Skeleton Ideals of Certain Graphs, Standard Monomials and Spherical Parking Functions

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Abstract

Let G be a graph on the vertex set $V = \{0, 1, \ldots, n\}$ with root 0. Postnikov and Shapiro were the first to consider a monomial ideal $\mathcal{M}_G$, called the *G-parking function ideal*, in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$ and explained its connection to the chip-firing game on graphs. The standard monomials of the Artinian quotient $\frac{R}{\mathcal{M}_G}$ correspond bijectively to G-parking functions. Dochtermann introduced and studied *skeleton ideals of the graph G*, which are subideals of the G-parking function ideal with an additional parameter k $(0 \leqslant k \leqslant n-1)$. A *$k$-skeleton ideal* $\mathcal{M}_G^{(k)}$ of the graph G is generated by monomials corresponding to non-empty subsets of the set of non-root vertices $[n]$ of size at most $k+1$. Dochtermann obtained many interesting homological and combinatorial properties of these skeleton ideals. In this paper, we study the k-skeleton ideals of graphs and for certain classes of graphs provide explicit formulas and combinatorial interpretation of standard monomials and the Betti numbers.

Mathematics Subject Classifications: 05E40, 13D02

1 Introduction

Let G be a graph on the vertex set $V = \{0, 1, \ldots, n\}$ with a root 0. The graph G is completely determined by a symmetric $(n+1) \times (n+1)$ matrix $A(G) = [a_{ij}]_{0 \leqslant i,j \leqslant n}$, called its *adjacency matrix*, where a_{ij} is the number of edges from i to j. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the standard polynomial ring in n variables over a field $\mathbb{K}$. The *G-parking function ideal* $\mathcal{M}_G$ of G is a monomial ideal in R given by the generating set

$$\mathcal{M}_G = \langle m_A : \emptyset \neq A \subseteq [n] = \{1, \ldots, n\} \rangle ,$$

where $m_A = \prod_{i\in A} x_i^{d_A(i)}$ and $d_A(i) = \sum_{j\in V\setminus A} a_{ij}$ is the number of edges from i to a vertex outside the set A in G. The standard monomial basis $\{\mathbf{x}^{\mathbf{b}} = \prod_{i=1}^{n} x_i^{b_i}\}$ of the Artinian quotient $\frac{R}{\mathcal{M}_G}$ is determined by the set

$$\mathrm{PF}(G) = \{\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n : \mathbf{x}^{\mathbf{b}} \notin \mathcal{M}_G\}$$

of G-parking functions. Further, $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_G}\right)$ is the number of spanning trees of G, given by the determinant $\det(L_G)$ of the reduced Laplacian matrix L_G of G. Let $\mathrm{SPT}(G)$ be the set of spanning trees of G. The edges of a spanning tree of G are given orientation so that all paths in the spanning tree are directed away from the root. As $|\mathrm{PF}(G)| = |\mathrm{SPT}(G)|$, one would like to construct an explicit bijection $\phi : \mathrm{PF}(G) \to \mathrm{SPT}(G)$. Using the Depth-First-Search version of burning algorithm, an algorithmic bijection $\phi : \mathrm{PF}(G) \to \mathrm{SPT}(G)$ for simple graphs G, preserving *reverse sum* $\mathrm{rsum}(\mathcal{P})$ of G-parking function $\mathcal{P}$ and the number $\kappa(G, \phi(\mathcal{P}))$ of κ-*inversions* of the spanning tree $\phi(\mathcal{P})$, is constructed by Perkinson, Yang and Yu [13]. A similar bijection for multigraphs G is constructed by Gaydarov and Hopkins [5].

Postnikov and Shapiro [15] introduced the G-parking function ideal $\mathcal{M}_G$ and derived many of its combinatorial and homological properties. In particular, they showed that the cellular free complex supported on the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex Δ_{n-1} is a free resolution of $\mathcal{M}_G$. Further, the cellular resolution of $\mathcal{M}_G$ is minimal, provided the graph G is saturated (i.e., $a_{ij} > 0$ for $i \neq j$). The minimal resolution of the parking function ideal $\mathcal{M}_G$ for any graph G is described in [2, 10, 12].

In a series of papers, Dochtermann [3, 4] introduced and studied subideals of the G-parking function ideal $\mathcal{M}_G$ described by k-dimensional 'skeleta'. For an integer k $(0 \leqslant k \leqslant n-1)$, the *$k$-skeleton ideal* $\mathcal{M}_G^{(k)}$ of the graph G is defined as the subideal

$$\mathcal{M}_G^{(k)} = \langle m_A : \emptyset \neq A \subseteq [n];\ |A| \leqslant k+1 \rangle$$

of the monomial ideal $\mathcal{M}_G$. For $k = 0$, the ideal $\mathcal{M}_G^{(0)}$ is generated by powers of variables $x_1, \ldots, x_n$. Hence, its minimal free resolution and the number of standard monomials can be easily determined. For $k = 1$ and $G = K_{n+1}$, the minimal resolution of the one-skeleton ideal $\mathcal{M}_{K_{n+1}}^{(1)}$ is a cocellular resolution supported on the labelled polyhedral complex induced by any generic arrangement of two tropical hyperplanes in $\mathbb{R}^n$ and the i^{th} Betti number

$$\beta_i\left(\frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}}\right) = \sum_{j=1}^{n} j\binom{j-1}{i-1} \quad \text{for} \quad 1 \leqslant i \leqslant n-1$$

(see [3]). Also, the number of standard monomials of $\frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}}$ is given by

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}}\right) = (2n-1)(n-1)^{n-1} = \det(Q_{K_{n+1}}),$$

where $Q_{K_{n+1}}$ is the reduced signless Laplacian matrix of K_{n+1}.

In this paper, we determine all the Betti numbers of the k-skeleton ideal $\mathcal{M}_{K_{n+1}}^{(k)}$ of the complete graph K_{n+1}. The crucial observation is an identification of the ideal $\mathcal{M}_{K_{n+1}}^{(k)}$ with an Alexander dual of some multipermutohedron ideal. We first describe a permutohedron and an associated permutohedron ideal. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ such that $u_1 < u_2 < \cdots < u_n$ and let $\mathfrak{S}_n$ be the set of permutations of $[n]$. For a permutation σ of $[n]$, let $\sigma\mathbf{u} = (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$ and $\mathbf{x}^{\sigma\mathbf{u}} = \prod_{i=1}^{n} x_i^{u_{\sigma(i)}}$. The convex hull of all permutations $\sigma\mathbf{u}$ of $\mathbf{u}$ in $\mathbb{R}^n$ is an $(n-1)$-dimensional polytope $\mathbf{P}(\mathbf{u})$, called a *permutohedron*. Also, the monomial ideal $I(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in \mathfrak{S}_n \rangle$ of R is called a *permutohedron ideal*. If some coordinates of $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ are allowed to be equal, then the polytope $\mathbf{P}(\mathbf{u})$ is called a *multipermutohedron* and the monomial ideal $I(\mathbf{u})$ is called a *multipermutohedron ideal*.

The multigraded Betti numbers of multipermutohedron ideals are described in [7]. Also, a combinatorial description of multigraded Betti numbers of Alexander duals of multipermutohedron ideals is given in [8]. Now from the identification of $\mathcal{M}_{K_{n+1}}^{(k)}$ with an Alexander dual of some multipermutohedron ideal, we obtain a combinatorial expression for the $(i-1)^{th}$ Betti number $\beta_{i-1}\left(\mathcal{M}_{K_{n+1}}^{(k)}\right)$ (Theorem 12). In particular, for $n \geqslant 3$, we show that $\beta_{i-1}\left(\mathcal{M}_{K_{n+1}}^{(1)}\right) = i\binom{n+1}{i+1}$ and $\beta_{i-1}\left(\mathcal{M}_{K_{n+1}}^{(n-2)}\right)$ as in Corollary 13.

The main object of study in this paper are spherical G-parking functions. A finite sequence $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a *G-parking function* if $\mathbf{x}^{\mathcal{P}} = \prod_{i=1}^{n} x_i^{p_i} \notin \mathcal{M}_G$, on the other hand, the sequence $\mathcal{P} = (p_1, \ldots, p_n)$ is called a *spherical G-parking function* if $\mathbf{x}^{\mathcal{P}} \in \mathcal{M}_G \setminus \mathcal{M}_G^{(n-2)}$. A G-parking or a spherical G-parking function $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ can be equivalently thought of as a function $\mathcal{P} : [n] \to \mathbb{N}$ with $\mathcal{P}(i) = p_i \quad (1 \leqslant i \leqslant n)$. The *sum* (or *degree*) of $\mathcal{P}$ is given by $\mathrm{sum}(\mathcal{P}) = \sum_{i\in[n]} \mathcal{P}(i)$. Let

$$\mathrm{PF}(G) = \{\mathcal{P} \in \mathbb{N}^n : \mathbf{x}^{\mathcal{P}} \notin \mathcal{M}_G\} \quad \text{and} \quad \mathrm{sPF}(G) = \{\mathcal{P} \in \mathbb{N}^n : \mathbf{x}^{\mathcal{P}} \in \mathcal{M}_G \setminus \mathcal{M}_G^{(n-2)}\}$$

be the sets of G-parking functions and spherical G-parking functions, respectively. The standard monomials of $\frac{R}{\mathcal{M}_G^{(n-2)}}$ are of the form $\mathbf{x}^{\mathcal{P}}$ for $\mathcal{P} \in \mathrm{PF}(G)$ or $\mathcal{P} \in \mathrm{sPF}(G)$. Thus,

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_G^{(n-2)}}\right) = \dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_G}\right) + \dim_{\mathbb{K}}\left(\frac{\mathcal{M}_G}{\mathcal{M}_G^{(n-2)}}\right) = |\mathrm{PF}(G)| + |\mathrm{sPF}(G)|.$$

A notion of spherical K_{n+1}-parking functions is introduced in [4]. We recall that a K_{n+1}-parking function $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is an *ordinary parking function* of length n, i.e., a non-decreasing rearrangement $p_{i_1} \leqslant p_{i_2} \leqslant \cdots \leqslant p_{i_n}$ of $\mathcal{P} = (p_1, \ldots, p_n)$ satisfies $p_{i_j} < j$, for all j. It can be easily checked that $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is a spherical K_{n+1}-parking function if a non-decreasing rearrangement $p_{i_1} \leqslant p_{i_2} \leqslant \cdots \leqslant p_{i_n}$ of $\mathcal{P} = (p_1, \ldots, p_n)$ satisfies $p_{i_1} = 1$ and $p_{i_j} < j$ for $2 \leqslant j \leqslant n$. The notion of spherical K_{n+1}-parking function has appeared earlier in the literature (see [16]) as *prime parking functions* of length n. Prime parking functions were defined and enumerated by Ira Gessel. The number of spherical K_{n+1}-parking functions is $(n-1)^{n-1}$, which is same as the number of *uprooted trees* on the vertex set $[n]$. A (labelled) rooted tree T on the vertex set $[n]$ is called *uprooted* if the root is bigger than all its children. Let $\mathcal{U}_n$ be the set of uprooted trees on the vertex set $[n]$. Dochtermann conjectured existence of a bijection $\phi_n : \mathrm{sPF}(K_{n+1}) \to \mathcal{U}_n$ such that $\mathrm{sum}(\mathcal{P}) = \binom{n}{2} - \kappa(K_n, \phi_n(\mathcal{P})) + 1$, where $\kappa(K_n, \phi_n(\mathcal{P}))$ is the κ-number of the uprooted tree $\phi_n(\mathcal{P})$ in the complete graph $K_n = K_{n+1} - \{0\}$ on the vertex set $[n]$.

For a simple graph G on the vertex set V whose root 0 is connected to all other vertices, we construct an injective map $\phi_G : \mathrm{sPF}(G) \to \mathcal{U}(G')$, where $G' = G - \{0\}$ and $\mathcal{U}(G')$ is the set of uprooted spanning trees of G'. Moreover, the injective map ϕ_G satisfies

$$\mathrm{sum}(\mathcal{P}) = g(G) - \kappa(G', \phi_G(\mathcal{P})) + 1 \quad \text{for all} \quad \mathcal{P} \in \mathrm{sPF}(G),$$

where $g(G)$ is the *genus* of the graph G (Theorem 20). We have determined the image of ϕ_G for many simple graphs G. In particular, we show that the map $\phi_{K_{n+1}} = \phi_n : \mathrm{sPF}(K_{n+1}) \to \mathcal{U}_n$ is a bijection and establish a conjecture of Dochtermann on spherical K_{n+1}-parking functions.

If e is an edge of G, then $G - \{e\}$ is the graph obtained from G by deleting the edge e. We show that $|\mathrm{sPF}(G)| = |\mathrm{sPF}(G - \{e_0\})|$ (Lemma 17), where e_0 is an edge from the root to another vertex. As an application, we observe that $|\mathrm{sPF}(K_{m+1,n})| = |\mathrm{sPF}(K_{n+1,m})|$ for complete bipartite graphs (Proposition 33). If e_1 is an edge in the complete graph K_{n+1}, not through the root, we show that $|\mathrm{sPF}(K_{n+1} - \{e_1\})| = (n-1)^{n-3}(n-2)^2$ (Theorem 31). In this case, spherical $(K_{n+1} - \{e_1\})$-parking functions correspond bijectively with some specified subset of uprooted trees on the vertex set $[n]$ (Theorem 23).

Some extensions of these results for the complete multigraph $K_{n+1}^{a,b}$ and the complete bipartite multigraph $K_{m+1,n}^{a,b}$ $(a, b \geqslant 1)$ are also obtained.

Remark 1. This paper is motivated by [3] and an earlier version of [4] posted on the arXiv. In the new version of [4], Dochtermann and King identify the standard monomials of k-skeleton ideals $\mathcal{M}_{K_{n+1}}^{(k)}$ with the vector parking functions and using a Breadth-First-Search burning algorithm, they construct a bijection from spherical K_{n+1}-parking functions to uprooted spanning trees of K_n that takes degree to an inversion statistic. In this paper, we obtain the standard monomials and the Betti numbers of $\mathcal{M}_{K_{n+1}}^{(k)}$ by identifying it with an Alexander dual of some multipermutohedron ideal. For constructing bijection, we use a Depth-First-Search variant of burning algorithm.

