On prime-valent symmetric Cayley graphs of finite simple groups

Jing Jian Li*

College of Mathematics and Information Science
Guangxi University
Nanning, P. R. China

lijjhx@163.com

Jicheng Ma†

Chongqing Key Lab. of Group & Graph Theories and Applications
Chongqing University of Arts and Sciences
Chongqing, P. R. China

ma_jicheng@hotmail.com

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Abstract

We give a characterization of the automorphism groups of connected prime-valent symmetric Cayley graphs on finite (non-abelian) simple groups.

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1 Introduction

Throughout this paper, all graphs are assumed to be finite, simple and undirected. For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ its vertex set, edge set, arc set and (full) automorphism group, respectively. A graph Γ is said to be *X-arc-transitive* or *X-symmetric* if $X \leqslant \mathrm{Aut}(\Gamma)$ acts transitively on $A(\Gamma)$. Especially, when $X = \mathrm{Aut}(\Gamma)$, an X-arc-transitive (or X-symmetric) graph is simply called an *arc-transitive* (or *symmetric*) graph.

Let G be a group and an inverse-closed subset S of $G\setminus\{1\}$. A *Cayley graph* $\mathrm{Cay}(G, S)$ of G with *connection set* S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Clearly, $\mathrm{Cay}(G, S)$ has valency $|S|$, and it is connected if and only if $\langle S\rangle = G$.

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†The corresponding author.

Moreover, each $g \in G$ induces an automorphism of $\mathrm{Cay}(G, S)$ by right multiplication on vertices, and so G can be regarded as a regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. In this way, if G is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$, then $\mathrm{Cay}(G, S)$ is called a *normal* Cayley graph, otherwise it is called a *non-normal* Cayley graph.

Define

$$\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}.$$

Then it is easy to see that $G{:}\mathrm{Aut}(G, S) \leqslant \mathrm{Aut}(\mathrm{Cay}(G, S))$. In fact, $G{:}\mathrm{Aut}(G, S)$ is the normalizer of G in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ (see for example [10, 24]). Thus normal Cayley graphs are precisely those $\mathrm{Cay}(G, S)$ with $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G{:}\mathrm{Aut}(G, S)$. Hence, the normality is crucial in determining the full automorphism group of a Cayley graph.

The normality of Cayley graphs of finite non-abelian simple groups has received considerable attention [5, 6, 7, 9, 17, 25, 26]. In this paper, we focus on symmetric Cayley graphs of prime valency on non-abelian simple groups. This work is motivated by the study of the case when the graph is cubic or pentavalent started by Li [17] and Fang et al. [7], respectively. In 1996, Li [17] listed all possible finite non-abelian simple groups on which a connected cubic symmetric Cayley graph might be non-normal. Li's list was later made explicit by Xu, Fang, Wang and Xu [25], who showed that there exists a connected cubic symmetric non-normal Cayley graph on a finite non-abelian simple group G if and only if $G = \mathrm{A}_{47}$. For connected pentavalent symmetric non-normal Cayley graphs on finite non-abelian groups, Fang, Ma and Wang first gave a characterization in 2011 [7]. Then recently Du, Feng and Zhou [5] obtained a list of all possible such non-abelian simple groups. To extend the above results to symmetric Cayley graphs of prime valency p on finite simple groups, we deal with the case when prime $p \geqslant 7$. Note that, if the regular simple group is abelian, say $\mathbf{Z}_q$ with prime q. Then as each symmetric Cayley graph of prime valency is of even order, thus $q = 2$, which implies that $p = 1$, a contradiction. Hence, one can only consider the non-abelian simple groups. Our main theorem in the following is a characterization of those possible non-normal ones. For undefined terms, see Section 2.

Theorem 1. *Let G be a finite non-abelian simple group, let $\Gamma = \mathrm{Cay}(G, S)$ be a connected p-valent symmetric Cayley graph on G with prime $p \geqslant 7$. Then, for $\alpha \in V(\Gamma)$, we have either* $\mathrm{Aut}(\Gamma) = G \rtimes \mathrm{Aut}(G, S)$ *or one of the following holds:*

(a) $\mathrm{Aut}(\Gamma)$ *is an almost simple group with socle $L > G$, and L is either a classical simple group or (L, G, L_α) lies in Table* 1*; or*

(b) $\mathrm{Aut}(\Gamma)$ *has an intransitive non-trivial normal subgroup K such that* $\mathrm{Aut}(\Gamma)/K$ *is almost simple with socle $\overline{L} \geqslant GK/K \cong G$. Moreover, we have $\overline{L}$ is either a classical simple group or $(\overline{L}, G, \overline{L}_{\overline{\alpha}})$ lies in Table* 2*, where $\overline{\alpha}$ is a vertex of the quotient graph Γ_K; or* $(\mathrm{Aut}(\Gamma), G, \mathrm{Aut}(\Gamma)_\alpha)$ *lies in Table* 3.

Remark 2. For line 1 of Table 1, we shall see in Example 3 that there exists a connected non-normal symmetric Cayley graph on M_{22} of valency 23. For line 2 in Table 1, it is shown in [7, Theorem 1.3] that there exists a connected non-normal symmetric Cayley graph on A_{p-1} of valency p for each prime $p \geqslant 7$.

Table 1:

	L	G	L_α	remark
1	M_{23}	M_{22}	C_{23}	$p = 23$
2	A_n	A_{n-1}	$[n]$	p divides n
3	A_{p+1}	$[p+1]$	A_p	$\Gamma = \mathsf{K}_{p+1}$
4	A_{p+3}	$\mathrm{PSL}(2,q)$	S_p	$p = q-2$ for q odd

Table 2:

	$\overline{L}$	G	$\overline{L}_{\overline{\alpha}}$	K	remark
1	A_n	A_{n-1}	has a subgroup of index n		p divides n
2	A_p	A_{p-2}	$\mathrm{PGL}(d,q).\langle\sigma\rangle$		$p = (q^d-1)/(q-1)$, σ divides f
3	A_p	A_{p-3}	$\mathrm{PGL}(2,q).\langle\sigma\rangle$		$p = q+1$, σ divides f
4	A_p	A_{p-3}	$\mathrm{AGL}(d,2)$		$p = 2^d-1$ for d odd
5	A_{p+1}	has a subgroup of index $p+1$	A_p		$\Gamma_K = \mathsf{K}_{p+1}$
6	A_{p+3}	$\mathrm{PSL}(2,p+2)$	S_p		$p \equiv 1 \pmod 4$
7	A_{p+k}	$\mathrm{PSL}(d,q)$	A_p or S_p		$\frac{q^d-1}{q-1} = p+k$, $k = 2$ or 3
8	A_{23} A_{24}	M_{23} M_{24}	A_{19} S_{19}	$[48]$ $[96]$	

Table 3:

	$\mathrm{Aut}(\Gamma)$	G	$\mathrm{Aut}(\Gamma)_\alpha$	Γ_K
1	$\mathrm{PSL}(2,11) \times \mathrm{M}_{12}$	M_{11}	M_{11}	K_{12}
2	$(\mathrm{C}_{11}{:}\mathrm{C}_5) \times \mathrm{M}_{12}$	$\mathrm{PSL}(2,11)$	M_{11}	K_{12}
3	$\mathrm{C}_5 \times \mathrm{M}_{12}$	A_5	M_{11}	K_{12}
4	$\mathrm{C}_{11} \times \mathrm{M}_{23}$	M_{22}	$\mathrm{C}_{23}{:}\mathrm{C}_{11}$	
5	$(\mathrm{C}_{23}{:}\mathrm{C}_{11}) \times \mathrm{M}_{24}$	$\mathrm{PSL}(2,23)$	M_{23}	K_{24}
6	$(\mathrm{C}_7{:}\mathrm{C}_3) \times \mathrm{AGL}(3,2)$	$\mathrm{PSL}(2,7)$	$\mathrm{SL}(3,2)$	K_8

2 Preliminaries

Let G be a finite group, denote by $\pi(G)$ the set of prime divisors of $|G|$, by $M(G)$ the Schur multiplier of G, and by $\mathrm{Soc}(G)$ the socle (that is, the product of all the minimal normal subgroups) of G. We say G is *almost simple* if $\mathrm{Soc}(G)$ is non-abelian simple. Let n be a positive integer, denote by $[n]$ an (unspecified) group of order n, by F_n a Frobenius group of order n, by D_{2n} the dihedral group of order $2n$, and by K_n the complete graph of order n. For a prime number r, let n_r be the largest power of r dividing n, let $n_{r'} = n/n_r$, and let $\mathbf{O}_r(G)$ be the largest normal r-subgroup of G.

Given a group X, let H be a core-free subgroup of (X of) finite index. Take g of $X \setminus H$ such that $g^2 \in H$, define a *coset graph* $\Gamma(X, H, g)$ to be the graph with the set of right cosets of H in X as vertex set, and join two vertices Hx and Hy an edge whenever $xy^{-1} \in HgH$. It is easy to see that $\Gamma(X, H, g)$ has valency $|H : H \cap g^{-1}Hg|$, and it is connected if and only if $\langle H, g\rangle = X$. Moreover, X acts on the right cosets by multiplication induces an arc-transitive subgroup of the automorphism group of $\Gamma(X, H, g)$.

