

On self-Mullineux and self-conjugate partitions

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Abstract

The Mullineux involution is a relevant map that appears in the study of the modular representations of the symmetric group and the alternating group. The fixed points of this map are certain partitions of particular interest. It is known that the cardinality of the set of these *self-Mullineux* partitions is equal to the cardinality of a distinguished subset of self-conjugate partitions. In this work, we give an explicit bijection between the two families of partitions in terms of the Mullineux symbol.

Mathematics Subject Classifications: 05E10, 20C30, 20C20

1 Introduction

Let n be a non negative integer. It is well known that the isomorphism classes of complex irreducible representations of the symmetric group \mathfrak{S}_n can be indexed by the set of partitions of n . Let λ be a partition of $n \geq 2$ (written $\lambda \vdash n$) and S^λ the associated irreducible $\mathbb{C}\mathfrak{S}_n$ -module. Tensoring S^λ with the sign representation ε of \mathfrak{S}_n results in the irreducible representation $S^{\lambda'}$ of \mathfrak{S}_n , where λ' is the conjugate partition of λ ([JK81, 2.1.8]). This procedure allows to understand, by Clifford theory, all complex irreducible representations of the alternating group A_n . Indeed, let λ be a partition of $n \geq 2$,

- If $\lambda \neq \lambda'$ then $S^\lambda \downarrow_{A_n} \simeq S^{\lambda'} \downarrow_{A_n}$ is irreducible.
- If $\lambda = \lambda'$ then $S^\lambda \downarrow_{A_n}$ splits into two irreducible, non-isomorphic $\mathbb{C}A_n$ -modules S_+^λ and S_-^λ

$$S^\lambda \downarrow_{A_n} \simeq S_+^\lambda \oplus S_-^\lambda,$$

and

$$\{S^\lambda \downarrow_{A_n} \mid \lambda \vdash n \text{ and } \lambda \neq \lambda'\} \sqcup \{S_+^\lambda, S_-^\lambda \mid \lambda \vdash n \text{ and } \lambda = \lambda'\}$$

is a complete set of non-isomorphic irreducible $\mathbb{C}A_n$ -modules, considering only one partition λ for each couple $\{\lambda, \lambda'\}$ with $\lambda \neq \lambda'$ ([JK81, 2.5.7]).

A natural question is then what happens when we change the characteristic of the field. Let p be an odd prime and F an algebraically closed field of characteristic p . It is well known that the number of isomorphism classes of irreducible representations of the symmetric group \mathfrak{S}_n over F , is equal to the number of conjugacy classes of p -regular elements of \mathfrak{S}_n ([Isa06, 15.11]), which in turn is in bijection with the p -regular partitions of n ([JK81, 6.1.2]). Also in this setting, understanding the tensor product with the sign representation allows to obtain a classification of irreducible FA_n -modules. However, the conjugate of a p -regular partition is not necessarily p -regular, so tensoring with the sign representation in this case does not amount to conjugating the corresponding p -regular partition.

In [Mul79a], G. Mullineux defined a bijection m on the set of p -regular partitions of n , which is an involution, and conjectured that for a p -regular partition λ with associated irreducible $F\mathfrak{S}_n$ -module D^λ we have

$$D^\lambda \otimes \varepsilon = D^{m(\lambda)}.$$

Later, in [Kle96], A. Kleshchev described a different algorithm to compute $m(\lambda)$ and in [FK97], B. Ford and A. Kleshchev proved Mullineux conjecture to be true. Mullineux conjecture was also proven to be true in [BO98] by C. Bessenrodt and J.B. Olsson by using yet another description of the Mullineux bijection m . Other properties of this map have been studied for example in [Mul79b], [BO94], [BOX99]. Hence, tensoring with the sign representation in the modular case amounts to applying m on partitions, which makes the Mullineux map a p -analogue of conjugation of partitions. This way we have a classification of irreducible representations of A_n in characteristic p as follows. Let λ be a p -regular partition of $n \geq 2$,