2 Parking functions and Depth-First-Search algorithms

In this section, we briefly describe some known results on parking functions and the Depth-First-Search algorithms. Most of the known results are stated without proof. These results and notions will be used in the subsequent sections of this paper.

2.1 Parking functions

A sequence $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called an *ordinary parking function* of length n, if a non-decreasing rearrangement $p_{i_1} \leqslant p_{i_2} \leqslant \cdots \leqslant p_{i_n}$ of $\mathcal{P}$ satisfies $p_{i_j} < j$ for $1 \leqslant j \leqslant n$.

We denote the set of ordinary parking functions of length n by $\mathrm{PF}(n)$. The notion of ordinary parking function has a nice generalization.

Definition 2. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 1$. A finite sequence $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a λ-*parking function* of length n, if a non-decreasing rearrangement $p_{i_1} \leqslant p_{i_2} \leqslant \cdots \leqslant p_{i_n}$ of $\mathcal{P}$ satisfies $p_{i_j} < \lambda_{n-j+1}$ for $1 \leqslant j \leqslant n$. Let $\mathrm{PF}(\lambda)$ be the set of λ-parking functions.

Clearly, the ordinary parking functions of length n are precisely λ-parking functions of length n for $\lambda = (n, n-1, \ldots, 2, 1) \in \mathbb{N}^n$. The number of λ-parking functions is given by the 'so-called' Steck determinantal formula (see [14]). Let

$$\Lambda(\lambda_1, \ldots, \lambda_n) = \left[\frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!} \right]_{1 \leqslant i,j \leqslant n}.$$

In other words, the $(i,j)^{th}$ entry of the $n \times n$ matrix $\Lambda(\lambda_1, \ldots, \lambda_n)$ is $\frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!}$, where by convention, $\frac{1}{(j-i+1)!} = 0$ for $i > j+1$. The determinant $\det(\Lambda(\lambda_1, \ldots, \lambda_n))$ is called a *Steck determinant*.

Theorem 3 (Pitman-Stanley)**.** *The number of λ-parking functions is given by*

$$|\mathrm{PF}(\lambda)| = (n!)\ \det(\Lambda(\lambda_1, \ldots, \lambda_n)) = n! \quad \det \left[\frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!} \right]_{1 \leqslant i,j \leqslant n}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 1$, Postnikov and Shapiro [15] considered the monomial ideal

$$\mathcal{M}_\lambda = \left\langle \left(\prod_{j \in A} x_j \right)^{\lambda_{|A|}} : \emptyset \neq A \subseteq [n] \right\rangle$$

in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. A monomial $\mathbf{x}^{\mathbf{b}} = \prod_{j=1}^n x_j^{b_j} \notin \mathcal{M}_\lambda$ is called a *standard monomial* of $\frac{R}{\mathcal{M}_\lambda}$ or $\mathcal{M}_\lambda$. Clearly, $\mathbf{x}^{\mathbf{b}} = \prod_{j=1}^n x_j^{b_j}$ is a standard monomial of $\mathcal{M}_\lambda$ if and only if $\mathbf{b} = (b_1, \ldots, b_n) \in \mathrm{PF}(\lambda)$. In other words, a monomial basis of the $\mathbb{K}$-vector space $\frac{R}{\mathcal{M}_\lambda}$ correspond bijectively with the λ-parking functions.

Theorem 4 (Pitman-Stanley, Postnikov-Shapiro)**.** *The dimension of $\frac{R}{\mathcal{M}_\lambda}$ is given by*

$$\dim_{\mathbb{K}} \left(\frac{R}{\mathcal{M}_\lambda} \right) = |\mathrm{PF}(\lambda)| = \sum_{(b_1, \ldots, b_n) \in \mathrm{PF}(n)} \prod_{i=1}^n \left(\lambda_{n-b_i} - \lambda_{n-b_i+1} \right),$$

where the summation runs over ordinary parking functions of length n and $\lambda_{n+1} = 0$.

A closed formula for the number of λ-parking functions for various specific values of λ is given in [14, 17]. For more on parking functions, we refer to an excellent survey article by Yan [18].

2.2 Graph theoretic notions and G-parking functions

Let G be a connected graph on the vertex set $V(G) = V = \{0, 1, \ldots, n\}$. Suppose $A(G) = [a_{ij}]_{0 \leqslant i,j \leqslant n}$ is the (symmetric) adjacency matrix of G. We assume that G is a loopless graph, i.e., $a_{ii} = 0$ for all i. Let $E(i, j) = E(j, i)$ be the set of edges between distinct pair of vertices $i, j \in V$. If $E(i, j) \neq \emptyset$, then i and j are called *adjacent vertices* and we write $i \sim j$. On the other hand, if i and j are non-adjacent, we write $i \nsim j$. We have $|E(i, j)| = a_{ij}$. The graph G is called a *simple graph* if $|E(i, j)| = a_{ij} \leqslant 1$ for $i, j \in V$. Otherwise, G is called a *multigraph.* The set $E(G) = \bigcup_{i,j \in V} E(i, j)$ is the set of edges of G.

If $v \in V$, then $G - \{v\}$ denotes the graph on the vertex set $V \setminus \{v\}$ obtained from G by deleting the vertex v and all the edges through v. If $e \in E(G)$ is an edge of G, then $G - \{e\}$ denotes the graph on the vertex set V obtained from G by deleting the edge e. If $E(i, j) \neq \emptyset$, then $G - E(i, j)$ denotes the graph on vertex set V obtained from G on deleting all the edges between i and j.

Fix a root $r \in V$ of G (usually, we take $r = 0$). Set $\widetilde{V} = V \setminus \{r\}$. Let $\mathrm{SPT}(G)$ be the set of spanning trees of G rooted at r. We orient spanning tree $T \in \mathrm{SPT}(G)$ so that all paths in T are directed away from the root r. For every $j \in \widetilde{V}$, there is a unique oriented path in T from the root r to j. An $i \in \widetilde{V}$ lying on this unique path in T is called an *ancestor* of j in T. Equivalently, we say that j is a *descendent* of i in T. If in addition, i and j are adjacent in T, then we say that i is a *parent* of its *child* j. Every child j has a unique parent $\mathrm{par}_T(j)$ in T.

Definition 5. By an *inversion* of $T \in \mathrm{SPT}(G)$, we mean an ordered pair (i, j) of vertices such that i is an ancestor of j in T with $i > j$. The total number of inversions of a spanning tree T is denoted by $\mathrm{inv}(T)$. An inversion (i, j) of T is called a *κ-inversion* of T if i is not the root r and $\mathrm{par}_T(i)$ is adjacent to j in G.

The invariant $g(G) = |E(G)| - |V(G)| + 1$ is called the *genus* of the graph G. The *κ-number* $\kappa(G, T)$ of T in G is given by

$$\kappa(G, T) = \sum_{\substack{i,j \in \widetilde{V};\\ i > j}} |E(\mathrm{par}_T(i), j)|.$$

For a simple graph G, the total number of κ-inversions of T is $\kappa(G, T)$. If $G = K_{n+1}$ with root 0, then $\kappa(K_{n+1}, T) = \mathrm{inv}(T)$ for every $T \in \mathrm{SPT}(K_{n+1})$.

Definition 6. Let G be a graph on the vertex set $V = \{0, 1, \ldots, n\}$ with the adjacency matrix $A(G) = [a_{ij}]_{0 \leqslant i,j \leqslant n}$. Let $r \in V$ be the root of G and $\widetilde{V} = V \setminus \{r\}$. A function $\mathcal{P} : \widetilde{V} \to \mathbb{N}$ is called a *G-parking function* (with respect to the root r) if for every non-empty set $A \subseteq \widetilde{V}$, there exists $i \in A$ such that $\mathcal{P}(i) < d_A(i) = \sum_{j \in V \setminus A} a_{ij}$.

Note that, if root $r = 0$, then $\mathcal{P}$ is a G-parking function if and only if $\mathbf{x}^{\mathcal{P}} \notin \mathcal{M}_G$, i.e., $\mathbf{x}^{\mathcal{P}}$ is a standard monomial of the G-parking function ideal $\mathcal{M}_G$. For a G-parking function $\mathcal{P} : \widetilde{V} \to \mathbb{N}$, the *sum* $\text{sum}(\mathcal{P})$ and the *reverse sum* $\text{rsum}(\mathcal{P})$ of $\mathcal{P}$ are respectively given by

$$\text{sum}(\mathcal{P}) = \sum_{i \in \widetilde{V}} \mathcal{P}(i) \quad \text{and} \quad \text{rsum}(\mathcal{P}) = g(G) - \text{sum}(\mathcal{P}) = g(G) - \sum_{i \in \widetilde{V}} \mathcal{P}(i).$$

Definition 7. A rooted tree on the vertex set $[n]$ is called an *uprooted tree* if the root is bigger than all its children.

Let $\mathcal{U}_n$ be the set of uprooted trees on the vertex set $[n]$. Then it is well known that $|\mathcal{U}_n| = (n-1)^{n-1}$. For certain graphs G on the vertex set V, we shall show that the spherical G-parking functions correspond to uprooted spanning trees of $G' = G - \{0\}$.

2.3 Depth-First-Search Algorithms

We now describe the Depth-First-Search burning algorithm of Perkinson-Yang-Yu [13] for simple graphs. Let G be a simple graph on the vertex set V with a root $r \in V$. Applied to an input function $\mathcal{P} : V \setminus \{r\} \to \mathbb{N}$, the Depth-First-Search algorithm of Perkinson-Yang-Yu [13] gives a subset `burnt_vertices` of burnt vertices and a subset `tree_edges` of tree edges as an output. We imagine that a fire starts at the root r and spread to other vertices of G according to the depth-first rule. The value $\mathcal{P}(j)$ of the input function $\mathcal{P}$ can be considered as the number of water droplets available at vertex j that prevents spread of fire to j. If i is a burnt vertex, then consider the largest non-burnt vertex j adjacent to i. If $\mathcal{P}(j) = 0$, then fire from i will spread to j. In this case, add j in `burnt_vertices` and include the edge (i, j) in `tree_edges`. Now the fire spreads from the burnt vertex j. On the other hand, if $\mathcal{P}(j) > 0$, then one water droplet available at j will be used to prevent fire from reaching j through the edge (i, j). In this case, the dampened edge (i, j) is removed from G, number of water droplets available at j is reduced to $\mathcal{P}(j) - 1$ and the fire continue to spread from the burnt vertex i through non-dampened edges. If all the edges from i to unburnt vertices get dampened, then the search backtracks. At the start, `burnt_vertices` $= \{r\}$ and `tree_edges` $= \{\}$.

Perkinson, Yang and Yu [13] constructed a bijection $\phi : \text{PF}(G) \to \text{SPT}(G)$ using their Depth-First-Search algorithm.

Theorem 8 (Perkinson-Yang-Yu)**.** *Let G be a simple graph on the vertex set V with root r. After applying Depth-First-Search burning algorithm to $\mathcal{P} : V \setminus \{r\} \to \mathbb{N}$, if* `burnt_vertices` $= V$*, then $\mathcal{P}$ is a G-parking function and tree edges in the set* `tree_edges` *form a spanning tree $\phi(\mathcal{P})$ of G. If* `burnt_vertices` $\neq V$*, then $\mathcal{P}$ is not a G-parking function. Further, the mapping $\mathcal{P} \mapsto \phi(\mathcal{P})$ given by the Depth-First-Search algorithm induces a bijection $\phi : \text{PF}(G) \to \text{SPT}(G)$ such that*

$$\text{rsum}(\mathcal{P}) = g(G) - \text{sum}(\mathcal{P}) = \kappa(G, \phi(\mathcal{P})) \quad \text{for all} \quad \mathcal{P} \in \text{PF}(G).$$

Let $\sum_{\mathcal{P}\in \mathrm{PF}(G)} q^{\mathrm{rsum}(\mathcal{P})}$ be the *reversed sum enumerator* for G-parking functions. Theorem 8 establishes the identity

$$\sum_{\mathcal{P}\in \mathrm{PF}(G)} q^{\mathrm{rsum}(\mathcal{P})} = \sum_{T\in \mathrm{SPT}(G)} q^{\kappa(G,T)},$$

that extends a similar identity obtained by Kreweras [6] for the complete graph K_{n+1}.

We now describe the Depth-First-Search burning algorithm of Gaydarov-Hopkins [5] for multigraphs. Consider a connected multigraph G on the vertex set V with root r. Let $E(i,j) = E(j,i)$ be the set of edges between distinct pair of vertices i and j. Fix a total order on $E(i,j)$ for all distinct pairs $\{i,j\}$ of vertices and write $E(i,j) = \{e_{ij}^0, e_{ij}^1, \ldots, e_{ij}^{a_{ij}-1}\}$, where $|E(i,j)| = a_{ij}$. Thus we assume that edges of the multigraph G are labelled. Applied to an input function $\mathcal{P} : V \setminus \{r\} \to \mathbb{N}$, the Depth-First-Search algorithm for multigraphs gives a subset `burnt_vertices` of burnt vertices and a subset `tree_edges` of tree edges with nonnegative labels on them as an output. As in the case of Depth-First-Search algorithm for simple graphs, we imagine that a fire starts at the root r and spread to other vertices of G according to the depth-first rule. If i is a burnt vertex, then consider the largest non-burnt vertex j adjacent to i. If $\mathcal{P}(j) < a_{ij} = |E(i,j)|$, then $\mathcal{P}(j)$ edges with higher labels, namely $e_{ij}^{a_{ij}-1}, \ldots, e_{ij}^{a_{ij}-\mathcal{P}(j)}$ will get dampened, the edge $e_{ij}^{a_{ij}-\mathcal{P}(j)-1}$ with label $a_{ij} - \mathcal{P}(j) - 1$ will be added to `tree_edges` and j in included in `burnt_vertices`. Now fire will spread from the burnt vertex j. On the other hand, if $\mathcal{P}(j) \geqslant a_{ij}$, then all the edges in $E(i,j)$ get dampened and $\mathcal{P}(j)$ reduced to $\mathcal{P}(j) - a_{ij}$. The fire continue to spread from the burnt vertex i through non-dampened edges. If all the edges from i to unburnt vertices get dampened, then the search backtracks. At the start, `burnt_vertices` $= \{r\}$ and `tree_edges` $= \{\}$. Gaydarov and Hopkins [5] extended Theorem 8 to multigraphs using the Depth-First-Search burning algorithm for multigraph.

