if its polar is again a lattice polygon. Lattice polytopes (in arbitrary dimension) whose polar is again a lattice polytope are called reflexive, and there are exactly sixteen of them in dimension two; that is, there are sixteen LDP-polygons of index 1. They include the five nonsingular ones (see [4, Theorem 8.3.7]).

It is known that there are finitely many toric log del Pezzo surfaces of any fixed index (see [10, Corollary 4.5]). Dais [5, 6] classified toric log del Pezzo surfaces of Picard number one (that is, LDP-triangles) and of index at most three, and Kasprzyk–Kreuzer–Nill [10] gave two independent algorithms that enumerate all toric log del Pezzo surfaces up to any given index.

In this paper, we focus on the number of singular points on toric log del Pezzo surfaces. Partial results on this exist even in the general, perhaps non-toric case: Belousov [2, 3] proved that a log del Pezzo surface of Picard number one has at most four singular points, and Kojima [11] classified log del Pezzo surfaces of Picard number one with unique singular points. Dais has recently classified all toric log del Pezzo surfaces with unique singular points:

Theorem 2 ([7, Theorem 1.4]). *An LDP-polygon has exactly one singular edge if and only if it is equivalent to one of the following:*

(1) $\operatorname{conv}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\}$ *for some positive integer p.*

(2) *A quadrilateral or pentagon obtained by blowing up one or both of the nonsingular edges of an LDP-triangle in part (1).*

The purpose of this paper is to extend this classification for LDP-polygons with two or three singular points:

Theorem 3. *An LDP-polygon has exactly two singular edges if and only if it is equivalent to one of the following:*

(1) $\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -p \\ -q \end{pmatrix}\right\}$ *for some $p, q \in \mathbb{Z}$ with $p, q \geqslant 2$ and* $\gcd(p, q) = 1$.

(2) $\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ p \end{pmatrix}, \begin{pmatrix} q \\ r \end{pmatrix}\right\}$ *for some $p, q, r \in \mathbb{Z}$ with*

$$p \leqslant 1, r \leqslant \min\{-pq-2, -2, -q-1, q-pq-1\}$$

and $\gcd(q, r) = 1$.

(3) $\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ p+1 \end{pmatrix}, \begin{pmatrix} -1 \\ p \end{pmatrix}, \begin{pmatrix} q \\ r \end{pmatrix}\right\}$ *for some $p, q, r \in \mathbb{Z}$ with $p \leqslant 0, 1 \leqslant q \leqslant -r-1$ and* $\gcd(q, r) = 1$.

Note that part (3) in Theorem 3 is the blow-up of an LDP-quadrilateral in part (2) at a nonsingular edge.

Theorem 4. *An LDP-polygon has exactly three singular edges if and only if it is equivalent to one of the following:*

(1) $\operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}p\\q\end{pmatrix},\begin{pmatrix}r\\s\end{pmatrix}\right\}$ *for some* $p,q,r,s\in\mathbb{Z}$ *with* $q\geqslant 2, s\leqslant -2, ps-qr\geqslant 2$ *and* $\gcd(p,q)=\gcd(r,s)=1$.

(2) $\operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}p\\q\end{pmatrix},\begin{pmatrix}r\\s\end{pmatrix}\right\}$ *for some* $p,q,r,s\in\mathbb{Z}$ *with*

$$p\leqslant\min\{-2,-q\}, s\leqslant\min\{-2,-r\}, ps-qr\geqslant\max\{2,p-r+1,s-q+1\}$$

and $\gcd(p,q)=\gcd(r,s)=1$.

(3) $\operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\p\end{pmatrix},\begin{pmatrix}-q\\pq-r\end{pmatrix},\begin{pmatrix}s\\-t\end{pmatrix}\right\}$ *for some*

$$p,q,r,s,t\in\mathbb{Z}$$

with

$$\begin{aligned}p&\leqslant 1,\\ r&\geqslant\max\{2,q+1\},\\ t&\geqslant\max\{2,s+1\},\\ qt+rs-pqs&\geqslant\max\{2,t-r-ps+1,r-t-pq+1\},\\ pq-r&\leqslant -1\end{aligned}$$

and $\gcd(q,r)=\gcd(s,t)=1$.

(4) *The blow-up of an LDP-pentagon in part (3) at a nonsingular edge.*

Our proofs of Theorems 3 and 4 use the following general properties of LDP-polygons:

Theorem 5. *Let* Q *be an LDP-polygon with at least one singular edge. Then:*

(1) *The nonsingular edges of* Q *(and hence also the singular ones) form a consecutive sequence. (See Proposition 16).*

(2) *If* Q *has at least two singular edges, then it has at most three nonsingular edges. (See Corollary 17).*

(3) *If* Q *is not a blow-up of a smaller LDP-polygon at a nonsingular edge, then* Q *has at most two nonsingular edges. (See Lemma 11).*

Theorems 2 and 5(2) imply that any LDP-polygon Q has at most four nonsingular edges except if Q is the polar of the unique smooth reflexive pentagon or hexagon. Furthermore, Theorem 5(3) implies that any LDP-polygon Q with more than two nonsingular edges is a blow-up of a smaller LDP-polygon except if Q is equivalent to

$$\operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\right\} \text{ or } \operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix}\right\}.$$

The structure of the paper is as follows: In Section 2, we recall the bijection between toric log del Pezzo surfaces and LDP-polygons, and we fix some notation. In Sections 3 and 4, we prove Theorems 3 and 4, respectively. The three parts of Theorem 5 are proved as we go along.