- If $\lambda \neq m(\lambda)$ then $D^\lambda \downarrow_{A_n} \simeq D^{m(\lambda)} \downarrow_{A_n}$ is irreducible.
- If $\lambda = m(\lambda)$ then $D^\lambda \downarrow_{A_n}$ splits into two irreducible, non-isomorphic FA_n -modules D_+^λ and D_-^λ

$$D^\lambda \downarrow_{A_n} \simeq D_+^\lambda \oplus D_-^\lambda,$$

and

$$\begin{aligned} & \{D^\lambda \downarrow_{A_n} \mid \lambda \vdash n, \lambda \text{ is } p\text{-regular and } \lambda \neq m(\lambda)\} \\ & \sqcup \{D_+^\lambda, D_-^\lambda \mid \lambda \vdash n, \lambda \text{ is } p\text{-regular and } \lambda = m(\lambda)\} \quad (1) \end{aligned}$$

is a complete set of non-isomorphic irreducible FA_n -modules, considering only one partition λ for each couple $\{\lambda, m(\lambda)\}$ with $\lambda \neq m(\lambda)$ ([For97, 2.1]). Following such an indexing of irreducible modular representations of A_n , it is natural to inquire about the set of p -regular partitions such that $\lambda = m(\lambda)$. The definition of the Mullineux map m is quite complicated combinatorially, as are the different descriptions mentioned above,

even if they are explicit. Therefore describing its fixed points is not easy. So that, in characteristic p , it is not straightforward to obtain a reasonably simple indexing set for the irreducible FA_n -modules.

In fact, the number of fixed points of the Mullineux map, or *self-Mullineux* partitions, is equal to the number of partitions of n with different odd parts, none of them divisible by p ([AO91, Proposition 2]). This number is, in turn, equal to the number of self-conjugate partitions with diagonal hook-length not divisible by p ([JK81, 2.5.11]). We refer to the latter as *BG-partitions* (see Definition 8 for details). There is an elementary algebraic argument to see this (Appendix A). Thus, it is natural to ask for an explicit bijection between the self-Mullineux partitions and the BG-partitions.

The Mullineux map can be defined in terms of a symbol called the Mullineux symbol, defined on p -regular partitions. In this work we introduce a new symbol, defined on self-conjugate partitions. From such a symbol, associated to a BG-partition, we describe how to reconstruct a BG-partition and a self-Mullineux partition, and this algorithm provides our bijection.

A further motivation for finding an explicit bijection can also be given in the context of the representation theory of the symmetric group and of the alternating group. In [BG10], O. Brunat and J.-B. Gramain have shown the existence of a p -basic set for the symmetric group, which, by restriction, gives a p -basic set for the alternating group. However, this set, which provides a natural indexing set for the modular irreducible representations is not explicit and it would be ideal to give a complete description of it. One thing we know about such a set is that it always contains the set of BG-partitions. Hence, it is convenient to have a better understanding of them. More generally, this work can be seen as a first step to give a new natural way to label the modular irreducible representations of the symmetric group, for which tensoring with the sign representation is easier to describe combinatorially. We hope to come back to this problem later.

A bijection between the set of self-Mullineux partitions of n and partitions of n with different odd parts, none of them divisible by p can alternatively be derived from a bijection between two more general sets defined by C. Bessenrodt in [Bes91]. However, the two approaches are quite different because our bijection is defined directly between the sets of our interest. Moreover, we obtain a different bijection (see Remark 24).

The paper is organized as follows. In Section 2 we recall some definitions about partitions and the definition of the Mullineux map. Section 3 contains the main result of this paper; we define a symbol on self-conjugate partitions and we show how through this symbol we obtain the mentioned explicit bijection. Finally, in Section 4 we prove that the inverse procedure of reconstructing a unique BG-partition from a self-Mullineux partition is well defined, which confirms that this is a one to one correspondence without the need of knowing beforehand that there exists a bijection between both sets of partition.

2 Preliminaries: the Mullineux map

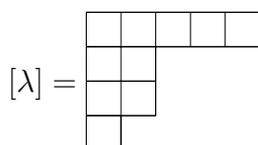
In this section we recall some general definitions about partitions and the definition of the Mullineux map, as defined by G. Mullineux in [Mul79a]. We follow closely definitions

in [FK97].

