Uniquely D-colourable digraphs with large girth II: simplification via generalization

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Submitted: Jul 3, 2020; Accepted: Feb 25, 2021; Published: Mar 12, 2021

Abstract

We prove that for every digraph D and every choice of positive integers k, ℓ there exists a digraph D^* with girth at least ℓ together with a surjective acyclic homomorphism $\psi\colon D^* \to D$ such that: (i) for every digraph C of order at most k, there exists an acyclic homomorphism $D^* \to C$ if and only if there exists an acyclic homomorphism $D \to C$; and (ii) for every D-pointed digraph C of order at most k and every acyclic homomorphism $\varphi\colon D^* \to C$ there exists a unique acyclic homomorphism $f\colon D \to C$ such that $\varphi = f \circ \psi$. This implies the main results in [A. Harutyunyan et al., Uniquely D-colourable digraphs with large girth, *Canad. J. Math.*, **64(6)** (2012), 1310–1328; MR2994666] analogously with how the work [J. Nešetřil and X. Zhu, On sparse graphs with given colorings and homomorphisms, *J. Combin. Theory Ser. B*, **90(1)** (2004), 161–172; MR2041324] generalizes and extends [X. Zhu, Uniquely H-colorable graphs with large girth, *J. Graph Theory*, **23(1)** (1996), 33–41; MR1402136].

Mathematics Subject Classifications: Primary 05C15; Secondary 05C20, 05C60, 60C05

1 Introduction

In 1959, Paul Erdős, in a landmark paper [7]—now known as one of the most pleasing uses of the probabilistic method—proved the existence of graphs with arbitrarily large girth

*Partially supported by a grant from the Simons Foundation (#279367 to Mark Kayll).
†Partially supported by a 2017 University of Montana Graduate Student Summer Research Award funded by the George and Dorothy Bryan Endowment and partially supported by a grant from the Simons Foundation (#279367 to Mark Kayll). This work forms part of the author's PhD dissertation [17].

and chromatic number. His technique has been extended in a number of ways, e.g., by Bollobás and Sauer [5] to prove that for all $k \geqslant 2$ and $\ell \geqslant 3$ there is a uniquely k-colourable graph whose girth is at least ℓ. It would be difficult to overstate the influence of this one [7] of Erdős' thousands of results. Indeed, one authoritative combinatorialist went so far as to assert that "All interesting combinatorics flows from the existence of graphs with large girth and chromatic number."[1] Of course, we interpret Thomassé's remark as somewhat tongue-in-cheek, but as they say, *many a truth is said in jest.* In the present article, we follow the flow, from colourings to homomorphisms and from graphs to digraphs. This work is a sequel to [9], with which we assume some familiarity. For example, because the introduction of [9] is more extensive than this one, we refer the reader there for more background. Also, some of the arguments from [9]—e.g. the statement/proof of Lemma 7 and Lemma 8 (both below)—prove useful here. We try to balance the conflicting goals of not duplicating earlier work while allowing our new results to stand on their own.

Erdős' argument in [7] was probabilistic, hence nonconstructive. To help answer the question of what graphs with large girth and chromatic number actually look like, in 1968 Lovász [13] constructed hypergraphs with arbitrarily large girth and chromatic number. Müller [15] also worked in this domain. More than twenty years after Lovász's contribution, Kříž [12] produced the first purely graph-theoretic construction of graphs with arbitrarily large girth and chromatic number. And more recently (2016), Alon et al. [1] constructed such graphs that also satisfy a side condition on maximum average degree. The time intervals separating these results offer some hint of the delicacy of their constructions.

Graph homomorphisms, as vertex mappings that preserve adjacency, naturally generalize graph colouring. In 1996, working in this realm, Zhu [22] proved that for every 'core' graph H and every positive integer $\ell \geqslant 3$ there exists a uniquely H-colourable graph with girth at least ℓ. Because complete graphs are cores, Zhu's result generalized [5] and [7]. Almost ten years later, Nešetřil and Zhu [16] further generalized the results in the sequence [7, 5, 22] using the notion of 'pointed' graphs.

Let us shift now to digraphs. Their circular chromatic number was first studied in [4], where Bokal et al. showed that the colouring theory for digraphs is similar to that for undirected graphs when stable vertex sets are replaced by acyclic sets. For example, using an analogue of Erdős' original argument from [7], they showed that there exist digraphs of arbitrarily large (directed) girth and circular chromatic number. Almost a decade later, in [9], a subset of these authors together with their doctoral students established analogues of Zhu's results from [22] in a digraph setting; namely, for a suitable digraph D, there exist digraphs of arbitrarily large girth that are uniquely D-colourable. Severino [19] presented a construction of highly chromatic digraphs without short cycles and another construction [20] of uniquely n-colourable digraphs (for arbitrary n) with arbitrarily large girth. The latter two articles, based on [21], give constructive proofs of results in [4] and [9] that were originally proved probabilistically.

This paper analogizes the results of Nešetřil and Zhu [16] to the realm of digraphs.

[1] Stéphan Thomassé included the assertion in his plenary CanaDAM lecture, 2 June 2011, Victoria, Canada.

Just as [16] puts the final icing on the sequence [7, 5, 22], so too does our main result—Theorem 1 below—provide a fitting capstone for the sequence [4, 9]. Postponing definitions for another minute (until Section 2), let us state our main result and lay bare its connection with [9].

