Symmetric polynomials in the symplectic alphabet and the change of variables $z_j = x_j + x_j^{-1}$

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Abstract

Given a symmetric polynomial P in $2n$ variables, there exists a unique symmetric polynomial Q in n variables such that

$$P(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) = Q(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}).$$

We denote this polynomial Q by $\Phi_n(P)$ and show that Φ_n is an epimorphism of algebras. We compute $\Phi_n(P)$ for several families of symmetric polynomials P: symplectic and orthogonal Schur polynomials, elementary symmetric polynomials, complete homogeneous polynomials, and power sums. Some of these formulas were already found by Elouafi (2014) and Lachaud (2016).

The polynomials of the form $\Phi_n(\mathrm{s}_{\lambda/\mu}^{(2n)})$, where $\mathrm{s}_{\lambda/\mu}^{(2n)}$ is a skew Schur polynomial in $2n$ variables, arise naturally in the study of the minors of symmetric banded Toeplitz matrices, when the generating symbol is a palindromic Laurent polynomial, and its roots can be written as $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$. Trench (1987) and Elouafi (2014) found efficient formulas for the determinants of symmetric banded Toeplitz matrices. We show that these formulas are equivalent to the result of Ciucu and Krattenthaler (2009) about the factorization of the characters of classical groups.

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1 Introduction and main results

In this paper we study symmetric polynomials P in $2n$ variables evaluated at the symplectic alphabet:

$$P(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}). \tag{1}$$

We show how to rewrite such expressions in terms of the "Dickson–Zhukovsky variables" $z_j \coloneqq x_j + x_j^{-1}$. The function $t \mapsto t + t^{-1}$ is widely known as the Zhukovsky transform; some authors [15] relate it with the name of Dickson. Let us start with two examples to illustrate the main idea of the paper.

Example 1. The complete homogeneous polynomial $\mathrm{h}_2(y_1,y_2,y_3,y_4)$ is an element in Sym_4 and is equal to

$$y_1^2 + y_1y_2 + y_1y_3 + y_1y_4 + y_2^2 + y_2y_3 + y_2y_4 + y_3^2 + y_3y_4 + y_4^2.$$

Thus,

$$\begin{aligned}\mathrm{h}_2(x_1,x_2,x_1^{-1},x_2^{-1}) &= x_1^2 + x_1x_2 + 1 + x_1x_2^{-1} + x_2^2 + x_1^{-1}x_2 + 1 + x_1^{-2} + x_1^{-1}x_2^{-1} + x_2^{-2}\\ &= (x_1+x_1^{-1})^2 + (x_2+x_2^{-1})^2 + (x_1+x_1^{-1})(x_2+x_2^{-1}) - 2\\ &= z_1^2 + z_1z_2 + z_2^2 - 2\\ &= \mathrm{h}_2(z_1,z_2) - 2\,\mathrm{h}_0(z_1,z_2).\end{aligned}$$

See Theorem 1.5 for the general case $\mathrm{h}_m(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})$.

Example 2. The power sum polynomial $\mathrm{p}_3(y_1,y_2,y_3,y_4)$ is defined as

$$y_1^3 + y_2^3 + y_3^3 + y_4^3.$$

Thus,

$$\begin{aligned}\mathrm{p}_3(x_1,x_2,x_1^{-1},x_2^{-1}) &= x_1^3 + x_2^3 + x_1^{-3} + x_2^{-3}\\ &= (x_1+x_1^{-1})^3 + (x_2+x_2^{-1})^3 - 3\big((x_1+x_1^{-1}) + (x_2+x_2^{-1})\big)\\ &= z_1^3 + z_2^3 - 3(z_1+z_2)\\ &= \mathrm{p}_3(z_1,z_2) - 3\,\mathrm{p}_1(z_1,z_2).\end{aligned}$$

See Theorem 1.6 for the general case $\mathrm{p}_m(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1})$.

The symplectic alphabet $x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}$ naturally arises as the list of the roots of palindromic (i.e. self-reciprocal) polynomials, see more details in Section 3. In particular, expressions of the form (1) appear in the following situations.

1. If A is a symplectic matrix or a special orthogonal matrix of even order, then the characteristic polynomial of A is palindromic. If P is a symmetric polynomial in $2n$ variables, then P evaluated at the eigenvalues of A is an expression of the form (1).

2. The characters of symplectic groups or special orthogonal groups of even orders are particular cases of (1). See more details in Section 5.

3. Given a palindromic Laurent polynomial a, consider the banded symmetric Toeplitz matrices $T_m(a)$ generated by a. The minors of $T_m(a)$, expressed in terms of the roots of a, are of the form (1). See more details in Section 8.

This paper is inspired by Elouafi's article [11] on the determinants of banded symmetric Toeplitz matrices. In the process of preparation of the paper, we found a paper by Lachaud [20] which contains "our" Proposition 4.2.

We work over the field $\mathbb{C}$, though some results can be extended to other fields of characteristic 0. Let n be a fixed natural number. We denote by Sym_n the algebra of symmetric polynomials in n variables.

Theorem 1.1. *Let $P \in \operatorname{Sym}_{2n}$. Then there exists a unique Q in Sym_n such that*

$$P\left(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\right) = Q\left(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}\right).$$

An analog (and also a corollary) of Theorem 1.1 for the odd symplectic alphabet

$$x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, 1,$$

is stated below.

Theorem 1.2. *Let $P \in \operatorname{Sym}_{2n+1}$. Then there exists a unique Q in Sym_n such that*

$$P\left(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, 1\right) = Q\left(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}\right).$$

Chebyshev polynomials of the first, second, third, and fourth kind, denoted by $\mathcal{T}_m$, $\mathcal{U}_m$, $\mathcal{V}_m$, $\mathcal{W}_m$, respectively, play an important role in this paper. In particular, the polynomials $2\mathcal{T}_m(t/2)$ and $\mathcal{U}_m(t/2)$ convert the Dickson–Zhukovsky variable $z_j := x_j + x_j^{-1}$ into the power sum and the complete homogeneous polynomial in x_j and x_j^{-1}:

$$2\mathcal{T}_m(z_j/2) = x_j^m + x_j^{-m} = \mathrm{p}_m(x_j, x_j^{-1}), \qquad \mathcal{U}_m(z_j/2) = \sum_{k=0}^{m} x_j^{m-2k} = \mathrm{h}_m(x_j, x_j^{-1}).$$

Section 2 lists necessary properties of Chebyshev polynomials, Section 3 considers palindromic univariate polynomials and their roots, and Section 4 contains proofs of Theorem 1.1 and 1.2.

In the situations of Theorems 1.1 and 1.2, we denote Q by $\Phi_n(P)$ and $\Phi_n^{\mathrm{odd}}(P)$, respectively. Clearly, the functions $\Phi_n\colon \operatorname{Sym}_{2n} \to \operatorname{Sym}_n$ and $\Phi_n^{\mathrm{odd}}\colon \operatorname{Sym}_{2n+1} \to \operatorname{Sym}_n$, defined by these rules, are linear and multiplicative, i.e. Φ_n and Φ_n^{odd} are homomorphisms of algebras. Example 4 shows that Φ_n and Φ_n^{odd} are not injective.

In this paper we freely apply some well-known properties of symmetric polynomials, see [24] or [13, Appendix A] as reference. Let $\mathcal{P}$ be the set of all integer partitions. Given a partition λ, $\ell(\lambda)$ and $|\lambda|$ denote the length and the weight of λ, respectively. Let $\mathcal{P}_n$ be the set of all integer partitions λ with $\ell(\lambda) \leqslant n$. We denote by Sym the algebra of symmetric functions and use the following bases of Sym.

Symbol	Family
e_λ	elementary functions
h_λ	complete homogeneous functions
p_λ	power sum functions
s_λ	Schur functions
sp_λ	symplectic Schur functions
o_λ	orthogonal Schur functions

The functions sp_λ and o_λ are defined by Jacobi–Trudi formulas, see Section 5.

For the symmetric functions introduced above, we write the super-index (n) to indicate their restrictions to n variables, i.e. the corresponding elements of Sym_n. For example, $\mathrm{s}_\lambda^{(n)}$ is the Schur polynomial in n variables associated to the partition λ. Furthermore, put

$$\mathrm{ez}_\lambda^{(n)} \coloneqq \Phi_n(\mathrm{e}_\lambda^{(2n)}), \quad \mathrm{sz}_\lambda^{(n)} \coloneqq \Phi_n(\mathrm{s}_\lambda^{(2n)}), \quad \mathrm{spz}_\lambda^{(n)} \coloneqq \Phi_n(\mathrm{sp}_\lambda^{(2n)}), \quad \mathrm{oz}_\lambda^{(n)} \coloneqq \Phi_n(\mathrm{o}_\lambda^{(2n)}),$$

etc. In other words, $\mathrm{sz}_\lambda^{(n)}$ is obtained from $\mathrm{s}_\lambda^{(2n)}$ by applying Theorem 1.1 and passing from the "symplectic alphabet" $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$ to the "Dickson–Zhukovsky variables" $z_j = x_j + x_j^{-1}$:

$$\mathrm{sz}_\lambda^{(n)}(z_1, \ldots, z_n) \coloneqq \mathrm{s}_\lambda^{(2n)}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}).$$

Similarly, put

$$\mathrm{sz}_\lambda^{\mathrm{odd},(n)} \coloneqq \Phi_n^{\mathrm{odd}}(\mathrm{s}_\lambda^{(2n+1)}), \quad \mathrm{spz}_\lambda^{\mathrm{odd},(n)} \coloneqq \Phi_n^{\mathrm{odd}}(\mathrm{sp}_\lambda^{(2n+1)}), \quad \mathrm{oz}_\lambda^{\mathrm{odd},(n)} \coloneqq \Phi_n^{\mathrm{odd}}(\mathrm{o}_\lambda^{(2n+1)}),$$

etc. For the sake of brevity, we will omit the superindex (n), when indicating the list of variables $z = (z_1, \ldots, z_n)$.

The next theorem, proven in Section 5, yields convenient bialternant formulas for spz_λ, $\mathrm{spz}_\lambda^{\mathrm{odd}}$, oz_λ, and $\mathrm{oz}_\lambda^{\mathrm{odd}}$, representing them as "Schur–Chebyshev quotients". We denote by $\mathcal{T}_m^{\mathrm{monic}}$ the monic integer version of the Chebyshev $\mathcal{T}_m$ polynomial, i.e. $\mathcal{T}_m^{\mathrm{monic}}(u) \coloneqq 2\mathcal{T}_m(u/2)$ for $m > 0$ and $\mathcal{T}_0^{\mathrm{monic}}(u) \coloneqq 1$. The polynomial $\mathcal{U}_m^{(1)}$ is defined as $\sum_{k=0}^m \mathcal{U}_k$.

Theorem 1.3. *For every λ in $\mathcal{P}_n$,*

$$\mathrm{spz}_\lambda(z_1, \ldots, z_n) = \frac{\det\bigl[\mathcal{U}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{U}_{n-j}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{U}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\prod_{1\leqslant j<k\leqslant n}(z_j - z_k)}, \tag{2}$$

$$\mathrm{oz}_\lambda(z_1, \ldots, z_n) = \frac{\det\bigl[\mathcal{T}_{\lambda_j+n-j}^{\mathrm{monic}}(z_k)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{T}_{n-j}^{\mathrm{monic}}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{T}_{\lambda_j+n-j}^{\mathrm{monic}}(z_k)\bigr]_{j,k=1}^n}{\prod_{1\leqslant j<k\leqslant n}(z_j - z_k)}, \tag{3}$$

$$\mathrm{spz}_\lambda^{\mathrm{odd}}(z_1, \ldots, z_n) = \frac{\det\bigl[\mathcal{U}_{\lambda_j+n-j}^{(1)}(z_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{U}_{n-j}^{(1)}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{U}_{\lambda_j+n-j}^{(1)}(z_k/2)\bigr]_{j,k=1}^n}{\prod_{1\leqslant j<k\leqslant n}(z_j - z_k)}, \tag{4}$$

$$\mathrm{oz}^{\mathrm{odd}}_{\lambda}(z_1,\ldots,z_n) = \frac{\det\bigl[\mathcal{W}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^{n}}{\det\bigl[\mathcal{W}_{n-j}(z_k/2)\bigr]_{j,k=1}^{n}} = \frac{\det\bigl[\mathcal{W}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^{n}}{\prod_{1\leqslant j<k\leqslant n}(z_j-z_k)}, \quad (5)$$

$$(-1)^{|\lambda|}\,\mathrm{oz}^{\mathrm{odd}}_{\lambda}(-z_1,\ldots,-z_n) = \frac{\det\bigl[\mathcal{V}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^{n}}{\det\bigl[\mathcal{V}_{n-j}(z_k/2)\bigr]_{j,k=1}^{n}} = \frac{\det\bigl[\mathcal{V}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^{n}}{\prod_{1\leqslant j<k\leqslant n}(z_j-z_k)}. \quad (6)$$

Our main results, stated below as Theorems 1.4, 1.5, 1.6, 1.7 and proven in Section 6, are properties of $\mathrm{hz}_m^{(n)}$, $\mathrm{ez}_m^{(n)}$, and $\mathrm{pz}_m^{(n)}$. Formula (7) was found by Lachaud [20, Theorem A.2 and proof of Lemma A.3], and formula (13) is similar to one part of Elouafi [11, Lemma 3].