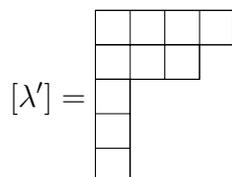
A *partition* λ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, \dots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \dots$, containing only finitely many non-zero terms. Let $n \in \mathbb{N}$ be such that $|\lambda| = \sum \lambda_i = n$. We say that λ is a partition of n , which we write $\lambda \vdash n$. Let $\text{Par}(n)$ denote the set of partitions of n . We call $|\lambda|$ the *size* of λ . The integers λ_i are called the *parts* of the partition λ . If there is a part that repeats k times, say $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+k}$, we denote λ as $(\dots, \lambda_i^k, \dots)$. The number of non-zero parts is the *length* of λ and is denoted $l(\lambda)$. The *Young diagram* of a partition λ is the set

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i \geq 1 \text{ and } 1 \leq j \leq \lambda_i\},$$

whose elements are called *nodes*. We represent the diagram as an array of boxes in the plane with the convention that i increases downwards and j increases from left to right. A partition is often identified with its Young diagram. The Young diagram of $\lambda = (5, 2^2, 1)$ is



The *conjugate* (or *transpose*) partition of $\lambda = (\lambda_1, \dots, \lambda_k)$ is the partition λ' of n defined as $\lambda'_j = \#\{i \mid \lambda_i \leq j\}$, which amounts to transposing the Young diagram $[\lambda]$ with respect to its main diagonal which consists of the nodes of the form (i, i) with $1 \leq i \leq k(\lambda)$. If $\lambda = (5, 2^2, 1)$, as above, then $\lambda' = (4, 3, 1^3)$ and its Young diagram is



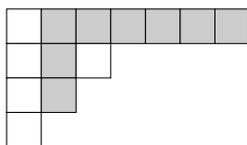
If $\lambda = \lambda'$ we say that λ is *self-conjugate*. For a positive integer p , λ is said to be *p-regular* if it does not contain p parts $\lambda_i \neq 0$ which are equal. The partition $\lambda = (5, 2^2, 1)$ above is not 2-regular but it is 3-regular. We denote by $\text{Reg}_p(n)$ the set of p -regular partitions of n .

Let (i, j) be a node of a partition λ . We define the (i, j) -th *hook* of λ , or the $(i, j)_\lambda$ -th hook, as the set of nodes in $[\lambda]$ to the right or below the node (i, j) , that is, the nodes (k, l) such that $k = i$ and $j \leq l \leq \lambda_i$, or $l = j$ and $i \leq k \leq \lambda'_j$. The *hook-length* of λ at $(i, j) \in [\lambda]$, denoted here h_{ij}^λ is the number of nodes in the $(i, j)_\lambda$ -th hook, that is

$$h_{ij}^\lambda = \lambda_i + \lambda'_j - i - j + 1.$$

We omit λ from the notation when there is no ambiguity. A partition which is equal to its $(1, 1)$ -th hook is called a *hook*.

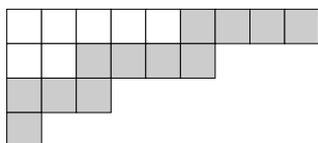
Example 1. Let $\lambda = (7, 3, 2, 1)$. The $(1, 2)$ -th hook of λ is represented by shaded boxes in the following diagram



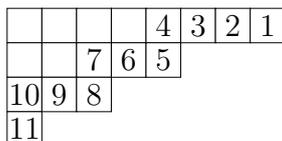
and its length is $h_{1,2} = 8$.

For any positive integer m , a m -hook (respectively (m) -hook) is a hook of length m (respectively divisible by m). We denote $k(\lambda) = \max\{i \mid \lambda_i \geq i\}$. We call a node (i, i) , for $1 \leq i \leq k(\lambda)$, a *diagonal* node and the set of diagonal nodes is the *diagonal* of λ . A (i, i) -th hook is referred to as a *diagonal hook*.