Theorem 1. *For every digraph D and every choice of positive integers k, ℓ there exists a digraph D^* together with a surjective acyclic homomorphism $\psi\colon D^* \to D$ with the following properties:*

(i). *$girth(D^*) \geqslant \ell$;*

(ii). *for every digraph C with at most k vertices, there exists an acyclic homomorphism $D^* \to C$ if and only if there exists an acyclic homomorphism $D \to C$;*

(iii). *for every D-pointed digraph C with at most k vertices and for every acyclic homomorphism $\varphi\colon D^* \to C$ there exists a unique acyclic homomorphism $f\colon D \to C$ such that $\varphi = f \circ \psi$.*

The precursor [9] established two main results:

Theorem 2. *If D and C are digraphs such that D is not C-colourable, then for every positive integer ℓ, there exists a digraph D^* of girth at least ℓ that is D-colourable but not C-colourable.*

Theorem 3. *For every core D and every positive integer ℓ, there is a digraph D^* of girth at least ℓ that is uniquely D-colourable.*

To see that Theorem 1 implies Theorem 2, let us be given a positive integer ℓ and two digraphs C, D with D not C-colourable (as in the hypotheses of Theorem 2). Taking k to be the order of C, we can put this C in the role of the digraph C in conclusion (ii) of Theorem 1, which delivers a digraph D^* with $D^* \to D$. As $D \not\to C$, the same conclusion shows that also $D^* \not\to C$, and conclusion (i) gives the girth requirement on D^*.

Before deriving Theorem 3 from Theorem 1, observe that if D is a core, then every acyclic homomorphism from D to itself must be an automorphism, and so if any two such homomorphisms agree on all but one vertex, they must also agree on that vertex. Therefore, cores D are D-pointed.

Now let us be given a positive integer ℓ and a core D (as in the hypotheses of Theorem 3). If we here take $k = |V(D)|$, then Theorem 1 delivers a large-girth digraph D^* together with a D-colouring $\psi\colon D^* \to D$. The preceding paragraph foreshadows that we can put D in the role of C in conclusion (iii), which shows that every acyclic homomorphism $\varphi\colon D^* \to D$ yields an acyclic homomorphism $f\colon D \to D$ such that $\varphi = f \circ \psi$. But D being a core implies that such an f is an automorphism, so we've shown that φ and ψ differ by an automorphism, i.e., that D^* is uniquely D-colourable.

Notice that being D-pointed is a necessary condition in part (iii) of Theorem 1. For consider two acyclic homomorphisms $f', f''\colon D \to C$ satisfying (for some vertex x_0 of D) $f'(x) = f''(x)$ for all $x \neq x_0$ and $f'(x_0) \neq f''(x_0)$, and assume that there is an arc between

$f'(x_0)$ and $f''(x_0)$ in C. Typically, the set $\psi^{-1}(x_0)$ can be split into two nonempty sets A, B and we can define $\varphi\colon D^* \to C$ by $f' \circ \psi(y)$ for $y \in V(D^*) \setminus B$ and $f'' \circ \psi(y)$ for $y \in B$. Now this φ sends A and B to two different points while $f \circ \psi$, for any given $f\colon D \to C$, sends these sets to a single point. Therefore, the acyclic homomorphism φ cannot be written as $\varphi = f \circ \psi$ for an acyclic homomorphism $f\colon D \to C$.

Remarks

As hinted above, Nešetřil's and Zhu's article [16] was in a sense a crowning achievement for a body of work initiated by Erdős in [7]. For any given graph G, they produced a high-girth graph G^* characterizing the small-order graphs admitting a homomorphism from G and furthermore, via G-pointedness, wound unique colourability into their tapestry. Their results generalized [5], [22] and moreover some other major contemporary theorems (e.g., the Sparse Incomparability Lemma and Müller's Theorem—see [22] and the discussion in [16]).

Because our Theorem 1 likewise characterizes when the high directed girth, high digraph chromatic number (for unique colourability) phenomenon occurs—phrased in terms of acyclic homomorphisms—it too reaches a satisfying destination, now for the sequence [4, 9]. And because this level of generality has actually shortened the proofs from [9], perhaps we've arrived at the 'right' vantage point for viewing these results.

2 Terminology, notation, and an auxiliary result

Without being overly encyclopedic, we attempt to include the required definitions. For basic notation and terminology concerning graphs and digraphs, we mainly follow [6] and [3], respectively, and we refer the reader there for any omissions. For a more (most) thorough treatment of graph homomorphisms, the reader could consult [8] ([10]). For probabilistic concerns, see, e.g., [2] or [14].

All our digraphs are finite and *simple*—i.e. loopless and without multiple arcs—however, we do allow two vertices u, v to be joined by two oppositely directed arcs uv, vu. *Cycles* in digraphs mean directed ones, and the *girth* of a digraph D is the length of a shortest cycle in D.

Just as graph homomorphisms generalize graph colouring, so too do acyclic homomorphisms of digraphs generalize (one variant of) digraph colouring. So we begin by recalling the definition of these sorts of homomorphisms from [4]; see [9] for background. An *acyclic homomorphism* of a digraph D to a digraph C is a function $\rho\colon V(D) \to V(C)$ such that:

(i). for every arc $uv \in A(D)$, either $\rho(u) = \rho(v)$, or $\rho(u)\rho(v)$ is an arc of C; and

(ii). for every vertex $x \in V(C)$, the subdigraph of D induced by $\rho^{-1}(x)$ is acyclic.

Acyclic homomorphisms can also be viewed as a generalization of (ordinary) homomorphisms of undirected graphs; again, see [9].