2 Toric log del Pezzo surfaces and LDP-polygons

We fix a notation and recall some basic facts from toric geometry which will be used in this paper. (See [4] for details.) Let Δ be a complete fan in $\mathbb{R}^2$. We list the primitive generators of rays, that is, one-dimensional cones, in Δ as $v_1, \ldots, v_d$ in counterclockwise order around the origin in $\mathbb{R}^2$, and we define $v_0 = v_d$ and $v_{d+1} = v_1$. For $i = 1, \ldots, d$, we write $v_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$. Observe that a complete fan Δ in $\mathbb{R}^2$ is completely described by its set of primitive ray generators $v_1, \ldots, v_d$, and that any such set defines a complete fan as long as all the v_i are primitive (that is, $\gcd(x_i, y_i) = 1$), no two are positive multiples of one another, and every open half-plane contains at least one v_i. We denote by $X(\Delta)$ the associated complete toric surface. Since Δ is simplicial, the Picard number $\rho(X(\Delta))$ of $X(\Delta)$ equals $d - 2$.

Definition 6. Let Δ be a complete fan in $\mathbb{R}^2$. We define the map $f : \{1, \ldots, d\} \to \mathbb{Z}$ by

$$f(i) = \det(v_{i-1}, v_i) + \det(v_i, v_{i+1}) + \det(v_{i+1}, v_{i-1}).$$

The equivalence of parts (1) and (2) in the following statement is Remark 6.7 in [5] (see also Exercise 8.3.9 in [4]). The equivalence of (2) and (3) is elementary.

Proposition 7. *Let Δ be a complete fan in $\mathbb{R}^2$ and let $v_1, \ldots, v_d$ be its primitive ray generators, in counterclockwise order. Then, the following are equivalent:*

(1) *The toric surface $X(\Delta)$ is log del Pezzo.*

(2) *Every v_i is a vertex of* $\operatorname{conv}\{v_1, \ldots, v_d\}$.

(3) $f(i) > 0$ *for every i.*

Let $Q \subset \mathbb{R}^2$ be an LDP-polygon and let $v_1, \ldots, v_d$ be its vertices, in counterclockwise order. We denote by $\operatorname{Sing}(Q)$ the set of singular edges of Q and denote by F_i the edge $v_i v_{i+1}$ of Q for $i = 1, \ldots, d$. For each edge F_i, we define the rational strongly convex polyhedral cone $\sigma_i = \{\lambda x \mid \lambda \geqslant 0, x \in F_i\} \subset \mathbb{R}^2$. Then the set Δ_Q of all such cones and their faces forms a complete fan in $\mathbb{R}^2$ whose primitive ray generators are precisely the vertices of Q. We define X_Q to be the associated toric surface $X(\Delta_Q)$.

Corollary 8 ([7, Proposition 4.2]). *The correspondence $Q \mapsto X_Q$ induces a bijection between equivalence classes of LDP-polygons and isomorphism classes of toric log del Pezzo surfaces.*

Example 9. Let p be a positive integer and let

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ -p \end{pmatrix}.$$

Then the convex hull $Q = \operatorname{conv}\{v_1, v_2, v_3\}$ is an LDP-triangle. The fan Δ_Q consists of the cones

$$\mathbb{R}_{\geqslant 0}v_1 + \mathbb{R}_{\geqslant 0}v_2, \mathbb{R}_{\geqslant 0}v_2 + \mathbb{R}_{\geqslant 0}v_3, \mathbb{R}_{\geqslant 0}v_3 + \mathbb{R}_{\geqslant 0}v_1$$

and their faces. The associated toric log del Pezzo surface X_Q is the weighted projective plane $\mathbb{P}(1, 1, p)$ (see [4, Example 3.1.17]).

3 Proof of Theorem 3

In this section, we give a proof of Theorem 3. We will use the notation introduced in Section 2 freely. The following lemmas play key roles in the proof of Theorems 3 and 4.

Lemma 10. *Let Q be an LDP-polygon with $d \geqslant 4$ edges. Suppose that there exists i with $1 \leqslant i \leqslant d$ such that F_{i-1} and F_{i+1} are singular while F_i is nonsingular. Then the following hold:*

(1) *We have $\det(v_{i+2}, v_{i-1}) \geqslant 2$. In particular, the cones $\sigma_{i-1}, \sigma_i, \sigma_{i+1}$ cover more than a half-plane.*

(2) *Q has at least three singular edges.*

Proof. Without loss of generality, we may assume that $i = 2, v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Then $y_1 \geqslant 2$ and $x_4 \geqslant 2$.

(1) Proposition 7 and $f(2) = x_1 + y_1 + 1$ imply $x_1 \geqslant -y_1$. Since $y_1 \geqslant 2$ and $v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ is primitive, we have $x_1 \geqslant 1 - y_1$ and $x_1 \neq 0$. A similar argument shows that $y_4 \geqslant 1 - x_4$ and $y_4 \neq 0$.