Theorem 9 (Gaydarov-Hopkins)**.** *Let G be a multigraph on V with root r. After applying Depth-First-Search burning algorithm to $\mathcal{P} : V \setminus \{r\} \to \mathbb{N}$, if* `burnt_vertices` $= V$*, then $\mathcal{P}$ is a G-parking function and tree edges with labels in the set* `tree_edges` *form a labelled spanning tree $\phi(\mathcal{P})$ of G. If* `burnt_vertices` $\neq V$*, then $\mathcal{P}$ is not a G-parking function. Suppose $\ell(e)$ is the label on an edge e of $\phi(\mathcal{P})$. Then the mapping $\mathcal{P} \mapsto \phi(\mathcal{P})$ given by Depth-First-Search burning algorithm induces a bijection $\phi : \mathrm{PF}(G) \to \mathrm{SPT}(G)$ such that*

$$\mathrm{rsum}(\mathcal{P}) = \kappa(G,T) + \sum_{e\in E(T)} \ell(e) \quad \text{for all} \quad \mathcal{P} \in \mathrm{PF}(G), \quad \text{where} \quad T = \phi(\mathcal{P}).$$

The bijective map induced by the Depth-First-Search algorithms is always denoted by ϕ in this paper ignoring its dependence on the graph and the root.

3 *k*-skeleton ideals of complete graphs

Let $0 \leqslant k \leqslant n-1$. Consider the k-skeleton ideal $\mathcal{M}^{(k)}_{K_{n+1}}$ of the complete graph K_{n+1} on the vertex set $V = \{0, 1, \ldots, n\}$. As stated in the Introduction, we have

$$\mathcal{M}^{(k)}_{K_{n+1}} = \left\langle \left(\prod_{j \in A} x_j\right)^{n-|A|+1} : \emptyset \neq A \subseteq [n];\ |A| \leqslant k+1 \right\rangle.$$

For $k = 0$, $\mathcal{M}^{(0)}_{K_{n+1}} = \langle x_1^n, \ldots, x_n^n \rangle$ is a monomial ideal in R generated by n^{th} power of variables. Thus, its minimal free resolution is given by the Koszul complex associated to the regular sequence $x_1^n, \ldots, x_n^n$ in R. Also, $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(0)}_{K_{n+1}}}\right) = n^n$. For $k = n-1$, $\mathcal{M}^{(n-1)}_{K_{n+1}} = \mathcal{M}_{K_{n+1}}$. The minimal free resolution of the K_{n+1}-parking function ideal $\mathcal{M}_{K_{n+1}}$ is the cellular resolution supported on the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1} and

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_{K_{n+1}}}\right) = |\mathrm{PF}(K_{n+1})| = |\mathrm{SPT}(K_{n+1})| = (n+1)^{n-1}.$$

For $k = 1$, the 1-skeleton ideal $\mathcal{M}^{(1)}_{K_{n+1}}$ has a minimal cocellular resolution supported on the labelled polyhedral complex induced by any generic arrangement of two tropical hyperplanes in $\mathbb{R}^{n-1}$ (see Theorem 4.6 of [3]) and $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(1)}_{K_{n+1}}}\right) = (2n-1)(n-1)^{n-1}$.

3.1 Betti numbers of $\mathcal{M}^{(k)}_{K_{n+1}}$

We now express the k-skeleton ideal $\mathcal{M}^{(k)}_{K_{n+1}}$ of K_{n+1} as an Alexander dual of a multipermutohedron ideal. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ such that $u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n$. Set $\mathbf{m} = (m_1, \ldots, m_s)$ such that the smallest entry in $\mathbf{u}$ is repeated exactly m_1 times, second smallest entry in $\mathbf{u}$ is repeated exactly m_2 times, and so on. Then $\sum_{j=1}^{s} m_j = n$ and $m_j \geqslant 1$ for all j. In this case, we write $\mathbf{u}(\mathbf{m})$ for $\mathbf{u}$. The monomial ideal $I(\mathbf{u}(\mathbf{m})) = \langle \mathbf{x}^{\sigma \mathbf{u}(\mathbf{m})} : \sigma \in \mathfrak{S}_n \rangle$ of R is called a *multipermutohedron ideal*. If $\mathbf{m} = (1, \ldots, 1) \in \mathbb{N}^n$, then $I(\mathbf{u}(\mathbf{m}))$ is a permutohedron ideal.

Let $\mathbf{u}(\mathbf{m}) = (1, 2, \ldots, k, k+1, \ldots, k+1) \in \mathbb{N}^n$, where $\mathbf{m} = (1, \ldots, 1, n-k) \in \mathbb{N}^{k+1}$. For $k = 0$, $\mathbf{u}(\mathbf{m}) = (1, \ldots, 1) \in \mathbb{N}^n$, while for $k = n-1$, $\mathbf{u}(\mathbf{m}) = (1, 2, \ldots, n) \in \mathbb{N}^n$. Let $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{n}]}$ be the Alexander dual of the multipermutohedron ideal $I(\mathbf{u}(\mathbf{m}))$ with respect to $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^n$.

Theorem 10. *For* $0 \leqslant k \leqslant n-1$, $\mathcal{M}^{(k)}_{K_{n+1}} = I(\mathbf{u}(\mathbf{m}))^{[\mathbf{n}]}$.

Proof. Using Proposition 5.23 of [11], it follows from the Lemma 2.3 of [8]. □

Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$. The $(i-1)^{th}$ multigraded Betti number $\beta_{i-1,\mathbf{b}}(\mathcal{M}^{(k)}_{K_{n+1}})$ of $\mathcal{M}^{(k)}_{K_{n+1}}$ in degree $\mathbf{b}$ is given by

$$\beta_{i-1,\mathbf{b}}(\mathcal{M}^{(k)}_{K_{n+1}}) = \dim_{\mathbb{K}} \tilde{H}^{|\mathrm{Supp}(\mathbf{b})|-i-1}\left(K_{\mathbf{b}}(\mathcal{M}^{(k)}_{K_{n+1}});\ \mathbb{K}\right) \quad \text{for} \quad i \geqslant 1,$$

where $K_{\mathbf{b}}(\mathcal{M}_{K_{n+1}}^{(k)})$ is the *lower Koszul simplicial complex* of $\mathcal{M}_{K_{n+1}}^{(k)}$ in degree $\mathbf{b}$ and $\mathrm{Supp}(\mathbf{b}) = \{j : b_j > 0\}$ (see Theorem 5.11 of [11]). Since $\mathcal{M}_{K_{n+1}}^{(k)} = I(\mathbf{u}(\mathbf{m}))^{[\mathbf{n}]}$, a combinatorial description of all multidegrees $\mathbf{b}$ such that $\beta_{i-1,\mathbf{b}}(\mathcal{M}_{K_{n+1}}^{(k)}) \neq 0$ is given in terms of dual $\mathbf{m}$-isolated subsets (see Definition 3.1 and Theorem 3.2 of [8]). For the particular case of $\mathbf{m} = (1, \ldots, 1, n-k) \in \mathbb{N}^{k+1}$, the notion of dual $\mathbf{m}$-isolated subsets can be easily described. Let $J = \{j_1, \ldots, j_t\} \subseteq [n]$ be a non-empty subset with $0 = j_0 < j_1 < \cdots < j_t$.

1. J is a dual $\mathbf{m}$-*isolated subset of type*-1 if $J \subseteq [k+1]$ and its dual weight $\mathrm{dwt}(J) = t-1$. Let $\mathcal{I}_{\mathbf{m}}^{*,1}$ be the set of dual $\mathbf{m}$-isolated subsets of type-1 and let $\mathcal{I}_{\mathbf{m}}^{*,1}\langle i\rangle = \{J \in \mathcal{I}_{\mathbf{m}}^{*,1} : \mathrm{dwt}(J) = i\}$.

2. $J = \{j_1, \ldots, j_t\}$ is a dual $\mathbf{m}$-*isolated subset of type*-2 if $J \setminus \{j_t\} \subseteq [k]$, $k+1 < j_t \leqslant n$ and its dual weight $\mathrm{dwt}(J) = (t-2)+(j_t-k)$. Let $\mathcal{I}_{\mathbf{m}}^{*,2}$ be the set of dual-$\mathbf{m}$ isolated subsets of type-2 and let $\mathcal{I}_{\mathbf{m}}^{*,2}\langle i\rangle = \{J \in \mathcal{I}_{\mathbf{m}}^{*,2} : \mathrm{dwt}(J) = i\}$.

Let $\mathcal{I}_{\mathbf{m}}^{*} = \mathcal{I}_{\mathbf{m}}^{*,1} \coprod \mathcal{I}_{\mathbf{m}}^{*,2}$ be the set of all dual $\mathbf{m}$-isolated subsets and $\mathcal{I}_{\mathbf{m}}^{*}\langle i\rangle = \mathcal{I}_{\mathbf{m}}^{*,1}\langle i\rangle \coprod \mathcal{I}_{\mathbf{m}}^{*,2}\langle i\rangle$.

Consider $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i = n-i+1$ for $1 \leqslant i \leqslant k$ and $\lambda_i = n-k$ for $k+1 \leqslant i \leqslant n$. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$. For $0 \leqslant i < j \leqslant n$, we set $\varepsilon(i,j) = \sum_{l=i+1}^{j} e_l$. For any $J = \{j_1, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m}}^{*}$, let $\mathbf{b}(J) = \sum_{\alpha=1}^{t} \lambda_{j_\alpha}\, \varepsilon(j_{\alpha-1}, j_\alpha) \in \mathbb{N}^n$.

We illustrate the concept of dual $\mathbf{m}$-isolated subsets and its relation with multigraded Betti numbers with an example.

Example 11. Let $n = 6$ and $k = 2$. Take $\mathbf{u}(\mathbf{m}) = (1,2,3,3,3,3)$. Then $\mathbf{m} = (1,1,4)$ and $\lambda = (6,5,4,4,4,4)$. Consider the multipermutohedron ideal $I(\mathbf{u}(\mathbf{m}))$ and the 2-skeleton ideal $\mathcal{M}_{K_{6+1}}^{(2)}$. Set $\mathbf{6} = (6,6,6,6,6,6)$. The Alexander dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{6}]} = \mathcal{M}_{K_{6+1}}^{(2)}$. A subset $J \subseteq [3]$ is a dual $\mathbf{m}$-isolated subset of type-1. For example, $J = \{2\}$ and $\tilde{J} = \{1,3\}$ are dual $\mathbf{m}$-isolated subsets of type-1 with dual weights 0 and 1, respectively. Also, the associated multidegrees are $\mathbf{b}(J) = (5,5,0,0,0,0)$ and $\mathbf{b}(\tilde{J}) = (6,4,4,0,0,0)$. The lower Koszul simplicial complex $K_{\mathbf{b}}(\mathcal{M}_{K_{6+1}}^{(2)})$ for $\mathbf{b} = \mathbf{b}(J)$ or $\mathbf{b}(\tilde{J})$ is isomorphic to the 0-dimensional simplicial complex consisting of two points. Thus $\beta_{0,\mathbf{b}(J)}(\mathcal{M}_{K_{6+1}}^{(2)}) = 1$ and $\beta_{1,\mathbf{b}(\tilde{J})}(\mathcal{M}_{K_{6+1}}^{(2)}) = 1$. Further, the subsets $J' = \{4\}$ and $J'' = \{1,5\}$ are examples of dual $\mathbf{m}$-isolated subsets of type-2 with dual weights 1 and 3, respectively. We have $\mathbf{b}(J') = (4,4,4,4,0,0)$ and $\mathbf{b}(J'') = (6,4,4,4,4,0)$. The lower Koszul simplicial complex $K_{\mathbf{b}(J')}(\mathcal{M}_{K_{6+1}}^{(2)})$ is isomorphic to the 0-skeleton of a 3-simplex, while $K_{\mathbf{b}(J'')}(\mathcal{M}_{K_{6+1}}^{(2)})$ is isomorphic to the 1-skeleton of a 3-simplex. Therefore $\beta_{1,\mathbf{b}(J')}(\mathcal{M}_{K_{6+1}}^{(2)}) = 3$ and $\beta_{3,\mathbf{b}(J'')}(\mathcal{M}_{K_{6+1}}^{(2)}) = 3$.

Theorem 12. *For $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ and $1 \leqslant i \leqslant n$, let $\beta_{i-1,\mathbf{b}}\left(\mathcal{M}_{K_{n+1}}^{(k)}\right)$ be the $(i-1)^{th}$ multigraded Betti number of $\mathcal{M}_{K_{n+1}}^{(k)}$ in degree $\mathbf{b}$. Then the following statements hold.*

(i) *For $J = \{j_1, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m}}^{*,1}\langle i-1\rangle$, $\beta_{i-1,\mathbf{b}(J)}\left(\mathcal{M}_{K_{n+1}}^{(k)}\right) = 1$, where $t = i$.*

(ii) *For* $J=\{j_1,\ldots,j_t\}\in\mathcal{I}^{*,2}_{\mathbf{m}}\langle i-1\rangle$, $\beta_{i-1,\mathbf{b}(J)}\left(\mathcal{M}^{(k)}_{K_{n+1}}\right)=\binom{j_t-j_{t-1}-1}{k-j_{t-1}}$, *where* $t+j_t-k=i+1$.

(iii) *If* $\mathbf{b}=\pi\mathbf{b}(J)$ *is a permutation of* $\mathbf{b}(J)$ *for some* $J\in\mathcal{I}^{*}_{\mathbf{m}}\langle i-1\rangle$ *and some* $\pi\in\mathfrak{S}_n$, *then* $\beta_{i-1,\mathbf{b}}\left(\mathcal{M}^{(k)}_{K_{n+1}}\right)=\beta_{i-1,\mathbf{b}(J)}\left(\mathcal{M}^{(k)}_{K_{n+1}}\right)$. *Otherwise,* $\beta_{i-1,\mathbf{b}}\left(\mathcal{M}^{(k)}_{K_{n+1}}\right)=0$.