If there exists an acyclic homomorphism of D to C, we say that D is *homomorphic* to C and write $D \to C$. Motivated by the connection to 'acyclic digraph colouring', we sometimes call an acyclic homomorphism of D to C a *C-colouring* of D and say that D is *C-colourable*. A digraph D is *uniquely C-colourable* if it is surjectively C-colourable, and for any two C-colourings ψ, φ of D, there is an automorphism f of C such that $\varphi = f \circ \psi$; when this occurs, we say that φ and ψ *differ by an automorphism* of C. A digraph D is a *core* if the only acyclic homomorphisms of D to itself are automorphisms. Given two digraphs C, D, we say that C is *D-pointed* if there do not exist two C-colourings ρ, φ of D such that $\rho(v) \neq \varphi(v)$ holds for exactly one vertex v of D. As noted following the statement of Theorem 3, digraph cores D are D-pointed.

Probabilistic tools

Our proof of Theorem 1 invokes several standard probabilistic tools. Aside from the First Moment Method (Markov's Inequality)—which is explicitly invoked a handful of times—Inclusion-Exclusion and the Janson Inequalities also make an implicit appearance through their use (in [9]) in proving Lemma 8 below. We shall not restate these standard results here; however, for convenience, we do include a version of Chernoff's famous bound(s) on the tail distributions of binomial random variables. Though more technical versions are available—see, e.g., [11]—this one will suffice for our main proof in Section 4:

Theorem 4. *If X is a binomial random variable and $0 < \gamma < 3/2$, then*

$$P(|X - E(X)| \geqslant \gamma E(X)) \leqslant 2e^{-\gamma^2 E(X)/3}.$$

3 Set-up for the proof of Theorem 1

We begin at the starting point for the main proof in [9], namely specifying a random digraph model, which needs no change here. Suppose that the digraph D is given with $V(D) = \{1, 2, \ldots, a\}$ and $|A(D)| = q$. Let n be a positive integer and $V_1, V_2, \ldots, V_a$ be pairwise-disjoint ordered n-sets $V_i = \{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$, for $i = 1, 2, \ldots, a$. Next let D_0 be the digraph with vertex set $V := V_1 \cup V_2 \cup \cdots \cup V_a$ and

$$\begin{aligned} A(D_0) := &\Big\{xy : x \in V_i, y \in V_j \text{ with } ij \in A(D), \text{for some } i, j \in \{1, 2, \ldots, a\}\Big\} \\ &\bigcup_{i=1}^{a} \Big\{v_{i_k} v_{i_t} : k, t \in \{1, 2, \ldots, n\} \text{ and } k < t\Big\}; \end{aligned}$$

so D_0 has na vertices and $a\binom{n}{2} + qn^2$ arcs.

Now fix an ϵ with $0 < \epsilon < 1/4\ell$. Our random digraph model $\mathcal{D}(n, p)$ consists of all spanning subdigraphs of D_0 in which the arcs are chosen randomly and independently with probability $p := n^{\epsilon - 1}$. Through the following three lemmas we prove essential technical facts about digraphs in $\mathcal{D}(n, p)$. Throughout the discussion n is assumed to be sufficiently large to support our assertions.

Our first aim is to show that most digraphs in $\mathcal{D}(n, p)$ have few short cycles which are pairwise vertex-disjoint.

Lemma 5.

(i). *The expected number of cycles of length less than ℓ in a digraph $\widehat{D} \in \mathcal{D}(n,p)$ is bounded from above by $n^{\epsilon\ell}n^{-\epsilon/2}$;*

(ii). *the expected number of pairs of cycles of length less than ℓ in a digraph $\widehat{D} \in \mathcal{D}(n,p)$ which intersect in at least one vertex is bounded from above by $n^{-1/2}$.*

By Markov's Inequality, Lemma 5 implies that asymptotically almost all digraphs from $\mathcal{D}(n,p)$ have at most $n^{\epsilon\ell}$ cycles of length less than ℓ, and these cycles are all vertex-disjoint. The ideas in the proofs of (i) and (ii) are contained, respectively, in the "Proof of (2.1)" and "Proof of (3.1)" in [9]; we include the proofs here for context, completeness, and consolidation.

Proof. (i) Let $\widehat{D} \in \mathcal{D}(n,p)$ and let the random variables X_i, X count, respectively, the number of cycles of length i, for $2 \leqslant i < \ell$, and of length less than ℓ in $\widehat{D}$. Then

$$E(X_i) \leqslant \binom{na}{i}(i-1)!p^i = \frac{na(na-1)\cdots(na-i+1)}{i}p^i < \frac{(na)^i}{i}p^i.$$

Hence

$$E(X) = \sum_{i=2}^{\ell-1} E(X_i) \leqslant \sum_{i=2}^{\ell-1} \frac{(na)^i}{i}p^i \leqslant \sum_{i=2}^{\ell-1} \frac{(n^\epsilon a)^i}{i},$$

recalling that $p = n^{\epsilon-1}$ for the last step. Now, the inequality $\sum_{i=2}^{\ell-1}(n^\epsilon a)^i/i < a^{\ell-1}n^{(\ell-1)\epsilon}$ (which can be proved by induction on ℓ) shows that

$$E(X) < a^{\ell-1}n^{(\ell-1)\epsilon} = a^{\ell-1}n^{-\epsilon}n^{\epsilon\ell} < n^{\epsilon\ell}n^{-\epsilon/2},$$

for sufficiently large values of n.

To prove part (ii), we need the following definition from [9] which in turn had its roots in [22]. For integers $\ell_1, \ell_2 < \ell$, we call a digraph an (ℓ_1,ℓ_2)-*double cycle* if it consists of a directed cycle C_{ℓ_1} of length ℓ_1 and a directed path of length ℓ_2 joining two (not necessarily distinct) vertices of C_{ℓ_1}. An (ℓ_1,ℓ_2)-double cycle contains $\ell_1+\ell_2$ arcs and $\ell_1+\ell_2-1$ vertices.