Case 1. Suppose that $x_1 \leqslant -1$ or $y_4 \leqslant -1$. We may assume that $x_1 \leqslant -1$. Then $y_4 \geqslant 1 - x_4$ implies $x_1 y_4 \leqslant x_1(1 - x_4)$. Since $1 - x_4 \leqslant 1 - 2 = -1$ and $x_1 \geqslant 1 - y_1$, we have $x_1(1 - x_4) \leqslant (1 - x_4)(1 - y_1)$. Thus, $x_1 y_4 \leqslant (1 - x_4)(1 - y_1)$. Therefore,

$$\det(v_4, v_1) = x_4 y_1 - x_1 y_4 \geqslant x_4 y_1 - (1 - x_4)(1 - y_1) = x_4 + y_1 - 1 \geqslant 3.$$

Case 2. Suppose that $x_1 \geqslant 1$ and $y_4 \geqslant 1$. Then both v_4 and v_1 are in

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2; x, y \geqslant 0 \right\}.$$

Since $v_1, \ldots, v_d$ are arranged in counterclockwise order, we must have $\det(v_4, v_1) = x_4y_1 - x_1y_4 \geqslant 1$. To deduce a contradiction we assume that $\det(v_4, v_1) = 1$. We see that $(x_4 + 1)/x_1 > 0$ and

$$\begin{aligned}&\frac{x_4+1}{x_1}\left(\left(1-\frac{x_4}{x_4+1}\right)v_3+\frac{x_4}{x_4+1}v_1\right)\\&=\frac{1}{x_1}\begin{pmatrix} x_1x_4 \\ -1+x_4y_1\end{pmatrix}=\frac{1}{x_1}\begin{pmatrix} x_1x_4 \\ x_1y_4\end{pmatrix}=v_4.\end{aligned}$$

Since v_4 is a vertex of Q, we must have $(x_4 + 1)/x_1 > 1$, so $x_4 \geqslant x_1$. The assumption $\det(v_4, v_1) = 1$ and $x_4 \geqslant 2$ imply $x_4 - 1 \geqslant x_1$. A similar argument shows that $y_1 - 1 \geqslant y_4$. Thus, $x_1y_4 \leqslant (x_4 - 1)(y_1 - 1)$. Therefore,

$$1 = \det(v_4, v_1) = x_4y_1 - x_1y_4 \geqslant x_4y_1 - (x_4 - 1)(y_1 - 1) = x_4 + y_1 - 1 \geqslant 3,$$

which is a contradiction.

In every case, we obtain $\det(v_4, v_1) \geqslant 2$.

(2) If $d = 4$, then F_4 is a singular edge by (1). Assume $d \geqslant 5$. Then by (1), there exists a lattice point $v \in (\mathrm{conv}\{0, v_4, v_1\} \cap \mathbb{Z}^2) \setminus \{0, v_4, v_1\}$. Since

$$\bigcup_{j=4}^{d} \mathrm{conv}\{0, v_j, v_{j+1}\} \supsetneq \mathrm{conv}\{0, v_4, v_1\},$$

there exists $j \in \{4, \ldots, d\}$ such that v is an interior point of $\mathrm{conv}\{0, v_j, v_{j+1}\}$. In particular, F_j is a singular edge and thus $|\mathrm{Sing}(Q)| \geqslant 3$. □

Lemma 11. *Let Q be an LDP-polygon with at least one singular edge. Suppose that Q cannot be obtained by blowing up a nonsingular edge of another LDP-polygon, and there exists i with $1 \leqslant i \leqslant d$ such that F_i and F_{i+1} are nonsingular. Then F_j is singular for every $j \in \{1, \ldots, d\} \setminus \{i, i + 1\}$.*

Proof. It is obvious for $d = 3$. We may assume that $d \geqslant 4, i = 2$ and

$$v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ a \end{pmatrix}$$

for $a \in \mathbb{Z}$. To deduce a contradiction we assume that there exists $j \in \{1, 4, 5, \ldots, d\}$ such that F_j is nonsingular. Proposition 7 and $f(3) = a + 2$ imply $a \geqslant -1$. Since Q is not a blow-up of a smaller LDP-polygon at a nonsingular edge, we have $v_2 + v_4 \neq v_3$. Thus, $a \geqslant 0$ and $y_5, \ldots, y_d, y_1 \geqslant 1$. In particular, $\sigma_2 \cup \sigma_3 \supset \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2; y \leqslant 0 \right\}$.

Case 1. Suppose that $5 \leqslant j \leqslant d$. If F_{j-1} (resp. F_{j+1}) is nonsingular, then $a = 0$ and $\sigma_{j-1} \cup \sigma_j$ (resp. $\sigma_j \cup \sigma_{j+1}$) coincides with $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2; y \geqslant 0 \right\}$, a contradiction. Hence, both F_{j-1} and F_{j+1} are singular. However, by Lemma 10 (1), the cones $\sigma_{j-1}, \sigma_j, \sigma_{j+1}$ cover more than a half-plane. This is a contradiction.

Case 2. Suppose that $j = 1$ or $j = 4$. We may assume that $j = 4$. Since Q has at least one singular edge, we may further assume that F_1 is singular. Then $y_1 \geqslant 2$ and $v_5 = \begin{pmatrix} b \\ ab+1 \end{pmatrix}$ for some $b \in \mathbb{Z}$. Note that $ab = y_5 - 1 \geqslant 0$. Proposition 7 and $f(2) = x_1 + y_1 + 1$ imply $x_1 \geqslant -y_1$. Since $y_1 \geqslant 2$ and v_1 is primitive, we have $x_1 \geqslant 1 - y_1$ and $x_1 \neq 0$. Furthermore, $1 \leqslant f(4) = 2 - b$ and $v_3 + v_5 \neq v_4$ imply $b \leqslant 0$. There are three subcases to consider.

Subcase 2.1. Suppose that $x_1 \geqslant 1$. Then $d \geqslant 5$ since $x_5 = b \leqslant 0$. Obviously, $\det(v_5, v_1) = by_1 - (ab+1)x_1 \geqslant 1$. On the other hand, if $a = 0$, then $by_1 \leqslant 0$ and $-x_1 \leqslant -1$, which means that $\det(v_5, v_1) \leqslant -1$. But even if $a \geqslant 1$, we have $ab \geqslant 0$ and $b \leqslant 0$, meaning that $b = 0$, and again $\det(v_5, v_1) \leqslant -1$. This is a contradiction.