Example 3. Let $X \cong \mathrm{M}_{23}$, $N \cong \mathrm{C}_{23}{:}\mathrm{C}_{11}$ be a maximal subgroup of X (see [4]), $H \cong \mathrm{C}_{23}$ be a normal subgroup of N and g be an involution of X. As N is the only maximal subgroup of X up to conjugation of order divisible by 23, it follows that $\langle H, g\rangle = X$ and $N = \mathbf{N}_X(H)$. Consequently, $g \notin \mathbf{N}_X(H)$ and so $H \cap g^{-1}Hg = 1$. Thus $\Gamma(X, H, g)$ is a connected X-symmetric graph of valency 23. Moreover, X has a subgroup $G \cong \mathrm{M}_{22}$. Since $|G||H| = |X|$ and $\gcd(|G|, |H|) = 1$, we see that G acts regularly by right multiplication. Hence $\Gamma(X, H, g)$ is a Cayley graph on G. As G is not normal in X, this is a non-normal Cayley graph on $G = \mathrm{M}_{22}$.

The following result is well-known (see for example [18, Theorem 1.1]).

Lemma 4. *Let X be a transitive permutation group of prime degree p. Then one of the following holds:*

(a) $\mathrm{C}_p \leqslant X \leqslant \mathrm{AGL}(1, p)$*;*

(b) $X = \mathrm{A}_p$ *or* S_p *with* $p \geqslant 5$*;*

(c) $\mathrm{PGL}(d, q) \leqslant X \leqslant \mathrm{P\Gamma L}(d, q)$ *and* $p = (q^d - 1)/(q - 1)$*, where* $d \geqslant 2$ *and* q *is a prime power;*

(d) $(X, p) = (\mathrm{PSL}(2, 11), 11), (\mathrm{M}_{11}, 11)$ *or* $(\mathrm{M}_{23}, 23)$.

A permutation group X on a set Ω is said to be *quasiprimitive* if its non-trivial normal subgroups are all transitive on Ω. For a graph Γ and a subgroup K of $\mathrm{Aut}(\Gamma)$, the *quotient graph* Γ_K of Γ by K is defined to be the graph with vertices the K-orbits on $V(\Gamma)$ such that two vertices $\overline{\alpha}$ and $\overline{\beta}$ of Γ_K are adjacent if and only if there exist $\alpha \in \overline{\alpha}$ and $\beta \in \overline{\beta}$ adjacent in Γ.

Proposition 5. ([9, Theorem 1.1]) *Let G be a finite non-abelian simple group, $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph on G, and M be a subgroup of $\mathrm{Aut}(\Gamma)$ containing $G{:}\mathrm{Aut}(G, S)$. Then either $M = G{:}\mathrm{Aut}(G, S)$ or one of the following holds:*

(a) *M is almost simple, and $\mathrm{Soc}(M) > G$ is transitive on $V(\Gamma)$;*

(b) *$G.\mathrm{Inn}(G) \leqslant M = (G{:}\mathrm{Aut}(G, S)).\mathrm{C}_2$ and S is a self-inverse union of G-conjugacy classes;*

(c) *M is not quasiprimitive and there is a maximal intransitive normal subgroup K of M such that one of the following holds:*

  (c.1) *M/K is almost simple, and $\mathrm{Soc}(M/K) \geqslant GK/K \cong G$ is transitive on $V(\Gamma_K)$;*

  (c.2) *$M/K = \mathrm{AGL}(3, 2)$, $G = \mathrm{PSL}(2, 7)$, and $\Gamma_K = \mathsf{K}_8$;*

  (c.3) *$\mathrm{Soc}(M/K) \cong T \times T$, and $GK/K \cong G$ is a diagonal subgroup of $\mathrm{Soc}(M/K)$, where T and G are given in Table 4.*

Table 4: Product action possibilities

	G	T	m	$\lvert V(\Gamma_K)\rvert$
1	A_6	G	6	m^2
2	M_{12}	G or A_m	12	m^2
3	$\mathrm{Sp}_4(q)(q = 2^a)$	G or A_m or $\mathrm{Sp_{4r}(q_0)(q = q_0^r)}$	$\frac{q^2(q^2-1)}{2}$	m^2
4		$\mathrm{Sp_{4r}(q_0)(q = q_0^r)}$	$\frac{q^2(q^2-1)}{2}$	$2m^2$
5	$\mathrm{P\Omega_8^+(q)}$	G or A_m or $\mathrm{Sp}_8(2)$ (if $q = 2$)	$\frac{q^3(q^4-1)}{(2,q-1)}$	m^2

Let Γ be a graph, $X \leqslant \mathrm{Aut}(\Gamma)$ and $\{\alpha, \beta\} \in E(\Gamma)$, let $\Gamma(\alpha)$ denote the neighborhood of α. Let $X_\alpha^{[1]}$ be the kernel of the vertex-stabilizer X_α acting on $\Gamma(\alpha)$, and let $X_{\alpha\beta}^{[1]} = X_\alpha^{[1]} \cap X_\beta^{[1]}$. For a positive integer s, an $(s+1)$-sequence $(\alpha_0, \alpha_1, \cdots, \alpha_s)$ of vertices of Γ is called an *s-arc* if $\{\alpha_{i-1}, \alpha_i\} \in E(\Gamma)$ for $i = 1, \ldots, s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, \ldots, s-1$. The graph Γ is said to be (X, s)-arc-transitive if X acts transitively on the set of s-arcs of Γ, and is said to be (X, s)-transitive if it is (X, s)-arc-transitive but not $(X, s+1)$-arc-transitive.

Proposition 6. ([13, Theorem 1.1]) *Let Γ be a connected X-symmetric graph of valency 7. Then for $\alpha \in V(\Gamma)$, X_α lies in Table* 5.

The next proposition follows from [12] and [20].

Proposition 7. *Let Γ be a connected (X, s)-transitive graph of prime valency $p > 7$ and let $\{\alpha, \beta\}$ be an edge of Γ. If X_α is solvable, then $X_\alpha \cong (\mathrm{C}_p{:}\mathrm{C}_m) \times \mathrm{C}_n$ for some m dividing $(p-1)$ and n dividing m. If X_α is nonsolvable, then $|X_\alpha|_p = p$, and either (s, p, X_α) lies in Table* 6*, or one of the following statements* (a)–(c) *holds, where $d \geqslant 2$ is an integer and $q = r^f$ for some prime r and positive integer f such that $p = (q^d - 1)/(q - 1)$.*

Table 5:

$\lvert X_\alpha\rvert_2$	X_α
1	C_7, F_{21}, $\mathrm{F}_{21}\times\mathrm{C}_3$
2	D_{14}, F_{42}, $\mathrm{F}_{42}\times\mathrm{C}_3$
2^2	D_{28}, $\mathrm{F}_{42}\times\mathrm{C}_2$, $\mathrm{F}_{42}\times\mathrm{C}_6$
2^3	$\mathrm{SL}(3,2)$, A_7
2^4	S_7
2^6	$\mathrm{C}_2^3{:}\mathrm{SL}(3,2)$, $\mathrm{SL}(3,2)\times\mathrm{S}_4$, $\mathrm{A}_7\times\mathrm{A}_6$
2^7	$\mathrm{C}_2^4{:}\mathrm{SL}(3,2)$, $(\mathrm{A}_7\times\mathrm{A}_6){:}\mathrm{C}_2$
2^8	$\mathrm{S}_6\times\mathrm{S}_7$
2^{10}	$\mathrm{C}_2^6{:}(\mathrm{SL}(3,2)\times\mathrm{S}_3)$
2^{24}	$[2^{20}]{:}(\mathrm{SL}(3,2)\times\mathrm{S}_3)$

Table 6:

s	p	X_α
2	p	A_p, S_p
2	11	$\mathrm{PSL}(2,11)$, M_{11}
2	23	M_{23}
3	p	$\mathrm{A}_{p-1}\times\mathrm{A}_p$, $(\mathrm{A}_{p-1}\times\mathrm{A}_p){:}\mathrm{C}_2$, $\mathrm{S}_{p-1}\times\mathrm{S}_p$
3	11	$\mathrm{A}_5\times\mathrm{PSL}(2,11)$, $\mathrm{A}_6\times\mathrm{M}_{11}$, $\mathrm{M}_{10}\times\mathrm{M}_{11}$
3	23	$\mathrm{M}_{22}\times\mathrm{M}_{23}$

(a) $s=2$ *and one of the following holds:*

- (a.1) $d=2$, $r=2$, $\mathrm{PSL}(2,q)\leqslant X_\alpha\leqslant\mathrm{P\Gamma L}(2,q)$ *and* $X_{\alpha\beta}^{[1]}=1$*;*
- (a.2) $d\geqslant 3$, $X_\alpha=((\mathrm{C}_r^{f(d-1)}{:}\mathrm{C}_\ell)\times\mathrm{PSL}(d,q)).\mathcal{O}$ *and* $X_{\alpha\beta}^{[1]}=1$*, where* $\mathcal{O}\leqslant\mathrm{C}_f$ *and* $\mathrm{C}_\ell\leqslant\mathrm{C}_{q-1}$*;*
- (a.3) $d\geqslant 3$, $X_\alpha=\mathbf{O}_r(X_\alpha).\mathrm{C}_\ell.\mathrm{PSL}(d,q).\mathcal{O}$ *and* $X_{\alpha\beta}^{[1]}\neq 1$*, where* $\mathcal{O}\leqslant\mathrm{C}_f$ *and* $\mathrm{C}_\ell\leqslant\mathrm{C}_{q-1}$.