A moment's reflection shows that if two cycles of length less than ℓ intersect in at least one vertex, then they contain (as a subdigraph) an (ℓ_1,ℓ_2)-double cycle for some $\ell_1,\ell_2 < \ell$. Hence in a random $\widehat{D} \in \mathcal{D}(n,p)$ the expected number of pairs of cycles of length less than ℓ that intersect in at least one vertex is at most the expected number of all (ℓ_1,ℓ_2)-double cycles for $\ell_1,\ell_2 < \ell$.

Let the random variable Y count the number of all (ℓ_1,ℓ_2)-double cycles for some $\ell_1,\ell_2 < \ell$ in a random $\widehat{D} \in \mathcal{D}(n,p)$. For fixed $\ell_1,\ell_2 < \ell$, let $Y(\ell_1,\ell_2)$ be the number of (ℓ_1,ℓ_2)-double cycles. Then

$$\begin{aligned}
E(Y(\ell_1,\ell_2)) &< 2\binom{an}{\ell_1}(\ell_1-1)!p^{\ell_1}(\ell_1)(\ell_1)\binom{an}{\ell_2-1}(\ell_2-1)!p^{\ell_2} \\
&< \ell_1(na)^{\ell_1}(na)^{\ell_2-1}p^{\ell_1+\ell_2} \\
&< \ell_1 a^{\ell_1+\ell_2}n^{\epsilon(\ell_1+\ell_2)}n^{-1}.
\end{aligned}$$

As $\epsilon(\ell_1+\ell_2) < 2\ell\epsilon < 1/2$ (because $\ell_1, \ell_2 \leqslant \ell$ and $\epsilon < 1/4\ell$), for large enough n we have

$$E(Y) = \sum_{\substack{2\leqslant \ell_1<\ell \\ 1\leqslant \ell_2<\ell}} E(Y(\ell_1,\ell_2)) < n^{-1/2}. \qquad \square$$

To state the second lemma we need the following definition (which leans on the parameters D and k of Theorem 1). This set-up and the ensuing analysis in Lemma 6 is modelled after an analogous discussion in [16]. Following these authors, we call a set $\mathcal{A} \subseteq V$ *large* if there are distinct $i, j \in [a]$, with ij an arc of D, such that both $|\mathcal{A} \cap V_i| \geqslant n/k$ and $|\mathcal{A} \cap V_j| \geqslant n/k$, and the D-arc ij in this case is a *good* arc for $\mathcal{A}$. For a large set $\mathcal{A}$, denote by $|\widehat{D}/\mathcal{A}|$ the minimum number of arcs of (a random) $\widehat{D}$ which lie in a set $\{xy : x \in \mathcal{A} \cap V_i,\ y \in \mathcal{A} \cap V_j\}$, with ij a good arc for $\mathcal{A}$.

Lemma 6. *If $\widehat{D} \in \mathcal{D}(n,p)$ and $\mathcal{A}$ is large, then $P(|\widehat{D}/\mathcal{A}| \geqslant n) = 1 - o(1)$.*

Thus asymptotically most digraphs from $\mathcal{D}(n,p)$ enjoy the property of all good arcs (of D) for large sets $\mathcal{A}$ inducing at least n arcs (of $\widehat{D} \in \mathcal{D}(n,p)$).

Proof. Let $\widehat{D} \in \mathcal{D}(n,p)$ and $\mathcal{A} \subseteq V$ be a large set and set $\alpha = P(|\widehat{D}/\mathcal{A}| \geqslant n)$. Essentially following [16, Proof (of Claim 2)], we have

$$\begin{aligned} 1-\alpha = P(|\widehat{D}/\mathcal{A}| < n) &\leqslant \sum_{\mathcal{B} \text{ large}} P(|\widehat{D}/\mathcal{B}| < n) \\ &\leqslant 2^{na}\binom{qn^2}{n}(1-p)^{n^2/k^2-n} \\ &< e^{cn\ln n - c'n^{1+\epsilon}} = o(1) \qquad (1) \end{aligned}$$

for some positive constants c and c' that are independent of n (with the estimates in (1) being borrowed from [16]). Thus we get $\alpha = 1 - o(1)$. $\square$

The last lemma of this section addresses a technical situation also encountered at the end of Section 3 of [9]. We repeat part of the proof here for completeness and also to facilitate fleshing out more of its details. See also [16, Claim 3] for an analogous statement (for graphs and homomorphisms) and an alternate proof approach (via enumeration).

Lemma 7. *Almost all digraphs from $\mathcal{D}(n,p)$ do not contain two nonempty sets $\mathcal{A} \subset V_{i_0}$, $\mathcal{B} \subset V_{j_0}$, for some $i_0, j_0 \in [a]$, with $i_0j_0 \in A(D)$ (resp. $j_0i_0 \in A(D)$), $|\mathcal{A}| = n-(k-1)|\mathcal{B}|$, $|\mathcal{B}| \leqslant n/k$, such that the set $\mathcal{A}\cup\mathcal{B}$ contains at most $\min\{|\mathcal{B}|, n^{\epsilon\ell}\}$ arcs from $\mathcal{A}$ to $\mathcal{B}$ (resp. from $\mathcal{B}$ to $\mathcal{A}$) and these arcs form a matching (i.e. a set of independent arcs).*