Subcase 2.2. Suppose that $x_1 \leqslant -1$ and $a = 0$. Then $v_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_5 = \begin{pmatrix} b \\ 1 \end{pmatrix}$ and $d \geqslant 5$. We see that $(\det(v_5, v_1) + 1)/y_1 > 0$ and

$$\frac{\det(v_5, v_1) + 1}{y_1} \left(\left(1 - \frac{1}{\det(v_5, v_1) + 1} \right) v_4 + \frac{1}{\det(v_5, v_1) + 1} v_1 \right) = v_5.$$

However,

$$\det(v_5, v_1) + 1 = by_1 - x_1 + 1 \leqslant by_1 + y_1 \leqslant y_1$$

and thus $0 < (\det(v_5, v_1) + 1)/y_1 \leqslant 1$, which contradicts the fact that v_5 is a vertex of Q.

Subcase 2.3. Suppose that $x_1 \leqslant -1$ and $a \geqslant 1$. Since $ab \geqslant 0$ and $b \leqslant 0$, we must have $b = 0$. Hence $v_4 = \begin{pmatrix} 1 \\ a \end{pmatrix}$, $v_5 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $d \geqslant 5$. We see that $(1 - x_1)/\det(v_4, v_1) > 0$ and

$$\frac{1 - x_1}{\det(v_4, v_1)} \left(\left(1 - \frac{1}{1 - x_1} \right) v_4 + \frac{1}{1 - x_1} v_1 \right) = v_5.$$

However,

$$\det(v_4, v_1) = y_1 - ax_1 \geqslant (1 - x_1) - ax_1 \geqslant 1 - x_1 + a > 1 - x_1$$

and thus $0 < (1 - x_1)/\det(v_4, v_1) < 1$, which contradicts the fact that v_5 is a vertex of Q.

Thus, we have reached a contradiction in every case. Hence, $F_1, F_4, F_5, \ldots, F_d$ are singular. $\square$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let Q be an LDP-polygon with two singular edges. Then there exists at least one i with $1 \leqslant i \leqslant d$ such that F_i is nonsingular. Hence we may assume that $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The case where $d = 3$. Since $|\mathrm{Sing}(Q)| = 2$, the edges F_2 and F_3 are singular. Hence $v_3 = \begin{pmatrix} -p \\ -q \end{pmatrix}$ for some $p, q \in \mathbb{Z}$ with $p, q \geqslant 2$ and $\gcd(p, q) = 1$. Conversely, for any

$p, q \in \mathbb{Z}$ with $p, q \geqslant 2$ and $\gcd(p, q) = 1$, the convex hull $\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -p \\ -q \end{pmatrix}\right\}$ is an LDP-triangle with two singular edges.

The case where $d = 4$. By Lemma 10 (2), either F_2 or F_4 is nonsingular. We may assume that F_2 is nonsingular. Then $v_3 = \begin{pmatrix} -1 \\ p \end{pmatrix}, v_4 = \begin{pmatrix} q \\ r \end{pmatrix}$ for some $p, q, r \in \mathbb{Z}$ with $\gcd(q, r) = 1$. Hence, it suffices to show that for $p, q, r \in \mathbb{Z}$ with $\gcd(q, r) = 1$, the sequence

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ p \end{pmatrix}, v_4 = \begin{pmatrix} q \\ r \end{pmatrix}$$

determines an LDP-polygon $Q = \operatorname{conv}\{v_1, \ldots, v_4\}$ (that is, $v_1, \ldots, v_4$ go counterclockwise around the origin exactly once in this order, and every v_i is a vertex of Q) and $|\operatorname{Sing}(Q)| = 2$ if and only if $p \leqslant 1$ and $r \leqslant \min\{-pq-2, -2, -q-1, q-pq-1\}$. We first observe that if the sequence determines Q, then we have

- F_1 and F_2 are nonsingular;
- $f(2) \geqslant 1 \Leftrightarrow p \leqslant 1$;
- F_3 is singular if and only if $r \leqslant -pq - 2$;
- F_4 is singular if and only if $r \leqslant -2$;
- $f(1) \geqslant 1 \Leftrightarrow r \leqslant -q$;
- $f(3) \geqslant 1 \Leftrightarrow r \leqslant q - pq$.

Suppose that the sequence determines Q and $|\operatorname{Sing}(Q)| = 2$. Since $r \leqslant -2$ and v_4 is primitive, we have $r \leqslant -q - 1$. If $r = q - pq$, then $q = \pm 1$ since v_4 is primitive, which contradicts that $r \leqslant -pq - 2$. Hence, $r \leqslant q - pq - 1$. Therefore, $p \leqslant 1$ and $r \leqslant \min\{-pq-2, -2, -q-1, q-pq-1\}$.

Conversely, suppose that $p \leqslant 1$ and $r \leqslant \min\{-pq-2, -2, -q-1, q-pq-1\}$. It suffices to show that $f(4) \geqslant 1$, that is, $p(1-q) - 2r \geqslant 1$. Since $r \leqslant -2$ and v_4 is primitive, we have $q \neq 0$.