(b) $s=3$ *and one of the following holds:*

- (b.1) $d=2$, $r=2$, $X_\alpha=((\mathrm{C}_2^f.\mathcal{O}_1)\times\mathrm{PSL}(2,q)).\mathcal{O}$ *and* $X_{\alpha\beta}^{[1]}=1$*, where* $\mathcal{O}\leqslant\mathrm{C}_f$ *and* $\mathcal{O}_1\leqslant\mathrm{C}_{q-1}.\mathcal{O}$*;*
- (b.2) $d\geqslant 3$, $X_\alpha=((\mathrm{C}_r^{f(d-1)}{:}\mathrm{C}_\ell.\mathrm{PSL}(d-1,q).\mathcal{O}')\times\mathrm{PSL}(d,q)).\mathcal{O}$ *and* $X_{\alpha\beta}^{[1]}=1$*, where* $\mathcal{O}\leqslant\mathrm{C}_f$, $\mathrm{C}_\ell\leqslant\mathrm{C}_{q-1}$ *and* $\mathcal{O}'\leqslant\mathrm{C}_{\gcd(d-1,q-1)}.\mathcal{O}$*;*

(b.3) $d \geqslant 3$, $X_\alpha = \mathbf{O}_r(X_\alpha).\mathrm{C}_\ell.((\mathrm{PSL}(d-1,q).\mathcal{O}') \times \mathrm{PSL}(d,q)).\mathcal{O}$ *and* $X_{\alpha\beta}^{[1]} \neq 1$, *where* $\mathcal{O} \leqslant \mathrm{C}_f$, $\mathrm{C}_\ell \leqslant \mathrm{C}_{q-1}$ *and* $\mathcal{O}' \leqslant \mathrm{C}_{\gcd(d-1,q-1)}.\mathcal{O}$; *moreover, if* $r \geqslant 5$ *then* $|\mathbf{O}_r(X_\alpha)|$ *divides* $q^{d(d-1)}$.

(c) $s = 5$, $d = 2$, $r = 2$, $X_\alpha = ([q^3]{:}\mathrm{GL}(2,q)).\mathcal{O}$ *and* $X_{\alpha\beta}^{[1]} = 1$, *where* $\mathcal{O} \leqslant \mathrm{C}_f$.

Recall that a permutation group is called k-homogeneous if it is transitive on the k-sets of permuted points. The following result is about the k-homogeneous groups which can be get from [15, Theorem 1].

Lemma 8. *Let G be a group k-homogeneous but not k-transitive on a finite set Ω of n points, where $n \geqslant 2k$. Then, up to permutation isomorphism, one of the following holds:*

(a) $k = 2$ *and* $G \leqslant \mathrm{A\Gamma L}(1,q)$ *with* $n = q \equiv 3 \mod 4$;

(b) $k = 3$ *and* $\mathrm{PSL}(2,q) \leqslant G \leqslant \mathrm{P\Gamma L}(2,\mathrm{q})$, *where* $n - 1 = q \equiv 3 \mod 4$;

(c) $k = 3$ *and* $G = \mathrm{AGL}(1,8)$, $\mathrm{A\Gamma L}(1,8)$ *or* $\mathrm{A\Gamma L}(1,32)$;

(d) $k = 4$ *and* $G = \mathrm{PSL}(2,8)$, $\mathrm{P\Gamma L}(2,8)$ *or* $\mathrm{P\Gamma L}(2,32)$.

3 Proof of the main result

In the following section, we give the proof of our main theorem.

Lemma 9. *Let X be a permutation group on a set Ω, let G be a transitive subgroup of X. Let $\alpha \in \Omega$, suppose that both X and G are non-abelian simple and X_α is as described in Proposition 6 or 7. Then either X is a classical simple group or (X, G, X_α) lies in Table 7.*

Proof. From Propositions 6 and 7 we see that there exists a prime $p \geqslant 7$ such that $|X_\alpha|_p = p$. As G is transitive, we have $X = GX_\alpha$. Suppose that X is not a classical simple group. Then X is an alternating group or a simple group of exceptional Lie type or a sporadic simple group.

First assume that X is a simple group of exceptional Lie type. Since $X = GX_\alpha$ with G non-abelian simple, it follows from [14, Theorem 1] that (X, G, X_α) lies in Table 8. In line 1 of Table 8, X_α has a composition factor $\mathrm{PSU}(3,4)$, which is not as described in Proposition 6 or 7, a contradiction. Similarly one may exclude lines 2 and 6–8 of Table 8. For the line 4 or 5, X_α has a composition factor $\mathrm{PSL}(3,q)$ with q a 3-power, and has no non-trivial solvable normal subgroup. It can be seen that only cases (a.2)–(a.3) and (b.2)–(b.3) of Proposition 7 satisfy that X_α has a composition factor $\mathrm{PSL}(3,q)$. However, in those cases X_α has a non-trivial solvable normal subgroup, a contradiction. Similarly one may exclude line 3 of Table 8. Hence, none of the triples (X, G, X_α) in Table 8 happens.

Next, assume that X is a sporadic simple group. By [11, Theorem 1.1], we know that (X, G, X_α) lies in Table 9. As X_α is described in Proposition 6 or 7, thus (X, G, X_α)

Table 7:

	X	G	X_α	conditions
1	A_n	A_{n-1}	transitive permutation group of degree n	$n \geqslant 6$
2	A_p	A_{p-2}	$PGL(d,q).\langle\sigma\rangle$	$p = \frac{q^d-1}{q-1}$, $\sigma \mid f$
3	A_p	A_{p-3}	$PGL(2,q).\langle\sigma\rangle$ $AGL(d,2)$	$p = q+1$, $\sigma \mid f$ $p = 2^d - 1$, d odd
4	A_{11}	A_9 A_7	$PSL(2,11)$ M_{11}	$p = 11$ $p = 11$
5	A_{23}	A_{19}	M_{23}	$p = 23$
6	A_{p+1}	transitive permutation group of degree $p+1$	A_p	p prime
7	A_{p+3}	$PSL(2,p+2)$	S_p	$p \equiv 1 \pmod 4$
8	A_{11} A_{12}	M_{11} M_{12}	A_7 or S_7	$p = 7$
9	A_{23} A_{24}	M_{23} M_{24}	A_{19} or S_{19}	$p = 19$
10	A_{p+k}	$PSL(d,q)$	A_p or S_p	$\frac{q^d-1}{q-1} = p+k$, $k = 2$ or 3
11	A_8	A_5 A_k	$AGL(3,2)$ $SL(3,2)$, $AGL(3,2)$	$p = 7$ $p = 7$, $k \in \{6,7\}$
12	M_{12}	M_{11}	M_{11}, $PSL(2,11)$	$p = 11$
13	M_{12}	$PSL(2,11)$	M_{11}	$p = 11$
14	M_{12}	A_5	M_{11}	$p = 11$
15	M_{23}	M_{22}	C_{23}, $C_{23}{:}C_{11}$	$p = 23$
16	M_{24}	M_{23}	$SL(3,2)$, $C_2^6{:}(SL(3,2) \times S_3)$	$p = 7$
17	M_{24}	$PSL(2,23)$	M_{23}	$p = 23$

Table 8:

	X	G	X_α
1	$\mathrm{G}_2(4)$	J_2	$\mathrm{PSU}(3,4)$, $\mathrm{PSU}(3,4).\mathrm{C}_2$
2	$\mathrm{G}_2(4)$	$\mathrm{PSU}(3,4)$	J_2
3	$\mathrm{G}_2(3^f)$	$\mathrm{PSL}(3,3^f)$	$\mathrm{PSU}(3,3^f)$, $\mathrm{PSU}(3,3^f).\mathrm{C}_2$
4	$\mathrm{G}_2(3^f)$	$\mathrm{PSU}(3,3^f)$	$\mathrm{PSL}(3,3^f)$, $\mathrm{PSL}(3,3^f).\mathrm{C}_2$
5	$\mathrm{G}_2(3^{2e+1})$	${}^2G_2(3^{2e+1})$	$\mathrm{PSL}(3,3^{2e+1})$, $\mathrm{PSL}(3,3^{2e+1}).\mathrm{C}_2$
6	$\mathrm{G}_2(3^{2e+1})$	$\mathrm{PSL}(3,3^{2e+1})$	${}^2G_2(3^{2e+1})$
7	$\mathrm{F}_4(2^f)$	$\mathrm{Sp}(8,2^f)$	${}^3\mathrm{D}_4(2^f)$, ${}^3\mathrm{D}_4(2^f).\mathrm{C}_3$
8	$\mathrm{F}_4(2^f)$	${}^3\mathrm{D}_4(2^f)$	$\mathrm{Sp}(8,2^f)$

Table 9:

	X	G	X_α
1	M_{12}	M_{11}	M_{11}, $\mathrm{PSL}(2,11)$
2	M_{12}	$\mathrm{PSL}(2,11)$	M_{11}
3	M_{12}	A_5	M_{11}
4	M_{23}	M_{22}	C_{23}, $\mathrm{C}_{23}{:}\mathrm{C}_{11}$
5	M_{24}	M_{23}	$\mathrm{M}_{12}.\mathrm{C}_2$, $\mathrm{C}_2^3{:}\mathrm{F}_{21}$, $\mathrm{C}_2^6{:}\mathrm{C}_{21}$, $\mathrm{C}_2^6{:}\mathrm{F}_{21}$, $\mathrm{C}_2^6{:}\mathrm{C}_7{:}\mathrm{S}_3$, $\mathrm{C}_2^6{:}(\mathrm{F}_{21}\times\mathrm{C}_3)$, $\mathrm{C}_2^6{:}(\mathrm{F}_{21}\times\mathrm{S}_3)$, $\mathrm{C}_2^6{:}(\mathrm{SL}(3,2)\times\mathrm{C}_3)$, $\mathrm{C}_2^6{:}(\mathrm{SL}(3,2)\times\mathrm{S}_3)$, $\mathrm{SL}(3,2)$, $\mathrm{SL}(3,2)\times\mathrm{C}_3$, $\mathrm{SL}(3,2)\times\mathrm{S}_3$, $\mathrm{PGL}(2,11)$, $\mathrm{PSL}(2,23)$
6	M_{24}	$\mathrm{PSL}(2,23)$	$\mathrm{P\Sigma L}(3,4)$, $\mathrm{PSL}(3,4).\mathrm{S}_3$, $\mathrm{C}_2^4{:}\mathrm{A}_7$, $\mathrm{C}_2^4{:}\mathrm{A}_8$, $\mathrm{M}_{22}.\mathrm{C}_2$, M_{22}, M_{23}
7	M_{24}	$\mathrm{PSL}(2,7)$	M_{23}
8	HS	M_{22}	$\mathrm{PSU}(3,5).\mathrm{C}_2$
9	Ru	$\mathrm{PSL}(2,29)$	${}^2\mathrm{F}_4(2)$
10	Suz	$\mathrm{G}_2(4)$	$\mathrm{PSU}(5,2)$, $\mathrm{C}_3^5{:}\mathrm{M}_{11}$
11	Suz	$\mathrm{PSU}(5,2)$	$\mathrm{G}_2(4)$
12	Fi_{22}	${}^2\mathrm{F}_4(2)'$	$\mathrm{C}_2.\mathrm{PSU}(6,2)$
13	Co_1	Co_2	$(\mathrm{C}_3.\mathrm{Suz}).\mathrm{C}_2$, $\mathrm{C}_3.\mathrm{Suz}$
14	Co_1	Co_2	$\mathrm{G}_2(4)\leqslant X_\alpha\leqslant(\mathrm{A}_4\times\mathrm{G}_2(4)).\mathrm{C}_2$
15	Co_1	$\mathrm{G}_2(4)$	Co_2
16	Co_1	Co_3	$(\mathrm{C}_3.\mathrm{Suz}).\mathrm{C}_2$, $\mathrm{C}_3.\mathrm{Suz}$
17	Co_1	Co_3	$\mathrm{G}_2(4).\mathrm{C}_2\leqslant X_\alpha\leqslant(\mathrm{A}_4\times\mathrm{G}_2(4)).\mathrm{C}_2$

cannot be lines 8–17 of Table 9. If (X, G, X_α) lies in lines 1–4 of Table 9, then one of lines 12–15 of Table 7 holds. If (X, G, X_α) lies in line 5–6 of Table 9, then $p = 7$, 11 or 23. Furthermore, by Proposition 6 and 7, we have lines 16–17 of Table 7 hold. If (X, G, X_α) lies in line 7 of Table 9, then $(X, G, X_\alpha) \cong (\mathrm{M}_{24}, \mathrm{PSL}(2, 27), \mathrm{M}_{23})$. Note that $|G \cap X_\alpha| = \frac{|\mathrm{PSL}(2,7)||\mathrm{M}_{23}|}{|\mathrm{M}_{24}|} = 7$. It follows that $X_\alpha \cong \mathrm{M}_{23}$, which has no subgroup of index 7, a contradiction. Hence this case cannot happen.

Finally, let X be the alternating group A_n naturally acts on a set Θ of n points with $n \geqslant 5$. Again, as $X = GX_\alpha$, we derive from [22, Theorem D and Remark 2] (which gave the maximal factorizations of the alternating groups) that one of the following holds:

(i) $G = \mathrm{A}_{n-k}$ for some $1 \leqslant k \leqslant 5$ and X_α is k-homogenous on Θ;

(ii) G is k-homogenous on Θ and $\mathrm{A}_{n-k} \trianglelefteq X_\alpha \leqslant (\mathrm{S}_{n-k} \times \mathrm{S}_k) \cap \mathrm{A}_n$ for some $1 \leqslant k \leqslant 5$;

(iii) $n = 6$, $G = \mathrm{PSL}(2, 5)$, $X_\alpha \leqslant \mathrm{S}_3 \wr \mathrm{S}_2$ and X_α is transitive on Θ;

(iv) $n = 10$, $G = \mathrm{PSL}(2, 8)$, $\mathrm{A}_5 \times \mathrm{A}_5 \trianglelefteq X_\alpha \leqslant \mathrm{S}_5 \wr \mathrm{S}_2$ and X_α is transitive on Θ.

To finish the proof, in the following, we analyze the above four cases (i)–(iv) one by one.

Case (i). Suppose that $G = \mathrm{A}_{n-k}$ for some $1 \leqslant k \leqslant 5$ and X_α is k-homogenous on Θ. If $k = 1$, then $n \geqslant 6$ and X_α is transitive on Θ, as in line 1 of Table 7. Henceforth assume $k \geqslant 2$. Since $G = \mathrm{A}_{n-k}$ is non-abelian simple group, $n - k \geqslant 5$, i.e., $n \geqslant 5 + k$. Note that, if $1 \leqslant k \leqslant 5$, then $n \geqslant 2k$.

Assume that X_α is k-homogeneous but not k-transitive. Then X_α is one of the four cases in Lemma 8, and especially we have $k \leqslant 4$. In the following, we will analyze these four cases one by one. Note that we have $|X : G| = |\mathrm{A}_n : \mathrm{A}_{n-k}| = n(n-1)\cdots(n-k+1)$ and $|X : G| \bigm| |X_\alpha|$.

Let $q = r^f$ for some prime r and positive integer f. If $k = 2$, then $X_\alpha \leqslant \mathrm{A\Gamma L}(1, q)$ with $n = q \equiv 3 \mod 4$. Note that $G \cong \mathrm{A}_{q-2}$, $|X{:}G| \bigm| |X_\alpha|$ with $|X{:}G| = q(q-1)$ and $X_\alpha \leqslant A\Gamma\mathrm{L}(1, q) \cong \mathrm{C}_r^f{:}(\mathrm{C}_{q-1}{:}\mathrm{C}_f)$ for $q = r^f$. It follows that $X_\alpha \cong \mathrm{C}_r^f{:}(\mathrm{C}_{q-1}{:}\mathrm{C}_\ell)$ for $\ell \bigm| f$, and so X_α is 2-transitive on Ω, a contradiction. Suppose that $k = 3$. Then $G \cong \mathrm{A}_{n-3}$ for $n \geqslant 8$, and $|X{:}G| = n(n-1)(n-2)$ is a factor of $|X_\alpha|$. On the other hand, Lemma 8 shows that either $\mathrm{PSL}(2, q) \leqslant X_\alpha \leqslant \mathrm{P\Gamma L}(2, q)$ with $n - 1 = q \equiv 3 \mod 4$, or $X_\alpha = \mathrm{AGL}(1, 8)$, $\mathrm{A\Gamma L}(1, 8)$ or $\mathrm{A\Gamma L}(1, 32)$. For the latter case, a calculation of the order for these candidates of X_α shows that this case cannot occur. Suppose that the former case occurs. Then $n = q + 1$, $n(n-1)(n-2) = (q-1)q(q+1)$ is a factor of $|X_\alpha|$, and $X_\alpha \cong \mathrm{PSL}(2, q).(C_2 \times \mathrm{C}_l)$ for $l \bigm| f$ (see). Upon to Lemma 7, $p = \frac{q^2-1}{q-1} = q + 1$, and so $n = p$. It is clear that X_α is 2-transitive on $\Theta = \{1, \cdots, p\}$, a contradiction. Assume that $k = 4$. Then $G \cong \mathrm{A}_{n-4}$ for $n \geqslant 9$ and $|X{:}X_\alpha| = n(n-1)(n-2)(n-3)$ is a factor of $|X_\alpha|$. In particular, Lemma 8 shows that $X_\alpha \cong \mathrm{PSL}(2, 8)$, $\mathrm{P\Gamma L}(2, 8)$ or $\mathrm{P\Gamma L}(2, 32)$. A calculation of the orders for those candidates of X_α shows that this case cannot occur.

Now we suppose that X_α is k-transitive on Θ for $k \geqslant 2$. Note that $(\mathrm{C}_p{:}\mathrm{C}_{p-1}){:}\mathrm{C}_\ell$ with $n = p \geqslant 7$ prime and $\ell \bigm| (p-1)$, is not isomorphic to a subgroup of A_p as C_{p-1} contains an element of odd permutation. Then since X_α is a k-transitive subgroup of A_n and is also as described in Proposition 6 or 7, one can get that either $\mathrm{PSL}(d, q) \leqslant X_\alpha \leqslant \mathrm{P\Gamma L}(d, q)$

with $n = p = (q^d - 1)/(q - 1) \geqslant 7$ prime for some integer $d \geqslant 2$ and prime power q, or $(X_\alpha, n) = (\mathrm{PSL}(2, 11), 11), (\mathrm{M}_{11}, 11)$ or $(\mathrm{M}_{23}, 23)$. For the latter case, one can deduce that line 4-5 of Table 7 hold. Now assume that the former case occurs. Then since X_α is a k-transitive permutation group for $k \geqslant 2$, by [3, Theorem 4.11], we have $k \leqslant 3$. Again, as X_α is described in Proposition 6 or 7, we deduce that if $k = 2$, then line 2 of Table 7 holds. For $k = 3$, X_α is a 3-transitive permutation group, and so $X_\alpha \cong \mathrm{PGL}(2, q).\langle\sigma\rangle$ for $p = q + 1$ and $\sigma \bigm| f$, or $\mathrm{AGL}(d, 2)$ for $p = 2^d - 1$ (see [3, Table 7.3, 7.4] for example). For $p = 2^d - 1$ is prime, then d is odd. Hence line 3 of Table 7 holds.