The *rim* of λ is the set of nodes $(i, j) \in [\lambda]$ such that $(i + 1, j + 1) \notin [\lambda]$, in other words, it is the south-east border of the Young diagram $[\lambda]$. For example, the rim of $\lambda = (9, 6, 3, 1)$ is represented in the following diagram by shaded boxes

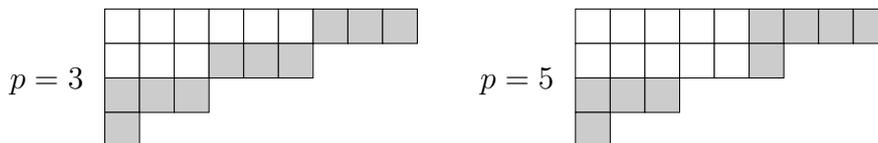


Let us label the nodes of the rim with positive integers from the top right to the bottom left, as shown in the following figure



Let p be an odd prime. The first p -segment of the rim consists of the nodes corresponding to integers less or equal than p . If the last node (i, j) of the first p -segment is in the last row of $[\lambda]$, then $[\lambda]$ only has one p -segment. If not, let l be the smallest label on row $i + 1$. The second p -segment of the rim consists of the nodes labelled by $l \leq m \leq l + p - 1$. Repeating this procedure we will eventually reach the bottom row of the diagram and it is clear that all p -segments have p nodes, except possibly the last one. The p -rim of λ is defined as the union of all the p -segments.

Example 2. The following two diagrams illustrate the p -rim of $\lambda = (9, 6, 3, 1)$ for $p = 3$ and $p = 5$.



We denote a_λ the number of nodes in the p -rim of λ . Define diagrams $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(l)}$ as follows. Put $\lambda^{(0)} = \lambda$ and for $i \geq 1$ put

$$\lambda^{(i)} = \lambda^{(i-1)} \setminus \{p\text{-rim of } \lambda^{(i-1)}\},$$

where we choose l maximal with respect to $\lambda^{(l)} \neq \emptyset$; so $\lambda^{(l+1)} = \emptyset$. We call the p -rim of $\lambda^{(i)}$ the i -th p -rim of λ . Let $a_i = a_{\lambda^{(i)}}$ be the number of nodes of the i -th p -rim of λ and r_i the number of rows of $\lambda^{(i)}$, that is, $r_i = l(\lambda^{(i)})$. The *Mullineux symbol* of λ (introduced in [BO94]) is

$$G_p(\lambda) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}. \quad (2)$$

Example 3. Let $p = 5$ and $\lambda = (9, 6, 3, 1)$. In the following diagram we represent the i -th p -rim of λ with label i on its nodes

2	2	2	2	1	0	0	0	0
2	1	1	1	1	0			
0	0	0						
0								

$$G_5(\lambda) = \begin{pmatrix} 9 & 5 & 5 \\ 4 & 2 & 2 \end{pmatrix}.$$

The following proposition is a reformulation ([AO91, §5]) of a result proved in [Mul79a, 3.6].

Proposition 4. *Let p be an odd prime and λ a p -regular partition of a non-negative integer n . Set*

$$\varepsilon_i = \begin{cases} 0 & \text{if } p \mid a_i, \\ 1 & \text{if } p \nmid a_i. \end{cases}$$

The entries of $G_p(\lambda)$ satisfy

1. $\varepsilon_i \leq r_i - r_{i+1} < p + \varepsilon_i$ for $0 \leq i < l$,
2. $1 \leq r_l < p + \varepsilon_l$,
3. $r_i - r_{i+1} + \varepsilon_{i+1} \leq a_i - a_{i+1} < p + r_i - r_{i+1} + \varepsilon_{i+1}$ for $0 \leq i < l$,
4. $r_l \leq a_l < p + r_l$,
5. $\sum_{i=0}^l a_i = n$.

Moreover, if $a_0, \dots, a_l, r_0, \dots, r_l$ are positive integers such that these inequalities are satisfied then there exists exactly one p -regular partition λ of n such that

$$G_p(\lambda) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}.$$

Remark 5. It is easy to recover the p -regular partition λ from its Mullineux symbol $G_p(\lambda)$; start with the hook $\lambda^{(l)}$ of size a_l and length r_l , and for $i = l - 1, l - 2, \dots, 0$, add the i -th p -rim (consisting of a_i nodes) to $\lambda^{(i+1)}$ from the bottom to the top, starting by placing a node on the first free placement in row r_i . Then, adding nodes either on top (whenever it is possible) or to the right of the last added node until having added the last node of the p -segment and add the following p -segment starting on the first free placement of the row on top of the last added node. This procedure finishes at the first row. This algorithm is more precisely described in [FK97, §1].