Theorem 1.4. *For m in $\{0,\ldots,2n\}$,*

$$\mathrm{ez}_m(z_1,\ldots,z_n) = \sum_{k=\max\{0,m-n\}}^{\lfloor m/2\rfloor} \binom{n-m+2k}{k}\,\mathrm{e}_{m-2k}(z_1,\ldots,z_n). \quad (7)$$

For m in $\{0,\ldots,n\}$,

$$\mathrm{ez}_m(z_1,\ldots,z_n) = \mathrm{oz}_{(1^m)}(z_1,\ldots,z_n), \quad (8)$$

$$\mathrm{ez}_m(z_1,\ldots,z_n) = \frac{1}{\prod_{1\leqslant j<k\leqslant n}(z_j-z_k)} \begin{vmatrix} \mathcal{T}_n^{\mathrm{monic}}(z_1) & \ldots & \mathcal{T}_n^{\mathrm{monic}}(z_n) \\ \vdots & \ddots & \vdots \\ \mathcal{T}_{n-m+1}^{\mathrm{monic}}(z_1) & \ldots & \mathcal{T}_{n-m+1}^{\mathrm{monic}}(z_n) \\ \mathcal{T}_{n-m-1}^{\mathrm{monic}}(z_1) & \ldots & \mathcal{T}_{n-m-1}^{\mathrm{monic}}(z_n) \\ \vdots & \ddots & \vdots \\ \mathcal{T}_0^{\mathrm{monic}}(z_1) & \ldots & \mathcal{T}_0^{\mathrm{monic}}(z_n) \end{vmatrix}, \quad (9)$$

$$\mathrm{ez}_m(z_1,\ldots,z_n) = \sum_{k=0}^{\lfloor m/2\rfloor} \mathrm{spz}_{(1^{m-2k})}(z_1,\ldots,z_n), \quad (10)$$

$$\mathrm{e}_m(z_1,\ldots,z_n) = \sum_{k=0}^{\lfloor m/2\rfloor} (-1)^k \tau_{n-m+2k,k}\,\mathrm{ez}_{m-2k}(z_1,\ldots,z_n), \quad (11)$$

where

$$\tau_{s,k} \coloneqq \begin{cases} \dfrac{(s-k-1)!\,s}{k!\,(s-2k)!}, & s\in\mathbb{N},\ 0\leqslant k\leqslant s; \\ 1, & k=s=0. \end{cases} \quad (12)$$

In the notation for partitions (in particular, in formula (8)), p^q is the number p repeated q times. For example, $(3^2,0^4)$ means $(3,3,0,0,0,0)$.

Theorem 1.5. *Let $m \in \mathbb{N}_0$. Then*

$$\operatorname{hz}_m(z_1,\ldots,z_n) = \sum_{j=1}^{n} \frac{\mathcal{U}_{m+n-1}(z_j/2)}{\prod\limits_{k\in\{1,\ldots,n\}\setminus\{j\}} (z_j - z_k)}, \tag{13}$$

$$\operatorname{hz}_m(z_1,\ldots,z_n) = \frac{1}{\prod_{1\leqslant j<k\leqslant n}(z_j - z_k)} \begin{vmatrix} \mathcal{U}_{m+n-1}(z_1/2) & \ldots & \mathcal{U}_{m+n-1}(z_n/2) \\ \mathcal{U}_{n-2}(z_1/2) & \ldots & \mathcal{U}_{n-2}(z_n/2) \\ \vdots & \ddots & \vdots \\ \mathcal{U}_0(z_1/2) & \ldots & \mathcal{U}_0(z_n/2) \end{vmatrix}, \tag{14}$$

$$\operatorname{hz}_m(z_1,\ldots,z_n) = \sum_{k=0}^{\lfloor m/2\rfloor} (-1)^k \binom{n+m-k-1}{k} \operatorname{h}_{m-2k}(z_1,\ldots,z_n), \tag{15}$$

$$\operatorname{hz}_m(z_1,\ldots,z_n) = \sum_{\substack{\alpha_1,\ldots,\alpha_n\in\mathbb{N}_0\\ \alpha_1+\cdots+\alpha_n=m}} \prod_{j=1}^{n} \mathcal{U}_{\alpha_j}(z_j/2), \tag{16}$$

$$\operatorname{hz}_m(z_1,\ldots,z_n) = \operatorname{spz}_{(m)}(z_1,\ldots,z_n), \tag{17}$$

$$\operatorname{hz}_m(z_1,\ldots,z_n) = \sum_{k=0}^{\lfloor m/2\rfloor} \operatorname{oz}_{(m-2k)}(z_1,\ldots,z_n), \tag{18}$$

$$\operatorname{h}_m(z_1,\ldots,z_n) = \sum_{k=0}^{\lfloor m/2\rfloor} \frac{(m+n-1)!\,(m+n-2k)}{k!\,(m+n-k)!} \operatorname{hz}_{m-2k}(z_1,\ldots,z_n). \tag{19}$$

Theorem 1.6. *Let $m \in \mathbb{N}_0$. Then*

$$\operatorname{pz}_m(z_1,\ldots,z_n) = \sum_{j=1}^{n} 2\mathcal{T}_m(z_j/2), \tag{20}$$

$$\operatorname{pz}_m(z_1,\ldots,z_n) = \begin{cases} m \displaystyle\sum_{j=0}^{\lfloor m/2\rfloor} \frac{(-1)^j}{m-j}\binom{m-j}{j} \operatorname{p}_{m-2j}(z_1,\ldots,z_n), & m\in\mathbb{N}; \\ 2\operatorname{p}_0(z_1,\ldots,z_n), & m=0, \end{cases} \tag{21}$$

$$\operatorname{p}_m(z_1,\ldots,z_n) = \sum_{k=0}^{\lfloor m/2\rfloor} \alpha_{m,k} \operatorname{pz}_{m-2k}(z_1,\ldots,z_n), \tag{22}$$

where the coefficients $\alpha_{m,k}$ are defined by

$$\alpha_{m,k} \coloneqq \begin{cases} \dbinom{m}{k}, & \text{if } k < \frac{m}{2}; \\ \dfrac{1}{2}\dbinom{m}{m/2}, & \text{if } k = \frac{m}{2}. \end{cases} \tag{23}$$

Notice that the coefficients in (7), (11), (15), and (19) depend on n. For example,

$$\operatorname{hz}_2(z_1,\ldots,z_n) = \operatorname{h}_2(z_1,\ldots,z_n) - n.$$

Therefore, $\mathrm{hz}_2^{(n)}$ is well-defined only as an element of Sym_n, i.e., there is no function in Sym that could be denoted by hz_2. In contrast to this, the coefficients in (21) and (22) do not depend on n, but in these formulas $\mathrm{p}_0^{(n)} := n$ and $\mathrm{pz}_0^{(n)} := 2n$.

Remark 1. We can avoid $\mathrm{p}_0^{(n)}$ and $\mathrm{pz}_0^{(n)}$, i.e. replace them by the explicit constants n and $2n$, respectively. Then in some formulas, one needs to separate the corresponding summands and to multiply them by the conditional factor $[m \text{ is even}]$. So, some formulas are nicer-looking with $\mathrm{p}_0^{(n)}$ and $\mathrm{pz}_0^{(n)}$.

Theorem 1.7. *Each of the sets* $\{\mathrm{ez}_m^{(n)}\}_{m=1}^n$, $\{\mathrm{hz}_m^{(n)}\}_{m=1}^n$, $\{\mathrm{pz}_m^{(n)}\}_{m=1}^n$ *is an algebraically independent generating subset of the unital algebra* Sym_n, *and the homomorphism* Φ_n *is surjective.*

As a consequence of Theorems 1.1 and 1.7, the algebra Sym_n is isomorphic to the algebra of the expressions of the form $P(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$, where $P \in \mathrm{Sym}_{2n}$, via the isomorphism $Q \mapsto Q(x + x^{-1})$. This fact is close to some ideas from [20, Appendix A].

Theorem 1.8. *Each of the families* $(\mathrm{sz}_\lambda^{(n)})_{\lambda\in\mathcal{P}_n}$, $(\mathrm{spz}_\lambda^{(n)})_{\lambda\in\mathcal{P}_n}$, $(\mathrm{oz}_\lambda^{(n)})_{\lambda\in\mathcal{P}_n}$, $(\mathrm{sz}_\lambda^{\mathrm{odd},(n)})_{\lambda\in\mathcal{P}_n}$, $(\mathrm{spz}_\lambda^{\mathrm{odd},(n)})_{\lambda\in\mathcal{P}_n}$, $(\mathrm{oz}_\lambda^{\mathrm{odd},(n)})_{\lambda\in\mathcal{P}_n}$ *is a basis of the vector space* Sym_n.

Next we state analogs of the Cauchy and dual Cauchy identities.

Theorem 1.9 (Cauchy identities)**.** *For* $z = (z_1, \ldots, z_n)$ *and* $y = (y_1, \ldots, y_m)$,

$$\sum_{\lambda\in\mathcal{P}} \mathrm{sz}_\lambda(z)\,\mathrm{s}_\lambda(y) = \prod_{j=1}^{n}\prod_{k=1}^{m} \frac{1}{1 - z_j y_k + y_k^2}, \tag{24}$$

$$\sum_{\lambda\in\mathcal{P}} \mathrm{sz}_\lambda(z)\,\mathrm{s}_{\lambda'}(y) = \prod_{j=1}^{n}\prod_{k=1}^{m} \left(1 + z_j y_k + y_k^2\right), \tag{25}$$

$$\sum_{\lambda\in\mathcal{P}} \mathrm{sz}_\lambda(z)\,\mathrm{sz}_{\lambda'}(y) = \prod_{j=1}^{n}\prod_{k=1}^{m} (z_j + y_k)^2, \tag{26}$$

$$\sum_{\lambda\in\mathcal{P}} \mathrm{sz}_\lambda^{\mathrm{odd}}(z)\,\mathrm{s}_\lambda(y) = \left(\prod_{j=1}^{n}\prod_{k=1}^{m} \frac{1}{1 - z_j y_k + y_k^2}\right)\left(\prod_{k=1}^{m} \frac{1}{1 - y_k}\right), \tag{27}$$

$$\sum_{\lambda\in\mathcal{P}} \mathrm{sz}_\lambda^{\mathrm{odd}}(z)\,\mathrm{s}_{\lambda'}(y) = \left(\prod_{j=1}^{n}\prod_{k=1}^{m} (1 + z_j y_k + y_k^2)\right)\left(\prod_{k=1}^{m} (1 + y_k)\right), \tag{28}$$

$$\sum_{\lambda\in\mathcal{P}} \mathrm{sz}_\lambda^{\mathrm{odd}}(z)\,\mathrm{sz}_{\lambda'}^{\mathrm{odd}}(y) = 2\left(\prod_{j=1}^{n}\prod_{k=1}^{m} (z_j + y_k)^2\right)\left(\prod_{j=1}^{n} (2 + z_j)\right)\left(\prod_{k=1}^{m} (2 + y_k)\right). \tag{29}$$

There are also analogs of Cauchy identities with $\mathrm{spz}_\lambda(z)$ or $\mathrm{oz}_\lambda(z)$ instead of $\mathrm{sz}_\lambda(z)$; we omit them for the sake of brevity.

Section 7 contains proofs of Theorems 1.8 and 1.9, and some other simple facts about $\mathrm{sz}_\lambda^{(n)}$. A natural problem is to find a bialternant formula for $\mathrm{sz}_\lambda^{(n)}$, similar to the formulas from Theorem 1.3. In Proposition 7.5 we prove that there is no bialternant formula for $\mathrm{sz}_{(2,1)}(z_1, z_2)$, with denominator $z_1 - z_2$. In Example 8 we give a bialternant formula for $\mathrm{sz}_{(2,1)}(z_1, z_2)$ with a more complicated denominator.

In Section 8 we show that there is a surjective (but non-injective) correspondence between minors of banded symmetric Toeplitz matrices and polynomials $\mathrm{sz}^{(n)}_{\lambda/\mu}$, obtained by applying Φ_n to skew Schur polynomials. In particular, the banded symmetric Toeplitz determinants correspond to $\mathrm{sz}^{(n)}_{(m^n)}$, and the factorization of $\mathrm{s}_{(m^n)}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$ proven in 2009 by Ciucu and Krattenthaler [9] is equivalent to the formulas for banded symmetric Toeplitz determinants, found independently by Trench [31] in 1987 and by Elouafi [11] in 2014. Elouafi's formula has been used in investigations about the eigenvalues of symmetric Toeplitz matrices [3].

All formulas from Theorems 1.3, 1.4, 1.5, 1.6, and 1.9 are thoroughly tested in Sagemath [29] for small values of parameters. We share the corresponding Sagemath code at the page http://www.egormaximenko.com/programs/tests_palindromic.html.

2 Necessary facts about Chebyshev polynomials

Most of the material in this section can be found in [25]. Four families of Chebyshev polynomials can be defined by the same recurrent formula

$$\mathcal{T}_m(u) = 2u\,\mathcal{T}_{m-1}(u) - \mathcal{T}_{m-2}(u), \qquad \mathcal{U}_m(u) = 2u\,\mathcal{U}_{m-1}(u) - \mathcal{U}_{m-2}(u), \qquad \ldots,$$

with the initial conditions $\mathcal{T}_0(u) = \mathcal{U}_0(u) = \mathcal{V}_0(u) = \mathcal{W}_0(u) = 1$,

$$\mathcal{T}_1(u) = u, \quad \mathcal{U}_1(u) = 2u, \quad \mathcal{V}_1(u) = 2u - 1, \quad \mathcal{W}_1(u) = 2u + 1.$$

The Chebyshev polynomials have the following important properties:

$$2\mathcal{T}_m\left(\frac{1}{2}(t + t^{-1})\right) = t^m + t^{-m}, \tag{30}$$

$$\mathcal{U}_m\left(\frac{1}{2}(t + t^{-1})\right) = \frac{t^{m+1} - t^{-m-1}}{t - t^{-1}}, \tag{31}$$

$$\mathcal{V}_m\left(\frac{1}{2}(t^2 + t^{-2})\right) = \frac{t^{2m+1} + t^{-2m-1}}{t + t^{-1}}, \tag{32}$$

$$\mathcal{W}_m\left(\frac{1}{2}(t^2 + t^{-2})\right) = \frac{t^{2m+1} - t^{-2m-1}}{t - t^{-1}}. \tag{33}$$

The generating functions of the sequences $(2\mathcal{T}_m(u/2))_{m=0}^{\infty}$ and $(\mathcal{U}_m(u/2))_{m=0}^{\infty}$ are given by

$$\frac{2 - tu}{1 - tu + t^2} = \sum_{m=0}^{\infty} 2\mathcal{T}_m(u/2)t^m, \tag{34}$$

$$\frac{1}{1 - tu + t^2} = \sum_{m=0}^{\infty} \mathcal{U}_m(u/2)t^m. \tag{35}$$

Using (34) and (35), it is easy to derive explicit formulas for $\mathcal{T}_m$ and $\mathcal{U}_m$:

$$2\mathcal{T}_m(u/2) = m \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k}{m-k} \binom{m-k}{k} u^{m-2k} \qquad (m \in \mathbb{N}), \tag{36}$$

$$\mathcal{U}_m(u/2) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{m-k}{k} u^{m-2k} \qquad (m \in \mathbb{N}_0). \tag{37}$$

With the notation $\tau_{m,k}$ defined by (12), we rewrite (36) in the form

$$\sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \tau_{m,k} u^{m-2k} = \mathcal{T}_m^{\text{monic}}(u) = \begin{cases} 2\mathcal{T}_m(u/2), & m \in \mathbb{N}; \\ \mathcal{T}_0(u/2), & m = 0. \end{cases} \tag{38}$$

The monomials u^m $(m \in \mathbb{N}_0)$ can be written as linear combinations of $\mathcal{T}_{m-2k}(u/2)$ or $\mathcal{U}_{m-2k}(u/2)$:

$$u^m = \sum_{k=0}^{\lfloor m/2 \rfloor} \alpha_{m,k} 2\mathcal{T}_{m-2k}(u/2), \tag{39}$$

$$u^m = \sum_{k=0}^{\lfloor m/2 \rfloor} \mathbf{C}_{m-k,k} \mathcal{U}_{m-2k}(u/2), \tag{40}$$

where $\alpha_{m,k}$ is defined by (23), and $\mathbf{C}_{m,k}$ are the elements of Catalan's triangle:

$$\mathbf{C}_{m,k} \coloneqq \frac{(m+k)!\,(m-k+1)}{k!\,(m+1)!}.$$

The forthcoming "duplication formulas" follow easily from (30)–(33) and can be found in [25, Section 1.2.4].