Case (ii). Assume that G is k-homogenous on Θ and $\mathrm{A}_{n-k} \trianglelefteq X_\alpha \leqslant (\mathrm{S}_{n-k} \times \mathrm{S}_k) \cap \mathrm{A}_n$ for some $1 \leqslant k \leqslant 5$. Note that X_α is given in Proposition 6 or 7. Then $X_\alpha \cong \mathrm{A}_{n-k}$ or S_{n-k} for $1 \leqslant k \leqslant 5$ and $n - k = p$. If $k = 1$, then $(X, X_\alpha) \cong (\mathrm{A}_{p+1}, \mathrm{A}_p)$ and G is a transitive permutation group of degree $n = p + 1$. Hence the line 6 of Table 7 holds.

For $k \geqslant 2$, assume that G is k-homogeneous but not k-transitive, then G is given in Lemma 8. Let $q = r^f$ for some prime r and positive integer f. Note that $n \geqslant 5 + k$ and $|X : X_\alpha| = n(n-1)\cdots(n-k+1)$ or $\frac{n(n-1)\cdots(n-k+1)}{2}$ respecting to $X_\alpha \cong \mathrm{A}_{n-k}$ or S_{n-k} for $n-k = p$. Since $|X : X_\alpha| \bigm| |G|$ and G is k-homogeneous but not k-transitive, by a careful analysis of the cases (a)–(d) in Lemma 8, we can draw that $k = 3$ and $G \cong \mathrm{PSL}(2, q)$ with $n - 1 = q \equiv 3 \mod 4$. Then $p = n - 3$ and $q = n - 1$ is odd, and so n is even. Therefore, $|G| = |\mathrm{PSL}(2, q)| = \frac{q(q-1)(q+1)}{(2,q-1)} = \frac{n(n-1)(n-2)}{(2,n-2)} = \frac{n(n-1)(n-2)}{2}$. Furthermore, since $|X{:}G| \bigm| |X_\alpha|$, we conclude that $X_\alpha \cong \mathrm{S}_p$. We derive from $q = n - 1$ and $p = n - 3$ that $q = p + 2$. It follows that $p \equiv 1 \mod 4$ as $q \equiv 3 \mod 4$. Hence the line 7 of Table 7 holds.

Now suppose that G is k-transitive on Θ. Note that $X_\alpha \cong \mathrm{A}_{n-k}$ or S_{n-k} with $n-k = p$. Since $G < X \cong \mathrm{A}_n$ is a non-abelian simple group and $k \geqslant 2$, by [3, Theorem 4.11], we conclude that $2 \leqslant k \leqslant 5$, and if $k = 4$ or 5, then $(G, n, k) = (\mathrm{M}_{11}, 11, 4)$, $(\mathrm{M}_{12}, 12, 5)$, $(\mathrm{M}_{23}, 23, 4)$ or $(\mathrm{M}_{24}, 24, 5)$. It follows that $(X, G, p) = (\mathrm{A}_{11}, \mathrm{M}_{11}, 7)$, $(\mathrm{A}_{12}, \mathrm{M}_{12}, 7)$, $(\mathrm{A}_{23}, \mathrm{M}_{23}, 19)$ or $(\mathrm{A}_{24}, \mathrm{M}_{24}, 19)$ respectively, and hence lines 8-9 of Table 7 hold. Now for $k = 2$ or 3. Since G is non-abelian simple, G is given in [3, Table 7.4]. Together with the conditions that $|X : X_\alpha| \bigm| |G|$ and $X_\alpha \cong \mathrm{A}_{n-k}$ or S_{n-k} for $n - k = p$, we can deduce that either $G \cong \mathrm{PSL}(d, q)$ for $n = (q^d - 1)/(q - 1)$, $d \geqslant 2$ and q being a prime power, or $G \cong \mathrm{Sp}(2d, 2)$ for $n = 2^{2d-1} \pm 2^{d-1}$ and $d \geqslant 3$. For the latter case, we derive from $n-k = p \geqslant 7$ is an odd prime that $2^{2d-1} \pm 2^{d-1} - 2 = p$ or $2^{2d-1} \pm 2^{d-1} - 3 = p$. However, $2^{2d-1} \pm 2^{d-1} - 2$ is even, which leads to that $2^{2d-1} \pm 2^{d-1} - 3 = p$. Noting that $2^{2d-1} + 2^{d-1} - 3 = 2^{2d-1} - 2 + 2^{d-1} - 1 = 2((2^{d-1})^2 - 1) + 2^{d-1} - 1 = 2(2^{d-1} - 1)(2^{d-1} + 1) + 2^{d-1} - 1 = (2^{d-1} - 1)(2(2^{d-1} + 1) + 1)$ is not a prime, we conclude that this case cannot occur. Then along the same lines as the previous case, we see that $2^{2d-1} - 2^{d-1} - 3 = (2^{d-1} + 1)(2(2^{d-1} - 1) - 1)$ is also not prime. It yields that $G \ncong \mathrm{Sp}(2d, 2)$ for $n = 2^{2d-1} \pm 2^{d-1}$ and $d \geqslant 3$, and hence line 10 of Table 7 holds.

Cases (iii) and (iv). Suppose that $n = 6$, $G = \mathrm{PSL}(2, 5)$, $X_\alpha \leqslant \mathrm{S}_3 \wr \mathrm{S}_2$ and X_α is transitive on Ω; or $n = 10$, $G = \mathrm{PSL}(2, 8)$, $\mathrm{A}_5 \times \mathrm{A}_5 \trianglelefteq X_\alpha \leqslant \mathrm{S}_5 \wr \mathrm{S}_2$ and X_α is transitive on Ω. Since X_α is given in Proposition 6 or 7, in particular, $|X_\alpha|_p = p \geqslant 7$, we can deduce that those cases cannot occur. □

In the rest of this section, we always let G be a finite non-abelian simple group, let $\Gamma = \mathrm{Cay}(G, S)$ be a connected symmetric Cayley graph on G of prime valency $p \geqslant 7$, and let $L = \mathrm{Soc}(\mathrm{Aut}\Gamma)$ and α be a vertex of Γ. Moreover, for short, let $A = \mathrm{Aut}\Gamma$. If $A = G.\mathrm{Aut}(G, S)$, then Γ is a normal Cayley graph. Now we assume that $A > G.\mathrm{Aut}(G, S)$.

Lemma 10. *Assume that A acts quasiprimitively on $V(\Gamma)$. Then either L is a classical simple group or Γ is isomorphic to one of the lines of Table* 1.

Proof. Since A is quasiprimitive on $V(\Gamma)$, then either (a) or (b) of Proposition 5 occurs.

Case (i). Suppose that (b) holds. Then the action of G on S by conjugation is either trivial or faithful as G is simple. If the action is trivial, then G is abelian as S generates G, a contradiction. Suppose that the action is faithful. Note that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ and $S^{\mathrm{Inn}(G)} = S$. Then $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G, S)$. Since $A = (G.\mathrm{Aut}(G, S)).\mathrm{C}_2$ and $|S|$ is odd prime, then $\mathrm{Aut}(G, S)$ acts transitively on S, and so it is primitive. It follows that $\mathrm{Inn}(G)$ is a transitive permutation group of degree p, so does G as $G \cong \mathrm{Inn}(G)$. Further, by Lemma 4, $G \cong \mathrm{PSL}(2, 11)$ for $p = 11$, M_{11} for $p = 11$ or M_{23} for $p = 23$, A_p, $\mathrm{PSL}(d, q)$ for $p = (q^d - 1)/(q - 1)$, where $d \geqslant 2$ and q is a prime power. On the other hand, since $G.\mathrm{Inn}(G) \leqslant A = (G.\mathrm{Aut}(G, S)).\mathrm{C}_2$, $|A| = |G||A_1|$ and $\mathrm{Aut}(G, S) \leqslant A_1$ for identity $1 \in G$ a vertex of Γ, then $|A_1| = 2|\mathrm{Aut}(G, S)|$ and $A_\alpha \cong A_1 \cong \mathrm{Aut}(G, S).\mathrm{C}_2$.

Assume that $G \cong \mathrm{PSL}(2, 11)$ for $p = 11$. Then $\mathrm{Inn}(G) \cong G \cong \mathrm{PSL}(2, 11)$ and $\mathrm{Aut}(G, S) \cong \mathrm{PSL}(2, 11)$ or $\mathrm{PSL}(2, 11).\mathrm{C}_2$. Since $A_1 \cong \mathrm{Aut}(G, S).\mathrm{C}_2$, by Proposition 7(b), we have $A_1 \cong \mathrm{PSL}(2, 11)$, and so $A_1 = \mathrm{Aut}(G, S)$, a contradiction. A similar argument excludes the case where $G \cong \mathrm{M}_{11}$ or M_{23}. Suppose that $G \cong \mathrm{A}_p$. Then $\mathrm{Inn}(G) \cong G \cong \mathrm{A}_p$. Since $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G, S)$, then $\mathrm{Aut}(G, S) \cong \mathrm{A}_p$ or S_p, and $|A_1| = 2|\mathrm{A}_p|$ or $2|\mathrm{S}_p|$ respectively. Noting $p \geqslant 7$ is prime, by Lemma 7, one can get that $A_1 \cong \mathrm{S}_p$, and then $\mathrm{Aut}(G, S) \cong \mathrm{A}_p$ and $A = (G.\mathrm{A}_p).\mathrm{C}_2 \cong (\mathrm{A}_p \times \mathrm{A}_p).\mathrm{C}_2$. Then $\mathrm{Soc}(A) = G \times \mathrm{A}_p \cong \mathrm{A}_p \times \mathrm{A}_p$. Since $\mathrm{Soc}(A)$ is a characteristic subgroup of A and $G \cong \mathrm{A}_p$ is a normal subgroup of $\mathrm{Soc}(A)$, then $G \trianglelefteq A$. However, it contradicts to the assumption that Γ is not a normal Cayley graph. Then $G \cong \mathrm{PSL}(d, q)$ for $p = (q^d - 1)/(q - 1)$, where $d \geqslant 2$. Along the same lines as the previous case, we can exclude this case.