Proof. Let $b \leqslant n/k$ and $s \leqslant \min\{b, \lceil n^{\epsilon\ell}\rceil\}$. We denote by $L(b,s)$ the expected number of pairs $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{A} \subseteq V_i$, $\mathcal{B} \subseteq V_j$, $ij \in A(D)$, $|\mathcal{A}| = n-(k-1)|\mathcal{B}|$, $|\mathcal{B}| = b$ and there are exactly s arcs joining a vertex in $\mathcal{A}$ to a vertex in $\mathcal{B}$. Then

$$\begin{aligned}
L(b,s) &\leqslant \binom{n}{n-(k-1)b}\binom{n}{b}\binom{(n-(k-1)b)b}{s}p^s(1-p)^{(n-(k-1)b)b-s} \\
&< n^{(k-1)b}n^b(nb)^s n^{s(\epsilon-1)}e^{-bn^\epsilon+n^{\epsilon-1}((k-1)b^2+s)} \\
&< n^{kb}b^s n^{\epsilon s}e^{-(bn^\epsilon)/2} \qquad (2)\\
&= b^s n^{\epsilon s}(n^k e^{-n^\epsilon/2})^b \\
&< b^s n^{\epsilon s}e^{-(bn^\epsilon)/3} \qquad (3)\\
&< e^{-n^\epsilon/4}. \qquad (4)
\end{aligned}$$

To help the reader through steps (2)–(4), we fill in the following estimates:
for (2):

$$-bn^\epsilon + n^{\epsilon-1}((k-1)b^2+s) = -bn^\epsilon + \frac{(k-1)b^2+s}{n^{1-\epsilon}} < -bn^\epsilon + \frac{bn^\epsilon}{2} = -\frac{bn^\epsilon}{2};$$

for (3): for large enough n, we have $n^k < e^{n^\epsilon/6}$, so that $n^k e^{-n^\epsilon/2} < e^{-n^\epsilon/3}$;
and lastly for (4):

$$(bn^\epsilon)^s = e^{s\ln(bn^\epsilon)} = (e^{\ln(bn^\epsilon)})^s < (e^{(1/12bn^\epsilon)^{1/s}})^s = e^{1/12bn^\epsilon},$$

and this implies that

$$b^s n^{\epsilon s}e^{-(bn^\epsilon)/3} < e^{-bn^\epsilon/4} < e^{-n^\epsilon/4}.$$

So with $L(b) := \sum_{s\leqslant \min\{b,\lceil n^{\epsilon\ell}\rceil\}} L(b,s)$, we find that

$$L(b) < \lceil n^{\epsilon\ell}\rceil e^{-n^\epsilon/4} < e^{-n^\epsilon/5},$$

and we finally obtain

$$\sum_{1\leqslant b\leqslant n/k} L(b) < (n/k)e^{-n^\epsilon/5} < e^{-n^\epsilon/6}.$$

An application of Markov's Inequality completes the proof. (Notice that we are getting a small upper estimate here even without the matching condition). □

4 Proof of Theorem 1

We continue to be guided by [16], but the argument here is complicated by the more technical definition of 'acyclic homomorphism' in our context compared to 'homomorphism' in the graph setting.

Choose a digraph D' in $\mathcal{D}(n,p)$ satisfying the properties asserted in Lemmas 5–7. So D' contains at most $n^{\epsilon\ell}$ (directed) cycles of length less than ℓ and these cycles are pairwise vertex-disjoint. Consequently (picking one arc from each cycle), there is a matching (an independent arc set) $M \subseteq A(D')$ of size at most $n^{\epsilon\ell}$ such that the digraph $D' - M =$

$(V(D'), A(D') \setminus M)$ has no cycles of length less than ℓ. We prove that this digraph—henceforth denoted $D^* := D' - M$—satisfies the conclusions of Theorem 1.

Define $\psi\colon V(D^*) \to V(D)$ by $\psi(x) = i$ if and only if $x \in V_i$, for $i \in [a]$. It is clear from the definition of $\mathcal{D}(n,p)$ that ψ is a surjective acyclic homomorphism. That $\mathrm{girth}(D^*) \geqslant \ell$ was arranged in our description of D^*, and this takes care of (i).

To prove part (ii) of Theorem 1, fix a digraph C of order at most k and consider an acyclic homomorphism $\varphi\colon D^* \to C$. We proceed to define a mapping $f\colon V(D) \to V(C)$. By the Pigeonhole Principle, for each $i \in V(D)$, there is a vertex $x \in V(C)$ such that $|V_i \cap \varphi^{-1}(x)| \geqslant n/k$. We let $f(i) = x$ (choosing x arbitrarily if more than one x has this property) and now prove that f is an acyclic homomorphism. To prove that f satisfies the first property of being an acyclic homomorphism, let ij be an arc of D with $f(i) = x$ and $f(j) = y$. If $x = y$, then we are done, so suppose that $x \neq y$. With $A_i = V_i \cap \varphi^{-1}(x)$ and $A_j = V_j \cap \varphi^{-1}(y)$, we have $|A_i| \geqslant n/k$ and $|A_j| \geqslant n/k$ from the definition of f. Hence $\mathcal{A} = A_i \cup A_j$ is a large set and ij is a good arc for $\mathcal{A}$, so we can invoke Lemma 6 to see that there exists an arc of D^* from A_i to A_j (Note that we deleted at most $n^{\epsilon\ell} < n^{1/4}$ arcs from D' to get D^*, but ij induces at least n arcs, so we did not delete all these arcs from A_i to A_j). Now, since φ is an acyclic homomorphism, we have $xy \in A(C)$ as required.