Proposition 2.1. *For every m in $\mathbb{N}_0$,*

$$\mathcal{T}_{2m}(t/2) = \mathcal{T}_m((t^2-2)/2), \tag{41}$$

$$\mathcal{U}_{2m+1}(t/2) = t\,\mathcal{U}_m((t^2-2)/2), \tag{42}$$

$$2\mathcal{T}_{2m+1}(t/2) = t\,\mathcal{V}_m((t^2-2)/2), \tag{43}$$

$$\mathcal{U}_{2m}(t/2) = \mathcal{W}_m((t^2-2)/2). \tag{44}$$

The polynomials $\mathcal{V}_m$ and $\mathcal{W}_m$ are related by

$$\mathcal{V}_m(-t) = (-1)^m \mathcal{W}_m(t). \tag{45}$$

We denote by $\mathcal{U}_m^{(1)}$ the polynomial $\sum_{k=0}^m \mathcal{U}_k$ and by $\sigma_{m,j}$ the coefficient of t^j in $\mathcal{U}_m^{(1)}(t/2)$:

$$\mathcal{U}_m^{(1)}(t/2) = \sum_{k=0}^m \mathcal{U}_k(t/2) = \sum_{j=0}^m \sigma_{m,j} t^j. \tag{46}$$

It follows from (37) that

$$\sigma_{m,k} = \sum_{j=k}^{\lfloor (m+k)/2 \rfloor} (-1)^{j-k} \binom{j}{k} = \sum_{j=0}^{\lfloor (m-k)/2 \rfloor} (-1)^j \binom{k+j}{j}. \tag{47}$$

The coefficients $\sigma_{m,k}$ form the sequence A128494 in the Online Encyclopedia of Integer Sequences [27]. It is easy to prove by induction that

$$\mathcal{U}_m^{(1)}(t/2) = \frac{\mathcal{V}_{m+1}(t/2) - 1}{t - 2}. \tag{48}$$

3 Palindromic univariate polynomials and their roots

A univariate polynomial $f(t) = \sum_{k=0}^{m} a_k t^k$, with $a_m \neq 0$, is called *palindromic* if $a_k = a_{m-k}$ for all k in $\{0, \ldots, m\}$. This condition is equivalent to the identity $f(t) = t^m f(1/t)$. In this section we review some known facts about palindromic polynomials of even degrees, then make a couple of remarks about palindromic polynomials of odd degrees. For $m = 2n$, the main observation is that $f(t)$ can be written as $t^n g(t + t^{-1})$, where g is a certain polynomial. This idea appears, for example, in Dickson [10, Chapter VIII], but without explicit formula for g. The explicit formula for g, i.e. Proposition 3.2 below, was discovered independently by various authors. See, for example, Elouafi [11, formula (2.2)], Lachaud [20, Remark A.4].

Remark 2. Wikipedia ("Reciprocal polynomial") mentions that the formula $f(t) = t^n g(t+t^{-1})$ was published by Durand in "Solutions numériques des équations algrébriques I" (1961), but we are unable to find that text. So, unfortunately, we cannot say who was first to discover the explicit formula for g in terms of $\mathcal{T}_m$.

Proposition 3.1. *Let $x_1, \ldots, x_n \in \mathbb{C}$. For each j in $\{1, \ldots, n\}$, put $z_j = x_j + x_j^{-1}$. Define univariate polynomials f and g by*

$$f(t) \coloneqq \prod_{j=1}^{n} \big((t - x_j)(t - x_j^{-1})\big), \qquad g(u) \coloneqq \prod_{j=1}^{n} (u - z_j).$$

Then

$$f(t) = t^n g\left(t + t^{-1}\right), \tag{49}$$

and the polynomial f is palindromic.

Proof. Formula (49) follows directly from the definition of f and g:

$$t^n g(t + t^{-1}) = t^n \prod_{j=1}^{n} \big(t + t^{-1} - x_j - x_j^{-1}\big) = \prod_{j=1}^{n} \big((t - x_j)(t - x_j^{-1})\big) = f(t).$$

The expression $g(t+t^{-1})$, being a linear combination of expressions of the form $(t+t^{-1})^k$, is a palindromic Laurent polynomial of the form $c_0 + \sum_{k=1}^{n} c_k (t^k + t^{-k})$. Now (49) implies that f is a palindromic polynomial. □

The next propositions are, in a certain sense, inverse to Proposition 3.1. Now we define g through the coefficients of f and make conclusions about the roots of f and g.

Proposition 3.2. *Let f be a palindromic univariate polynomial with complex coefficients:*

$$f(t) = \sum_{k=0}^{2n} a_k t^k,$$

where $a_{2n-k} = a_k$ for every k in $\{0, \ldots, n\}$. Define a univariate polynomial g by

$$g(u) \coloneqq \sum_{j=0}^{n} a_{n-j} \mathcal{T}_j^{\mathrm{monic}}(u) = a_n + \sum_{j=1}^{n} 2a_{n-j} \mathcal{T}_j(u/2).$$

Then

$$f(t) = t^n g\left(t + t^{-1}\right). \tag{50}$$

Proof. For every k in $\{0, \ldots, n-1\}$, write $a_k t^k + a_{2n-k} t^{2n-k}$ as $a_k t^n (t^{n-k} + t^{k-n})$, then apply (30). □

Proposition 3.3. *Let f and g be as in Proposition 3.2, $a_0 \neq 0$, and $z_1, \ldots, z_n$ be the roots of the polynomial g:*

$$g(u) = a_0 \prod_{j=1}^{n} (u - z_j).$$

For each j, denote by x_j a complex number satisfying $x_j + x_j^{-1} = z_j$. Then the numbers

$$x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$$

are the roots of the polynomial f, i.e.

$$f(t) = a_0 \prod_{j=1}^{n} \left((t - x_j)\left(t - x_j^{-1}\right)\right).$$

Proof. Follows directly from (50) and the definitions of z_j and x_j:

$$f(t) = t^n g\left(t + t^{-1}\right) = a_0 t^n \prod_{j=1}^{n} \left(t + t^{-1} - x_j - x_j^{-1}\right) = a_0 \prod_{j=1}^{n} \left((t - x_j)\left(t - x_j^{-1}\right)\right).$$

□

Here is a simple result about palindromic polynomials of odd degrees.

Proposition 3.4. *Let f be a palindromic polynomial of degree $2n+1$. Then there exists a unique palindromic polynomial g of degree $2n$ such that $f(t) = (t+1)g(t)$. The zeros of f can be written as*

$$x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, -1.$$

If $P \in \mathrm{Sym}_{2n+1}$ and P is homogeneous of degree d, then

$$P(x, x^{-1}, -1) = (-1)^d P(-x, -x^{-1}, 1).$$

In this situation, it is possible to work over the alphabet $x, x^{-1}, 1$, instead of $x, x^{-1}, -1$.

4 Construction of the morphisms Φ_n and Φ_n^{odd}

In what follows, we denote by x, x^{-1}, and z the lists of variables $x_1, \ldots, x_n$, $x_1^{-1}, \ldots, x_n^{-1}$, and $z_1, \ldots, z_n$, respectively, where $z_j = x_j + x_j^{-1}$.

The next proposition also appears in [19, eq. (4.4)].

Proposition 4.1. *For each m in $\{0, \ldots, 2n\}$,*

$$\mathrm{e}_{2n-m}(x, x^{-1}) = \mathrm{e}_m(x, x^{-1}). \tag{51}$$

Proof. We know that the polynomial f from Proposition 3.1 is palindromic. This fact and Vieta's formula yield (51). □

The next formula was recently published by Lachaud [20, Lemma A.3]. We found it independently, but with exactly the same proof. Therefore here we only give an idea of the proof.

Proposition 4.2. *For every m in $\{0, \ldots, 2n\}$,*

$$\mathrm{e}_m(x, x^{-1}) = \sum_{k=\max\{m-n,0\}}^{\lfloor m/2 \rfloor} \binom{n+2k-m}{k} \mathrm{e}_{m-2k}(z). \tag{52}$$

Proof. From Propositions 3.1, 4.1 and Vieta's formulas for the coefficients of the polynomials f and g,

$$\sum_{m=0}^{2n} (-1)^m \mathrm{e}_m(x, x^{-1}) t^m = \sum_{j=0}^{n} (-1)^{n-j} \mathrm{e}_{n-j}(z) t^n (t + t^{-1})^j. \tag{53}$$

Expanding $(t+t^{-1})^j$ by the binomial theorem and matching the coefficient of t^m yields (52). □

For every j in $\{1, \ldots, n\}$, let $\Omega_j(z)$ be defined as follows:

$$\Omega_j(z) \coloneqq \prod_{k \in \{1,\ldots,n\} \setminus \{j\}} (z_j - z_k).$$

Denote by $\mathrm{Van}(z)$ the Vandermonde polynomial in the variables $z_1, \ldots, z_n$:

$$\mathrm{Van}(z) \coloneqq \det\bigl[z_k^{n-j}\bigr]_{j,k=1}^{n} = \prod_{j=1}^{n} \Omega_1(z_j, \ldots, z_n) = \prod_{1 \leqslant j < k \leqslant n} (z_j - z_k). \tag{54}$$

Recall that

$$\mathrm{h}_m(y_1, \ldots, y_p) = \sum_{j=1}^{p} \frac{y_j^{m+p-1}}{\Omega_j(y_1, \ldots, y_p)}. \tag{55}$$

Lemma 4.3. *Let $0 \leqslant s < n-1$. Then*

$$\sum_{j=1}^{n} \frac{z_j^s}{\Omega_j(z)} = 0. \tag{56}$$

Proof. It is easy to see that the left-hand side of (56) equals $\det(A(z))/\operatorname{Van}(z)$, where $A(z)$ is the matrix with the entries

$$A_{j,k}(z) := \begin{cases} z_k^s, & j = 1 \\ z_k^{n-j}, & 2 \leqslant j \leqslant n. \end{cases}$$

Since the first row of $A(z)$ coincides with the $(n-s)$th, $\det(A(z)) = 0$. □

Proposition 4.4. *We have the identity*

$$\mathrm{h}_m(x, x^{-1}) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{n+m-k-1}{k} \mathrm{h}_{m-2k}(z_1, \ldots, z_n). \tag{57}$$

Proof. Apply (55) with the variables x, x^{-1}:

$$\mathrm{h}_m(x, x^{-1}) = \sum_{j=1}^{n} \frac{x_j^{2n+m-1}}{\Omega_j(x, x^{-1})} + \sum_{j=1}^{n} \frac{x_j^{-2n-m+1}}{\Omega_{n+j}(x, x^{-1})}. \tag{58}$$

Since

$$(x_j - x_k)(x_j - x_k^{-1}) = x_j(z_j - z_k), \qquad (x_j^{-1} - x_k)(x_j^{-1} - x_k^{-1}) = x_j^{-1}(z_j - z_k),$$

the denominators in (58) can be written as

$$\Omega_j(x, x^{-1}) = (x_j - x_j^{-1}) x_j^{n-1} \Omega_j(z), \qquad \Omega_{n+j}(x, x^{-1}) = -(x_j - x_j^{-1}) x_j^{-n+1} \Omega_j(z).$$

Applying (31) we arrive at

$$\mathrm{h}_m(x, x^{-1}) = \sum_{j=1}^{n} \frac{\mathcal{U}_{m+n-1}(z_j/2)}{\Omega_j(z)}. \tag{59}$$

From this and (37),

$$\mathrm{h}_m(x, x^{-1}) = \sum_{j=1}^{n} \frac{\mathcal{U}_{n+m-1}(z_j/2)}{\Omega_j(z)} = \sum_{k=0}^{\lfloor (n+m-1)/2 \rfloor} (-1)^k \binom{n+m-k-1}{k} \sum_{j=1}^{n} \frac{z_j^{m+n-1-2k}}{\Omega_j(z)}.$$

If $k > \lfloor m/2 \rfloor$, then $m + n - 1 - 2k < n - 1$, and the inner sum is zero by Lemma 4.3. So, the range of k in the outer sum can be restricted to $\{0, \ldots, \lfloor m/2 \rfloor\}$. Applying (55) yields (15). □

Proposition 4.5. *For every m in $\mathbb{N}_0$,*

$$\mathrm{p}_m(x, x^{-1}) = \begin{cases} m \displaystyle\sum_{j=0}^{\lfloor m/2 \rfloor} \frac{(-1)^j}{m-j} \binom{m-j}{j} \mathrm{p}_{m-2j}(z_1, \ldots, z_n), & m \in \mathbb{N}; \\ 2\,\mathrm{p}_0(z_1, \ldots, z_n), & m = 0. \end{cases} \tag{60}$$

Proof. The result is trivial for $m = 0$. Suppose $m \in \mathbb{N}$. By (30),

$$\mathrm{p}_m(x, x^{-1}) = \sum_{k=1}^{n} (x_k^m + x_k^{-m}) = \sum_{k=1}^{n} 2\mathcal{T}_m\big((x_k + x_k^{-1})/2\big),$$

i.e.

$$\mathrm{p}_m(x, x^{-1}) = \sum_{k=1}^{n} 2\mathcal{T}_m\,(z_k/2)\,. \tag{61}$$

Furthermore, applying (36),

$$\begin{aligned}\mathrm{p}_m(x, x^{-1}) &= m \sum_{k=1}^{n} \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{(-1)^j}{m-j} \binom{m-j}{j} z_k^{m-2j} = m \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{(-1)^j}{m-j} \binom{m-j}{j} \sum_{k=1}^{n} z_k^{m-2j} \\ &= m \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{(-1)^j}{m-j} \binom{m-j}{j} \mathrm{p}_{m-2j}(z). \end{aligned}$$

□

Proof of Theorem 1.1. Since the set $\{\mathrm{e}_m\}_{m=1}^{2n}$ generates the unital algebra Sym_{2n}, the existence in Theorem 1.1 follows from Proposition 4.2. Similarly, the existence also follows from Proposition 4.4 and from Proposition 4.5.