Case (ii). Now assume that (a) of Lemma 5 holds, that is A is an almost simple group with $G < L$ and L is transitive on $V(\Gamma)$. Note that the valency of Γ is prime p. Then, for $\alpha \in V(\Gamma)$, A_α is primitive on $\Gamma(\alpha)$, so is L_α as $L_\alpha \trianglelefteq A_\alpha$. It implies that Γ is L-locally-primitive. Then $\Gamma = \Gamma(A, A_\alpha, g) \cong \Gamma(L, L_\alpha, t)$. Let $L = HD$ be a maximal factorization of L for $G \leqslant H$ and $L_\alpha \leqslant D$, in particular, $L = GL_\alpha$, $G \cap L_\alpha = 1$ and $|L| = |G||L_\alpha|$. Now we assume that L is not a classical simple group. Then the triples (L, G, L_α) are given in Table 7 of Lemma 9.

Since $|L| = |G||L_\alpha|$, the calculation shows that only lines 1, 4, 7, 9, 10 or 15 of Table 7 of Lemma 9 hold. In the following, we will analyze them one by one. Assume that $L \cong \mathrm{A}_n$ and $G \cong \mathrm{A}_{n-1}$, just as in line 1 of Table 7. Then $|L_\alpha| = n$. It is shown in [7, Theorem 1.3] that there is a connected symmetric non-normal Cayley graph on A_{p-1} of valency p for each prime $p \geqslant 7$. Then line 2 of Table 1 holds.

Now consider the line 4 of Table 7. Since $|L| = |G||L_\alpha|$, a straight forward calculation shows that $(L, G, L_\alpha) \cong (\mathrm{A}_{11}, \mathrm{A}_7, \mathrm{M}_{11})$ and $p = 11$. With the help of Magma, no such graphs exist in this case. Assume that $(L, L_\alpha) \cong (\mathrm{A}_{p+1}, \mathrm{A}_p)$ and $(p+1) \bigm| |G|$, as the line 6 of Table 7. Then $|G| = p+1$, in particular, Γ is the complete graph K_{p+1}. Hence line 3 of Table 1 holds.

Assume that line 8 of Table 7 holds. Note that $|L| = |G||L_\alpha|$. Then $(L, G, L_\alpha) \cong (\mathrm{A}_{11}, \mathrm{M}_{11}, \mathrm{A}_7)$ or $(\mathrm{A}_{12}, \mathrm{M}_{12}, \mathrm{A}_7)$, in particular, $p = 7$ and Γ is 2-arc-transitive. With the help of Magma, neither $\Gamma(A_{11}, M_{11}, g)$ nor $\Gamma(A_{12}, M_{12}, g)$ exists. Suppose that line 10 of Table 7 holds. A straight forward calculation shows that $(L, G, L_\alpha) \cong (\mathrm{A}_{p+3}, \mathrm{PSL}(2,q), \mathrm{S}_p)$ for q odd and $p = q-2$, and so the line 4 of Table 1 holds.

Suppose that the line 15 of Table 7 holds. Then $(L, G, L_\alpha) \cong (\mathrm{M}_{23}, \mathrm{M}_{22}, \mathrm{C}_{23})$. Moreover, $p = 23$ and Γ is 1-regular. By Example 3, there does exist graph in this case. Hence line 1 of Table 1 holds, and thus Lemma 10 holds. □

In the following Lemma, we will consider the case where $\mathrm{Aut}\Gamma$ is not quasiprimitive on $V(\Gamma)$.

Lemma 11. *Assume that A is not quasiprimitive on $V(\Gamma)$. Then there exists an intransitive non-trivial normal subgroup K of A such that A/K is almost simple with socle $\overline{L} \geqslant GK/K \cong G$. Moreover, for $\overline{\alpha} \in V(\Gamma_K)$, we have*

(a) *$\overline{L}$ is a classical simple group or $(\overline{L}, G, \overline{L}_{\overline{\alpha}})$ lies in Table* 2*; or*

(b) *(A, G, A_α) lies in Table* 3.

Proof. Since A is not quasiprimitive, there exists a non-trivial maximal intransitive normal subgroup, say K. If K has two orbits on $V(\Gamma)$, then Γ is bipartite. Since G is transitive on $V(\Gamma)$, then G has a normal subgroup of index 2, a contradiction. It follows that K has at least $p+1$ orbits on $V(\Gamma)$. Let Γ_K be the quotient graph of Γ relative to K. Clearly, Γ_K is arc-transitive with valency p. According to the maximum of K and Γ is locally primitive, we can conduct that the action of A/K on $V(\Gamma_K)$ is quasiprimitive and Γ_K is A/K-locally primitive, in particular, $(A/K)_{\overline{\alpha}}$ is given in Lemma 6 and 7 for $\overline{\alpha} \in V(\Gamma_K)$. Especially, Proposition 5 shows that there are only three cases in this situation:

(i). A/K is almost simple, and $\mathrm{Soc}(A/K)$ contains GK/K and is transitive on $V(\Gamma_K)$;

(ii). $A/K \cong \mathrm{AGL}(3,2)$, $G \cong \mathrm{PSL}(2,7)$ and $\Gamma_K \cong \mathsf{K}_8$; or

(iii). $\mathrm{Soc}(A/K) = T \times T$, and $GK/K \cong G$ is a diagonal subgroup of $\mathrm{Soc}(A/K)$, where T and G are given in [9, Table 1].

Case (i). Now suppose that A/K is almost simple, just as (i). Write $\mathrm{Soc}(A/K) = \overline{L}$, which is a finite non-abelian simple group containing $\overline{G} = GK/K \cong G$. If $\overline{L}$ is regular on $V(\Gamma_K)$, then $\overline{L} = \overline{G}$ as $\overline{G} \leqslant \overline{L}$ is transitive on Γ_K. So $\overline{G}$ is regular on $V(\Gamma_K)$. It follows that $|V(\Gamma)| = |G| = |\overline{G}| = |V(\Gamma_K)|$, a contradiction. Hence $\overline{L}$ is not regular on $V(\Gamma_K)$. We claim that $\overline{L} \neq \overline{G}$. If not, then $\overline{L} = \overline{G}$, i.e., GK/K is a characteristic subgroup of A/K, and so GK is a characteristic subgroup of A. Noting that $G \trianglelefteq GK$, we have $G \trianglelefteq A$, and then Γ is normal, a contradiction. Hence the claim holds. Then Γ_K is $\overline{L}$-arc-transitive, and so $\overline{L}$ is locally primitive as the valency of Γ_K is prime, in particular, $\overline{L}_{\overline{\alpha}}$ is isomorphic to a group of Proposition 6 or 7, where $\alpha \in V(\Gamma)$ and $\overline{\alpha} \in V(\Gamma_K)$. Further, $\overline{L} = \overline{L}_{\overline{\alpha}}\overline{G}$ with $\overline{G} \cap \overline{L}_{\overline{\alpha}} = \overline{G}_{\overline{\alpha}}$. Since G is regular on $V(\Gamma)$ and K is semiregular on $V(\Gamma)$, we have $K \cong \overline{G}_{\overline{\alpha}}$. Thus $|\overline{L}{:}\overline{G}| = |\overline{L}_{\overline{\alpha}}{:}\overline{G}_{\overline{\alpha}}| = |\overline{L}_{\overline{\alpha}}|/|K|$. We claim that $(A/K)_{\overline{\alpha}} \cong A_\alpha$. Note that $A_{\overline{\alpha}}/K = (A/K)_{\overline{\alpha}}$. By the Frattini argument, we have $A_{\overline{\alpha}} = K{:}A_\alpha$, i.e., $A_{\overline{\alpha}}/K \cong A_\alpha$. Hence $(A/K)_{\overline{\alpha}} \cong A_\alpha$, and so the claim holds. By Lemma 9, $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}})$ are given in Table 7. Since $|A| = |GA_\alpha| = |G||A_\alpha|$ and $|K| = |A|/|A/K|$, we have $|A_\alpha|/|K| = |A/K|/|G|$. Note that $A_\alpha \cong (A/K)_{\overline{\alpha}}$ and $G \cong \overline{G}$. Then $|(A/K)_{\overline{\alpha}}|/|K| = |A/K|/|\overline{G}|$.

Suppose that $\overline{L}$ is not a classical simple group. Then by Lemma 9, $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}})$ are given in Table 7. In the following, we will analyze them one by one.

(1). Assume that $\overline{L} \cong \mathrm{A}_n$ and $\overline{G} \cong \mathrm{A}_{n-1}$, in particular, $n \bigm| |\overline{L}_{\overline{\alpha}}|$ and $p \bigm| n$ for $n \geqslant 6$, as line 1 of Table 7. In [21, Theorem 1.1], it is shown that there exists a graph with $n = p = 7$ and $\overline{L}_{\overline{\alpha}} \cong \mathrm{C}_7$. Hence line 1 of Table 2 holds. For a similar reason, lines 2-3 of Table 7 lead to that line 2-4 of Table 2 hold.