(iv) *The* $(i-1)^{th}$*-Betti number* $\beta_{i-1}\left(\mathcal{M}^{(k)}_{K_{n+1}}\right)$ *of* $\mathcal{M}^{(k)}_{K_{n+1}}$ *is given by,*

$$\beta_{i-1}\left(\mathcal{M}^{(k)}_{K_{n+1}}\right)=\beta_i\left(\frac{R}{\mathcal{M}^{(k)}_{K_{n+1}}}\right)=\sum_{J\in\mathcal{I}^{*,1}_{\mathbf{m}}\langle i-1\rangle}\beta^{J}_{i-1}+\sum_{\tilde{J}\in\mathcal{I}^{*,2}_{\mathbf{m}}\langle i-1\rangle}\beta^{\tilde{J}}_{i-1},$$

where $\beta^{J}_{i-1}=\prod_{\alpha=1}^{i}\binom{j_{\alpha+1}}{j_\alpha}$ *and* $\beta^{\tilde{J}}_{i-1}=\left[\prod_{\alpha=1}^{t}\binom{l_{\alpha+1}}{l_\alpha}\right]\binom{l_t-l_{t-1}-1}{k-l_{t-1}}$ *for* $J=\{j_1,\ldots,j_i\}\in\mathcal{I}^{*,1}_{\mathbf{m}}\langle i-1\rangle$ *and* $\tilde{J}=\{l_1,\ldots,l_t\}\in\mathcal{I}^{*,2}_{\mathbf{m}}\langle i-1\rangle$. *Here,* $j_{i+1}=l_{t+1}=n$ *and* $l_0=0$.

Proof. Since $\mathcal{M}^{(k)}_{K_{n+1}}=I(\mathbf{u}(\mathbf{m}))^{[\mathbf{n}]}$, theorem follows from Theorem 3.2 and Corollary 3.4 of [8]. □

Theorem 12 describes all multigraded Betti numbers of $\mathcal{M}^{(k)}_{K_{n+1}}$. We hope that it could be helpful in constructing a concrete minimal resolution of $\mathcal{M}^{(k)}_{K_{n+1}}$.

Corollary 13. *Assume that* $n\geqslant 3$ *and* $1\leqslant i\leqslant n$. *Then* $\beta_{i-1}\left(\mathcal{M}^{(1)}_{K_{n+1}}\right)=i\binom{n+1}{i+1}$ *and*

$$\beta_{i-1}\left(\mathcal{M}^{(n-2)}_{K_{n+1}}\right)=\sum_{\mathbf{j}}\frac{n!}{j_1!(j_2-j_1)!\cdots(n-j_i)!}+\sum_{\ell}\frac{n!(n-l_{i-2}-1)}{l_1!(l_2-l_1)!\cdots(n-l_{i-2})!},$$

where the first and second summations run over all sequences of integers $\mathbf{j}=(j_1,\ldots,j_i)$ *with* $0<j_1<\cdots<j_i<n$ *and* $\ell=(l_0,l_1,\ldots,l_{i-2})$ *with* $0=l_0<l_1<\cdots<l_{i-2}<n-1$, *respectively.*

Proof. For $k=1$, we have $\mathbf{m}=(1,n-1)\in\mathbb{N}^2$. We can easily see that $\mathcal{I}^{*}_{\mathbf{m}}\langle i-1\rangle=\{\{1,i\},\{i+1\}\}$ for $i\geqslant 2$ and $\mathcal{I}^{*}_{\mathbf{m}}\langle 0\rangle=\{\{1\},\{2\}\}$. Thus, $\beta_0(\mathcal{M}^{(1)}_{K_{n+1}})=\beta^{\{1\}}_0+\beta^{\{2\}}_0=\binom{n}{1}+\binom{n}{2}=\binom{n+1}{2}$. For $i\geqslant 2$,

$$\begin{aligned}\beta_{i-1}(\mathcal{M}^{(1)}_{K_{n+1}}) &= \beta^{\{1,i\}}_{i-1}+\beta^{\{i+1\}}_{i-1}=\binom{i}{1}\binom{n}{i}\binom{i-2}{0}+\binom{n}{i+1}\binom{i}{1}\\ &= i\binom{n}{i}+i\binom{n}{i+1}=i\binom{n+1}{i+1},\end{aligned}$$

which is same as $\beta_i\left(\frac{R}{\mathcal{M}^{(1)}_{K_{n+1}}}\right)=\sum_{j=1}^{n}j\binom{j-1}{i-1}=\sum_{j=1}^{n}i\binom{j}{i}=(i)\sum_{j=1}^{n}\binom{j}{i}=i\binom{n+1}{i+1}$ obtained in [3].

For $k = n-2$, $J = \{j_1, \dots, j_i\} \in \mathcal{I}^{*,1}_{\mathbf{m}}\langle i-1\rangle$ if and only if $J \subseteq [n-1]$ and $\beta^J_{i-1} = \prod_{\alpha=1}^{i} \binom{j_{\alpha+1}}{j_\alpha}$. Also, $\tilde{J} = \{l_1, \dots, l_t\} \in \mathcal{I}^{*,2}_{\mathbf{m}}\langle i-1\rangle$ if and only if $l_{t-1} \leqslant n-2$, $l_t = n$ and $t = i-1$. Since, $\beta^{\tilde{J}}_{i-1} = \left[\prod_{\alpha=1}^{i-2} \binom{l_{\alpha+1}}{l_\alpha}\right] \binom{n-l_{i-2}-1}{n-l_{i-2}-2}$, we get the desired expression for $\beta_{i-1}\left(\mathcal{M}^{(n-2)}_{K_{n+1}}\right)$. □

Consider the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1}. We construct a polyhedral cell complex $\mathbf{Bd}^{(k)}(\Delta_{n-1})$ whose vertices are the vertices of $\mathbf{Bd}(\Delta_{n-1})$ corresponding to subsets $A \subseteq [n]$ with $|A| \leqslant k+1$. An edge in $\mathbf{Bd}^{(k)}(\Delta_{n-1})$ corresponds either to a chain $A_1 \subsetneq A_2 \subseteq [n]$ with $|A_2| \leqslant k+1$ or a pair $\{A, B\}$ of subsets of $[n]$ with $|A| = |B| = k+1$ and $|A \setminus B| = 1$. The higher dimensional faces of $\mathbf{Bd}^{(k)}(\Delta_{n-1})$ are polytopes spanned by its edges. A vertex of $\mathbf{Bd}^{(k)}(\Delta_{n-1})$ corresponding to A with $|A| \leqslant k+1$ has a natural label $\left(\prod_{j\in A} x_j\right)^{n-|A|+1}$. The cellular resolution supported on the polyhedral cell complex $\mathbf{Bd}^{(k)}(\Delta_{n-1})$ is a non-minimal resolution of $\mathcal{M}^{(k)}_{K_{n+1}}$ if $1 \leqslant k \leqslant n-2$. The minimal cellular resolution of $\mathcal{M}^{(1)}_{K_{4+1}}$ constructed in [3] can be obtained by deleting certain edges of the polyhedral cell complex $\mathbf{Bd}^{(1)}(\Delta_3)$.

3.2 Standard monomials of $\mathcal{M}^{(k)}_{K_{n+1}}$

A monomial $\mathbf{x}^{\mathbf{b}} = \prod_{j=1}^{n} x_j^{b_j} \notin \mathcal{M}^{(k)}_{K_{n+1}}$ is called a *standard monomial* of $\frac{R}{\mathcal{M}^{(k)}_{K_{n+1}}}$ or $\mathcal{M}^{(k)}_{K_{n+1}}$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_i = n-i+1$ for $1 \leqslant i \leqslant k$ and $\lambda_j = n-k$ for $k+1 \leqslant j \leqslant n$. We have seen that $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{n}]} = \mathcal{M}^{(k)}_{K_{n+1}} = \mathcal{M}_\lambda$. In view of Theorem 4, the number of standard monomials of $\mathcal{M}^{(k)}_{K_{n+1}}$ is precisely the number of λ-parking functions and $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(k)}_{K_{n+1}}}\right) = |\mathrm{PF}(\lambda)| = n! \quad \det(\Lambda(n, n-1, \dots, n-k+1, n-k, \dots, n-k))$.

More generally, for $a, b \geqslant 1$, we consider the complete multigraph $K^{a,b}_{n+1}$ on the vertex set V with adjacency matrix $A(K^{a,b}_{n+1}) = [a_{ij}]_{0\leqslant i,j\leqslant n}$ given by $a_{0,i} = a_{i,0} = a$ and $a_{i,j} = b$ for $i, j \in V \setminus \{0\}$; $i \neq j$. In other words, $K^{a,b}_{n+1}$ has exactly a number of edges between the root 0 and any other vertex i, while it has exactly b number of edges between distinct non-root vertices i and j. Clearly, $K^{1,1}_{n+1} = K_{n+1}$. The k-skeleton ideal $\mathcal{M}^{(k)}_{K^{a,b}_{n+1}}$ of $K^{a,b}_{n+1}$ is given by

$$\mathcal{M}^{(k)}_{K^{a,b}_{n+1}} = \left\langle \left(\prod_{j\in A} x_j\right)^{a+(n-|A|)b} : \emptyset \neq A \subseteq [n];\ |A| \leqslant k+1 \right\rangle.$$

Let $\lambda^{a,b} = (\lambda^{a,b}_1, \dots, \lambda^{a,b}_n)$, where $\lambda^{a,b}_i = a+(n-i)b$ for $1 \leqslant i \leqslant k$ and $\lambda^{a,b}_j = a+(n-k-1)b$ for $k+1 \leqslant j \leqslant n$. Then, $\mathcal{M}^{(k)}_{K^{a,b}_{n+1}} = \mathcal{M}_{\lambda^{a,b}}$ and from Theorem 4,

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(k)}_{K^{a,b}_{n+1}}}\right) = n! \quad \det(\Lambda(\lambda^{a,b}_1, \dots, \lambda^{a,b}_n)).$$

We proceed to evaluate the Steck determinant and compute the number of standard monomials of $\mathcal{M}^{(k)}_{K^{a,b}_{n+1}}$. Consider the polynomial

$$f_n(x) = \det(\Lambda(x + (n-1)b, x + (n-2)b, \ldots, x + b, x))$$

in an indeterminate x. In other words, we have

$$f_n(x) = \det \begin{bmatrix} \frac{x}{1!} & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \\ 1 & \frac{x+b}{1!} & \frac{(x+b)^2}{2!} & \cdots & \frac{(x+b)^{n-2}}{(n-2)!} & \frac{(x+b)^{n-1}}{(n-1)!} \\ 0 & 1 & \frac{x+2b}{1!} & \cdots & \frac{(x+2b)^{n-3}}{(n-3)!} & \frac{(x+2b)^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(x+(n-2)b)}{1!} & \frac{(x+(n-2)b)^2}{2!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{x+(n-1)b}{1!} \end{bmatrix}.$$

The polynomial $f_n(x) = \frac{x(x+nb)^{n-1}}{n!}$ and $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_{K^{a,b}_{n+1}}}\right) = a(a+nb)^{n-1}$ (see [14, 15]). Also, for $1 \leqslant k \leqslant n-2$, consider another polynomial $g_{n;k}(x)$ in x given by

$$g_{n;k}(x) = \det(\Lambda(x + kb, x + (k-1)b, \ldots, x + b, x, \ldots, x)),$$

where the last $n-k$ coordinates in $(x + kb, x + (k-1)b, \ldots, x + b, x, \ldots, x)$ are x.

Proposition 14. *The polynomial $g_{n;k}(x)$ is given by*

$$g_{n;k}(x) = \sum_{j=0}^{k} \frac{1}{j!}\frac{x^{n-j}}{(n-j)!}(k-j+1)(k+1)^{j-1}b^j.$$

Proof. We first give a simple proof of $f_n(x) = \frac{x(x+nb)^{n-1}}{n!}$ as in [9]. Clearly, $f_1(x) = x$ and $f_2(x) = \frac{x(x+2b)}{2!}$. Proceeding by induction on n, we assume that $f_j(x) = \frac{x(x+jb)^{j-1}}{j!}$ for $1 \leqslant j \leqslant n-1$. Further, using properties of determinants, we observe that the derivative $f_n'(x)$ of $f_n(x)$ satisfies $f_n'(x) = f_{n-1}(x+b)$. This shows that $f_n'(x) = \frac{(x+b)(x+nb)^{n-2}}{(n-1)!}$. As $f_n(0) = 0$, on integrating $f_n'(x) = \frac{(x+b)(x+nb)^{n-2}}{(n-1)!}$ by parts, we get $f_n(x) = \frac{x(x+nb)^{n-1}}{n!}$.

Again using properties of determinants, we see that the $(n-k-1)^{th}$ derivative $g^{(n-k-1)}_{n;k}(x)$ of $g_{n;k}(x)$ satisfies

$$g^{(n-k-1)}_{n;k}(x) = f_{k+1}(x) = \frac{x(x+(k+1)b)^k}{(k+1)!} = \sum_{j=0}^{k}\binom{k}{j}x^{k-j+1}\frac{(k+1)^j b^j}{(k+1)!}.$$

Since $g_{n;k}(0) = g'_{n;k}(0) = \cdots = g^{(n-k-1)}_{n;k}(0) = 0$ and the $(n-k-1)^{th}$ derivative of $\frac{x^{n-j}}{(n-j)(n-j-1)\ldots(k-j+2)}$ is x^{k-j+1}, we get $g_{n;k}(x) = \sum_{j=0}^{k}\binom{k}{j}\frac{x^{n-j}}{(n-j)(n-j-1)\cdots(k-j+2)}\frac{(k+1)^j b^j}{(k+1)!}$. ☐

Theorem 15 (Yan)**.** *The number of standard monomials of* $\frac{R}{\mathcal{M}^{(k)}_{K^{a,b}_{n+1}}}$ *is given by*

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(k)}_{K^{a,b}_{n+1}}}\right) = \sum_{j=0}^{k}\binom{n}{j}(a+(n-k-1)b)^{n-j}(k-j+1)(k+1)^{j-1}b^j.$$

In particular, we have $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(1)}_{K^{a,b}_{n+1}}}\right) = (a+(n-2)b)^{n-1}(a+(2n-2)b)$ *for* $k=1$ *and* $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(n-2)}_{K^{a,b}_{n+1}}}\right) = a(a+nb)^{n-1}+(n-1)^{n-1}b^n$ *for* $k=n-2$.