For the uniqueness, suppose that $Q_1, Q_2 \in \mathrm{Sym}_n$ such that

$$Q_1\left(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}\right) = Q_2\left(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}\right).$$

Consider the difference $Q_3 = Q_1 - Q_2$. The assumptions on Q_1 and Q_2 imply that $Q_3 \in \mathrm{Sym}_n$ and the expression

$$Q_3\left(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}\right) \tag{62}$$

is zero. If Q_3 is a non-zero polynomial and $cz_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is one of the leading terms of Q_3, then it is easy to see that the expression (62) contains the summand $cx_1^{\alpha_1} \cdots x_n^{\alpha_n}$. This contradiction shows that Q_3 has to be the zero polynomial. □

Proof of Theorem 1.2. In order to prove the existence, define $R \in \mathrm{Sym}_{2n}$ by

$$R(y_1, \ldots, y_{2n}) \coloneqq P(y_1, \ldots, y_{2n}, 1).$$

Applying Theorem 1.1 to R we get Q in Sym_n which has the desired property:

$$\begin{aligned} Q(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}) &= R(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) \\ &= P(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, 1). \end{aligned}$$

The uniqueness also reduces to the uniqueness in Theorem 1.1. □

Remark 4.6. The two-valued inverse Dickson–Zhukovsky transform, i.e. the pair of the solutions of the equation $t + t^{-1} = u$, is given by $t = (u \pm \sqrt{u^2 - 4})/2$. So, Φ_n acts by the following explicit rule:

$$(\Phi_n(P))(z_1, \ldots, z_n) = P\Bigg(\frac{z_1 + \sqrt{z_1^2 - 4}}{2}, \ldots, \frac{z_n + \sqrt{z_n^2 - 4}}{2}, \frac{z_1 - \sqrt{z_1^2 - 4}}{2}, \ldots, \frac{z_n - \sqrt{z_n^2 - 4}}{2}\Bigg), \quad (63)$$

and Theorem 1.1 ensures that the right-hand side of (63) is a symmetric polynomial in $z_1, \ldots, z_n$.

Example 3. Let $n = 2$ and

$$P(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 5x_1x_2 + 5x_1x_3 + 5x_1x_4 + 5x_2x_3 + 5x_2x_4 + 5x_3x_4.$$

Then

$$P(x_1, x_2, x_1^{-1}, x_2^{-1}) = \left(x_1 + x_1^{-1}\right)^2 + \left(x_2 + x_2^{-1}\right)^2 + 5\left(x_1 + x_1^{-1}\right)\left(x_2 + x_2^{-1}\right) + 6,$$

i.e.

$$\Phi_2(P)(z_1, z_2) = z_1^2 + z_2^2 + 5z_1z_2 + 6.$$

In this example $\Phi_2(P)$ is not homogeneous, though P is homogeneous.

Example 4. For a general n, put

$$P_1(x_1, \ldots, x_{2n}) = \mathrm{e}_{2n}(x_1, \ldots, x_{2n}) = \prod_{j=1}^{2n} x_j, \qquad P_2(x_1, \ldots, x_{2n}) = 1,$$

$$P_3(x_1, \ldots, x_{2n+1}) = \mathrm{e}_{2n+1}(x_1, \ldots, x_{2n+1}) = \prod_{j=1}^{2n+1} x_j, \qquad P_4(x_1, \ldots, x_{2n+1}) = 1.$$

Then $P_1(x, x^{-1}) = P_2(x, x^{-1}) = 1$ and $P_3(x, x^{-1}, 1) = P_4(x, x^{-1}, 1) = 1$, i.e.

$$\Phi_n(P_1) = \Phi_n(P_2), \qquad \Phi_n^{\mathrm{odd}}(P_3) = \Phi_n^{\mathrm{odd}}(P_4).$$

This example shows that Φ_n and Φ_n^{odd} are not injective.

5 Symplectic and orthogonal Schur polynomials and Schur–Chebyshev quotients

In this section we recall various equivalent formulas for symplectic and orthogonal Schur polynomials and relate them with Schur–Chebyshev quotients.

The symplectic and orthogonal Schur functions sp_λ and o_λ appear naturally in representation theory in relation with the symplectic and orthogonal groups. See [13, Section 24 and Appendix A] for most of the following formulas for sp_λ and o_λ. An additional information can be found in [18].

For every λ in $\mathcal{P}$, the symplectic Schur function sp_λ and the orthogonal Schur function o_λ are elements of Sym defined by the following analogs of Jacobi–Trudi identities:

$$\mathrm{sp}_\lambda \coloneqq \frac{1}{2}\det\bigl[\mathrm{h}_{\lambda_j-j+k}+\mathrm{h}_{\lambda_j-j-k+2}\bigr]_{j,k=1}^{\ell(\lambda)}, \tag{64}$$

$$\mathrm{o}_\lambda \coloneqq \det\bigl[\mathrm{h}_{\lambda_j-j+k}-\mathrm{h}_{\lambda_j-j-k}\bigr]_{j,k=1}^{\ell(\lambda)}. \tag{65}$$

They also satisfy the following analogs of dual Jacobi–Trudi identities:

$$\mathrm{sp}_\lambda = \det\bigl[\mathrm{e}_{\lambda'_j-j+k}-\mathrm{e}_{\lambda'_j-j-k}\bigr]_{j,k=1}^{\lambda_1}, \tag{66}$$

$$\mathrm{o}_\lambda = \frac{1}{2}\det\bigl[\mathrm{e}_{\lambda'-j+k}+\mathrm{e}_{\lambda'-j-k+2}\bigr]_{j,k=1}^{\lambda_1}. \tag{67}$$

Example 5. By using the first Jacobi–Trudi type formulas (64) and (65), we have

$$\mathrm{sp}_{(2,2,1)} = \mathrm{h}_1+\mathrm{h}_{(2,1)}-\mathrm{h}_{(3,2)}+\mathrm{h}_{(4,1)}-\mathrm{h}_{(1,1,1)}+\mathrm{h}_{(2,2,1)}-\mathrm{h}_{(3,1,1)}, \tag{68}$$

$$\mathrm{o}_{(2,2,1)} = \mathrm{h}_3-\mathrm{h}_{(2,1)}-\mathrm{h}_{(3,2)}+\mathrm{h}_{(4,1)}+\mathrm{h}_{(2,2,1)}-\mathrm{h}_{(3,1,1)}\,. \tag{69}$$

The functions sp_λ and o_λ are traditionally evaluated at the *even symplectic alphabet*

$$(x,x^{-1}) = (x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}).$$

Since Φ_n is a morphism of algebras, (64)–(67) imply the following analogs of Jacobi–Trudi identities for spz_λ and oz_λ:

$$\mathrm{spz}_\lambda(z) = \frac{1}{2}\det\bigl[\mathrm{hz}_{\lambda_j-j+k}(z)+\mathrm{hz}_{\lambda_j-j-k+2}(z)\bigr]_{j,k=1}^{\ell(\lambda)}, \tag{70}$$

$$\mathrm{spz}_\lambda(z) = \det\bigl[\mathrm{ez}_{\lambda'_j-j+k}(z)-\mathrm{ez}_{\lambda'_j-j-k}(z)\bigr]_{j,k=1}^{\lambda_1}, \tag{71}$$

$$\mathrm{oz}_\lambda(z) = \det\bigl[\mathrm{hz}_{\lambda_j-j+k}(z)-\mathrm{hz}_{\lambda_j-j-k}(z)\bigr]_{j,k=1}^{\ell(\lambda)}, \tag{72}$$

$$\mathrm{oz}_\lambda(z) = \frac{1}{2}\det\bigl[\mathrm{ez}_{\lambda'_j-j+k}(z)+\mathrm{ez}_{\lambda'_j-j-k+2}(z)\bigr]_{j,k=1}^{\lambda_1}. \tag{73}$$

There are bialternant formulas for $\mathrm{sp}_\lambda(x,x^{-1})$ and $\mathrm{o}_\lambda(x,x^{-1})$:

$$\mathrm{sp}_\lambda(x,x^{-1}) = \frac{\det\bigl[x_k^{\lambda_j+n-j+1}-x_k^{-(\lambda_j+n-j+1)}\bigr]_{j,k=1}^{n}}{\det\bigl[x_k^{n-j+1}-x_k^{-(n-j+1)}\bigr]_{j,k=1}^{n}}, \tag{74}$$

$$\mathrm{o}_\lambda(x,x^{-1}) = \frac{\det\bigl[\zeta_{\lambda_j+n-j}(x_k)\bigr]_{j,k=1}^{n}}{\det\bigl[\zeta_{n-j}(x_k)\bigr]_{j,k=1}^{n}}, \tag{75}$$

where $\zeta_m(u) \coloneqq u^m + u^{-m}$ if $m > 0$ and $\zeta_0(u) \coloneqq 1$.

Characters of the odd orthogonal groups are related with the functions o_λ evaluated at the *odd symplectic alphabet*

$$(x, x^{-1}, 1) = (x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, 1).$$

It is known that

$$\mathrm{o}_\lambda(t_1^2, \ldots, t_n^2, t_1^{-2}, \ldots, t_n^{-2}, 1) = \frac{\det\left[t_k^{2\lambda_j+2n-2j+1} - t_k^{-(2\lambda_j+2n-2j+1)}\right]_{j,k=1}^n}{\det\left[t_k^{2n-2j+1} - t_k^{-(2n-2j+1)}\right]_{j,k=1}^n}. \tag{76}$$

Okada [28] proved the following bialternant formula for sp_λ evaluated at the generalized odd symplectic alphabet: if $\lambda \in \mathcal{P}_{n+1}$, then

$$\mathrm{sp}_\lambda(t_1^2, \ldots, t_n^2, t_1^{-2}, \ldots, t_n^{-2}, t_{n+1}) = \frac{\det A_\lambda(t_1, \ldots, t_n, 1)}{\det A_\varnothing(t_1, \ldots, t_n, 1)}, \tag{77}$$

where $(A_\lambda(t_1, \ldots, t_n, t_{n+1}))_{j,k}$ is defined as

$$\begin{cases} (t_k^{2\lambda_j+2n-2j+4} - t_k^{-(2\lambda_j+2n-2j+4)}) - t_{n+1}^{-1}(t_k^{2\lambda_j+2n-2j+2} - t_k^{-(2\lambda_j+2n-2j+2)}) & \text{if } 1 \leqslant k \leqslant n, \\ t_{n+1}^{2\lambda_j+2n-2j+2} & \text{if } k = n+1. \end{cases}$$

The bialternant formulas (74), (75), (76), and (77) are natural to rewrite in terms of the Chebyshev polynomials, as certain "Schur–Chebyshev quotients".

Proposition 5.1. *We have that*

$$\begin{aligned} \det\left[\mathcal{T}_{n-j}^{\mathrm{monic}}(z_k)\right]_{j,k=1}^n &= \det\left[\mathcal{U}_{n-j}(z_k/2)\right]_{j,k=1}^n = \det\left[\mathcal{V}_{n-j}(z_k/2)\right]_{j,k=1}^n \\ &= \det\left[\mathcal{W}_{n-j}(z_k/2)\right]_{j,k=1}^n = \det\left[\mathcal{U}_{n-j}^{(1)}(z_k/2)\right]_{j,k=1}^n = \mathrm{Van}(z). \end{aligned}$$

Proof. We sketch how to prove the formula for $\det\left[\mathcal{U}_{n-j}(z_k/2)\right]_{j,k=1}^n$; the other four determinants can be computed in a similar way. We apply the same ideas used to compute the classical Vandermonde determinant $\mathrm{Van}(z)$. One way is to use elementary transformations of the determinants and to obtain the following recursive formula:

$$\det\left[\mathcal{U}_{n-j}(z_k/2)\right]_{j,k=1}^n = \Omega_1(z_1, \ldots, z_n) \det\left[\mathcal{U}_{n-j}(z_{k+1}/2)\right]_{j,k=1}^{n-1}.$$

Another way is to notice that the determinant of $\left[\mathcal{U}_{n-j}(z_k/2)\right]_{j,k=1}^n$ vanishes when any two of the variables $z_1, \ldots, z_n$ coincide, therefore it must be a multiple of $\mathrm{Van}(z)$, i.e. there exists a polynomial $C(z)$ such that

$$\det\left[\mathcal{U}_{n-j}(z_k/2)\right]_{j,k=1}^n = C(z)\,\mathrm{Van}(z).$$

Comparing the coefficient of the term $z_1^{n-1} z_2^{n-2} \cdots z_n^0$ yields $C(z) = 1$. □

Proposition 5.2. *Let $z_k = x_k + x_k^{-1}$ for every k in $\{1,\ldots,n\}$. Then*

$$\mathrm{sp}_\lambda(x,x^{-1}) = \frac{\det\bigl[\mathcal{U}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{U}_{n-j}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{U}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\mathrm{Van}(z)}, \tag{78}$$

$$\mathrm{o}_\lambda(x,x^{-1}) = \frac{\det\bigl[\mathcal{T}^{\mathrm{monic}}_{\lambda_j+n-j}(z_k)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{T}^{\mathrm{monic}}_{n-j}(z_k)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{T}^{\mathrm{monic}}_{\lambda_j+n-j}(z_k)\bigr]_{j,k=1}^n}{\mathrm{Van}(z)}, \tag{79}$$

$$\mathrm{sp}_\lambda(x,x^{-1},1) = \frac{\det\bigl[\mathcal{U}^{(1)}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{U}^{(1)}_{n-j}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{U}^{(1)}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\mathrm{Van}(z)}, \tag{80}$$

$$\mathrm{o}_\lambda(x,x^{-1},1) = \frac{\det\bigl[\mathcal{W}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{W}_{n-j}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{W}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\mathrm{Van}(z)}, \tag{81}$$

$$\begin{aligned}\mathrm{o}_\lambda(x,x^{-1},-1) &= (-1)^{|\lambda|}\,\mathrm{o}_\lambda(-x,-x^{-1},1)\\ &= \frac{\det\bigl[\mathcal{V}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{V}_{n-j}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{V}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\mathrm{Van}(z)}.\end{aligned} \tag{82}$$

Proof. Let us prove the first equality in (78). Use the bialternant formula (74) and the property (31) of the polynomials $\mathcal{U}$:

$$\mathrm{sp}_\lambda(x,x^{-1}) = \frac{\prod_{k=1}^n (x_k - x_k^{-1})\det\bigl[\mathcal{U}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\prod_{k=1}^n (x_k - x_k^{-1})\det\bigl[\mathcal{U}_{n-j}(z_k/2)\bigr]_{j,k=1}^n} = \frac{\det\bigl[\mathcal{U}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{U}_{n-j}(z_k/2)\bigr]_{j,k=1}^n}.$$

The second equality in (78) follows from the first one transforming the denominator by Proposition 5.1. In a similar way, (79) follows from (75) and (30), and (81) follows from (76) and (33). Finally, (82) is a consequence of (81) and (45).