To finish the proof that f is an acyclic homomorphism, we need to show that $f^{-1}(x)$ induces an acyclic subdigraph in D for every $x \in V(C)$. We prove this by contradiction. Suppose that there is a vertex $v' \in V(C)$ such that the subdigraph induced by $f^{-1}(v')$ in D contains a cycle Q. Write $Q = i_1 i_2 \cdots i_t$ and observe that $2 \leqslant t \leqslant a$. Since $f(i_s) = v'$, for $s = 1, 2, \ldots, t$, we have $|V_{i_s} \cap \varphi^{-1}(v')| \geqslant n/k$, for $s = 1, 2, \ldots, t$ (from the definition of f). The fact that $n^{\epsilon\ell} \ll n/k$ implies that each set $V_{i_s} \cap \varphi^{-1}(v')$ contains a subset W_{i_s} of size $w := \lceil n/(2k) \rceil$ such that no arc in M has an end vertex in W_{i_s}. It follows from $W_{i_s} \subseteq V_{i_s} \cap \varphi^{-1}(v')$ that $\varphi(W_{i_1}) = \cdots = \varphi(W_{i_s}) = \{v'\}$. Since φ is an acyclic homomorphism, the subdigraph of D^* induced by $W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_s}$ is acyclic. We show that the event that $W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_s}$ induces an acyclic subdigraph in D^* is unlikely.

Let us consider a sequence of sets $U_{j_1}, U_{j_2}, \ldots, U_{j_r}$ such that for $i = 1, 2, \ldots, r$ we have $U_{j_i} \subseteq V_{j_i}$ and $|U_{j_i}| = w$, and the vertex sequence $j_1, j_2, \ldots, j_r$ is a cycle in D. We denote by P_r the probability that the subdigraph of D^* induced by $U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_r}$ is acyclic and call this sequence *bad* if it induces an acyclic subdigraph in D^*. Now, for the expected number N of bad sequences in D^*, we have

$$N \leqslant \sum_{r=2}^{a} \binom{a}{r} (r-1)! \binom{n}{w}^r P_r. \tag{5}$$

We pause to note that (5) is relation (2.6) from [9], adapted to our present context. The following result bounds the probabilities P_r; for a proof, see [9] (which actually contains two proofs).

Lemma 8 ([9, Lemma 2.1])**.** *For every integer $r \in \{2, \ldots, a\}$, we have $P_r \leqslant e^{-n^{1+\epsilon}/(10k^2)}$.*

As in [9, relations (2.19)], we see that for large enough n, the relation (5) and Lemma 8 give

$$N \leqslant \sum_{r=2}^{a} \binom{a}{r} (r-1)! \binom{n}{w}^{r} e^{-n^{(1+\epsilon)}/(10k^2)} < \sum_{r=2}^{a} \frac{e^{-n}}{2a} < \frac{e^{-n}}{2}.$$

So to finish this chain of reasoning as in [9], using Markov's Inequality, we find that

$$P(\exists \text{ a bad sequence}) < \frac{e^{-n}}{2}.$$

This achieves the goal stated before Lemma 8 which in turn contradicts our assumption that the subdigraph induced by $f^{-1}(v')$ in D contains a cycle. Thus the forward implication in part (ii) of Theorem 1 is proved.

For the converse in (ii), let $f\colon V(D) \to V(C)$ be an acyclic homomorphism. We define a mapping $\varphi\colon V(D^*) \to V(C)$ as $\varphi(x) = f(i)$, where $x \in V_i$. Each V_i induces an acyclic set in D^*. Arcs of D that are mapped to single vertices $f(i)$ in C do not lead to cycles in preimages $\varphi^{-1}(f(i))$ because f is itself an acyclic homomorphism. Furthermore, each arc $xy \in A(D^*)$ with $\varphi(x) \neq \varphi(y)$ is mapped to the arc $\varphi(x)\varphi(y) \in A(C)$ again because f is an acyclic homomorphism. Hence φ is an acyclic homomorphism. This completes the proof of part (ii).

We turn our attention to part (iii) of Theorem 1. Let C be a D-pointed digraph of order at most k and φ be an acyclic homomorphism from D^* to C. We want to show that there exists a unique acyclic homomorphism $f\colon V(D) \to V(C)$ such that $\varphi = f \circ \psi$. Note that for every $i \in V(D)$ there exists a unique $x_i \in V(C)$ such that $|\varphi^{-1}(x_i) \cap V_i| \geqslant n/k$. Existence follows from the Pigeonhole Principle. If there were $x_i' \neq x_i$ with the same property ($|\varphi^{-1}(x_i') \cap V_i| \geqslant n/k$), then our definition of f here would lead to another acyclic homomorphism $f'\colon V(D) \to V(C)$ such that $f(j) = f'(j)$ for all $j \in V(D) \smallsetminus \{i\}$. But then the D-pointedness of C would force $x_i = f(i) = f'(i) = x_i'$. Now, we define $f\colon V(D) \to V(C)$ as $f(i) = x_i$ for $i = 1, 2, \ldots, a$. Because f is defined as in part (ii), we again see that this function is an acyclic homomorphism. Hence, it remains to show that $\varphi = f \circ \psi$.

Remark Until now, parts of our proof have involved carefully piecing together ideas from [9] and [16]. The remainder of the argument follows quite a different path and underscores the extra complexity inherent in working with acyclic homomorphisms (of digraphs) compared to ordinary homomorphisms (of graphs).