Proof. The first part follows from $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(k)}_{K^{a,b}_{n+1}}}\right) = n!\ g_{n;k}(a+(n-k-1)b)$ using Proposition 14.

For $k=1$, $g^{(n-2)}_{n;1}(x) = f_2(x) = \frac{x(x+2b)}{2!} = \frac{x^2}{2!}+bx$. As $g^{(j)}_{n;1}(0)=0$ for $0 \leqslant j \leqslant n-2$, we obtain

$$g_{n;1}(x) = \frac{x^n}{n!}+\frac{bx^{n-1}}{(n-1)!} = \frac{x^{n-1}(x+nb)}{n!}.$$

Now $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(1)}_{K^{a,b}_{n+1}}}\right) = n!\ g_{n;1}(a+(n-2)b) = (a+(n-2)b)^{n-1}(a+(2n-2)b)$.

Also, for $k=n-2$, we have $g'_{n;n-2}(x) = f_{n-1}(x) = \frac{x(x+(n-1)b)^{n-2}}{(n-1)!}$. On integrating it by parts, we get $g_{n;n-2}(x) = \frac{x(x+(n-1)b)^{n-1}}{(n-1)!(n-1)} - \frac{(x+(n-1)b)^n}{n!(n-1)} + C$, where C is a constant of integration. Since $g_{n;n-2}(0)=0$, we get $C = \frac{(n-1)^{n-1}b^n}{n!}$. Hence,

$$g_{n;n-2}(x) = \frac{1}{n!}[(x-b)(x+(n-1)b)^{n-1}+(n-1)^{n-1}b^n].$$

Again, from $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(n-2)}_{K^{a,b}_{n+1}}}\right) = n!\ g_{n;n-2}(a+b)$, we get the desired result. □

Remark 16. The determinant $\det(Q_{K^{a,b}_{n+1}})$ of the reduced signless Laplacian matrix $Q_{K^{a,b}_{n+1}}$ of $K^{a,b}_{n+1}$ satisfies $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(1)}_{K^{a,b}_{n+1}}}\right) = (a+(n-2)b)^{n-1}(a+(2n-2)b) = \det(Q_{K^{a,b}_{n+1}})$. Also, we have $g'_{n;n-2}(x) = f_{n-1}(x) = \frac{x(x+(n-1)b)^{n-2}}{(n-1)!} = \sum_{j=0}^{n-2}\binom{n-2}{j}\frac{x^{n-1-j}}{(n-1)!}(n-1)^j b^j$. Thus on integrating $g'_{n;n-2}(x)$ in two ways, we get $g_{n;n-2}(x)$ and a polynomial identity

$$\frac{(x-b)(x+(n-1)b)^{n-1}+(n-1)^{n-1}b^n}{n!} = \frac{\sum_{j=0}^{n-2}\binom{n}{j}x^{n-j}(n-j-1)(n-1)^{j-1}b^j}{n!}.$$

On substituting $x = a + b$, we get an identity

$$\sum_{j=0}^{n-2} \binom{n}{j} (a+b)^{n-j}(n-j-1)(n-1)^{j-1}b^j = a(a+nb)^{n-1} + (n-1)^{n-1}b^n$$

for positive integers a and b. Taking $a = b = 1$, it justifies the equality

$$\sum_{j=0}^{n-2} \binom{n}{j} 2^{n-j}(n-j-1)(n-1)^{j-1} = (n+1)^{n-1} + (n-1)^{n-1}$$

described in [4](Corollary 3.7).

4 Spherical G-parking functions

Let G be a connected graph on the vertex set $V = \{0, 1, \ldots, n\}$ with root 0. As stated in the Introduction, $\mathcal{P} : [n] \to \mathbb{N}$ is a spherical G-parking function if $\mathbf{x}^{\mathcal{P}} = \prod_{i \in [n]} x_i^{\mathcal{P}(i)} \in \mathcal{M}_G \setminus \mathcal{M}_G^{(n-2)}$. Let $\mathrm{PF}(G)$ (or $\mathrm{sPF}(G)$) be the set of G-parking functions (respectively, spherical G-parking functions).

Let e_0 be an edge of G joining the root 0 to another vertex. We shall compare $\mathrm{sPF}(G)$ with $\mathrm{sPF}(\bar{G})$, where $\bar{G} = G - \{e_0\}$. After renumbering vertices, we may assume that $e_0 = e_{0,n}$ is an edge joining the root 0 with n.

Lemma 17. *Let G be a connected graph on the vertex set V and $\bar{G} = G - \{e_0\}$. Then*

$$\mathcal{M}_{\bar{G}} = (\mathcal{M}_G : x_n) = \{z \in R : zx_n \in \mathcal{M}_G\}.$$

Further, the multiplication map $\mu_{x_n} : \{\mathbf{x}^{\mathcal{P}} : \mathcal{P} \in \mathrm{sPF}(\bar{G})\} \to \{\mathbf{x}^{\mathcal{P}} : \mathcal{P} \in \mathrm{sPF}(G)\}$ induced by x_n is a bijection. In particular, $|\mathrm{sPF}(G)| = |\mathrm{sPF}(\bar{G})|$.

Proof. For $\emptyset \neq A \subseteq [n]$, let m_A and m'_A be the generators of $\mathcal{M}_G$ and $\mathcal{M}_{\bar{G}}$, respectively. Clearly, $m_A = m'_A$ if $n \notin A$ and $m_A = m'_A x_n$ if $n \in A$. This shows that $\mathcal{M}_{\bar{G}} = (\mathcal{M}_G : x_n)$. Also, $\mathcal{M}_{\bar{G}}^{(n-2)} = (\mathcal{M}_G^{(n-2)} : x_n)$. Thus the natural sequences of R-modules (or $\mathbb{K}$-vectors spaces)

$$0 \to \frac{R}{\mathcal{M}_{\bar{G}}} \overset{\mu_{x_n}}{\to} \frac{R}{\mathcal{M}_G} \to \frac{R}{\langle \mathcal{M}_G, x_n \rangle} \to 0 \text{ and } 0 \to \frac{R}{\mathcal{M}_{\bar{G}}^{(n-2)}} \overset{\mu_{x_n}}{\to} \frac{R}{\mathcal{M}_G^{(n-2)}} \to \frac{R}{\langle \mathcal{M}_G^{(n-2)}, x_n \rangle} \to 0$$

are short exact. Let $\alpha : \frac{R}{\mathcal{M}_{\bar{G}}^{(n-2)}} \to \frac{R}{\mathcal{M}_{\bar{G}}}$ and $\beta : \frac{R}{\mathcal{M}_G^{(n-2)}} \to \frac{R}{\mathcal{M}_G}$ be the natural projections. Since $\langle \mathcal{M}_G,\ x_n \rangle = \langle \mathcal{M}_G^{(n-2)},\ x_n \rangle$, the multiplication map μ_{x_n} induces an isomorphism $\ker(\alpha) \overset{\sim}{\to} \ker(\beta)$ between kernels $\ker(\alpha)$ and $\ker(\beta)$. Also $\{\mathbf{x}^{\mathcal{P}} : \mathcal{P} \in \mathrm{sPF}(\bar{G})\}$ and $\{\mathbf{x}^{\mathcal{P}} : \mathcal{P} \in \mathrm{sPF}(G)\}$ are monomial basis of $\ker(\alpha)$ and $\ker(\beta)$, respectively. Thus μ_{x_n} induces a bijection between the bases. ☐

We now give a few applications of the Lemma 17.

Proposition 18. *Let E be the set of all edges of K_{n+1} or $K_{n+1}^{a,b}$ through the root* 0*. Then*

(1) $|\mathrm{sPF}(K_{n+1}-E)| = |\mathrm{sPF}(K_{n+1})|$.

(2) $|\mathrm{sPF}(K_{n+1}^{a,b}-E)| = |\mathrm{sPF}(K_{n+1}^{a,b})|$.

(3) $|\mathrm{sPF}(K_{n+1}^{a,b})| = b^n(n-1)^{n-1}$.

Proof. By Lemma 17, we know that the number of spherical G-parking functions and the number of spherical $(G-\{e_0\})$-parking functions are the same for any edge e_0 of G through the root 0. Now, repeatedly applying Lemma 17, we see that (1) and (2) hold.

Let $\lambda = ((n-1)b, (n-2)b, \ldots, 2b, b, b)$. Consider the graph $K_{n+1}^{a,b}-E$ and its $(n-2)$-skeleton ideal $\mathcal{M}^{(n-2)}_{K_{n+1}^{a,b}-E}$. Clearly, $\mathcal{M}^{(n-2)}_{K_{n+1}^{a,b}-E} = \mathcal{M}_\lambda$. As $K_{n+1}^{a,b}-E$ is disconnected, $\mathrm{PF}(K_{n+1}^{a,b}-E) = \emptyset$. Thus

$$\begin{aligned} |\mathrm{sPF}(K_{n+1}^{a,b})| &= |\mathrm{sPF}(K_{n+1}^{a,b}-E)| = \dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}^{(n-2)}_{K_{n+1}^{a,b}-E}}\right) \\ &= |\mathrm{PF}(\lambda)| = (n!)g_{n;n-2}(b) = b^n(n-1)^{n-1}, \end{aligned}$$

where the polynomial $g_{n;n-2}(x)$ is given in the Remark 16. □

Note that the cardinality $|\mathrm{sPF}(K_{n+1}^{a,b})|$ is independent of a. As we have seen that $|\mathrm{PF}(K_{n+1}^{a,b})| = a(a+bn)^{n-1}$, $|\mathrm{sPF}(K_{n+1}^{a,b})| = b^n(n-1)^{n-1}$ also follows from Theorem 15.

4.1 A modified Depth-First-Search burning algorithm

Let G be a connected simple graph on the vertex set V with a root 0. Let $\mathcal{M}_G = \langle m_A : \emptyset \neq A \subseteq [n]\rangle$ be the G-parking function ideal. For a spherical G-parking function $\mathcal{P} \in \mathrm{sPF}(G)$, define $\widetilde{\mathcal{P}} : [n] \to \mathbb{N}$ so that $\mathbf{x}^{\widetilde{\mathcal{P}}} = \frac{\mathbf{x}^{\mathcal{P}}}{m_{[n]}}$, where $m_{[n]}$ is the generator of $\mathcal{M}_G$ corresponding to $[n]$. We say that $\widetilde{\mathcal{P}}$ is the *reduced spherical G-parking function* associated to $\mathcal{P} \in \mathrm{sPF}(G)$. Let $\widetilde{\mathrm{sPF}}(G) = \{\widetilde{\mathcal{P}} : \mathcal{P} \in \mathrm{sPF}(G)\}$ be the set of reduced spherical G-parking functions. We shall analyse the condition $\widetilde{\mathrm{sPF}}(G) \subseteq \mathrm{PF}(G)$. Since removing (or adding) edges from the root 0 to another vertex in G do not change the number of spherical G-parking functions (Lemma 17), we may assume that the root 0 is connected to all the other vertices in G. In this case, $m_{[n]} = x_1x_2\cdots x_n$ and $\widetilde{\mathcal{P}}(i) = \mathcal{P}(i) - 1$ for $i \in [n]$.

Lemma 19. *Let G be a connected simple graph on the vertex set V with a root* 0*. Suppose the root* 0 *is connected to all other vertices of G. Then*

(i) $\widetilde{\mathrm{sPF}}(G) \subseteq \mathrm{PF}(G)$.

(ii) *Let $\mathcal{P} \in \mathrm{sPF}(G)$ and $r \in [n]$ be the unique vertex such that $\widetilde{\mathcal{P}}(r) = 0$ but $\widetilde{\mathcal{P}}(j) \geqslant 1$ for $j > r$. Consider the graph $G' = G - \{0\}$ on the vertex set $[n]$ with root r. Then $\widehat{\mathcal{P}} = \widetilde{\mathcal{P}}|_{[n]\setminus\{r\}}$ is a G'-parking function.*

Proof. Let $\mathcal{P} \in \mathrm{sPF}(G)$ such that $\widetilde{\mathcal{P}} \notin \mathrm{PF}(G)$. Then there exists $\emptyset \neq A \subseteq [n]$ such that $m_A \mid \mathbf{x}^{\widetilde{\mathcal{P}}}$, i.e., m_A divides $\mathbf{x}^{\widetilde{\mathcal{P}}}$. Thus $m_A m_{[n]} \mid \mathbf{x}^{\mathcal{P}}$. If $A \neq [n]$, then $m_A \mid \mathbf{x}^{\mathcal{P}}$, a contradiction to $\mathcal{P} \in \mathrm{sPF}(G)$. Also, if $A = [n]$, then $(m_{[n]})^2 \mid \mathbf{x}^{\mathcal{P}}$. Since G is a simple graph and the root 0 is connected to all other vertices of G, $m_B \mid (x_1 x_2 \cdots x_n)^2$ for any $B \subseteq [n]$ with $|B| = n-1$. Again a contradiction. This proves the first part.

Let $\mathcal{P} \in \mathrm{sPF}(G)$. If $\widetilde{\mathcal{P}}(i) \geqslant 1$ for all $i \in [n]$, then $\mathcal{P}(i) \geqslant 2$ for all i. Thus $(m_{[n]})^2 \mid \mathbf{x}^{\mathcal{P}}$, which leads to a contradiction. Thus $\widetilde{\mathcal{P}}(i) = 0$ for some i. Let $r = \max\{i \in [n] : \widetilde{\mathcal{P}}(i) = 0\}$. Now consider the graph $G' = G - \{0\}$ on the vertex set $[n]$ with root r. When we emphasize the root r of G', we denote this graph by (G', r). Let $\mathcal{M}_{(G',r)} = \langle \bar{m}_A : \emptyset \neq A \subseteq [n] \setminus \{r\} \rangle$ be the G'-parking function ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, \widehat{x_r}, \ldots, x_n]$. We see that $\bar{m}_A = \frac{m_A}{\gcd(m_A, m_{[n]})}$. If $\widehat{\mathcal{P}} = \widetilde{\mathcal{P}}|_{[n]\setminus\{r\}}$ is not a G'-parking function, then $\bar{m}_A \mid \prod_{i \in [n]\setminus\{r\}} (x_i)^{\widehat{\mathcal{P}}(i)}$ for some non-empty subset $A \subseteq [n] \setminus \{r\}$. As $\widetilde{\mathcal{P}}(r) = 0$, $\mathbf{x}^{\widetilde{\mathcal{P}}} = \prod_{i \in [n]\setminus\{r\}} (x_i)^{\widehat{\mathcal{P}}(i)} = \frac{\mathbf{x}^{\mathcal{P}}}{m_{[n]}}$. Thus $m_A \mid \mathbf{x}^{\mathcal{P}}$, a contradiction to $\mathcal{P} \in \mathrm{sPF}(G)$. □

We now proceed to associate uprooted trees to spherical parking functions by modifying the Depth-First-Search burning algorithm. Let G be a connected simple graph satisfying the hypothesis of Lemma 19. Let $\mathcal{P} \in \mathrm{sPF}(G)$ and $\widetilde{\mathcal{P}}$ be the associated reduced spherical G-parking function. In the following three steps, an uprooted spanning tree of G' is associated to each $\mathcal{P} \in \mathrm{sPF}(G)$.