Case 1. Suppose $q \geqslant 1$. Then $p \leqslant 1$ implies $p(1-q) \geqslant 1 - q$. Since $r \leqslant -q - 1$, we have $-2r \geqslant 2q + 2$. Thus,

$$f(4) = p(1-q) - 2r \geqslant (1-q) + (2q+2) = q + 3 \geqslant 4.$$

Case 2. Suppose $q \leqslant -1$. It suffices to show $pq(1-q) - 2qr \leqslant q$. The assumption $pq \leqslant -r - 2$ implies $pq(1-q) \leqslant (q-1)(r+2)$. Since $r + 2 \leqslant 0$ and $q - 1 \geqslant 2q$, we have $(q-1)(r+2) \leqslant 2q(r+2)$. Thus,

$$pq(1-q) - 2qr \leqslant 2q(r+2) - 2qr = 4q < q.$$

In every case, we obtain $f(4) \geqslant 1$. Therefore, $Q = \operatorname{conv}\{v_1, \ldots, v_4\}$ is an LDP-quadrilateral with $|\operatorname{Sing}(Q)| = 2$.

The case where $d \geqslant 5$. First we show the following claims:

Claim 12. *Every LDP-polygon Q with exactly two singular edges and more than four edges in total is an iterated blow-up, at nonsingular edges, of an LDP-quadrilateral.*

Proof of Claim 12. We use induction on d. Assume $d \geqslant 5$. We may assume that F_1 and F_2 are nonsingular. If $Q = Q_d$ cannot be obtained by blowing up a nonsingular edge of another LDP-polygon, then $F_3, \dots, F_d$ are all singular by Lemma 11, which contradicts that $|\mathrm{Sing}(Q)| = 2$. Hence, Q_d is the blow-up of a smaller LDP-polygon at a nonsingular edge. We may further assume that $v_2 = v_1 + v_3$. Let $Q_{d-1} = \mathrm{conv}\{v_1, v_3, v_4, \dots, v_d\}$. Then Q_d is the blow-up of Q_{d-1} at the nonsingular edge v_1v_3. By induction hypothesis there is a sequence of LDP-polygons

$$Q = Q_d, Q_{d-1}, \dots, Q_4,$$

where Q_i is the blow-up of Q_{i-1} at a nonsingular edge for $5 \leqslant i \leqslant d$. □

Claim 13. *Every LDP-polygon with exactly two singular edges has five edges or less in total.*

Proof of Claim 13. Let Q' be an LDP-quadrilateral with $|\mathrm{Sing}(Q')| = 2$, and let $Q = \mathrm{conv}\{v_1, \dots, v_5\}$ be the blow-up of Q' at one nonsingular edge. Then we may assume that

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ p+1 \end{pmatrix}, v_4 = \begin{pmatrix} -1 \\ p \end{pmatrix}, v_5 = \begin{pmatrix} q \\ r \end{pmatrix}$$

for $p, q, r \in \mathbb{Z}$ with

$$p \leqslant 1, r \leqslant \min\{-pq-2, -2, -q-1, q-pq-1\}, \gcd(q, r) = 1.$$

By Claim 12, it suffices to show that the blow-up of Q at any nonsingular edge does not increase the number of vertices. Proposition 7 and $f(4) = q + 1$ imply $q \geqslant 0$. Since $r \leqslant -2$ and v_5 is primitive, we have $q \geqslant 1$. The nonsingular edges of Q are F_1, F_2, F_3. However, we see that

$$\begin{aligned}
&\det(v_5, v_1) + \det(v_1, v_1 + v_2) + \det(v_1 + v_2, v_5) = 1 - q \leqslant 0, \\
&\det(v_2 + v_3, v_3) + \det(v_3, v_4) + \det(v_4, v_2 + v_3) = 0, \\
&\det(v_2, v_3) + \det(v_3, v_3 + v_4) + \det(v_3 + v_4, v_2) = 0.
\end{aligned}$$

Hence, the blow-up of Q at any nonsingular edge does not increase the number of vertices. This completes the proof of Claim 13. □

By Claims 12 and 13, it suffices to show that for $p, q, r \in \mathbb{Z}$ with $\gcd(q, r) = 1$, the sequence

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ p+1 \end{pmatrix}, v_4 = \begin{pmatrix} -1 \\ p \end{pmatrix}, v_5 = \begin{pmatrix} q \\ r \end{pmatrix}$$

determines an LDP-polygon $Q = \mathrm{conv}\{v_1, \dots, v_5\}$ and $|\mathrm{Sing}(Q)| = 2$ if and only if $p \leqslant 0$ and $1 \leqslant q \leqslant -r - 1$. We observe that if the sequence determines Q, then we have

- F_1, F_2, F_3 are nonsingular;
- F_5 is singular if and only if $r \leqslant -2$;
- $f(1) \geqslant 1 \Leftrightarrow q \leqslant -r$;
- $f(2) \geqslant 1 \Leftrightarrow p \leqslant 0$;
- $f(3) = 1$;
- $f(4) \geqslant 1 \Leftrightarrow q \geqslant 0$.

If the sequence determines Q and $|\mathrm{Sing}(Q)| = 2$, then $p \leqslant 0$ and $1 \leqslant q \leqslant -r-1$, since $r \leqslant -2$ and v_5 is primitive. Conversely, suppose that $p \leqslant 0$ and $1 \leqslant q \leqslant -r-1$. We need to show that $f(5) \geqslant 1$ and $\det(v_4, v_5) \geqslant 2$. Since $p(1-q) \geqslant 0$ and $r \leqslant -2$, we have $f(5) = p(1-q) - 2r \geqslant 4$ and $\det(v_4, v_5) = -r - pq \geqslant 2$. Thus, $Q = \mathrm{conv}\{v_1, \ldots, v_5\}$ is an LDP-pentagon with $|\mathrm{Sing}(Q)| = 2$. This completes the proof of Theorem 3. $\square$

4 Proof of Theorem 4

In this section, we prove Theorem 4. First we show the following lemma:

Lemma 14. *If Q is an LDP-pentagon with exactly two nonsingular edges, then these two edges are consecutive.*

Proof. To deduce a contradiction we assume that there is an LDP-pentagon Q with $\mathrm{Sing}(Q) = \{F_1, F_3, F_5\}$. We may assume that $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Then $y_1 \geqslant 2, x_4 \geqslant 2, x_1 + y_1 \geqslant 1, x_4 + y_4 \geqslant 1$.