Let λ be a p -regular partition of n , with Mullineux symbol (2) and let ε_i be as in Proposition 4. Define $s_i = a_i + \varepsilon_i - r_i$. In [Mul79a, 4.1] it is shown that the array

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ s_0 & s_1 & \cdots & s_l \end{pmatrix},$$

corresponds to the Mullineux symbol of a p -regular partition. We are now able to define the Mullineux map m .

Definition 6. With the above notations, $m(\lambda)$ is defined as the unique p -regular partition such that

$$G_p(m(\lambda)) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ s_0 & s_1 & \cdots & s_l \end{pmatrix}.$$

Because of Proposition 4, $m(\lambda)$ is well defined, and from the definition we can see that m is an involution.

Remark 7. If $p > n$, then $\text{Reg}_p(n) = \text{Par}(n)$ and irreducible $F\mathfrak{S}_n$ -modules are therefore indexed by all partitions of n . In this case, the Mullineux map coincides with conjugation: $m(\lambda) = \lambda'$.

3 From BG-partitions to self-Mullineux partitions

Let p be an odd prime. Let us define the set of BG-partitions.

Definition 8. We call *BG-partition* any self-conjugate partition with no diagonal (p)-hooks, that is, any partition λ such that $p \nmid h_{ii}^\lambda$ for every $1 \leq i \leq k(\lambda)$ with $(i, i) \in [\lambda]$. We denote BG_p the set of BG-partitions and BG_p^n the set of BG-partitions of a non-negative integer n .

As said in the introduction, the aim of this work is to give an explicit bijection between BG-partitions and self-Mullineux partitions. In this section we describe such bijection: we define the *BG-symbol* associated to a BG-partition, which is a two-row array of positive integers. We prove that BG-symbols of BG-partitions are Mullineux symbols of self-Mullineux partitions and that this association is injective, resulting in a bijection.

3.1 BG-symbol

We introduce a symbol, defined in general for self-conjugate partitions. This symbol is in some way inspired by the Mullineux symbol. In a similar way as the Mullineux symbol, which is defined by counting nodes on the p -rim of a sequence of partitions, the BG-symbol is defined by counting elements in a set of nodes called the p -rim* which is a *symmetric* analogue of the p -rim.

Let λ be a self-conjugate partition and denote by $\text{Rim}_p(\lambda) \subseteq [\lambda]$ the set of nodes in the p -rim of $[\lambda]$. Set

$$U_\lambda = \{(i, j) \in \text{Rim}_p(\lambda) \mid i \leq j\},$$

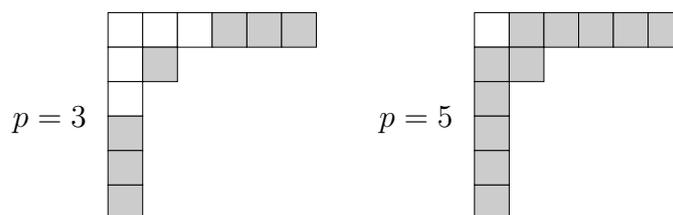
that is, U_λ consists of the nodes of the p -rim which are above (or on) the diagonal of $[\lambda]$. We denote $r_\lambda^* := \#U_\lambda$. Set

$$L_\lambda = \{(j, i) \mid (i, j) \in U_\lambda\}.$$

The set L_λ consists of the nodes in U_λ reflected across the diagonal of λ . Notice that $L_\lambda \subseteq [\lambda]$, since $\lambda = \lambda'$, so that $(i, j) \in [\lambda]$ if and only if $(j, i) \in [\lambda]$.

Definition 9. Let λ be a self-conjugate partition. The p -rim* of λ is the set $\text{Rim}_p^*(\lambda) = U_\lambda \cup L_\lambda$. We denote a_λ^* the number of nodes in the p -rim* of λ , that is, $a_\lambda^* = \#\text{Rim}_p^*(\lambda)$. Define ε_λ^* as $\varepsilon_\lambda^* = a_\lambda^* \bmod 2$. It is the parity of the number of nodes on the p -rim* of λ .

Example 10. The following two diagrams illustrate the p -rim* of $\lambda = (6, 2, 1^4)$ in shaded boxes, for $p = 3$ and $p = 5$.