1. Set $r = \max\{i \in [n] : \widetilde{\mathcal{P}}(i) = 0\}$ and consider the graph $G' = G - \{0\}$ with root r.
2. Let $\phi : \mathrm{PF}(G', r) \to \mathrm{SPT}(G', r)$ be the bijective map induced by Depth-First-Search algorithm (Theorem 8). As $\widehat{\mathcal{P}} = \widetilde{\mathcal{P}}|_{[n]\setminus\{r\}}$ is a (G', r)-parking function, $\phi(\widehat{\mathcal{P}})$ is a spanning tree of G'. Also, $\mathrm{sum}(\widehat{\mathcal{P}}) = g(G') - \kappa(G', \phi(\widehat{\mathcal{P}}))$.
3. Since $\widehat{\mathcal{P}} \in \mathrm{PF}(G', r)$ and $\widehat{\mathcal{P}}(j) \geqslant 1$ for all $j > r$, there exists $i < r$ such that $\widehat{\mathcal{P}}(i) = 0$. On applying the Depth-First-Search algorithm to $\widehat{\mathcal{P}}$, all the edges (r, j) for $j > r$ get dampened. Thus the spanning tree $\phi(\widehat{\mathcal{P}})$ is an uprooted spanning tree of G'.

Let $\mathcal{U}(G')$ be the set of uprooted spanning trees of the graph G'. We define a map $\phi_G : \mathrm{sPF}(G) \to \mathcal{U}(G')$ given by $\phi_G(\mathcal{P}) = \phi(\widehat{\mathcal{P}})$, where $\widehat{\mathcal{P}} = \widetilde{\mathcal{P}}|_{[n]\setminus\{r\}}$. We say that the map ϕ_G is induced by a *modified Depth-First-Search algorithm.*

Theorem 20. *Let G be a simple graph on the vertex set V with root 0 and $G' = G - \{0\}$. Suppose the root 0 is connected to all other vertices of G. Then there exists an injective map $\phi_G : \mathrm{sPF}(G) \to \mathcal{U}(G')$ such that $\mathrm{sum}(\mathcal{P}) = g(G) - \kappa(G', \phi_G(\mathcal{P})) + 1$ for all $\mathcal{P} \in \mathrm{sPF}(G)$.*

Proof. We have already constructed the map ϕ_G. Let $\mathcal{P}, \mathcal{P}' \in \mathrm{sPF}(G)$ such that $\phi_G(\mathcal{P}) = \phi_G(\mathcal{P}') = T \in \mathcal{U}(G')$. Let r be the root of T. Since $\phi : \mathrm{PF}(G', r) \to \mathrm{SPT}(G', r)$ is a bijection and $\phi(\widehat{\mathcal{P}}) = \phi(\widehat{\mathcal{P}'})$, we have $\widehat{\mathcal{P}} = \widehat{\mathcal{P}'}$ and hence $\mathcal{P} = \mathcal{P}'$. Note that $\mathrm{sum}(\mathcal{P}) = \mathrm{sum}(\widehat{\mathcal{P}}) + n$ and $g(G) = g(G') + n - 1$. Thus $\mathrm{sum}(\mathcal{P}) = g(G) - \kappa(G', \phi_G(\mathcal{P})) + 1$ follows from $\mathrm{sum}(\widehat{\mathcal{P}}) = g(G') - \kappa(G', \phi(\widehat{\mathcal{P}}))$. □

Let $\mathrm{Im}(\phi_G) = \{\phi_G(\mathcal{P}) : \mathcal{P} \in \mathrm{sPF}(G)\}$ be the image of ϕ_G in $\mathcal{U}(G')$. Theorem 20 shows that under some mild conditions on the simple graph G, the spherical G-parking functions correspond bijectively with the uprooted trees in $\mathrm{Im}(\phi_G)$. In general, it is not easy to give a combinatorial description for the image $\mathrm{Im}(\phi_G)$.

Let $T \in \mathcal{U}(G')$ be an uprooted spanning tree of $G' = G - \{0\}$. Suppose $\mathrm{root}(T) = r$. Consider the bijective map $\phi : \mathrm{PF}(G', r) \to \mathrm{SPT}(G', r)$. Then there exists a unique (G', r)-parking function $\mathcal{P}_T$ such that $\phi(\mathcal{P}_T) = T$. Let

$$\overline{\mathcal{U}}(G') = \{T \in \mathcal{U}(G') : \mathcal{P}_T(j) \geqslant 1 \text{ for } j > r = \mathrm{root}(T)\}.$$

Proposition 21. $\mathrm{Im}(\phi_G) \subseteq \overline{\mathcal{U}}(G') = \{T \in \mathcal{U}(G') : \mathcal{P}_T(j) \geqslant 1 \text{ for } j > r = \mathrm{root}(T)\}$.

Proof. Let $\phi_G(\mathcal{P}) = \phi(\widehat{\mathcal{P}}) = T$, where $\widehat{\mathcal{P}} = \widetilde{\mathcal{P}}|_{[n]\setminus\{r\}}$. As $\mathcal{P}_T = \widehat{\mathcal{P}}$ and the root is given by $\mathrm{root}(T) = \max\{i \in [n] : \widetilde{\mathcal{P}}(i) = 0\}$, the result follows. $\square$

4.2 Spherical parking functions for complete graphs

Let K_{n+1} be the complete graph on the vertex set V and $K_n = K_{n+1} - \{0\}$ be the complete graph on the vertex set $[n]$. Let $\mathcal{U}_n = \mathcal{U}(K_n)$ be the set of uprooted trees on the vertex set $[n]$. From Theorem 20, there exists an injective map $\phi_n = \phi_{K_{n+1}} : \mathrm{sPF}(K_{n+1}) \to \mathcal{U}_n$. We show that ϕ_n is a bijection and solve a conjecture of Dochtermann on spherical K_{n+1}-parking functions.

Theorem 22. *There exists a bijection $\phi_n : \mathrm{sPF}(K_{n+1}) \to \mathcal{U}_n$ such that*

$$\mathrm{sum}(\mathcal{P}) = \binom{n}{2} - \kappa(K_n, \phi_n(\mathcal{P})) + 1 \quad \text{for all} \quad \mathcal{P} \in \mathrm{sPF}(K_{n+1}).$$

Proof. The existence of injective map $\phi_n = \phi_{K_{n+1}} : \mathrm{sPF}(K_{n+1}) \to \mathcal{U}_n$ with the desired property follows from the Theorem 20. We just need to show that ϕ_n is surjective. Let $T \in \mathcal{U}_n$ and $\mathrm{root}(T) = r$. Consider the bijective map $\phi : \mathrm{PF}(K_n, r) \to \mathrm{SPT}(K_n, r)$ induced by Depth-First-Search algorithm and $\mathcal{P}_T$ is the unique (K_n, r)-parking function such that $\phi(\mathcal{P}_T) = T$. Since T is uprooted, $\mathcal{P}_T(j) \geqslant 1$ for $j > r$. Now consider ideals $\mathcal{M}_{K_{n+1}} = \langle m_A : \emptyset \neq A \subseteq [n] \rangle$ and $\mathcal{M}_{(K_n, r)} = \langle \bar{m}_B : \emptyset \neq B \subseteq [n] \setminus \{r\} \rangle$.

Suppose, if possible, $\mathcal{P}_T \neq \widehat{\mathcal{P}}$ for all $\mathcal{P} \in \mathrm{sPF}(K_{n+1})$. Then $m_{[n]} \prod_{j \in [n]\setminus\{r\}} x_j^{\mathcal{P}_T(j)}$ is not a standard monomial of $\mathcal{M}_{K_{n+1}}^{(n-2)}$. Thus there exists $\emptyset \neq A \subsetneq [n]$ such that m_A divides $m_{[n]} \prod_{j \in [n]\setminus\{r\}} x_j^{\mathcal{P}_T(j)}$. If $r \in A$, then x_r appearing in $m_A = (\prod_{j \in A} x_j)^{n-|A|+1}$ must have the multiplicity 1. This is possible, only if $A = [n]$, a contradiction. If $r \notin A$, then $\bar{m}_A = \frac{m_A}{\gcd(m_A, m_{[n]})}$ and $\bar{m}_A \mid \prod_{j \in [n]\setminus\{r\}} x_j^{\mathcal{P}_T(j)}$. This shows that $\mathcal{P}_T$ is not a (K_n, r)-parking function, again a contradiction. Hence ϕ_n is surjective.

The surjectivity of ϕ_n also follows from $|\mathrm{sPF}(K_{n+1})| = |\mathcal{U}_n| = (n-1)^{n-1}$. $\square$

We now study spherical G-parking functions for $G = K_{n+1} - \{e\}$, where e is an edge not through the root 0. Let $e = e_{p,q} = (p, q)$ be the edge in K_{n+1} joining p and

q with $1 \leqslant p < q \leqslant n$. Let $G' = G - \{0\}$ be the graph on the vertex set $[n]$ and $\mathcal{U}(G')$ be the set of uprooted spanning trees of G'. In fact, $\mathcal{U}_n^{(p \nsim q)} = \mathcal{U}(G')$ is the set of uprooted trees on the vertex set $[n]$ with no edge between p and q (i.e., $p \nsim q$). Let $\overline{\mathcal{U}}_n^{(p \nsim q)} = \overline{\mathcal{U}}(G') = \{T \in \mathcal{U}(G') : \mathcal{P}_T(j) \geqslant 1 \text{ for } j > r = \text{root}(T)\}$ as in Proposition 21 and set $\mathcal{U}'_n = \mathcal{U}_n^{(1 \nsim n)}$. In view of Theorem 20 and Proposition 21, there exists an injective map $\phi_G : \text{sPF}(G) \to \overline{\mathcal{U}}_n^{(p \nsim q)}$.

Theorem 23. *For $n \geqslant 3$ and $G = K_{n+1} - \{e_{p,q}\}$, the map $\phi_G : \text{sPF}(G) \to \overline{\mathcal{U}}_n^{(p \nsim q)}$ is a bijection such that* $\text{sum}(\mathcal{P}) = \binom{n}{2} - \kappa(G', \phi_G(\mathcal{P}))$ *for all* $\mathcal{P} \in \text{sPF}(G)$*, where $G' = G - \{0\}$.*

Proof. We only need to show that $\text{Im}(\phi_G) = \overline{\mathcal{U}}(G')$. This proof is similar to the proof of Theorem 22. Let $T \in \overline{\mathcal{U}}(G') = \overline{\mathcal{U}}_n^{(p \nsim q)}$ and $\text{root}(T) = r$. Consider the bijective map $\phi : \text{PF}(G', r) \to \text{SPT}(G', r)$ induced by Depth-First-Search algorithm and $\mathcal{P}_T$ is the unique (G', r)-parking function such that $\phi(\mathcal{P}_T) = T$. Let $\mathcal{M}_G = \langle m_A : \emptyset \neq A \subseteq [n] \rangle$ and $\mathcal{M}_{(G',r)} = \langle \bar{m}_A : \emptyset \neq A \subseteq [n] \setminus \{r\} \rangle$ be the parking function ideals. Suppose, if possible, $\mathcal{P}_T \neq \widehat{\mathcal{P}}$ for all $\mathcal{P} \in \text{sPF}(G)$. Then $m_{[n]} \prod_{j \in [n] \setminus \{r\}} x_j^{\mathcal{P}_T(j)}$ is not a standard monomial of $\mathcal{M}_G^{(n-2)}$. Thus there exists $\emptyset \neq A \subsetneq [n]$ such that m_A divides $m_{[n]} \prod_{j \in [n] \setminus \{r\}} x_j^{\mathcal{P}_T(j)}$.

Let $r \in A$ but $r \notin \{p, q\}$. As $m_A \mid m_{[n]} \prod_{j \in [n] \setminus \{r\}} x_j^{\mathcal{P}_T(j)}$, x_r appearing in m_A must have multiplicity 1. Thus $A = [n]$, a contradiction. Now suppose $r = q \in A$ (or $r = p \in A$). Then $A \neq [n]$ implies that $A = [n] \setminus \{p\}$ (respectively, $A = [n] \setminus \{q\}$). In fact, $m_{[n] \setminus \{p\}} = (\prod_{j \in [n] \setminus \{p,q\}} x_j^2) x_q$ and $m_{[n] \setminus \{q\}} = (\prod_{j \in [n] \setminus \{p,q\}} x_j^2) x_p$. Clearly, in either of the cases, $\bar{m}_{[n] \setminus \{p,q\}} = \prod_{j \in [n] \setminus \{p,q\}} x_j$ divides $\prod_{j \in [n] \setminus \{r\}} x_j^{\mathcal{P}_T(j)}$, a contradiction to $\mathcal{P}_T$ being (G', r)-parking function.

Finally, if $r \notin A$, then $\bar{m}_A = \frac{m_A}{\gcd(m_A, m_{[n]})}$ and $\bar{m}_A$ divides $\prod_{j \in [n] \setminus \{r\}} x_j^{\mathcal{P}_T(j)}$. This shows that $\mathcal{P}_T$ is not a (G', r)-parking function, again a contradiction. This completes the proof. ☐

We now determine conditions so that $\mathcal{U}_n^{(p \nsim q)} = \overline{\mathcal{U}}_n^{(p \nsim q)}$.