Case 1. Suppose $y_5 \leqslant 0$. Then $x_5 \geqslant 1$. We will show that $x_5 + y_5 - x_4 - y_4 \geqslant 1$. Since $f(5) = (x_5 - x_4)y_1 - x_1(y_5 - y_4) + 1$ and $f(4) = x_4 - x_5 + 1$, Proposition 7 gives $(x_5 - x_4)y_1 - x_1(y_5 - y_4) \geqslant 0$ and $x_4 \geqslant x_5$. It follows that

$$1 = \det(v_4, v_5) = x_4 y_5 - x_5 y_4 \leqslant x_5 y_5 - x_5 y_4 = x_5(y_5 - y_4)$$

and thus $y_5 - y_4 \geqslant 1$. Since $x_1 \geqslant 1 - y_1$, we have $x_1(y_5 - y_4) \geqslant (1 - y_1)(y_5 - y_4)$. Hence,

$$\begin{aligned} 0 &\leqslant (x_5 - x_4)y_1 - x_1(y_5 - y_4) \\ &\leqslant (x_5 - x_4)y_1 - (1 - y_1)(y_5 - y_4) \\ &= (x_5 + y_5 - x_4 - y_4)y_1 - (y_5 - y_4) \\ &\leqslant (x_5 + y_5 - x_4 - y_4)y_1 - 1. \end{aligned}$$

Therefore, $x_5 + y_5 - x_4 - y_4 \geqslant 1$.

Let $v' = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. We calculate

$$\det(v_5, v') + \det(v', v_3) + \det(v_3, v_5) = (x_5 + y_5) + 1 + x_5$$

$$
\begin{aligned}
&\geqslant (x_4+y_4+1)+1+x_5 \geqslant 3+x_5 \geqslant 4,\\
\det(v',v_3)+\det(v_3,v_4)+\det(v_4,v') &= 1+x_4+(x_4+y_4) \geqslant 1+x_4+1 \geqslant 4,\\
\det(v_3,v_4)+\det(v_4,v_5)+\det(v_5,v_3) &= f(4) \geqslant 1,\\
\det(v_4,v_5)+\det(v_5,v')+\det(v',v_4) &= 1+x_5+y_5-x_4-y_4 \geqslant 2.
\end{aligned}
$$

Hence, $\mathrm{conv}\{v', v_3, v_4, v_5\}$ is an LDP-quadrilateral. However, $\det(v_3, v_4) \neq 1$ and

$$\det(v_5, v') = x_5 + y_5 \geqslant x_4 + y_4 + 1 \geqslant 2$$

while $\det(v', v_3) = \det(v_4, v_5) = 1$, which contradicts Theorem 3.

Case 2. Suppose $y_5 = 1$. Then $Q' = \mathrm{conv}\{v_2, v_3, v_4, v_5\}$ is an LDP-quadrilateral with $|\mathrm{Sing}(Q')| = 1$. By Theorem 2, we have either $v_2 = v_5 + v_3$ or $v_5 = v_4 + v_2$. If $v_2 = v_5 + v_3$, then $v_5 = v_2 - v_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and thus $2 \leqslant \det(v_5, v_1) = -x_1 - y_1 \leqslant -1$, a contradiction. If $v_5 = v_4 + v_2$, then $v_4 = v_5 - v_2 = \begin{pmatrix} x_5+1 \\ 1 \end{pmatrix}$ and thus

$$1 \leqslant \det(v_4, v_5) + \det(v_5, v_1) + \det(v_1, v_4) = 1 - y_1 \leqslant 1 - 2 = -1,$$

a contradiction.

Case 3. Suppose $y_5 \geqslant 2$. Then $Q' = \mathrm{conv}\{v_2, v_3, v_4, v_5\}$ is an LDP-quadrilateral with $|\mathrm{Sing}(Q')| = 2$. However, $\det(v_3, v_4) \neq 1$ and $\det(v_5, v_2) = y_5 \geqslant 2$ while $\det(v_2, v_3) = \det(v_4, v_5) = 1$, which contradicts Theorem 3.

Thus, we have reached a contradiction in every case. □

Proof of Theorem 4. Let Q be an LDP-polygon with three singular edges. We may assume that $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The case where $d = 3$. Since $|\mathrm{Sing}(Q)| = 3$, all edges of Q are singular. Hence $v_2 = \begin{pmatrix} p \\ q \end{pmatrix}$ and $v_3 = \begin{pmatrix} r \\ s \end{pmatrix}$ for some $p, q, r, s \in \mathbb{Z}$ with $q \geqslant 2, s \leqslant -2, ps - qr \geqslant 2$ and $\gcd(p, q) = \gcd(r, s) = 1$. Conversely, for any $p, q, r, s \in \mathbb{Z}$ with $q \geqslant 2, s \leqslant -2, ps - qr \geqslant 2$ and $\gcd(p, q) = \gcd(r, s) = 1$, the convex hull $\mathrm{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix}\right\}$ is an LDP-triangle with three singular edges.