Let us prove (80) using Okada's formula (77) with $\lambda_{n+1} = 0$, $x_k = t_k^2$ $(1 \leqslant k \leqslant n)$ and $t_{n+1} = 1$. For $1 \leqslant k \leqslant n$,

$$\begin{aligned}A_\lambda(t_1,\ldots,t_n,1)_{j,k} &= (x_k^{\lambda_j+n-j+2} - x_k^{-(\lambda_j+n-j+2)}) - (x_k^{\lambda_j+n-j+1} - x_k^{-(\lambda_j+n-j+1)})\\ &= (x_k - x_k^{-1})(\mathcal{U}_{\lambda_j+n-j+1}(z_j/2) - \mathcal{U}_{\lambda_j+n-j}(z_j/2)\\ &= (x_k - x_k^{-1})\mathcal{V}_{\lambda_j+n-j+1}(z_j/2).\end{aligned}$$

In particular, if $1 \leqslant k \leqslant n$, then the entry $(n+1,k)$ is $(x_k - x_k^{-1})\mathcal{V}_0(z_j/2)$. After factoring $x_k - x_k^{-1}$ from the kth column $(1 \leqslant k \leqslant n)$,

$$\det A_\lambda(t_1,\ldots,t_n,1) = \left(\prod_{k=1}^n (x_k - x_k^{-1})\right)\begin{vmatrix}\bigl[\mathcal{V}_{\lambda_j+n+1-j}(z_k/2)\bigr]_{j,k=1}^n & [1]_{j=1}^n\\ [1,\ldots,1] & 1\end{vmatrix}.$$

Now from each column $1,\ldots,n$ we subtract the column $n+1$, then we expand the determinant by the last row and use the formula (48).

$$\det A_\lambda(t_1,\ldots,t_n,1) = \left(\prod_{k=1}^n (x_k - x_k^{-1})\right)\det\bigl[\mathcal{V}_{\lambda_j+n+1-j}(z_k/2) - 1\bigr]_{j,k=1}^n$$

$$= \left(\prod_{k=1}^{n}(x_k - x_k^{-1})\right)\left(\prod_{k=1}^{n}(z_k - 2)\right)\det\bigl[\mathcal{U}^{(1)}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^{n}.$$

Thus,

$$\operatorname{spz}_\lambda(x, x^{-1}, 1) = \frac{\det A_\lambda(t_1,\ldots,t_n,1)}{\det A_\varnothing(t_1,\ldots,t_n,1)} = \frac{\det\bigl[\mathcal{U}^{(1)}_{\lambda_j+n-j}(z_k/2)\bigr]_{j,k=1}^{n}}{\det\bigl[\mathcal{U}^{(1)}_{n-j}(z_k/2)\bigr]_{j,k=1}^{n}}.$$

Taking into account Proposition 5.1, we obtain the second equality in (80). □

Example 6. For $n = 2$ and $\lambda = (3, 1)$, formula (74) yields

$$\operatorname{sp}_{(3,1)}(x_1, x_2, x_1^{-1}, x_2^{-1}) = \frac{\begin{vmatrix} x_1^5 - x_1^{-5} & x_2^5 - x_2^{-5} \\ x_1^2 - x_1^{-2} & x_2^2 - x_2^{-2} \end{vmatrix}}{\begin{vmatrix} x_1^2 - x_1^{-2} & x_2^2 - x_2^{-2} \\ x_1 - x_1^{-1} & x_2 - x_2^{-1} \end{vmatrix}}.$$

Now we factorize $x_1 - x_1^{-1}$ in the first column of the determinants and $x_2 - x_2^{-1}$ in the second one, and notice that

$$\frac{x_k^5 - x_k^{-5}}{x_k - x_k^{-1}} = x_k^4 + x_k^2 + x_k^0 + x_k^{-2} + x_k^{-4} = (x_k + x_k^{-1})^4 - 3(x_k + x_k^{-1})^2 + 1.$$

Thus,

$$\operatorname{sp}_{(3,1)}(x_1, x_2, x_1^{-1}, x_2^{-1}) = \frac{\begin{vmatrix} z_1^4 - 3z_1^2 + 1 & z_2^4 - 3z_2^2 + 1 \\ z_1 & z_2 \end{vmatrix}}{\begin{vmatrix} z_1 & z_2 \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \mathcal{U}_4(z_1/2) & \mathcal{U}_4(z_2/2) \\ \mathcal{U}_1(z_1/2) & \mathcal{U}_1(z_2/2) \end{vmatrix}}{z_1 - z_2},$$

which is a particular case of (78).

Example 7. For $n = 2$ and $\lambda = (3, 1)$, formula (75) yields

$$\operatorname{o}_{(3,1)}(x_1, x_2, x_1^{-1}, x_2^{-1}) = \frac{\begin{vmatrix} x_1^4 + x_1^{-4} & x_2^4 + x_2^{-4} \\ x_1^1 + x_2^{-1} & x_2^1 + x_2^{-1} \end{vmatrix}}{\begin{vmatrix} x_1 + x_1^{-1} & x_2 + x_2^{-1} \\ 1 & 1 \end{vmatrix}}.$$

Notice that

$$x_k^4 + x_k^{-4} = (x_k + x_k^{-1})^4 - 4(x_k + x_k^{-1})^2 + 2, \qquad x_k^2 + x_k^{-2} = (x_k + x_k^{-1})^2 - 2.$$

Hence,

$$\operatorname{o}_{(3,1)}(x_1, x_2, x_1^{-1}, x_2^{-1}) = \frac{\begin{vmatrix} z_1^4 - 4z_1^2 + 2 & z_2^4 - 4z_2^2 + 2 \\ z_1 & z_2 \end{vmatrix}}{\begin{vmatrix} z_1 & z_2 \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \mathcal{T}_4^{\text{monic}}(z_1) & \mathcal{T}_4^{\text{monic}}(z_2) \\ \mathcal{T}_1^{\text{monic}}(z_1) & \mathcal{T}_1^{\text{monic}}(z_2) \end{vmatrix}}{z_1 - z_2},$$

which is a particular case of (79).

Theorem 1.3 follows from Proposition 5.2 and the definitions of Φ_n and Φ_n^{odd}. Duplication formulas for the Chebyshev polynomials lead to another equivalent form of the bialternant formulas from Proposition 5.2.

Proposition 5.3. *For every λ in $\mathcal{P}_n$,*

$$\operatorname{spz}_\lambda(u_1^2-2,\ldots,u_n^2-2)=\frac{\det\bigl[\mathcal{U}_{2\lambda_j+2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{U}_{2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}, \tag{83}$$

$$\operatorname{oz}_\lambda(u_1^2-2,\ldots,u_n^2-2)=\frac{\det\bigl[\mathcal{T}^{\mathrm{monic}}_{2\lambda_j+2n-2j}(u_k)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{T}^{\mathrm{monic}}_{2n-2j}(u_k)\bigr]_{j,k=1}^n}, \tag{84}$$

$$\operatorname{oz}^{\mathrm{odd}}_\lambda(u_1^2-2,\ldots,u_n^2-2)=\frac{\det\bigl[\mathcal{U}_{2\lambda_j+2n}(u_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{U}_{2n-2j}(u_k/2)\bigr]_{j,k=1}^n}, \tag{85}$$

$$(-1)^{|\lambda|}\operatorname{oz}^{\mathrm{odd}}_\lambda(-u_1^2+2,\ldots,-u_n^2+2)=\frac{\det\bigl[\mathcal{T}_{2\lambda_j+2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}{\det\bigl[\mathcal{T}_{2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}. \tag{86}$$

Proof. Apply (2) and (42):

$$\begin{aligned}\operatorname{spz}_\lambda(u_1^2-2,\ldots,u_n^2-2)&=\frac{\det\bigl[u_k\,\mathcal{U}_{2\lambda_j+2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}{\det\bigl[u_k\,\mathcal{U}_{2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}\\&=\frac{\left(\prod_{k=1}^n u_k\right)\det\bigl[\mathcal{U}_{2\lambda_j+2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}{\left(\prod_{k=1}^n u_k\right)\det\bigl[\mathcal{U}_{2n-2j+1}(u_k/2)\bigr]_{j,k=1}^n}.\end{aligned}$$

After canceling $\prod_{k=1}^n u_k$ we obtain (83). Similarly, formula (84) follows from (3) and (41), (85) follows from (5) and (44), and (86) follows from (6) and (43). ☐

6 Properties of ez, hz, and pz

Denote by $\widetilde{\mathrm{H}}(z)(t)$ the generating series of the sequence $(\mathrm{hz}_k(z))_{k=0}^\infty$ and by $\widetilde{\mathrm{E}}(z)(t)$ the generating series of the sequence $(\mathrm{ez}_k(z))_{k=0}^\infty$:

$$\widetilde{\mathrm{H}}(z)(t)\coloneqq\sum_{k=0}^\infty \mathrm{hz}_k(z)t^k,\qquad \widetilde{\mathrm{E}}(z)(t)\coloneqq\sum_{k=0}^\infty \mathrm{ez}_k(z)t^k=\sum_{k=0}^{2n}\mathrm{ez}_k(z)t^k.$$

Proposition 6.1. *The generating series are given by*

$$\widetilde{\mathrm{E}}(z)(t)=\prod_{j=1}^n(1+z_jt+t^2), \tag{87}$$

$$\widetilde{\mathrm{H}}(z)(t)=\prod_{j=1}^n\frac{1}{1-z_jt+t^2}. \tag{88}$$

Proof. Given a list of variables $a = (a_1, \ldots, a_p)$, $\mathrm{E}(a)(t)$ stands for the generating series of the sequence $(\mathrm{e}_k(a))_{k=0}^{\infty}$ and $\mathrm{H}(a)(t)$ for the generating series of the sequence $(\mathrm{h}_k(a))_{k=0}^{\infty}$:

$$\mathrm{E}(a)(t) = \sum_{k=0}^{\infty} \mathrm{e}_k(a) t^k = \sum_{k=0}^{p} \mathrm{e}_k(a) t^k = \prod_{j=1}^{p} (1 + a_j t), \qquad \mathrm{H}(a)(t) = \sum_{k=0}^{\infty} \mathrm{h}_k(a) t^k.$$

Using the substitutions $z_j = x_j + x_j^{-1}$ we obtain

$$\begin{aligned}\widetilde{\mathrm{E}}(z)(t) &= \sum_{k=0}^{\infty} \mathrm{ez}_k(z) t^k = \sum_{k=0}^{\infty} \mathrm{e}_k(x, x^{-1}) t^k = \mathrm{E}(x, x^{-1})(t) \\ &= \prod_{j=1}^{n} \big((1 + x_j t)(1 + x_j^{-1} t)\big) = \prod_{j=1}^{n} (1 + z_j t + t^2).\end{aligned}$$

Since the series $\mathrm{H}(x, x^{-1})(t)$ is the reciprocal of $\mathrm{E}(x, x^{-1})(-t)$, the series $\widetilde{\mathrm{H}}(z)(t)$ is the reciprocal of $\widetilde{\mathrm{E}}(z)(t)$. □

Formulas for ez

Now we rewrite Proposition 4.1 using the notation ez.

Proposition 6.2. *For each m in $\{0, \ldots, 2n\}$,*

$$\mathrm{ez}_{2n-m}(z) = \mathrm{ez}_m(z). \tag{89}$$

Next we prove Theorem 1.4. Formula (7) follows from Proposition 4.2, and (8) follows from (73) with $\lambda = (1^m)$.

Proof of (11). Let f and g be as in Propositions 3.1, 3.2. By (38),

$$\begin{aligned} g(t + t^{-1}) &= (-1)^n \mathrm{e}_n(x, x^{-1}) + \sum_{j=1}^{n} (-1)^{n-j} \mathrm{e}_{n-j}(x, x^{-1}) 2\mathcal{T}_j\left(\frac{t + t^{-1}}{2}\right) \\ &= \sum_{j=0}^{n} \sum_{k=0}^{\lfloor j/2 \rfloor} (-1)^{n-j} \mathrm{e}_{n-j}(x, x^{-1}) (-1)^k \tau_{j,k} (t + t^{-1})^{j-2k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=2k}^{n} (-1)^{n-j+k} \tau_{j,k} \mathrm{e}_{n-j}(x, x^{-1}) (t + t^{-1})^{j-2k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{m=2k}^{n} (-1)^{m-k} \tau_{n-m+2k,k} \mathrm{ez}_{m-2k}(z) (t + t^{-1})^{n-m} \\ &= \sum_{m=0}^{n} (-1)^m \left(\sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \tau_{n-m+2k,k} \mathrm{ez}_{m-2k}(z) \right) (t + t^{-1})^{n-m}. \end{aligned}$$

On the other hand, by Vieta's formula,

$$g(t+t^{-1}) = \sum_{m=0}^{n} (-1)^m \operatorname{e}_m(z)(t+t^{-1})^{n-m}.$$

Matching the coefficients of the powers of $(t+t^{-1})$ we obtain (11). ☐

Proof of (10). If $\lambda = (1^q)$, then $\lambda' = (q)$, and (71) takes the form

$$\operatorname{spz}_{(1^q)}(z) = \operatorname{ez}_q(z) - \operatorname{ez}_{q-2}(z). \tag{90}$$

In particular, $\operatorname{spz}_{(1)}(z) = \operatorname{ez}_1(z)$ and $\operatorname{spz}_{(0)}(z) = \operatorname{ez}_0(z)$. Now we write $\operatorname{ez}_m(z)$ as a telescopic sum and apply (90):

$$\operatorname{ez}_m(z) = \sum_{k=0}^{\lfloor m/2\rfloor - 1} (\operatorname{ez}_{m-2k}(z) - \operatorname{ez}_{m-2k-2}(z)) + \operatorname{ez}_{m-2\lfloor m/2\rfloor}(z) = \sum_{k=0}^{\lfloor m/2\rfloor} \operatorname{spz}_{(1^{m-2k})}(z).$$

☐

In the proposition below we abbreviate $(z_1, \ldots, z_n, z_{n+1})$ as (z, z_{n+1}).