Proof of $\boldsymbol{\varphi = f \circ \psi}$

First, we show that φ and $f \circ \psi$ have the same range. It is clear that $\mathrm{Range}(f \circ \psi) \subseteq \mathrm{Range}(\varphi)$. To prove the reverse containment, suppose to the contrary that there is a vertex $y \in \mathrm{Range}(\varphi)$ that is not in $\mathrm{Range}(f \circ \psi)$. Since y is in the range of φ, the set $\varphi^{-1}(y) \cap V_i$ is not empty for some $i \in \{1, 2, \ldots, a\}$. On the other hand, the definition of f shows that $|\varphi^{-1}(f(i)) \cap V_i| \geqslant n/k$; in particular $\varphi^{-1}(f(i)) \cap V_i \neq \varnothing$. Because $f(i) \in \mathrm{Range}(f \circ \psi)$ while $y \notin \mathrm{Range}(f \circ \psi)$ we see that $V_i \smallsetminus \big(\varphi^{-1}(f(i)) \cap V_i\big) \neq \varnothing$ for some $i \in \{1, 2, \ldots, a\}$. We show that this leads to a contradiction.

Let $i_0 \in \{1, 2, \ldots, a\}$ be such that $t := |\varphi^{-1}(f(i_0)) \cap V_{i_0}|$ is minimum. It is easy to see that $t \geqslant n/k$. Our discussion in the preceding paragraph implies that $t < n$. We choose $x \in V(C)$ with $x \neq f(i_0)$ such that $b := |\varphi^{-1}(x) \cap V_{i_0}|$ is maximum. Using the Pigeonhole Principle we obtain $b \geqslant (n-t)/(k-1)$ which gives $t \geqslant n-(k-1)b$. Furthermore $b < n/k$ as there is only one vertex of $V(C)$ satisfying the negation ($f(i_0) \neq x$ already has this property). Now we define a mapping $f' \colon V(D) \to V(C)$ as

$$f'(i) = \begin{cases} f(i) & \text{for } i \neq i_0 \\ x & \text{for } i = i_0. \end{cases}$$

Since f and f' differ only at i_0 and C is D-pointed, the function f' cannot be an acyclic homomorphism. We distinguish two cases.

Case I: $x \notin \mathrm{Range}(f)$.

In this case, the only reason that f' is not an acyclic homomorphism is that there must be a vertex $v \neq i_0$ in $V(D)$ such that either $f(v)f(i_0) \in A(C)$ but $f(v)x \notin A(C)$ (and $vi_0 \in A(D)$), or $f(i_0)f(v) \in A(C)$ but $xf(v) \notin A(C)$ (and $i_0v \in A(D)$). Without loss of generality, assume that $f(v)f(i_0) \in A(C)$ but $f(v)x \notin A(C)$ (and $vi_0 \in A(D)$) occurs. We have $|\varphi^{-1}(f(v)) \cap V_v| \geqslant t \geqslant n-(k-1)b$, so we can choose a set $U \subseteq \varphi^{-1}(f(v)) \cap V_v$ with $|U| = n-(k-1)b$. Then there must be at most $\min\{b, n^{\epsilon\ell}\}$ arcs from U to $\mathcal{A} := \varphi^{-1}(x) \cap V_{i_0}$ in D'; otherwise after passing from D' to D^*, we have some arc(s) left between these two sets in D^* and since φ is an acyclic homomorphism, $f(v)x \in A(C)$ which is a contradiction. But the property just described is the rare property articulated in Lemma 7, and D' was chosen not to enjoy it, so Case I leads to this contradiction.

Case II: $x \in \mathrm{Range}(f)$.

In this case, there are two potential reasons for f' not to be an acyclic homomorphism. The reason we explained in Case I is still a potential reason in the present case, and it similarly leads to a contradiction. The other reason here is when $f'^{-1}(x)$ does not induce an acyclic subdigraph in D. We proceed to show that this also leads to a contradiction.

We know that $\varphi^{-1}(x) \cap V_{i_0} \neq \varnothing$. Since $x \in \mathrm{Range}(f)$, we have $x = f(j_0)$ for some $j_0 \in V(D)$ and $j_0 \neq i_0$. The reason for $j_0 \neq i_0$ is that $f(i_0) \neq x = f(j_0)$. We show that in this case $i_0j_0, j_0i_0 \in A(D)$. Suppose to the contrary that this is wrong. Without loss of generality, we assume that $i_0j_0 \notin A(D)$. First we claim that there exists a vertex $p_0 \in V(D)$ such that it has a different situation with respect to i_0 and j_0 in the sense of adjacency (like, for example, $p_0i_0 \in A(D)$, but $p_0j_0 \notin A(D)$). For if every $p_0 \in V(D)$ that is adjacent to i_0 is also adjacent to j_0 (preserving the directions), then we can define the mapping $g \colon V(D) \to V(C)$ by $g(i) = f(i)$ for $i \neq i_0$ and $g(i_0) = f(j_0)$. Then $f \neq g$ (but they differ only at i_0), and g is clearly an acyclic homomorphism; this contradicts the D-pointedness of C. We also claim that there exist $p_0 \in V(D)$ and $v \in V(C)$ such that $f(p_0) = v$, the arc $p_0i_0 \in A(D)$, $p_0j_0 \notin A(D)$, the arc $vf(i_0) \in A(C)$, and $vf(j_0) \notin A(C)$. For if every $p_0 \in V(D)$ with $p_0i_0 \in A(D)$, $p_0j_0 \notin A(D)$ satisfies both $vf(i_0) \in A(C)$ and $vf(j_0) \in A(C)$, then we can again define the mapping $g \colon V(D) \to V(C)$ by $g(i) = f(i)$ for $i \neq i_0$ and $g(i_0) = f(j_0)$, which again contradicts the fact that C is D-pointed.