The case where $d = 4$. We may assume that $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and F_2, F_3, F_4 are singular. Then $v_3 = \begin{pmatrix} p \\ q \end{pmatrix}$ and $v_4 = \begin{pmatrix} r \\ s \end{pmatrix}$ for some $p, q, r, s \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(r, s) = 1$. We observe the following:

- F_2 is singular if and only if $p \leqslant -2$;
- $f(2) \geqslant 1 \Leftrightarrow p \leqslant -q$;
- F_4 is singular if and only if $s \leqslant -2$;

- $f(1) \geqslant 1 \Leftrightarrow s \leqslant -r$;
- F_3 is singular if and only if $ps - qr \geqslant 2$; moreover, this inequality together with $p, s \leqslant -2$ guarantees that the four rays $(1,0)$, $(0,1)$, (p,q) and (r,s) are in counterclockwise order.
- $f(3) \geqslant 1 \Leftrightarrow ps - qr \geqslant p - r + 1$;
- $f(4) \geqslant 1 \Leftrightarrow ps - qr \geqslant s - q + 1$.

Hence, the assertion holds.

The case where $d \geqslant 5$. Firstly, we assume that $d = 5$. By Lemma 14, there exists i such that F_i and F_{i+1} are nonsingular. We may assume that $i = 1, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$v_3 = \begin{pmatrix} -1 \\ p \end{pmatrix}, v_4 = \begin{pmatrix} -q \\ pq - r \end{pmatrix}, v_5 = \begin{pmatrix} s \\ -t \end{pmatrix}$$

for some $p, q, r, s, t \in \mathbb{Z}$ with $\gcd(q, r) = \gcd(s, t) = 1$. Hence, it suffices to show that for $p, q, r, s, t \in \mathbb{Z}$ with $\gcd(q, r) = \gcd(s, t) = 1$, the sequence

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ p \end{pmatrix}, v_4 = \begin{pmatrix} -q \\ pq - r \end{pmatrix}, v_5 = \begin{pmatrix} s \\ -t \end{pmatrix}$$

determines an LDP-polygon $Q = \operatorname{conv}\{v_1, \ldots, v_5\}$ and $|\operatorname{Sing}(Q)| = 3$ if and only if

$$\begin{aligned} p &\leqslant 1, \\ r &\geqslant \max\{2, q + 1\}, \\ t &\geqslant \max\{2, s + 1\}, \\ qt + rs - pqs &\geqslant \max\{2, t - r - ps + 1, r - t - pq + 1\}, \\ pq - r &\leqslant -1. \end{aligned}$$

If the above sequence determines Q, then we observe the following:

- F_1 and F_2 are nonsingular;
- $f(2) \geqslant 1 \Leftrightarrow p \leqslant 1$;
- F_3 is singular if and only if $r \geqslant 2$;
- $f(3) \geqslant 1 \Leftrightarrow r \geqslant q$;
- F_5 is singular if and only if $t \geqslant 2$;
- $f(1) \geqslant 1 \Leftrightarrow t \geqslant s$;
- F_4 is singular if and only if $qt + rs - pqs \geqslant 2$;

- $f(4) \geqslant 1 \Leftrightarrow qt + rs - pqs \geqslant t - r - ps + 1$;
- $f(5) \geqslant 1 \Leftrightarrow qt + rs - pqs \geqslant r - t - pq + 1$.

Suppose that the same sequence determines Q and $|\mathrm{Sing}(Q)| = 3$. Since $t \geqslant 2$ and v_5 is primitive, we have $t \geqslant s + 1$. If $r = q$, then $q = \pm 1$ since v_4 is primitive, which contradicts that $r \geqslant 2$. Hence, $r \geqslant q + 1$. It remains to show that $pq - r \leqslant -1$. If $p \leqslant 0$, then $pq - r \leqslant -1$ holds obviously. If $p = 1$, then $pq - r = q - r \leqslant q - (q + 1) = -1$. We therefore obtain the required inequalities. The converse is obvious.

To prove the remaining part of the theorem, we need the following claim:

Claim 15. *Every LDP-polygon Q with three singular edges and more than five edges in total is an iterated blow-up, at nonsingular edges, of an LDP-pentagon.*

Proof of Claim 15. We use induction on d. Assume $d = 6$. To deduce a contradiction we assume that Q cannot be obtained by blowing up a nonsingular edge of another LDP-polygon. By Lemma 11, if F_i is nonsingular, then both F_{i-1} and F_{i+1} are singular. We may assume that $\mathrm{Sing}(Q) = \{F_1, F_3, F_5\}$. By Lemma 10 (1), we have $\det(v_4, v_1) \geqslant 2$. We consider the LDP-pentagon $\mathrm{conv}\{v_1, \ldots, v_5\}$. If $\det(v_5, v_1) = 1$, then this contradicts Theorem 3. Otherwise this contradicts Lemma 14. Hence, Q is the blow-up of a smaller LDP-polygon at a nonsingular edge.