Proposition 24. $\mathcal{U}_n^{(p \nsim q)} \setminus \overline{\mathcal{U}}_n^{(p \nsim q)} = \{T \in \mathcal{U}_n^{(p \nsim q)} : \text{root}(T) = p \text{ and } \mathcal{P}_T(q) = 0\}$.

Proof. Let $T \in \mathcal{U}_n^{(p \nsim q)}$ such that $\text{root}(T) = r \neq p$. Consider the unique (G', r)-parking function $\mathcal{P}_T$ such that $\phi(\mathcal{P}_T) = T$. As T is uprooted, all the edges (r, j) in G' for $j > r$ must get dampened. Thus $\mathcal{P}_T(j) \geqslant 1$ for all $j > r$ such that $r \sim j$ in G' or G. Since $G = K_{n+1} - \{e_{p,q}\}$, $T \in \overline{\mathcal{U}}_n^{(p \nsim q)}$. ☐

Since there are no uprooted tree T on the vertex set $[n]$ with $\text{root}(T) = 1$, it follows from Proposition 24 that $\mathcal{U}_n^{(p \nsim q)} = \overline{\mathcal{U}}_n^{(p \nsim q)}$ if and only if $p = 1$. The following corollary is immediate.

Corollary 25. *For $n \geqslant 3$ and $G = K_{n+1} - \{e_{1,n}\}$, the map $\phi_G : \text{sPF}(G) \to \mathcal{U}_n^{(1 \nsim n)} = \mathcal{U}'_n$ induces a bijection between the set of spherical G-parking functions and the set of uprooted trees on the vertex set $[n]$ with $1 \nsim n$.*

Remark 26. By renumbering vertices of G, we easily see that

$$|\mathrm{sPF}(K_{n+1} - \{e_{p,q}\})| = |\mathrm{sPF}(K_{n+1} - \{e_{1,n}\})| = |\mathcal{U}'_n|,$$

for any edge $e_{p,q}$ between vertices $p, q \in [n]$ with $p < q$. Thus, $|\overline{\mathcal{U}}_n^{(p\nsim q)}| = |\mathcal{U}'_n|$.

The bijection $\phi_n : \mathrm{sPF}(K_{n+1}) \to \mathcal{U}_n$ constructed in Theorem 22 can be extended to the case of the complete multigraph $K_{n+1}^{a,b}$ on the vertex set V.

Let $\mathrm{sPF}(K_{n+1}^{a,b})$ be the set of spherical $K_{n+1}^{a,b}$-parking functions. Let $\mathcal{U}_n^b$ be the set of uprooted tree T on the vertex set $[n]$ with label $\ell : E(T) \to \{0, 1, \ldots, b-1\}$ on the edges of T and a weight $\omega(r) \in \{0, 1, \ldots, b-1\}$ assigned to the root r of T. Clearly, $|\mathcal{U}_n^b| = b^n|\mathcal{U}_n| = b^n(n-1)^{n-1}$. Also, $|\mathrm{sPF}(K_{n+1}^{a,b})| = b^n(n-1)^{n-1}$ is independent of a. We may assume that $a \geqslant b$. As an application of the Depth-First-Search algorithm for multigraph (Theorem 9), we construct a bijection

$$\phi_n^b : \mathrm{sPF}(K_{n+1}^{a,b}) \to \mathcal{U}_n^b.$$

The *reduced spherical* $K_{n+1}^{a,b}$*-parking function* $\widetilde{\mathcal{P}}$ associated to $\mathcal{P} \in \mathrm{sPF}(K_{n+1}^{a,b})$ is given by $\widetilde{\mathcal{P}}(i) = \mathcal{P}(i) - a$ for all $i \in [n]$. Let $\widetilde{\mathrm{sPF}}(K_{n+1}^{a,b}) = \{\widetilde{\mathcal{P}} : \mathcal{P} \in \mathrm{sPF}(K_{n+1}^{a,b})\}$. Then as $a \geqslant b$, we can verify that $\widetilde{\mathrm{sPF}}(K_{n+1}^{a,b}) \subseteq \mathrm{PF}(K_{n+1}^{a,b})$. Let $K_n^b = K_{n+1}^{a,b} - \{0\}$ be the complete multigraph on the vertex set $[n]$ such that $|E(i,j)| = b$ for every distinct pair $\{i, j\}$ of vertices.

Theorem 27. *There exists a bijection* $\phi_n^b : \mathrm{sPF}(K_{n+1}^{a,b}) \to \mathcal{U}_n^b$ *such that*

$$\mathrm{rsum}(\mathcal{P}) + \omega(r) + 1 = \kappa(K_n^b, T) + \sum_{e \in E(T)} \ell(e) \quad \text{for all} \quad \mathcal{P} \in \mathrm{sPF}(K_{n+1}^{a,b}),$$

where $T = \phi_n^b(\mathcal{P})$ *and weight* $\omega(r) \in \{0, 1, \ldots, b-1\}$ *at the* $\mathrm{root}(T) = r$.

Proof. Let $\mathcal{P} \in \mathrm{sPF}(K_{n+1}^{a,b})$. Then $\widetilde{\mathcal{P}} \in \mathrm{PF}(K_{n+1}^{a,b})$. Choose the largest vertex r of $K_n^b = K_{n+1}^{a,b} - \{0\}$ such that $\widetilde{\mathcal{P}}(r) < b$. We claim that $\widetilde{\mathcal{P}}(j) < b$ for some $j < r$. Otherwise, $\mathcal{P}(i) \geqslant a + b$, for all $i \in [n] \setminus \{r\}$, a contradiction to $\mathcal{P} \in \mathrm{sPF}(K_{n+1}^{a,b})$. Now consider r to be the root of the complete multigraph K_n^b on the vertex set $[n]$. Then $\widehat{\mathcal{P}} = \widetilde{\mathcal{P}}\,|_{[n]\setminus\{r\}}$ is a (K_n^b, r)-parking function. On applying the Depth-First-Search algorithm for multigraph (Theorem 9), we get $\phi(\widehat{P}) \in \mathcal{U}_n^b$ with root r and weight $\omega(r) = \widetilde{\mathcal{P}}(r)$. The mapping $\phi_n^b : \mathrm{sPF}(K_{n+1}^{a,b}) \to \mathcal{U}_n^b$ given by $\phi_n^b(\mathcal{P}) = \phi(\widehat{P})$ is clearly injective. Since $|\mathrm{sPF}(K_{n+1}^{a,b})| = |\mathcal{U}_n^b| = b^n(n-1)^{n-1}$, the map ϕ_n^b is a bijection. Also,

$$g(K_n^b) - \sum_{i \in [n]\setminus\{r\}} \widetilde{\mathcal{P}}(i) = \mathrm{rsum}(\widehat{P}) = \kappa(K_n^b, \phi(\widehat{P})) + \sum_{e \in E(\phi(\widehat{P}))} \ell(e).$$

Since $\mathrm{rsum}(\mathcal{P}) = g(K_{n+1}^{a,b}) - \sum_{i\in[n]} \mathcal{P}(i)$, we verify that $\mathrm{rsum}(\widehat{P}) = \mathrm{rsum}(\mathcal{P}) + \omega(r) + 1$. $\square$

4.3 Counting uprooted trees

In this subsection, we determine the number $|\mathcal{U}'_n|$ of uprooted trees on the vertex set $[n]$ with $1 \nsim n$. Let $\mathcal{T}_{n,0}$ be the set of labelled trees on the vertex set $[n]$ such that the root has no child (or son) with smaller labels. Let $\mathcal{A}_n$ be the set of labelled rooted-trees on the vertex set $[n]$ with a non-rooted leaf n. Chauve, Dulucq and Guibert [1] constructed a bijection $\eta : \mathcal{T}_{n,0} \to \mathcal{A}_n$. As earlier, let $\mathcal{U}_n$ be the set of uprooted trees on the vertex set $[n]$. Also, let $\mathcal{B}_n$ be the set of labelled rooted-trees on the vertex set $[n]$ with a non-rooted leaf 1. We see that there are bijections $\mathcal{U}_n \to \mathcal{T}_{n,0}$ and $\mathcal{B}_n \to \mathcal{A}_n$ obtained by simply changing label i to $n-i+1$ for all i. The bijection $\eta : \mathcal{T}_{n,0} \to \mathcal{A}_n$ induces a bijection $\psi : \mathcal{U}_n \to \mathcal{B}_n$. For sake of completeness, we briefly describe construction of the bijection ψ essentially as in [1].

Let $T \in \mathcal{U}_n$ with root r. Note that $r \neq 1$.

Step (1) : Consider a maximal increasing subtree T_0 of T containing 1. Let $T_1, \ldots, T_l$ be the subtrees (with at least one edge) of T obtained by deleting edges in T_0. Let r_i be the root of T_i for $1 \leqslant i \leqslant l$. The root r of T must be a root of one of the subtrees T_i. Let $r_j = r$. Then 1 is a leaf of T_j.

Step (2) : If T_0 has m vertices, then T_0 is determined by an increasing tree $\overline{T_0}$ on the vertex set $[m]$ and a set S_0 of labels on T_0. We write $T_0 = (\overline{T_0}, S_0)$.

Step (3) : Let $\overline{S_0} = (S_0 \setminus \{1\}) \cup \{r\}$. Then $(\overline{T_0}, \overline{S_0})$ determines an increasing subtree $\widetilde{T_0}$ with root $r' = \min\{\overline{S_0}\}$. Graft T_j on the increasing subtree $\widetilde{T_0}$ at the root r and obtain a tree T'_j. Now graft T_i $(i \neq j)$ on T'_j at r_i and obtain a tree T' with root r'. Also note that 1 is a non-rooted leaf of T'.

All the above steps can be reversed, thus $\psi(T) = T'$ defines a bijection $\psi : \mathcal{U}_n \to \mathcal{B}_n$.

Lemma 28. $|\mathcal{U}_n| = |\mathcal{B}_n| = (n-1)^{n-1}$.

Proof. The bijection $\psi : \mathcal{U}_n \to \mathcal{B}_n$ gives $|\mathcal{U}_n| = |\mathcal{B}_n|$. The number of labelled rooted-trees on the vertex set $\{2, 3, \ldots, n\}$ by Cayley's formula is $(n-1)^{n-2}$. Any tree in $\mathcal{B}_n$ is obtained uniquely by attaching 1 to any node i of a labelled rooted tree on the vertex set $\{2, 3, \ldots, n\}$. Since there are exactly $n-1$ possibilities for i, we have $|\mathcal{B}_n| = (n-1)^{n-2}(n-1) = (n-1)^{n-1}$. □

For $n \geqslant 3$, let $\mathcal{U}'_n = \{T \in \mathcal{U}_n : 1 \nsim n \text{ in } T\}$. We shall determine the image $\psi(\mathcal{U}'_n) \subseteq \mathcal{B}_n$ of $\mathcal{U}'_n$ under the bijection $\psi : \mathcal{U}_n \to \mathcal{B}_n$. Let $\mathcal{B}'_n = \{T' \in \mathcal{B}_n : 1 \nsim n \text{ in } T'\}$. Set

$$\begin{aligned}
\mathcal{A} &= \{T' \in \mathcal{B}'_n : \ \text{root}(T') = r' = n\}, \\
\mathcal{B}' &= \{T' \in \mathcal{B}'_n : \ \text{root}(T') = r' \neq n \text{ with } r' \sim n \text{ and 1 is a descendent of } n\}, \\
\mathcal{B}'' &= \{T' \in \mathcal{B}'_n : \text{root}(T') = r' \neq n \text{ with } r' \nsim n\}.
\end{aligned}$$

Lemma 29. $\psi(\mathcal{U}'_n) = \mathcal{A} \coprod \mathcal{B}' \coprod \mathcal{B}''$.

Proof. Let $T' \in \mathcal{B}_n$. Then there is a unique $T \in \mathcal{U}_n$ such that $T' = \psi(T)$. Let r and r' be the roots of T and T', respectively. Clearly, $r \neq 1$. Let $\text{Son}_T(1)$ be the set of sons of 1 in T. Then from the construction of $T' = \psi(T)$, $r' = \min\{\{r\} \cup \text{Son}_T(1)\}$. Also, the leaf 1 in T' is adjacent to j if and only if $j = \mathrm{par}_T(1)$ is the parent of 1 in T. This shows that $1 \nsim n$ in T if and only if $1 \nsim n$ in T'. Hence, $\psi(\mathcal{U}'_n) \subseteq \mathcal{B}'_n$. Further, we see that $r' = n$ if and only if 1 is already a leaf in T, and in this case, $T' = \psi(T) = T$. In other words, $\mathcal{A} \subseteq \mathcal{U}'_n$ and $\psi(T) = T$ for all $T \in \mathcal{A}$.

If $T' \in \mathcal{B}''$, then the unique $T \in \mathcal{U}_n$ with $\psi(T) = T'$ must have $1 \nsim n$ in T, that is, $T \in \mathcal{U}'_n$. Now we consider the remaining case. Let $T' \in \mathcal{B}'_n$ with $\mathrm{root}(T') = r' \neq n$ and $r' \sim n$ in T'. We shall show that $\psi(T) = T'$ for $T \in \mathcal{U}'_n$ if and only if 1 is a descendent of n in T' (or equivalently, $T' \in \mathcal{B}'$). Consider the maximal increasing subtree T'_0 of T' containing the root r'. If 1 is a descendent of a leaf r'_j of T'_0, then the maximal increasing subtree T_0 of T containing 1 is obtained by replacing r'_j with 1 in the vertex set of T'_0 and labeling it as indicated in Step (2) of the construction of ψ. Clearly, $r'_j = r$ is the root of T. If $r'_j = r \neq n$, then $1 \sim n$ in T as $r' \sim n$ in T'. Thus, if $r'_j \neq n$, i.e., 1 is not a descendent of n in T', then $T' \notin \psi(\mathcal{U}'_n)$. On the other hand, if $r'_j = n$, i.e., 1 is a descendent of n in T' with $1 \nsim n$, then $\mathrm{root}(T) = r = n$ and $1 \nsim n$ in T. ☐

Proposition 30. *For $n \geqslant 3$, we have $|\mathcal{U}'_n| = (n-1)^{n-3}(n-2)^2$.*

Proof. By Lemma 29, we have $|\mathcal{U}'_n| = |\psi(\mathcal{U}'_n)| = |\mathcal{A}| + |\mathcal{B}'| + |\mathcal{B}''|$. First we enumerate the subset $\mathcal{A} = \{T' \in \mathcal{B}'_n : \mathrm{root}(T') = r' = n\}$. The number of labelled trees on the vertex set $\{2, 3, \ldots, n\}$ with root n is $(n-1)^{n-3}$. Since any tree in $\mathcal{A}$ is uniquely obtained by attaching 1 to any node $i \in \{2, \ldots, n-1\}$ of a labelled tree on the vertex set $\{2, \ldots, n\}$ with root n, we have $|\mathcal{A}| = (n-1)^{n-3}(n-2)$.