Proposition 6.3. *For every m in $\mathbb{N}_0$,*

$$\operatorname{ez}_{m+2}(z, z_{n+1}) = \operatorname{ez}_{m+2}(z) + z_{n+1} \operatorname{ez}_{m+1}(z) + \operatorname{ez}_m(z), \tag{91}$$

$$\operatorname{ez}_{m+1}(z, z_{n+1}) - \operatorname{ez}_{m+1}(z, z_{n+2}) = (z_{n+1} - z_{n+2}) \operatorname{ez}_m(z). \tag{92}$$

Proof. By (87),

$$\widetilde{\operatorname{E}}(z, z_{n+1})(t) = (1 + z_{n+1}t + t^2)\widetilde{\operatorname{E}}(z)(t), \tag{93}$$

$$\widetilde{\operatorname{E}}(z, z_{n+1})(t) - \widetilde{\operatorname{E}}(z, z_{n+2})(t) = (z_{n+1} - z_{n+2})t\widetilde{\operatorname{E}}(z)(t). \tag{94}$$

Matching the coefficient of t^m in (93) yields (91), and matching the coefficient of t^m in (94) yields (92). ☐

Formulas for hz

In this subsection we prove Theorem 1.5 and some simple relations between the expressions hz with different lists of arguments (Proposition 6.6). Notice that (13) is another form of (59), and (15) is another form of (57).

Lemma 6.4. *Let $s \in \mathbb{N}_0$. Then*

$$\frac{\det \begin{bmatrix} \mathcal{U}_{n+s-1}(z_1/2) & \ldots & \mathcal{U}_{n+s-1}(z_n/2) \\ \mathcal{U}_{n-2}(z_1/2) & \ldots & \mathcal{U}_{n-2}(z_n/2) \\ \ldots & \ldots & \ldots \\ \mathcal{U}_0(z_1/2) & \ldots & \mathcal{U}_0(z_n/2) \end{bmatrix}}{\operatorname{Van}(z)} = \sum_{j=1}^{n} \frac{\mathcal{U}_s(z_j/2)}{\Omega_j(z)}. \tag{95}$$

Proof. Expand the determinant in the numerator along the first row and simplify the cofactors using Proposition 5.1. □

Lemma 6.5. *Let* $0 \leqslant s < n-1$. *Then*

$$\sum_{j=1}^{n} \frac{\mathcal{U}_s(z_j/2)}{\Omega_j(z)} = 0. \tag{96}$$

Proof. Represent the left-hand side of (96) by (95). Since $0 \leqslant s < n-1$, the first row of the determinant coincides with one of the other rows, and the determinant is zero. □

Proof of (16). Use (88) and (35):

$$\sum_{m=0}^{\infty} \mathrm{hz}_m(z) t^m = \widetilde{\mathrm{H}}(z)(t) = \prod_{j=1}^{n} \frac{1}{1 - z_j t + t^2} = \prod_{j=1}^{n} \sum_{q=0}^{\infty} \mathcal{U}_q(z_j/2) t^q.$$

Equating the coefficients of t^m we arrive at (16). □

Proof of (17). Follows from (64) and (70) with $\lambda = (m)$. The equivalence of (13) and (17) also follows from (2) and Lemma 6.4. □

Proof of (18). Identity (72) for $\lambda = (s)$ yields

$$\mathrm{oz}_{(s)}(z) = \mathrm{hz}_s(z) - \mathrm{hz}_{s-2}(z),$$

i.e. $\mathrm{hz}_s(z) = \mathrm{oz}_{(s)}(z) + \mathrm{hz}_{s-2}(z)$. By induction over m, we obtain (18). □

Proof of (19). By (55) and (40),

$$\mathrm{h}_m(z) = \sum_{k=0}^{\lfloor (m+n-1)/2 \rfloor} \mathbf{C}_{m+n-1-k,k} \left(\sum_{j=1}^{n} \frac{\mathcal{U}_{m+n-1-2k}(z_j/2)}{\Omega_j(z)} \right).$$

Lemma 6.5 allows us to restrict the upper limit in the sum over k; after that we apply (13):

$$\mathrm{h}_m(z) = \sum_{k=0}^{\lfloor m/2 \rfloor} \mathbf{C}_{m+n-1-k,k} \left(\sum_{j=1}^{n} \frac{\mathcal{U}_{m+n-1-2k}(z_j/2)}{\Omega_j(z)} \right) = \sum_{k=0}^{\lfloor m/2 \rfloor} \mathbf{C}_{m+n-1-k,k}\, \mathrm{hz}_{m-2k}(z).$$

□

Formula (98) below is inspired by [22, Lemma 1]. We abbreviate $(z_1, \ldots, z_n, z_{n+1})$ as (z, z_{n+1}).

Proposition 6.6. *For every* m *in* $\mathbb{N}_0$,

$$\mathrm{hz}_{m+2}(z, z_{n+1}) = \mathrm{hz}_{m+2}(z) + z_{n+1}\, \mathrm{hz}_{m+1}(z, z_{n+1}) - \mathrm{hz}_m(z, z_{n+1}), \tag{97}$$

$$\mathrm{hz}_{m+1}(z, z_{n+1}) - \mathrm{hz}_{m+1}(z, z_{n+2}) = (z_{n+1} - z_{n+2})\, \mathrm{hz}_m(z, z_{n+1}, z_{n+2}). \tag{98}$$

Proof. We start with a simple expression:

$$\widetilde{\mathrm{E}}(z, z_{n+1})(-t) = (1 - z_{n+1}t + t^2)\widetilde{\mathrm{E}}(z)(-t).$$

Divide by the product $\widetilde{\mathrm{E}}(z, z_{n+1})(-t)\widetilde{\mathrm{E}}(z)(-t)$:

$$\widetilde{\mathrm{H}}(z)(t) = (1 - z_{n+1}t + t^2)\widetilde{\mathrm{H}}(z, z_{n+1})(t).$$

By equating the coefficients of t^{m+2} in both sides,

$$\mathrm{hz}_{m+2}(z) = \mathrm{hz}_{m+2}(z, z_{n+1}) - z_{n+1}\,\mathrm{hz}_{m+1}(z, z_{n+1}) + \mathrm{hz}_m(z, z_{n+1}).$$

The obtained formula is equivalent to (97). Now consider the difference

$$\widetilde{\mathrm{E}}(z, z_{n+2})(-t) - \widetilde{\mathrm{E}}(z, z_{n+1})(-t) = (z_{n+1} - z_{n+2})t\widetilde{\mathrm{E}}(z)(-t).$$

Divide over the product $\widetilde{\mathrm{E}}(z, z_{n+1})(-t)\widetilde{\mathrm{E}}(z, z_{n+2})(-t)$:

$$\widetilde{\mathrm{H}}(z, z_{n+1})(t) - \widetilde{\mathrm{H}}(z, z_{n+2})(t) = \frac{(z_{n+1} - z_{n+2})t\widetilde{\mathrm{H}}(z)(t)}{(1 - z_{n+1}t + t^2)(1 - z_{n+2}t + t^2)},$$

i.e.

$$\widetilde{\mathrm{H}}(z, z_{n+1})(t) - \widetilde{\mathrm{H}}(z, z_{n+2})(t) = (z_{n+1} - z_{n+2})t\widetilde{\mathrm{H}}(z, z_{n+1}, z_{n+2})(t).$$

Equating the coefficients of t^{m+1} we get (98). □

Formulas for pz

In this subsection we prove Theorem 1.6. Formula (21) is another form of (60), and (20) is another form of (61).

Proof of (22). Use (39) and change the order of the sums:

$$\begin{aligned}\mathrm{p}_m(z) &= \sum_{j=1}^{n} z_j^m = \sum_{j=1}^{n}\sum_{k=0}^{\lfloor m/2\rfloor} \alpha_{m,k} 2\mathcal{T}_{m-2k}\left(z_j/2\right) = \sum_{k=0}^{\lfloor m/2\rfloor}\alpha_{m,k}\sum_{j=1}^{n}\left(x_j^{m-2k} + x_j^{-(m-2k)}\right)\\ &= \sum_{k=0}^{\lfloor m/2\rfloor}\alpha_{m,k}\,\mathrm{p}_{m-2k}(x, x^{-1}) = \sum_{k=0}^{\lfloor m/2\rfloor}\alpha_{m,k}\,\mathrm{pz}_{m-2k}(z).\end{aligned}$$

□

Recall that the involution $\omega\colon \mathrm{Sym} \to \mathrm{Sym}$ can be defined by

$$\omega(\mathrm{p}_m) = (-1)^m\,\mathrm{p}_m \qquad (m \geqslant 1). \tag{99}$$

In order to define $\omega_n\colon \mathrm{Sym}_n \to \mathrm{Sym}_n$, we use the monomial embedding $\mathrm{Sym}_n \to \mathrm{Sym}$. In other words, given f in Sym_n, we expand f in the monomial basis and then apply ω.

Proposition 6.7. *We have that*

$$\omega_n(\mathrm{pz}_m)(z) = \begin{cases} (-1)^{m-1}\,\mathrm{pz}_m(z), & m \text{ is odd,} \\ (-1)^{m-1}\,\mathrm{pz}_m(z) + \frac{1}{2}(-1)^{m/2}\binom{m}{m/2}\,\mathrm{pz}_0(z), & m \text{ is even.} \end{cases} \tag{100}$$

Proof. Apply (21) and (99). For odd m,

$$\omega_n(\mathrm{pz}_m)(z) = \sum_{k=0}^{\lfloor m/2 \rfloor} \alpha_{m,k}(-1)^{m-2k-1}(-1)^k\,\mathrm{p}_{m-2k}(z) = (-1)^{m-1}\,\mathrm{pz}_m(z).$$

For even m,

$$\mathrm{pz}_m(z) = \sum_{k=0}^{\frac{m}{2}-1} \alpha_{m,k}(-1)^k\,\mathrm{p}_{m-2k}(z) + \alpha_{m,m/2}(-1)^{m/2}\,\mathrm{p}_0(z).$$

Thus,

$$\begin{aligned} \omega_n(\mathrm{pz}_m)(z) &= \sum_{k=0}^{\frac{m}{2}-1} \alpha_{m,k}(-1)^{m-1}(-1)^k\,\mathrm{p}_{m-2k}(z) + \alpha_{m,m/2}(-1)^{m/2}\,\mathrm{p}_0(z) \\ &= (-1)^{m-1}\sum_{k=0}^{m/2} \alpha_{m,k}(-1)^{m-1}(-1)^k\,\mathrm{p}_{m-2k}(z) + 2\alpha_{m,m/2}(-1)^{m/2}\,\mathrm{p}_0(z). \end{aligned}$$

The last expression yields the second case of (100). □

Unfortunately, in general ω_n does not convert $\mathrm{hz}_\lambda^{(n)}$ into $\mathrm{ez}_\lambda^{(n)}$, even with the restriction $\ell(\lambda) \leqslant n$.

Algebraically independent generating subsets

Here we consider Sym_n as a complex algebra with identity. Given a subset S of Sym_n, we denote by $\langle S \rangle$ the subalgebra (with identity) generated by S. It is known [17, Theorem 5.9 and Proposition 5.10] that if A is a finitely generated commutative associative algebra with identity, $S \subseteq A$, and $\langle S \rangle = A$, then its (Krull) dimension is

$$\dim(A) = \sup\{\#T\colon\ T \subseteq S,\ T \text{ is finite and algebraically independent}\}. \tag{101}$$

It is well known that $\{\mathrm{e}_m\}_{m=1}^n$ is an algebraically independent generating subset of Sym_n. As a consequence, $\dim(\mathrm{Sym}_n) = n$. So, if S is a generating subset of Sym_n consisting of n elements, then S is algebraically independent. Moreover, in this situation, S is a minimal by size (and minimal by inclusion) generating subset of Sym_n.

Proof of Theorem 1.7. Since $\{\mathrm{e}_m\}_{m=1}^n$ is a generating subset of Sym_n, the relation

$$\mathrm{E}(z)(-t)\,\mathrm{H}(z)(t) = 1$$

implies that $\{\mathrm{h}_m\}_{m=1}^n$ is also a generating subset of Sym_n. Newton's identity yields a similar conclusion for $\{\mathrm{p}_m\}_{m=1}^n$. Now Theorem 1.7 follows from the identities (7), (11), (15), (19), (21), and (22), combined with the reasoning before this proof. □

Formulas for $\mathrm{ez}^{\mathrm{odd}}$, $\mathrm{hz}^{\mathrm{odd}}$, and $\mathrm{pz}^{\mathrm{odd}}$

With the generating series for the sequences $(\mathrm{e}_m)_{m=0}^{\infty}$ and $(\mathrm{h}_m)_{m=0}^{\infty}$ it is easy to obtain the following recurrence relations:

$$\mathrm{e}_{m+1}(y_1,\ldots,y_p,y_{p+1}) = \mathrm{e}_{m+1}(y_1,\ldots,y_p) + y_{p+1}\,\mathrm{e}_m(y_1,\ldots,y_p), \tag{102}$$

$$\mathrm{h}_{m+1}(y_1,\ldots,y_p,y_{p+1}) = \mathrm{h}_{m+1}(y_1,\ldots,y_p) + y_{p+1}\,\mathrm{h}_m(y_1,\ldots,y_p,y_{p+1}). \tag{103}$$

The definition of p_m immediately yields

$$\mathrm{p}_m(y_1,\ldots,y_p,y_{p+1}) = \mathrm{p}_m(y_1,\ldots,y_p) + y_{p+1}^m. \tag{104}$$

Apply these formulas to the odd symplectic alphabet $x, x^{-1}, 1$, then use the morphisms Φ_n and Φ_n^{odd}:

$$\mathrm{ez}_{m+1}^{\mathrm{odd}}(z) = \mathrm{ez}_{m+1}(z) + \mathrm{ez}_m(z), \tag{105}$$

$$\mathrm{hz}_{m+1}^{\mathrm{odd}}(z) = \mathrm{hz}_{m+1}(z) + \mathrm{hz}_m^{\mathrm{odd}}(z), \tag{106}$$

$$\mathrm{pz}_m^{\mathrm{odd}}(z) = \mathrm{pz}_m(z) + 1. \tag{107}$$

These identities and Theorems 1.4, 1.5, 1.6 yield the following formulas for $\mathrm{ez}_m^{\mathrm{odd}}(z)$, $\mathrm{hz}_m^{\mathrm{odd}}(z)$, and $\mathrm{pz}_m^{\mathrm{odd}}(z)$.