In the case $d \geqslant 7$, we may assume that F_1 and F_2 are nonsingular. The rest of the proof is as for Claim 12. $\square$

Let Q' be an LDP-pentagon with $|\mathrm{Sing}(Q')| = 3$, and let $Q = \mathrm{conv}\{v_1, \ldots, v_6\}$ be the blow-up of Q' at one nonsingular edge. Then we may assume that

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ p+1 \end{pmatrix}, \\ v_4 &= \begin{pmatrix} -1 \\ p \end{pmatrix}, v_5 = \begin{pmatrix} -q \\ pq - r \end{pmatrix}, v_6 = \begin{pmatrix} s \\ -t \end{pmatrix} \end{aligned}$$

for $p, q, r, s, t \in \mathbb{Z}$ with

$$\begin{aligned} p &\leqslant 1, \\ r &\geqslant \max\{2, q+1\}, \\ t &\geqslant \max\{2, s+1\}, \\ qt + rs - pqs &\geqslant \max\{2, t - r - ps + 1, r - t - pq + 1\}, \\ pq - r &\leqslant -1 \end{aligned}$$

and $\gcd(q, r) = \gcd(s, t) = 1$. By Claim 15, to prove the remaining part it suffices to show that the blow-up of Q at any nonsingular edge does not increase the number of vertices. Proposition 7 and $f(4) = 1 - q$ imply $q \leqslant 0$. Since $r - pq \geqslant 1$ and $(r - pq)s = rs - pqs \geqslant 2 - qt \geqslant 2$, we have $s \geqslant 1$. The nonsingular edges of Q are F_1, F_2, F_3. However, we see that

$$\det(v_6, v_1) + \det(v_1, v_1 + v_2) + \det(v_1 + v_2, v_6) = 1 - s \leqslant 0,$$

$$\det(v_2+v_3, v_3) + \det(v_3, v_4) + \det(v_4, v_2+v_3) = 0,$$
$$\det(v_2, v_3) + \det(v_3, v_3+v_4) + \det(v_3+v_4, v_2) = 0.$$

Hence, the blow-up of Q at any nonsingular edge does not increase the number of vertices. This completes the proof of Theorem 4. □

The following proposition holds even for $|\mathrm{Sing}(Q)| \geqslant 4$.

Proposition 16. *In every LDP-polygon Q the nonsingular edges (hence also all the singular edges) form a consecutive sequence.*

Proof. We may assume that Q cannot be obtained by blowing up a nonsingular edge of another LDP-polygon. If $d \leqslant 5$, then the assertion follows from Theorems 3 and 4.

Assume $d = 6$. If $|\mathrm{Sing}(Q)| \geqslant 5$, then the assertion is obvious. If $|\mathrm{Sing}(Q)| \leqslant 3$, then the assertion follows from Theorems 3 and 4. Hence, we may further assume that $|\mathrm{Sing}(Q)| = 4$ and F_2 is nonsingular. To deduce a contradiction we assume that the other nonsingular edge is one of F_4, F_5, F_6.

Case 1. Suppose that either F_4 or F_6 is nonsingular. We may assume that F_4 is nonsingular. By Lemma 10 (1), we have $\det(v_4, v_1) \geqslant 2$. We consider the LDP-pentagon $\mathrm{conv}\{v_1, \dots, v_5\}$. If $\det(v_5, v_1) = 1$, then this contradicts Theorem 3. Otherwise this contradicts Theorem 4.

Case 2. Suppose that F_5 is nonsingular. By Lemma 10 (1), the cones $\sigma_1, \sigma_2, \sigma_3$ cover more than a half-plane. Similarly, $\sigma_4, \sigma_5, \sigma_6$ cover more than a half-plane. This is a contradiction.

Hence, the other nonsingular edge is either F_1 or F_3. Therefore, the assertion holds for $d = 6$.

We prove the assertion for $d \geqslant 7$. We use induction on d. If there exists i such that F_i and F_{i+1} are nonsingular, then the remaining edges are all singular by Lemma 11. Hence, we may assume that if F_i is nonsingular, then both F_{i-1} and F_{i+1} are singular. To deduce a contradiction we assume that there are i and j with $1 \leqslant i < j \leqslant d$ and $j \geqslant i+2$ such that F_i and F_j are nonsingular. By Lemma 10 (1), we must have

$$(i, j) \in \{(1,3), (2,4), \dots, (d-2, d), (1, d-1), (2, d)\}.$$

We may further assume that $(i, j) = (2, 4)$. Then $\det(v_4, v_1) \geqslant 2$ by Lemma 10 (1). We consider the LDP-polygon $\mathrm{conv}\{v_1, \dots, v_{d-1}\}$. We have $\det(v_1, v_2) \neq 1$ and $\det(v_3, v_4) \neq 1$ while $\det(v_2, v_3) = \det(v_4, v_5) = 1$, which contradicts the hypothesis of induction. Hence, $|\mathrm{Sing}(Q)| = d - 1$. Therefore, the assertion holds for d. This completes the proof. □

Finally, we give an upper bound for the number of nonsingular edges of an LDP-polygon.

Corollary 17. *Let Q be an LDP-polygon. If $|\mathrm{Sing}(Q)| \geqslant 2$, then Q has at most three nonsingular edges.*

Proof. By Proposition 16, we may assume that there exists an integer n with $0 \leqslant n \leqslant d-2$ such that F_i is nonsingular for $1 \leqslant i \leqslant n$ and F_i is singular for $n+1 \leqslant i \leqslant d$. To deduce a contradiction we assume that $n \geqslant 4$. We may further assume that $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We consider the LDP-polygon $Q' = \operatorname{conv}\{v_1, \ldots, v_n, v_{n+1}, v_d\}$. Then Q' has more than five vertices. Since $y_5, y_6, \ldots, y_{n+1}, y_d \leqslant -1$, we have $\det(v_{n+1}, v_d) \geqslant 1$. If $\det(v_{n+1}, v_d) = 1$, then Q' has exactly one singular edge, which contradicts Theorem 2. Otherwise Q' has exactly two singular edges, which contradicts Theorem 3. Hence, we must have $n \leqslant 3$. This completes the proof. ☐

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