Thus let p_0, v as above satisfy $f(p_0) = v$, the arc $p_0 i_0 \in A(D)$, $p_0 j_0 \notin A(D)$, the arc $vf(i_0) \in A(C)$, and $vf(j_0) \notin A(C)$. The sets $W_{p_0} := \varphi^{-1}(v) \cap V_{p_0}$ and $\mathcal{B}' := \varphi^{-1}(f(j_0)) \cap V_{i_0}$ satisfy $|W_{p_0}| \geqslant t = |\varphi^{-1}(f(i_0)) \cap V_{i_0}|$ and $n/k \geqslant |\mathcal{B}'| = b \geqslant (n-t)/(k-1)$. Hence, there exists a set $\mathcal{A}' \subseteq W_{p_0}$ such that $|\mathcal{A}'| = n - |\mathcal{B}'|(k-1)$ with the property that there is no arc from $\mathcal{A}'$ to $\mathcal{B}'$ in D (as $\varphi(\mathcal{B}') = f(j_0)$ and $\varphi(\mathcal{A}') = v$ and $vf(j_0) \notin A(C)$). However, this contradicts Lemma 7. Thus, $i_0 j_0, j_0 i_0 \in A(D)$. Using this important fact, we proceed to show that (the second reason in) Case II also leads to a contradiction.

The definition of f gives us $|\varphi^{-1}(f(j_0)) \cap V_{j_0}| \geqslant n/k$. Since $n^{\epsilon\ell} \leqslant n^{1/4} \ll n/k$ we can choose $\mathcal{A} \subseteq \varphi^{-1}(f(j_0)) \cap V_{j_0}$ with $|\mathcal{A}| = \lfloor n/2k \rfloor$ such that no arc of M (the matching defined at the start of Section 4) has an end vertex in $\mathcal{A}$. Let $\mathcal{B} = \{z\} \subset \varphi^{-1}(f(j_0)) \cap V_{i_0}$. Since all arcs of M are independent, at most one arc of M is incident with z. Since $\varphi(\mathcal{A} \cup \mathcal{B}) = \{x\}$ and φ is an acyclic homomorphism, the subdigraph of D^* induced by $\mathcal{A} \cup \mathcal{B}$ is acyclic. To show that this is unlikely, we first estimate the expected number N of ways to select a vertex $z \in V_{i_0}$ and a subset $U \subseteq V_{j_0}$ of cardinality $\lfloor n/2k \rfloor$ so that the subdigraph $H_{z,U}$ of D^* they induce is acyclic and no arc of M is incident with a vertex in U. If $P_{z,U}$ denotes the probability that $H_{z,U}$ is acyclic, then

$$N \leqslant n \binom{n}{\lfloor n/2k \rfloor} P_{z,U} < n^n P_{z,U}. \tag{6}$$

In order to bound $P_{z,U}$, we employ Chernoff's Inequality (Theorem 4). Let Ω be the set of all potential arcs in the subdigraph $D'_{z,U}$, of D_0 induced by $\{z\} \cup U$. Each arc in Ω appears in $H_{z,U}$ with probability p. Let $\tau > (2+\epsilon)/\epsilon$ be a fixed integer. We index (by positive integers) those cycles of $D'_{z,U}$ that are of length $\tau + 1$. For $i \geqslant 1$, let S_i be the arc set of the ith such cycle and B_i be the event that the arcs in S_i all appear (i.e., the cycle determined by S_i is present in $H_{z,U}$). Let the random variable Y count the B_i's that occur. Since $P(Y = 0)$ is an upper bound for $P_{z,U}$, we can bound $P_{z,U}$ by bounding $P(Y = 0)$. Using Theorem 4 with $\gamma = 1$, we have

$$P(Y = 0) \leqslant P\big(|Y - E(Y)| \geqslant E(Y)\big) \leqslant 2e^{-E(Y)/3}. \tag{7}$$

Since the arcs of $D'_{z,U}$ within U are acyclically oriented, each choice of τ vertices within U determines exactly one potential $(\tau+1)$-cycle. It follows that

$$E(Y) = \binom{\lfloor n/2k \rfloor}{\tau} p^{\tau+1} > \Big(\frac{\lfloor n/2k \rfloor}{\tau}\Big)^{\tau} p^{\tau+1} > \frac{n^{\epsilon\tau+\epsilon-1}}{(4k\tau)^{\tau}}. \tag{8}$$

Using (7) and (8) we find that

$$P(Y = 0) \leqslant 2e^{-n^{\epsilon\tau+\epsilon-1}/3(4k\tau)^{\tau}},$$

and recalling our choice of τ (as exceeding $(2+\epsilon)/\epsilon$), we see that

$$P(Y = 0) \leqslant 2e^{-n^{1+2\epsilon}/3(4k\tau)^{\tau}} < e^{-n^{1+\epsilon}}. \tag{9}$$

Returning to (6), we have

$$N < n^n P_{z,U} < n^n e^{-n^{1+\epsilon}} = (ne^{-n^\epsilon})^n < e^{-n^{1+\epsilon}/2}.$$

By Markov's Inequality, the probability that there exists such a set $\{z\} \cup U$ that induces an acyclic subdigraph is less than $e^{-n^{1+\epsilon}/2}$, which means it is unlikely as desired.

Our discussion in Cases I and II implies that φ and $f \circ \psi$ have the same range. It is now evident that $\varphi = f \circ \psi$, for otherwise the same situation as in the proof that $\mathrm{Range}(\varphi) = \mathrm{Range}(f \circ \psi)$ occurs and similarly leads to a contradiction. Hence $\varphi = f \circ \psi$ as desired and therefore the proof of part (iii) of Theorem 1 is complete. $\square$

Acknowledgements

The authors thank Liam Rafferty (who suggested this line of research soon after completing [18]) and Michael Morris (who pointed out a gap in Case II within the manuscript). Thanks also to two referees (A and B) for their careful reading and constructive comments.

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