(2). Assume that $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{A}_{11}, \mathrm{A}_9, \mathrm{PSL}(2, 11))$ or $(\mathrm{A}_{11}, \mathrm{A}_7, \mathrm{M}_{11})$ and $p = 11$, just as line 4 of Table 7. For $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{A}_{11}, \mathrm{A}_9, \mathrm{PSL}(2, 11))$. Since $\mathrm{Soc}(A/K) = \overline{L} \cong \mathrm{A}_{11}$, we can conduct that $A/K \cong \mathrm{A}_{11}$ or S_{11}, and so $(A/K)_{\overline{\alpha}} \cong \mathrm{PSL}(2, 11)$ or $\mathrm{PSL}(2, 11){:}\mathrm{C}_2$ respectively. On the other hand, $(A/K)_{\overline{\alpha}}$ is given in Proposition 7(b), which gives that $(A/K)_{\overline{\alpha}} \cong \mathrm{PSL}(2, 11)$, and so $A/K \cong \mathrm{A}_{11}$, i.e., $A/K = \overline{L}$. Furthermore, Γ_K is $(\overline{L}, 2)$-arc transitive. By Magma, we have that the graph Γ_K does not exist, a contradiction. Then along the same lines as the previous case we can exclude the cases when $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{A}_{11}, \mathrm{A}_7, \mathrm{M}_{11})$ (which corresponding to line 4 of Table 7), and $(\mathrm{A}_{23}, \mathrm{A}_{19}, \mathrm{M}_{23})$ (which corresponding to line 5 of Table 7).

(3). Assume that $(\overline{L}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{A}_{p+1}, \mathrm{A}_p)$ and $(p+1) \bigm| |\overline{G}|$, just as line 6 of Table 7. It is clear that $\Gamma_K \cong \mathsf{K}_{p+1}$. Hence line 5 of Table 2 holds. For a similar reason, line 7 and 10 of Table 7 gives line 6 and 7 of Table 2 respectively.

(4). Assume that $(\overline{L}, \overline{G}) \cong (\mathrm{A}_{11}, \mathrm{M}_{11})$ or $(\mathrm{A}_{12}, \mathrm{M}_{12})$, and $\overline{L}_{\overline{\alpha}} \cong \mathrm{A}_7$ or S_7, in particular, $p = 7$, just as line 8 of Table 7. It is clear that A_7 is 2-transitive on $\Gamma_K(\overline{\alpha})$. By Magma, we have that the graph Γ_K does not exist, a contradiction. Hence this case does not occur.

(5). Assume that $(\overline{L}, \overline{G}) \cong (\mathrm{A}_{23}, \mathrm{M}_{23})$ or $(\mathrm{A}_{24}, \mathrm{M}_{24})$ and $\overline{L}_{\overline{\alpha}} \cong \mathrm{A}_{19}$ or S_{19}, in particular, $p = 19$, just as line 9 of Table 7. Suppose that $(\overline{L}, \overline{G}) \cong (\mathrm{A}_{23}, \mathrm{M}_{23})$. Note that $|\overline{L}_{\overline{\alpha}}|/|K| = |\overline{L}|/|\overline{G}| = 1267136462592000$ denoted by m, and $K \cong \overline{G}_{\overline{\alpha}} \leqslant (\overline{L})_{\overline{\alpha}}$, we have that K is isomorphic to a subgroup of A_{19} or S_{19} with index m. Thus $|K| = 48$ or 96 respects to $\overline{L}_{\overline{\alpha}} \cong \mathrm{A}_{19}$ or S_{19}. Hence the first line of line 8 in Table 2 holds. For the same reason, the case where $(\overline{L}, \overline{G}) \cong (\mathrm{A}_{24}, \mathrm{M}_{24})$ implies that line 8 of Table 2 holds.

(6). Assume that $(\overline{L}, \overline{G}) \cong (\mathrm{A}_8, \mathrm{A}_k)$ for $k \in \{5, 6, 7\}$, and $\overline{L}_{\overline{\alpha}} \cong \mathrm{SL}(3, 2)$ or $\mathrm{AGL}(3, 2)$, in particular, $p = 7$, as line 11 of Table 7. By [13, Theorem 1.1], we have Γ_K is $(\overline{L}, 2)$-arc transitive. With the help of Magma, there is no such graph Γ_K exists, a contradiction.

(7). Assume that $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{M}_{12}, \mathrm{M}_{11}, \mathrm{M}_{11})$ or $(\mathrm{M}_{12}, \mathrm{M}_{11}, \mathrm{PSL}(2, 11))$, and $p = 11$ as line 12 of Table 7. Suppose that $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{M}_{12}, \mathrm{M}_{11}, \mathrm{M}_{11})$. Then Γ_K is isomorphic to a complete graph K_{12}. Note that $\mathrm{Soc}(A/K) = \overline{L} \cong \mathrm{M}_{12}$. Then $A/K \cong \mathrm{M}_{12}$ or $\mathrm{M}_{12}.\mathrm{C}_2$, and so $(A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$ or $\mathrm{M}_{11}.\mathrm{C}_2$. On the other hand, since Γ_K is A/K-arc transitive graph of valency 11, then $(A/K)_{\overline{\alpha}}$ is given in Proposition 7(b), which shows that $A_\alpha \cong (A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$. It follows that $A/K = \overline{L} \cong \mathrm{M}_{12}$. Since $|\overline{L}_{\overline{\alpha}}|/|K| = |\overline{L}|/|\overline{G}| = 12$ and $K \cong \overline{G}_{\overline{\alpha}} \leqslant (\overline{L})_{\overline{\alpha}} \cong \mathrm{M}_{11}$, we have that K is isomorphic to a subgroup of M_{11} of index 12. By [4, Page 18], $K \cong \mathrm{PSL}(2, 11)$. On the other hand, the Schur multiplier $M(\mathrm{M}_{12}) \cong \mathrm{C}_2$

(see [4, Page 31] for example). Hence $A \cong K.A/K \cong \mathrm{PSL}(2,11).\mathrm{M}_{12} \cong \mathrm{PSL}(2,11) \times \mathrm{M}_{12}$. Hence $(A, G, A_\alpha) \cong (\mathrm{PSL}(2,11) \times \mathrm{M}_{12}, \mathrm{M}_{11}, \mathrm{M}_{11})$, just as line 1 of Table 3. Assume that $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{M}_{12}, \mathrm{M}_{11}, \mathrm{PSL}(2,11))$. By Proposition 7(b), we have Γ_K is $(\overline{L}, 2)$-arc transitive. With the help of MAGMA, the graph Γ_K does not exist, a contradiction.

(8). Suppose that $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{M}_{12}, \mathrm{PSL}(2,11), \mathrm{M}_{11})$ and $p = 11$ as line 13 of Table 7. Then Γ_K is isomorphic to a complete graph K_{12}. Note that $\overline{L}_{\overline{\alpha}} \cong \mathrm{M}_{11}$ and $\mathrm{Soc}(A/K) = \overline{L} \cong \mathrm{M}_{12}$. Then $A/K \cong \mathrm{M}_{12}$ or $\mathrm{M}_{12}.\mathrm{C}_2$, and so $(A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$ or $\mathrm{M}_{11}.\mathrm{C}_2$ respectively. On the other hand, since Γ_K is A/K-arc transitive graph of valency 11, then $(A/K)_{\overline{\alpha}}$ is given in Proposition 7(b), which shows that $(A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$. It follows that $A_\alpha \cong (A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$ and $A/K \cong \overline{L} \cong \mathrm{M}_{12}$. Noting $G \cong \overline{G} \cong \mathrm{PSL}(2,11)$, we have $|(A/K)_{\overline{\alpha}}|/|K| = |A/K|/|\overline{G}| = 144$. Since $K \cong \overline{G}_{\overline{\alpha}} \leqslant (A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$, we have K is isomorphic to a subgroup of M_{11} of index 144. By [4, Page 18], $K \cong \mathrm{C}_{11}{:}\mathrm{C}_5$ and the Schur multiplier $M(\mathrm{M}_{12}) \cong \mathrm{C}_2$, and hence $A \cong K.A/K \cong (\mathrm{C}_{11}{:}\mathrm{C}_5).\mathrm{M}_{12}$. With the help of GAP, we have $\mathrm{Aut}(\mathrm{C}_{11}{:}\mathrm{C}_5) \cong (\mathrm{C}_{11}{:}\mathrm{C}_5){:}\mathrm{C}_2$. Thus $(\mathrm{C}_{11}{:}\mathrm{C}_5).\mathrm{M}_{12} \cong (\mathrm{C}_{11}{:}\mathrm{C}_5) \times \mathrm{M}_{12}$. Thereby, $(A, G, A_\alpha) \cong ((\mathrm{C}_{11}{:}\mathrm{C}_5) \times \mathrm{M}_{12}, \mathrm{PSL}(2,11), \mathrm{M}_{11})$, just as line 2 of Table 3.