Proposition 6.8. *For every m in $\{0,\ldots,2n+1\}$,*

$$\mathrm{ez}_m^{\mathrm{odd}}(z) = \sum_{k=\max\{0,2m-2n-1\}}^{m} \binom{n-m+k}{\lfloor k/2\rfloor} \mathrm{e}_{m-k}(z). \tag{108}$$

Proposition 6.9. *For every m in $\mathbb{N}_0$,*

$$\mathrm{hz}_m^{\mathrm{odd}}(z) = \sum_{k=0}^{m} \mathrm{hz}_k(z), \tag{109}$$

$$\mathrm{hz}_m^{\mathrm{odd}}(z) = \sum_{k=0}^{m} \mathrm{spz}_{(k)}(z), \tag{110}$$

$$\mathrm{hz}_m^{\mathrm{odd}}(z) = \sum_{j=1}^{n} \frac{\mathcal{U}_{m+n-1}^{(1)}(z_j/2)}{\Omega_j(z)}, \tag{111}$$

$$\mathrm{hz}_m^{\mathrm{odd}}(z) = \sum_{k=0}^{m} \sigma_{m+n-1,k+n-1}\,\mathrm{h}_k(z), \tag{112}$$

where $\sigma_{m,k}$ is defined by (47).

Proposition 6.10. *We have*

$$\mathrm{pz}_m^{\mathrm{odd}}(z) = \begin{cases} 1 + m\displaystyle\sum_{j=0}^{\lfloor m/2\rfloor} \frac{(-1)^j}{m-j}\binom{m-j}{j}\,\mathrm{p}_{m-2j}(z_1,\ldots,z_n), & m\in\mathbb{N};\\ 1 + 2\,\mathrm{p}_0(z_1,\ldots,z_n), & m=0. \end{cases} \tag{113}$$

7 Schur polynomials in symplectic variables

We denote by $\mathrm{s}_{\lambda/\mu}$ the skew Schur function associated to a skew partition λ/μ. It is given by the following Jacobi–Trudi formulas:

$$\mathrm{s}_{\lambda/\mu} \coloneqq \det\left[\mathrm{h}_{\lambda_j-\mu_k-j+k}\right]_{j,k=1}^{\ell(\lambda)}, \qquad \mathrm{s}_{\lambda/\mu} = \det\left[\mathrm{e}_{\lambda'_j-\mu'_k-j+k}\right]_{j,k=1}^{\lambda_1}.$$

Put $\mathrm{sz}_{\lambda/\mu}^{(n)} \coloneqq \Phi_n(\mathrm{s}_{\lambda/\mu}^{(2n)})$. Since Φ_n is a homomorphism of algebras, the polynomials $\mathrm{sz}_{\lambda/\mu}^{(n)}$ inherit some properties of the classical Schur polynomials. In particular, $\mathrm{sz}_{\lambda/\mu}^{(n)}$ can be computed by the following analogs of the Jacobi–Trudi formulas:

$$\mathrm{sz}_{\lambda/\mu}(z) = \det\left[\mathrm{hz}_{\lambda_j-\mu_k-j+k}(z)\right]_{j,k=1}^{\ell(\lambda)}, \tag{114}$$

$$\mathrm{sz}_{\lambda/\mu}(z) = \det\left[\mathrm{ez}_{\lambda'_j-\mu'_k-j+k}(z)\right]_{j,k=1}^{\lambda_1}. \tag{115}$$

Furthermore,

$$\mathrm{sz}_\lambda(z)\,\mathrm{sz}_\mu(z) = \sum_\nu \mathrm{LR}_{\lambda,\mu}^\nu\,\mathrm{sz}_\nu(z),$$

where $\mathrm{LR}_{\lambda,\mu}^\nu$ are the usual Littlewood–Richardson coefficients for the Schur functions.

In this section we prove Theorems 1.8 and 1.9, and state a few other properties of the polynomials $\mathrm{sz}_\lambda^{(n)}$.

Proposition 7.1. *Let $\lambda \in \mathcal{P}$ and $\ell(\lambda) > 2n$. Then $\mathrm{sz}_\lambda(z_1, \ldots, z_n) = 0$.*

Proof. Since $\ell(\lambda) > 2n$, the Jacobi–Trudi formula easily implies that

$$\mathrm{s}_\lambda(y_1, \ldots, y_{2n}) = 0.$$

Now $\mathrm{sz}_\lambda(z_1, \ldots, z_n) = 0$ by definition of the morphism Φ_n. □

The following "symmetry property" of sz is equivalent to [19, Lemma 1] and follows from the dual Jacobi–Trudi identity (115) and Proposition 6.2.

Proposition 7.2. *For $1 \leqslant k \leqslant n$*

$$\mathrm{sz}_{(\lambda_1,\ldots,\lambda_{n+k})}(z_1, \ldots, z_n) = \mathrm{sz}_\mu(z_1, \ldots, z_n),$$

where $\mu = (\lambda_1^{n-k}, \lambda_1 - \lambda_{n+k}, \lambda_1 - \lambda_{n+k-1}, \ldots, \lambda_1 - \lambda_2)$.

To prove Theorem 1.8, we need an elementary lemma from linear algebra that can be proved by induction.

Lemma 7.3. *Let V be a vector space, $(b_\lambda)_{\lambda\in J}$ be a basis of V, and the index set J be the union of the sequence $(J_w)_{w=0}^\infty$ of finite sets J_w, such that $J_w \subseteq J_{w+1}$ for every w in $\mathbb{N}_0$. Put $J_{-1} \coloneqq \varnothing$. Denote by V_w the subspace generated by b_λ with λ in J_w. Suppose that $(a_\lambda)_{\lambda\in J}$ is a family of vectors in V such that for every w in $\mathbb{N}_0$ and every λ in $J_w \setminus J_{w-1}$,*

$$a_\lambda - b_\lambda \in V_{w-1}.$$

Then $(a_\lambda)_{\lambda\in J}$ is a basis of V.

Proof of Theorem 1.8. For every w in $\mathbb{N}_0$, put $J_w := \{\lambda \in \mathcal{P}_n\colon\ |\lambda| \leqslant w\}$ and denote by V_w the subspace generated by $\{\mathrm{s}_\lambda^{(n)}\colon\ \lambda \in J_w\}$. It is well known that the family $(\mathrm{s}_\lambda^{(n)})_{\lambda\in\mathcal{P}_n}$ is a basis of the vector space Sym_n, and V_w consists of all symmetric polynomials in n variables of degree $\leqslant w$.

Given a partition λ in $\mathcal{P}_n$, write $\mathrm{sz}_\lambda(z)$ in the form (114) and expand $\mathrm{hz}_{\lambda_j-j+k}(z)$ into a linear combination of $\mathrm{h}_m(z)$ with $m \leqslant \lambda_j - j + k$, using (15). Thereby it can be shown that

$$\mathrm{sz}_\lambda(z) = \mathrm{s}_\lambda(z) + R(z),$$

where R is a symmetric polynomial of degree strictly less than $|\lambda|$ and with integer coefficients. So, the conditions of Lemma 7.3 are fulfilled, and $(\mathrm{sz}_\lambda^{(n)})_{\lambda\in\mathcal{P}_n}$ is a basis of the vector space Sym_n.

For the other families from Theorem 1.8, the proofs are similar. For the families $(\mathrm{spz}_\lambda^{(n)})_{\lambda\in\mathcal{P}_n}$, $(\mathrm{oz}_\lambda^{(n)})_{\lambda\in\mathcal{P}_n}$, the bialternant identities can be used instead of the Jacobi–Trudi formulas. □

Proof of Theorem 1.9. All formulas in Theorem 1.9 follow easily from the classical Cauchy and dual Cauchy identities. Let us verify only (24). Representing z_j as $x_j + x_j^{-1}$,

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}} \mathrm{sz}_\lambda(z)\,\mathrm{s}_\lambda(y) &= \sum_{\lambda\in\mathcal{P}} \mathrm{s}_\lambda(x, x^{-1})\,\mathrm{s}_\lambda(y) \\ &= \prod_{j=1}^{n}\prod_{k=1}^{m} \frac{1}{(1 - x_j y_k)(1 - x_j^{-1} y_k)} = \prod_{j=1}^{n}\prod_{k=1}^{m} \frac{1}{1 - z_j y_k + y_k^2}.\end{aligned}$$ □

Remark 7.4. The sum in the left-hand side of (24) can be reduced to the partitions λ with $\ell(\lambda) \leqslant \min\{2n, m\}$, and the sum in the left-hand side of (27) can be reduced to the partitions λ with $\ell(\lambda) \leqslant \min\{2n+1, m\}$. These sums can be treated in the formal sense. They are absolutely converging for small values of z_j and y_k. The sums in the left-hand sides of (25), (26), (28), and (29) are finite. For example, the sum in the left-hand side of (25) can be reduced to the partitions λ satisfying $\ell(\lambda) \leqslant 2n$ and $\lambda_1 \leqslant m$.

There are simple bialternant formulas for $\mathrm{sz}_{(m)}^{(n)}$ and $\mathrm{sz}_{(1^m)}^{(n)}$, see (9) and (14). The next proposition shows that there is no similar formula for $\mathrm{sz}_\lambda^{(n)}$ with general λ.

Proposition 7.5. *There do not exist univariate polynomials f and g such that*

$$\mathrm{sz}_{(2,1)}(z_1, z_2) = \frac{\begin{vmatrix} f(z_1) & f(z_2) \\ g(z_1) & g(z_2) \end{vmatrix}}{\mathrm{Van}(z_1, z_2)}. \tag{116}$$

Proof. Using (114) we obtain

$$\mathrm{sz}_{(2,1)}(z_1, z_2) = z_1^2 z_2 + z_1 z_2^2 + z_1 + z_2. \tag{117}$$

Suppose that f and g are univariate polynomials satisfying (116), and

$$f(t) = \sum_{j=0}^{p} f_j t^j, \qquad g(t) = \sum_{j=0}^{q} g_j t^j,$$

with $f_p \neq 0$, $g_q \neq 0$. If $p = q$, then an elementary operation with the rows of the determinant in the numerator allows to pass to the case $q = p - 1$. So, without loss of generality, we consider the case $p > q$. Then the leading terms of the numerator of (116) are $f_p g_p z_1^p z_2^q$ and $-f_p g_p z_1^q z_2^p$, and it is easy to conclude that $p = 3$, $q = 1$. Since $f_p g_p = 1$, we may consider the case $f_p = 1$ and $g_p = 1$. An elementary computation yields

$$\frac{\begin{vmatrix} f(z_1) & f(z_2) \\ g(z_1) & g(z_2) \end{vmatrix}}{z_1 - z_2} = z_1^2 z_2 + z_1 z_2^2 + g_0(z_1^2 + z_2^2) + (g_0 + f_2) z_1 z_2 + g_0 f_2 (z_1 + z_2) + g_0 f_1 - f_0.$$

From this and (117), $g_0 = 0$ and simultaneously $g_0 f_2 = 1$, which is impossible. □

However, there is a bialternant formula for $\mathrm{sz}_{(2,1)}(z_1, z_2)$ with a more complicated denominator.

Example 8. It can be verified directly that

$$\mathrm{sz}_{(2,1)}(z_1, z_2) = \frac{\begin{vmatrix} z_1^4 + 1 & z_2^4 + 1 \\ z_1^2 & z_2^2 \end{vmatrix}}{\begin{vmatrix} z_1^2 + 1 & z_2^2 + 1 \\ z_1 & z_2 \end{vmatrix}}. \tag{118}$$

Remark 7.6. A natural problem for future work is to generalize Example 8, i.e. for every n in $\mathbb{N}$ and λ in $\mathcal{P}_n$ find univariate polynomials $f_1, \ldots, f_n$, $g_1, \ldots, g_n$, with coefficients depending on λ and n, such that

$$\mathrm{sz}_\lambda(z_1, \ldots, z_n) = \frac{\det\big[f_j(z_k)\big]_{j,k=1}^n}{\det\big[g_j(z_k)\big]_{j,k=1}^n}.$$

Remark 3. Unfortunately, we are unable to solve this problem at the moment. We tried to solve this problem for $\mathrm{sz}_{(3,1)}(z_1, z_2)$. A simple reasoning for the leading terms shows that the degrees of f_1, f_2, g_1, g_2 should be of the form $s+4, s+1, s+1, s$. We have proved that there is no solution for $s = 0$ nor for $s = 1$ (the proof is uninteresting and involves many terms).

Littlewood [23, Appendix] expanded s_λ into linear combinations of sp_μ or o_μ:

$$\mathrm{s}_\lambda = \sum_{\mu \in \mathcal{P}} \left(\sum_{\substack{\nu \in \mathcal{P} \\ \nu' \text{ even}}} \mathrm{LR}_{\nu,\mu}^\lambda \right) \mathrm{sp}_\mu, \qquad \mathrm{s}_\lambda = \sum_{\mu \in \mathcal{P}} \left(\sum_{\substack{\nu \in \mathcal{P} \\ \nu \text{ even}}} \mathrm{LR}_{\nu,\mu}^\lambda \right) \mathrm{o}_\mu\,. \tag{119}$$

Here the restrictions "ν even" or "ν' even" mean that all parts of ν or ν', respectively, are even. Evaluating both sides of the identities (119) at the symplectic list of variables and applying the homomorphism Φ_n, yields expansions of $\mathrm{sz}_\lambda^{(n)}$ into linear combinations of $\mathrm{spz}_\mu^{(n)}$ or $\mathrm{oz}_\mu^{(n)}$, with coefficients not depending on n. Krattenthaler [19] studied particular cases of (119) for partitions λ of "nearly rectangular form". In that cases the coefficients are always 0 or 1.