Let us consider the subset $\mathcal{C} = \{T' \in \mathcal{B}'_n : \mathrm{root}(T') = r' \neq n\} \subseteq \mathcal{B}'_n$. Clearly, $\mathcal{B} = \mathcal{B}' \coprod \mathcal{B}'' \subseteq \mathcal{C}$. The enumeration of $\mathcal{C}$ is similar to that of $\mathcal{A}$, except now the root $r' \in \{2, \ldots, n-1\}$ can take any one of the $n-2$ values. Thus $|\mathcal{C}| = (n-1)^{n-3}(n-2)^2$. We can easily construct a bijective correspondence between $\mathcal{A}$ and $\mathcal{C} \setminus \mathcal{B}$. Let $T' \in \mathcal{A}$. Then $1 \nsim n$ in T' and $\mathrm{root}(T') = n$. Consider the unique path from the root n to the leaf 1 in T'. As $1 \nsim n$ in T', the child $\tilde{r}$ of n lying on this unique path is different from 1. Let $\tilde{T'}$ be rooted tree consisting of the tree T' with the new root $\tilde{r}$. As $\mathrm{root}(\tilde{T'}) = \tilde{r} \neq n$, $\tilde{r} \sim n$ and 1 is not a descendent of n in $\tilde{T'}$, we have $\tilde{T'} \in \mathcal{C} \setminus \mathcal{B}$. The mapping $T' \mapsto \tilde{T'}$ from $\mathcal{A}$ to $\mathcal{C} \setminus \mathcal{B}$ is clearly a bijection. If $\tilde{T'} \in \mathcal{C} \setminus \mathcal{B}$, then $\mathrm{root}(\tilde{T'}) = \tilde{r} \neq n$, $\tilde{r} \sim n$ and 1 is not a descendent of n in $\tilde{T'}$. Now unique $T' \in \mathcal{A}$ that maps to $\tilde{T'}$ is the rooted tree obtained from $\tilde{T'}$ by taking n as the new root. Thus $|\mathcal{A}| = |\mathcal{C} \setminus \mathcal{B}|$ and hence, $|\mathcal{U}'_n| = |\mathcal{C}| = (n-1)^{n-3}(n-2)^2$. ☐

Theorem 31. *Let $e_{p,q}$ be an edge of K_{n+1} joining distinct vertices $p, q \in [n]$. For $n \geqslant 3$, the number of spherical parking functions of $K_{n+1} - \{e_{p,q}\}$ is given by*

$$|\mathrm{sPF}(K_{n+1} - \{e_{p,q}\})| = |\mathcal{U}'_n| = (n-1)^{n-3}(n-2)^2.$$

Proof. In view of Theorem 23 and Remarks 26, the result follows. ☐

Let $F_l = \{e_{1,n}, e_{1,n-1}, \ldots, e_{1,n-l+1}\}$ be a set of l-edges through the vertex 1 in the complete graph K_{n+1}. We consider the graph $K_{n+1} - F_l$ and ask the following question.

Question 32. What is the number of spherical $(K_{n+1} - F_l)$-parking functions?

Computations for smaller values of n and l indicate that

$$|\mathrm{sPF}(K_{n+1} - F_l)| = (n-1)^{n-3}(n-l-1)^2.$$

5 Spherical $K_{m+1,n}$-parking functions

Let $K_{m+1,n}$ be the complete bipartite graph on the vertex set $V' = [0,m]\coprod[m+1,m+n]$, where $[0,m]=\{0,1,\ldots,m\}$ and $[m+1,m+n]=\{m+1,\ldots,m+n\}$. Let $K^{a,b}_{m+1,n}$ be the complete bipartite multigraph on V'. More precisely, there are a number of edges in $K^{a,b}_{m+1,n}$ between the root 0 and j, while b number of edges between i and j, where $i\in[m]$ and $j\in[m+1,m+n]$.

Proposition 33. *We have* $|\mathrm{sPF}(K^{a,b}_{m+1,n})| = |\mathrm{sPF}(K^{a,b}_{n+1,m})|$.

Proof. Let E and E' be the set of all edges of $K^{a,b}_{m+1,n}$ and $K^{a,b}_{n+1,m}$ through the root 0, respectively. On repeatedly applying the Lemma 17, we see that

$$|\mathrm{sPF}(K^{a,b}_{m+1,n})| = |\mathrm{sPF}(K^{a,b}_{m+1,n}-E)| \quad \text{and} \quad |\mathrm{sPF}(K^{a,b}_{n+1,m})| = |\mathrm{sPF}(K^{a,b}_{n+1,m}-E')|.$$

Since graphs $K^{a,b}_{m+1,n}-E$ and $K^{a,b}_{n+1,m}-E'$ are obtained from each other by interchanging vertices as $i\leftrightarrow n+i$ and $m+j\leftrightarrow j$ (for $i\in[m], j\in[n]$), $|\mathrm{sPF}(K^{a,b}_{m+1,n}-E)| = |\mathrm{sPF}(K^{a,b}_{n+1,m}-E')|$. □

Although the root 0 is not connected to all the other vertices in the simple complete bipartite graph $K_{m+1,n}$, we can construct a map $\phi_{K_{m+1,n}}:\mathrm{sPF}(K_{m+1,n})\to\mathcal{U}(K_{m,n})$ as in Theorem 20, where $\mathcal{U}(K_{m,n})$ is the set of uprooted spanning trees of $K_{m,n}=K_{m+1,n}-\{0\}$.

The reduced spherical $K_{m+1,n}$-parking function $\widetilde{\mathcal{P}}$ associated to $\mathcal{P}\in\mathrm{sPF}(K_{m+1,n})$ is given by $\widetilde{\mathcal{P}}(j)=\mathcal{P}(j)$ for $1\leqslant j\leqslant m$ and $\widetilde{\mathcal{P}}(j)=\mathcal{P}(j)-1$ for $m+1\leqslant j\leqslant m+n$. We see that $K_{m,n}=K_{m+1,n}-\{0\}$ is the complete bipartite graph on the vertex set $[m]\coprod[m+1,m+n]$. The following statements can be easily verified.

(i) $\widetilde{\mathrm{sPF}}(K_{m+1,n})\subseteq\mathrm{PF}(K_{m+1,n})$.

(ii) Let $r=\max\{i\in[m+n]:\widetilde{\mathcal{P}}(i)=0\}$. Then $m+1\leqslant r\leqslant m+n$.

(iii) $\widehat{\mathcal{P}}=\widetilde{\mathcal{P}}|_{[m+n]\setminus\{r\}}$ is a $(K_{m,n},r)$-parking function.

(iv) If $\phi:\mathrm{PF}(K_{m,n},r)\to\mathrm{SPT}(K_{m,n},r)$ is the bijection induced by Depth-First-Search algorithm, then $\phi(\widehat{\mathcal{P}})$ is an uprooted spanning tree of $K_{m,n}$.

Now define a map $\phi_{K_{m+1,n}}:\mathrm{sPF}(K_{m+1,n})\to\mathcal{U}(K_{m,n})$ given by $\phi_{K_{m+1,n}}(\mathcal{P})=\phi(\widehat{\mathcal{P}})$ for $\mathcal{P}\in\mathrm{sPF}(K_{m+1,n})$. For each $T\in\mathcal{U}(K_{m,n})$, let $\mathcal{P}_T$ be the unique $(K_{m,n},r)$-parking function such that $\phi(\mathcal{P}_T)=T$. Let $\overline{\mathcal{U}}(K_{m,n})=\{T\in\mathcal{U}(K_{m,n}):\mathcal{P}_T(j)\geqslant 1 \text{ for } j>\mathrm{root}(T)\}$.

Theorem 34. *The map* $\phi_{K_{m+1,n}}:\mathrm{sPF}(K_{m+1,n})\to\mathcal{U}(K_{m,n})$ *is injective with the image* $\overline{\mathcal{U}}(K_{m,n})$ *and* $\mathrm{sum}(\mathcal{P})=m(n-1)-\kappa(K_{m,n},\phi_{K_{m+1,n}}(\mathcal{P}))+1$ for all $\mathcal{P}\in\mathrm{sPF}(K_{m+1,n})$.

Proof. Proceed as in the proof of Theorems 20 and 22. □

Remark 35. The following three statements can be easily verified.
(1) $|\mathrm{sPF}(K_{m+1,1})| = 1 = |\mathrm{sPF}(K_{1+1,n})|$.
(2) Every spanning tree T of $K_{m,n}$ with $\mathrm{root}(T) = m+n$ lies in $\overline{\mathcal{U}}(K_{m,n})$. Thus

$$|\{\mathcal{P} \in \mathrm{sPF}(K_{m+1,n}) : \widetilde{\mathcal{P}}(m+n) = 0\}| = |\mathrm{PF}(K_{m,n})| = m^{n-1}n^{m-1}.$$

(3) We have $|\mathrm{sPF}(K^{a,b}_{m+1,n})| = b^{m+n}|\mathrm{sPF}(K_{m+1,n})|$.

We could not enumerate $\mathrm{sPF}(K_{m+1,n})$ or $\overline{\mathcal{U}}(K_{m,n})$. Thus we ask the following question.

Question 36. What is the number of spherical $K_{m+1,n}$-parking functions?

For $n = 2$, this question has an easy answer.

Proposition 37. *For $m \geqslant 1$, $|\mathrm{sPF}(K_{m+1,2})| = (m-1)2^m + 1$.*

Proof. We know that $|\mathrm{sPF}(K_{m+1,2})| = |\mathrm{sPF}(K_{m+1,2} - E)|$, where E is the set of all edges of $K_{m+1,2}$ through the root 0. Now the m-skeleton ideal of the (disconnected) graph $K_{m+1,2} - E$ is given by

$$\mathcal{M}^{(m)}_{K_{m+1,2}-E} = \left\langle x_i^2, y_j^m, y_1y_2, x_{i_1}x_{i_2}\cdots x_{i_s}y_j^{m-s} : i \in [m];\ j = 1,2 \text{ and } \{i_1,\dots,i_s\} \subseteq [m] \right\rangle,$$

where $y_j = x_{m+j}$ for $j = 1, 2$. The standard monomials of $\mathcal{M}^{(m)}_{K_{m+1,2}-E}$ are of the forms $x_{i_1}x_{i_2}\dots x_{i_s}y_1^{\alpha}$ with $0 \leqslant \alpha < m-s$ or $x_{i_1}x_{i_2}\cdots x_{i_s}y_2^{\beta}$ with $1 \leqslant \beta < m-s$. Thus the number of standard monomials of the first type is $\sum_{s=0}^{m}\binom{m}{s}(m-s) = m2^{m-1}$, while that of the second type is $\sum_{s=0}^{m-1}\binom{m}{s}(m-s-1) = (m-2)2^{m-1} + 1$. ☐

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References

[1] C. Chauve, S. Dulucq and O. Guibert. Enumeration of some labelled trees. Proceeding of SFCA/FPSAC 2000 (Moscow, June 2000), D. Krobo and A. Mikhalev eds., Springer, pages 146–157, 2000.

[2] A. Dochtermann and R. Sanyal. Laplacian ideals, arrangements, and resolutions. *J. Algebraic Combin.*, 40(3):805–822, 2014.

[3] A. Dochtermann. One-skeleta of G-parking function ideal : resolutions and standard monomials. arXiv:1708.04712v4, 2017.

[4] A. Dochtermann and W. King. Trees, parking functions, and standard monomials of skeleton ideals. arXiv:1806.04289v2, 2018.

[5] P. Gaydarov and S. Hopkins. Parking functions and tree inversions revisited. *Adv. in Appl. Math.*, 80:151–179, 2016.

[6] G. Kreweras. Une famille de polynômes ayant plusieurs propriétés énumeratives. *Period. Math. Hungar.*, 11(4):309–320, 1980.

[7] A. Kumar and C. Kumar. Multigraded Betti numbers of multipermutohedron ideals. *J. Ramanujan Math. Soc.*, 28(1):1–18, 2013.

[8] A. Kumar and C. Kumar. Alexander duals of multipermutohedron ideals. *Proc. Indian Acad. Sci.(Math Sci.)*, 124(1):1–15, 2014.

[9] C. Kumar. Steck determinants and parking functions. *Ganita*, 68(1):33–38, 2018.

[10] M. Manjunath, F.-O. Schreyer and J. Wilmes. Minimal free resolutions of the G-parking function ideal and the toppling ideal. *Trans. Amer. Math. Soc.*, 367(4):2853–2874, 2015.

[11] E. Miller and B. Sturmfels. Combinatorial commutative algebra. Graduate Texts in Mathematics 227, Springer-Verlag, New York, 2005.

[12] F. Mohammadi and F. Shokriech. Divisors on graphs, connected flags, and syzygies. *Int. Math. Res. Not. IMRN*, 24:6839–6905, 2014.

[13] D. Perkinson, Q. Yang and K. Yu. G-parking functions and tree inversions. *Combinatorica*, 37(2):269–282, 2017.

[14] J. Pitman and R. P. Stanley. A polytope related to empirical distributions, plane trees, parking functions, and the associahedron. *Discrete and Computational Geometry*, 27:603–634, 2002.

[15] A. Postnikov and B. Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math. Soc.*, 356:3109–3142, 2004.

[16] R. P. Stanley. Enumerative combinatorics: volume 2. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, 1999.

[17] C. H. Yan. On the enumeration of generalized parking functions. Proceedings of the 31-st Southeastern International Conference on Combinatorics, *Graph Theory and Computing* (Boca Raton, FL, 2000), *Congressus Numerantium*, 147:201–209, 2000.

[18] C. H. Yan. Parking functions. In *Handbook of enumerative combinatorics*, *Discrete math. Appl.* (Boca Raton), pages 853–893. CRC Press, Boca Raton, FL 2015.