(9). Assume that $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{M}_{12}, \mathrm{A}_5, \mathrm{M}_{11})$ as line 14 of Table 7, in particular, $p = 11$ and $\Gamma_K \cong \mathsf{K}_{12}$. Note that $\overline{L}_{\overline{\alpha}} \cong \mathrm{M}_{11}$ and $\mathrm{Soc}(A/K) = \overline{L} \cong \mathrm{M}_{12}$. Then $A/K \cong \mathrm{M}_{12}$ or $\mathrm{M}_{12}.\mathrm{C}_2$, and so $(A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$ or $\mathrm{M}_{11}.\mathrm{C}_2$. On the other hand, since Γ_K is A/K-arc transitive graph of valency 11, then $(A/K)_{\overline{\alpha}}$ is given in Proposition 7(b), which shows that $(A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$. It follows that $A_\alpha \cong (A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$ and $A/K = \overline{L} \cong \mathrm{M}_{12}$. Note that $G \cong \overline{G} \cong \mathrm{M}_{11}$, we have $|(A/K)_{\overline{\alpha}}|/|K| = |A/K|/|G| = |\mathrm{M}_{12}|/|\mathrm{A}_5| = 2^4 \cdot 3^2 \cdot 11$. Since $K \cong \overline{G}_{\overline{\alpha}} \leqslant (A/K)_{\overline{\alpha}} \cong \mathrm{M}_{11}$, we have K is isomorphic to a subgroup of M_{11} of index $2^4 \cdot 3^2 \cdot 11$. By [4, Page 18], $K \cong \mathrm{C}_5$ and $M(\mathrm{M}_{12}) \cong \mathrm{C}_2$, and hence $A \cong K.A/K \cong \mathrm{C}_5.\mathrm{M}_{12} \cong \mathrm{C}_5 \times \mathrm{M}_{12}$. Hence $(A, G, A_\alpha) \cong (\mathrm{C}_5 \times \mathrm{M}_{12}, \mathrm{A}_5, \mathrm{M}_{11})$, just as line 3 of Table 3.

(10). Assume that $(\overline{L}, \overline{G}) \cong (\mathrm{M}_{23}, \mathrm{M}_{22})$ and $\overline{L}_{\overline{\alpha}} \cong \mathrm{C}_{23}{:}\mathrm{C}_{11}$ or C_{23}, as line 15 of Table 7, in particular, $p = 23$. Note that $\mathrm{Soc}(A/K) = \overline{L} \cong \mathrm{M}_{23}$ and $\mathrm{Out}(\mathrm{M}_{23}) = 1$. Then $A/K \cong \mathrm{M}_{23}$, and so $A_\alpha \cong (A/K)_{\overline{\alpha}} = \overline{L}_{\overline{\alpha}} \cong \mathrm{C}_{23}{:}\mathrm{C}_{11}$ or C_{23}. In particular, $|(A/K)_{\overline{\alpha}}|/|K| = |A/K|/|\overline{G}| = 23$. Since $K \cong \overline{G}_{\overline{\alpha}} \leqslant (A/K)_{\overline{\alpha}}$ of index 23 and $K \neq 1$, then $\overline{L}_{\overline{\alpha}} \cong \mathrm{C}_{23}{:}\mathrm{C}_{11}$ and $K \cong \mathrm{C}_{11}$. By [4, Page 71], the Schur multiplier $M(\mathrm{M}_{23}) = 1$, and so $A \cong \mathrm{C}_{11}.\mathrm{M}_{23} \cong \mathrm{C}_{11} \times \mathrm{M}_{23}$. Hence line 4 of Table 3 holds.

(11). Assume that $(\overline{L}, \overline{G}) \cong (\mathrm{M}_{24}, \mathrm{M}_{23})$ and $\overline{L}_{\overline{\alpha}} \cong \mathrm{SL}(3,2)$ or $\mathrm{C}_2^6{:}(\mathrm{SL}(3,2) \times \mathrm{S}_3)$, as line 16 of Table 7, in particular, $p = 7$. Suppose that $\overline{L}_{\overline{\alpha}} \cong \mathrm{C}_2^6 \times (\mathrm{SL}(3,2) \times \mathrm{S}_3)$. By Lemma 6, we have Γ_K is $(\overline{L}, 2)$-arc transitive. However, by MAGMA, we have that the graph Γ_K does not exist, a contradiction. If $\overline{L}_{\overline{\alpha}} \cong \mathrm{SL}(3,2)$, then $P := \overline{L}_{\overline{\alpha}\overline{\beta}} \cong \mathrm{S}_4$ for $\overline{\beta} \in \Gamma_K(\overline{\alpha})$. With the help of GAP, we have $N := \mathbf{N}_{\overline{L}}(P) \cong \mathrm{S}_3 \times \mathrm{S}_4$ and $\langle \overline{L}_{\overline{\alpha}}, N \rangle < \mathrm{M}_{24}$. It contradicts to the assumption that Γ_K is connected.

(12). Assume that $(\overline{L}, \overline{G}, \overline{L}_{\overline{\alpha}}) \cong (\mathrm{M}_{24}, \mathrm{PSL}(2,23), \mathrm{M}_{23})$ as line 17 of Table 7, in particular, $p = 23$ and $\Gamma_K \cong \mathsf{K}_{24}$. Note that $\overline{L}_{\overline{\alpha}} \cong \mathrm{M}_{23}$ and $\mathrm{Soc}(A/K) = \overline{L} \cong \mathrm{M}_{24}$. Then $A/K \cong \mathrm{M}_{24}$ and so $A_\alpha \cong (A/K)_{\overline{\alpha}} \cong \mathrm{M}_{23}$. Note that $G \cong \overline{G} \cong \mathrm{PSL}(2,23)$, we have $|(A/K)_{\overline{\alpha}}|/|K| = |A/K|/|G| = |\mathrm{M}_{24}|/|\mathrm{PSL}(2,23)| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. Since $K \cong \overline{G}_{\overline{\alpha}} \leqslant (A/K)_{\overline{\alpha}} \cong \mathrm{M}_{23}$, we have K is isomorphic to a subgroup of M_{23} of index $2^7 \cdot 3^2 \cdot 5 \cdot 7$. By [4, Page 71], $K \cong \mathrm{C}_{23}{:}\mathrm{C}_{11}$ and $M(\mathrm{M}_{24}) = 1$, and hence $A \cong K.A/K \cong (\mathrm{C}_{23}{:}\mathrm{C}_{11}).\mathrm{M}_{24} \cong (\mathrm{C}_{23}{:}\mathrm{C}_{11}) \times \mathrm{M}_{24}$. Hence $(A, G, A_\alpha) \cong ((\mathrm{C}_{23}{:}\mathrm{C}_{11}) \times \mathrm{M}_{24}, \mathrm{PSL}(2,23), \mathrm{M}_{23})$, just as line 5 of

Table 3.

Case (ii). Assume that (ii) occurs, i.e., $A/K \cong \mathrm{AGL}(3,2)$, $G \cong \mathrm{PSL}(2,7)$ and $\Gamma_K \cong \mathsf{K}_8$. Since $\overline{G} \cong G$ is transitive on $V(\Gamma_K)$, then $|G|/8 = |\overline{G}_{\overline{\alpha}}|$, in particular, the index of $\overline{G}_{\overline{\alpha}}$ in G is 8. It follows that $\overline{G}_{\overline{\alpha}} \cong \mathrm{C}_7{:}\mathrm{C}_3$. Since G is regular and K is semiregular on $V(\Gamma)$, then $\overline{G}_{\overline{\alpha}} \cong K$. Hence $A \cong K.A/K \cong (\mathrm{C}_7{:}\mathrm{C}_3).\mathrm{AGL}(3,2)$. Note that $\mathrm{AGL}(3,2) \cong \mathrm{C}_2^3{:}\mathrm{SL}(3,2)$ with C_2^3 being the unique minimal normal subgroup. By [4], $M(\mathrm{SL}(3,2)) \cong \mathrm{C}_2$. Note that $\mathrm{Aut}(\mathrm{C}_7{:}\mathrm{C}_3) \cong (\mathrm{C}_7{:}\mathrm{C}_3){:}\mathrm{C}_2$, we have $A \cong (\mathrm{C}_7{:}\mathrm{C}_3) \times \mathrm{AGL}(3,2)$. On the other hand, since $|A/K| = 8|(A/K)_{\overline{\alpha}}|$ and $A/K = \mathrm{AGL}(3,2) \cong \mathrm{C}_2^3{:}\mathrm{SL}(3,2)$, then $(A/K)_{\overline{\alpha}} \cong \mathrm{SL}(3,2)$. It follows that $A_\alpha \cong (A/K)_{\overline{\alpha}} \cong \mathrm{SL}(3,2)$. Then Lemma 11 holds in this case.

Case (iii). Assume that (iii) occurs, i.e., $\mathrm{Soc}(A/K) = T \times T$, and $GK/K \cong G$ is a diagonal subgroup of $\mathrm{Soc}(A/K)$, where T and G are given in [9, Table 1]. Then Γ_K is $\overline{L}$-arc transitive and $\overline{L}_{\overline{\alpha}}$ is primitive on $\Gamma_K(\overline{\alpha})$. So $\overline{L}_{\overline{\alpha}}$ is given in Lemma 6 and 7, in particular, $(|\mathrm{Soc}(A/K)|/|V(\Gamma_K)|)_p = |\overline{L}_\alpha|_p = p$ for $p \geqslant 7$. However a calculation on the index of $|V(\Gamma_K)|$ in $\mathrm{Soc}(A/K)$ shows that $(|\mathrm{Soc}(A/K)|/|V(\Gamma_K)|)_p \geqslant p^2$, a contradiction.

This finishes the proof of Lemma 11. □

The proof of Theorem 1: The Theorem 1 follows immediately from Lemma 10 (which gives (a) of Theorem 10) and Lemma 11 (which gives (b) of Theorem 10). □

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