8 Connection to the determinants and minors of banded symmetric Toeplitz matrices

Recall that Toeplitz matrices are of the form $T_n(a) = [a_{j-k}]_{j,k=1}^n$, where a_j are some entries. Various equivalents formulas for banded Toeplitz determinants were found by Baxter and Schmidt [4], and Trench [30]. Bump and Diaconis [8], Lascoux [21], and other authors noticed relations between Toeplitz minors and skew Schur polynomials. It was shown explicitly in [1, 26] that every minor of the $m \times m$ Toeplitz matrix $T_m(a)$ generated by a Laurent polynomial a, can be written as a certain skew Schur polynomial evaluated at the roots of a. García-García and Tierz [14] studied the asymptotic behavior of Toeplitz minors (as the order of the matrix tends to infinity, while the indices of the struck-out rows and columns are fixed) using their connection to symmetric polynomials.

Many applications and investigations involve Hermitian Toeplitz matrices [16, 6, 7, 5]. In particular, an important object of study are symmetric banded Toeplitz matrices generated by palindromic Laurent polynomials

$$a(t) = \sum_{k=-n}^{n} a_k t^k = a_0 + \sum_{k=1}^{n} a_k (t^k + t^{-k}), \tag{120}$$

where $a_n \neq 0$ and $a_k = a_{-k}$ for all k. For $|k| > n$, we put $a_k = 0$.

In this section we consider minors and determinants of Toeplitz matrices $T_n(a)$ generated by such polynomials.

By Proposition 3.2, there exists a polynomial g of degree n such that $a(t) = g(t+1/t)$, and we denote by $z_1, \ldots, z_n$ the zeros of g:

$$g(u) = a_n + \sum_{k=1}^{n} a_{n-k} 2\mathcal{T}_k(u/2) = a_n \prod_{j=1}^{n} (u - z_j). \tag{121}$$

First, we notice that all minors of symmetric Toeplitz matrices can be expressed through $\mathrm{sz}_{\lambda/\mu}$ with a certain skew partition λ/μ. We write id_d for $(1, \ldots, d)$ and $\mathrm{rev}(\alpha_1, \ldots, \alpha_d)$ for $(\alpha_d, \ldots, \alpha_1)$.

Proposition 8.1. *Let a be a palindromic Laurent polynomial of the form* (120)*, and $z_1, \ldots, z_n$ are the zeros of the polynomial g defined by* (121)*. Furthermore, let $m \in \mathbb{N}$, $r \leqslant m$, $\rho_1, \ldots, \rho_r \in \{1, \ldots, m\}$, $\sigma_1, \ldots, \sigma_r \in \{1, \ldots, m\}$, such that $\rho_1 < \cdots < \rho_r$ and $\sigma_1 < \cdots < \sigma_r$. Then*

$$\det T_m(a)_{\rho,\sigma} = (-1)^{rn+|\rho|+|\sigma|} a_n^r \, \mathrm{sz}_{(r^n, \mathrm{rev}(\xi - \mathrm{id}_d))/(\mathrm{rev}(\eta - \mathrm{id}_d))}(z_1, \ldots, z_n) \tag{122}$$

$$\det T_m(a)_{\rho,\sigma} = (-1)^{rn+|\rho|+|\sigma|} a_n^r \operatorname{sz}_{(r^n, r^d + \mathrm{id}_d - \eta)/(r^d + \mathrm{id}_d - \xi)}(z_1, \ldots, z_n), \tag{123}$$

where $d = m - r$, $\{\xi_1, \ldots, \xi_d\} = \{1, \ldots, m\} \setminus \{\rho_1, \ldots, \rho_r\}$, $\{\eta_1, \ldots, \eta_d\} = \{1, \ldots, m\} \setminus \{\sigma_1, \ldots, \sigma_r\}$, $\xi_1 < \cdots < \xi_d$, $\eta_1 < \cdots < \eta_d$.

Proof. According to [26], for a general Laurent polynomial of the form

$$a(t) = \sum_{k=-q}^{p} a_k t^k = a_p t^{-q} \prod_{j=1}^{p+q} (t - x_j),$$

the minor $\det T_m(a)_{\rho,\sigma}$ can be expressed as the following skew Schur polynomial in the variables $x_1, \ldots, x_{p+q}$:

$$\det T_m(a)_{\rho,\sigma} = (-1)^{nr+|\rho|+|\sigma|} a_p^r \operatorname{s}_{(r^n, \operatorname{rev}(\xi - \mathrm{id}_d))/(\operatorname{rev}(\eta - \mathrm{id}_d))}(x_1, \ldots, x_{p+q}) \tag{124}$$

$$= (-1)^{nr+|\sigma|+|\rho|} a_p^r \operatorname{s}_{(r^n, r^d + \mathrm{id}_d - \eta)/(r^d + \mathrm{id}_d - \xi)}(x_1, \ldots, x_{p+q}). \tag{125}$$

In the palindromic case (120), we use notation sz and obtain the desired formula. □

In the three corollaries below, let a and $z_1, \ldots, z_n$ be as in Proposition 8.1, and $m \in \mathbb{N}$.

Corollary 8.2. *We have that*

$$\det T_m(a) = (-1)^{nm} a_n^m \operatorname{sz}_{(m^n)}(z).$$

Corollary 8.3. *Let* $p, q \in \{1, \ldots, m\}$*. Then the* (p, q)*th entry of the adjugate matrix of* $T_m(a)$ *is*

$$(\operatorname{adj}(T_m(a)))_{p,q} = (-1)^{n(m-1)} a_n^{m-1} \operatorname{sz}_{((m-1)^n, q-1)/(p-1)}(z).$$

Corollary 8.4. *Suppose that* $\det T_m(a) = 0$*. Then the vector* $v = [v_q]_{q=1}^m$ *with components*

$$v_q = \operatorname{sz}_{((m-1)^{n-1}, m-q)}(z)$$

belongs to the nullspace of $T_m(a)$*.*

Remark 8.5. Since the Toeplitz matrices are persymmetric, it is easy to select two different submatrices of a large Toeplitz matrix such that their determinants coincide. So, for fixed n and m, with m large enough, the correspondence $(\rho, \sigma) \mapsto \operatorname{sz}_{\lambda/\mu}^{(n)}$, defined in Proposition 8.1, is not injective.

Furthermore, we will show that every polynomial of the form $\operatorname{sz}_{\lambda/\mu}(z)$ can be written as a minor of a banded symmetric Toeplitz matrix. Notice that the skew partition λ/μ in (122) has a special form: the initial entries of λ coincide. The main idea is canceling them out with the initial entries of μ.

Proposition 8.6. *Let $\lambda, \mu \in \mathcal{P}_n$, $\mu \leqslant \lambda$, let a be the palindromic Laurent polynomial given by*

$$a(t) = \prod_{j=1}^{n} (t - z_j + t^{-1}),$$

and let $m \in \mathbb{N}$ with $m > n + \ell(\lambda)$. Then there exist $r \leqslant m$, $\rho_1, \ldots, \rho_r \in \{1, \ldots, m\}$, $\sigma_1, \ldots, \sigma_r \in \{1, \ldots, m\}$, such that $\rho_1 < \cdots < \rho_r$, $\sigma_1 < \cdots < \sigma_r$, and

$$\det T_m(a)_{\rho,\sigma} = (-1)^{rn+|\rho|+|\sigma|} \operatorname{sz}_{\lambda/\mu}(z_1, \ldots, z_n). \tag{126}$$

Proof. Put $q := \ell(\lambda)$, $d := n + q$, $r := m - d$,

$$\xi := (\mathrm{id}_n, n^q + \mathrm{id}_q + \mathrm{rev}(\lambda)), \qquad \eta := (\mathrm{id}_q + \mathrm{rev}(\mu), (m - n - q)^n + \mathrm{id}_n), \tag{127}$$

i.e.

$$\begin{aligned} &\xi_1 := 1, \ \ldots, \ \xi_n := n, && \xi_{n+1} := n + 1 + \lambda_q, \ \ldots, \ \xi_{n+q} = n + q + \lambda_1, \\ &\eta_1 := 1 + \mu_q, \ \ldots, \ \eta_q := q + \mu_1, && \eta_{q+1} := m - n - q + 1, \ \ldots, \ \eta_{q+n} := m - q. \end{aligned}$$

Moreover, define $\rho_1, \ldots, \rho_r$ to be the elements of $\{1, \ldots, m\} \setminus \{\xi_1, \ldots, \xi_d\}$, enumerated in the ascending order, and $\sigma_1, \ldots, \sigma_r$ to be the elements of $\{1, \ldots, m\} \setminus \{\eta_1, \ldots, \eta_d\}$ enumerated in the ascending order. Then

$$(r^n, \mathrm{rev}(\xi - \mathrm{id}_d))/(\mathrm{rev}(\eta - \mathrm{id}_d)) = ((m - n - q)^n, \lambda, 0^n)/((m - n - q)^n, \mu),$$

and by Proposition 8.1 we obtain (126). □

Corollary 8.2 relates banded symmetric Toeplitz determinants with Schur polynomials corresponding to rectangular partitions and evaluated at the symplectic alphabet. Efficient formulas for these polynomials were independently found in [31, 19, 11]. As before, write x, x^{-1}, and z instead of $x_1, \ldots, x_n$, $x_1^{-1}, \ldots, x_n^{-1}$, and $z_1, \ldots, z_n$, respectively.

Proposition 8.7. *We have*

$$\mathrm{s}_{(m^n)}(x, x^{-1}) = \begin{cases} \mathrm{sp}_{((p-1)^n)}(x, x^{-1}) \, \mathrm{o}_{(p^n)}(x, x^{-1}), & m = 2p - 1, \\ \mathrm{o}_{(p^n)}(x, x^{-1}, 1) \, \mathrm{o}_{(p^n)}(x, x^{-1}, -1), & m = 2p, \end{cases} \tag{128}$$

i.e.

$$\operatorname{sz}_{(m^n)}(z) = \begin{cases} \operatorname{spz}_{((p-1)^n)}(z) \operatorname{oz}_{(p^n)}(z), & m = 2p - 1, \\ (-1)^{pn} \operatorname{oz}^{\mathrm{odd}}_{(p^n)}(z) \operatorname{oz}^{\mathrm{odd}}_{(p^n)}(-z), & m = 2p. \end{cases} \tag{129}$$

Equivalently,

$$\operatorname{sz}_{(m^n)}(u_1^2 - 2, \ldots, u_n^2 - 2) = \frac{\det\bigl[2\mathcal{T}_{m+2j-1}(u_k/2)\bigr]_{j,k=1}^{n} \, \det\bigl[\mathcal{U}_{m+2j-2}(u_k/2)\bigr]_{j,k=1}^{n}}{\det\bigl[2\mathcal{T}_{2j-1}(u_k/2)\bigr]_{j,k=1}^{n} \, \det\bigl[\mathcal{U}_{2j-2}(u_k/2)\bigr]_{j,k=1}^{n}}. \tag{130}$$

The equivalence between (129) and (130) follows from Proposition 5.3.

Comparing to (128) or (129), formula (130) has a "defect" that it uses auxiliary variables $u_1, \ldots, u_n$, related with $z_1, \ldots, z_n$ by $z_j = u_j^2 - 2$. On the other hand, this defect is not important after applying trigonometric or hyperbolic changes of variables (say, $u_j = \cos(\theta_j/2)$ and $z_j = \cos(\theta_j)$), and formula (130) can be more convenient in some applications because it joins two cases appearing in (128) and (129).

Ciucu and Krattenthaler [9] proved (128). Trench and Elouafi did not use the language of symmetric polynomials. Trench [31] worked with symmetric Toeplitz matrices generated by rational functions. For the case of symmetric Toeplitz matrices, his result is equivalent to (130), after trigonometric changes of variables. Elouafi [11] worked with banded symmetric Toeplitz determinants; his result is equivalent to (129), with the right-hand side written in the bialternant form, see Theorem 1.3.

Recently Ayyer and Behrend [2, formulas (18) and (19)] generalized (128) to partitions of the symmetric form

$$(2\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_n, \lambda_1 - \lambda_n, \ldots, \lambda_1 - \lambda_2, \lambda_1 - \lambda_1)$$

or

$$(2\lambda_1 + 1, \lambda_1 + \lambda_2 + 1, \ldots, \lambda_1 + \lambda_n + 1, \lambda_1 - \lambda_n, \ldots, \lambda_1 - \lambda_2, \lambda_1 - \lambda_1).$$

Notice that the proof in [2] uses exactly the same ideas as the proof in [31]: in both cases one starts with the bialternant formula for $\mathrm{s}_{(m^n)}(x, x^{-1})$, then applies elementary transformations of determinants, and reduces them to block-triangular form.

For readers' convenience, we explain below the idea of the proof given by Elouafi, but in the language of symmetric polynomials.

Proof of (129) *following [11].* Start with the Jacobi–Trudi formula for $\mathrm{sz}_{(m^n)}(z)$:

$$\mathrm{sz}_{(m^n)}(z) = \det\big[\mathrm{hz}_{m-j+k}(z)\big]_{j,k=1}^{n}. \tag{131}$$

Using (17) and Lemma 6.5, it is possible to derive the following expansions of hz:

$$\mathrm{hz}_{2p+1-j+k}(z) = \sum_{s=1}^{n} \frac{2\mathcal{T}_{n+p+1-j}(z_s/2)\,\mathcal{U}_{p+k-1}(z_s/2)}{\Omega_s(z)}, \tag{132}$$

$$\mathrm{hz}_{2p-j+k}(z) = \sum_{s=1}^{n} \frac{\mathcal{W}_{n+p-j}(z_s/2)\,\mathcal{V}_{p+k-1}(z_s/2)}{\Omega_s(z)}. \tag{133}$$

Applying (132) in the case $m = 2p + 1$ or (133) in the case $m = 2p$, one can write the determinant in the right-hand side of (131) as a product of determinants. □

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