Remark 11. For a self-conjugate partition λ , from the definition of p -rim* we have

$$\begin{aligned} \varepsilon_\lambda^* = 0 &\Leftrightarrow a_\lambda^* \text{ is even.} \\ &\Leftrightarrow \text{Rim}_p^*(\lambda) \text{ has no diagonal nodes.} \end{aligned}$$

This way, the number of nodes in U_λ , that is, the number of nodes of the p -rim* of λ over (or on) the diagonal is

$$r_\lambda^* = \begin{cases} \frac{a_\lambda^*}{2} & \text{if } a_\lambda^* \text{ is even,} \\ \frac{a_\lambda^*+1}{2} & \text{otherwise,} \end{cases}$$

thus

$$r_\lambda^* = \frac{a_\lambda^* + \varepsilon_\lambda^*}{2}.$$

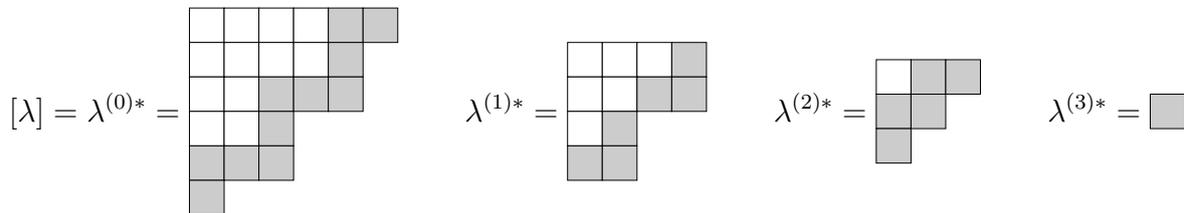
Let λ be a self-conjugate partition. We define diagrams $\lambda^{(0)*}, \lambda^{(1)*}, \dots, \lambda^{(l)*}$ in an analogue way as for the Mullineux symbol, by considering the p -rim* instead of the p -rim. Put $\lambda^{(0)*} = \lambda$ and for $i \geq 1$ put

$$\lambda^{(i)*} = \lambda^{(i-1)*} \setminus \{p\text{-rim}^* \text{ of } \lambda^{(i-1)*}\},$$

where we chose l maximal with respect to $\lambda^{(l)*} \neq \emptyset$; so $\lambda^{(l+1)*} = \emptyset$. We call the p -rim* of $\lambda^{(i)*}$ the i -th p -rim* of λ .

Remark 12. Notice that the p -rim* is only defined for self-conjugate partitions, but we claim that the diagrams $\lambda^{(i)*}$ are well defined, given the fact that $\text{Rim}_p^*(\lambda)$ is *symmetric* in the sense that $(u, v) \in \text{Rim}_p^*(\lambda)$ if and only if $(v, u) \in \text{Rim}_p^*(\lambda)$. Therefore, removing these nodes from $[\lambda]$ to obtain $\lambda^{(1)*}$ results again in a self-conjugate partition and then so it is for every $\lambda^{(i)*}$. In other words, if $\lambda^{(i)*}$ is self-conjugate, then $\lambda^{(i+1)*}$ is self-conjugate.

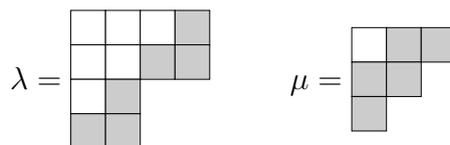
Example 13. Let $p = 3$ and $\lambda = (6, 5^2, 3^2, 1)$. Then



where shaded boxes represent the i -th p -rim* of λ .

For the partition $\lambda^{(i)*}$, obtained by succesively removing nodes on the p -rim*, starting with $\lambda = \lambda^{(0)*}$, we denote $a_{\lambda^{(i)*}}^*$ as a_i^* . Similarly $r_{\lambda^{(i)*}}^* = r_i^*$ and $\varepsilon_{\lambda^{(i)*}}^* = \varepsilon_i^*$. All these values associated to self-conjugate partitions may seem technical, and they are better understood by means of the Young diagram, see the following example.

Example 14. Let $p = 3$, $\lambda = (4^2, 2^2)$, and $\mu = (3, 2, 1)$



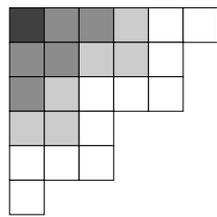
We have $a_\lambda^* = 6$, $\varepsilon_\lambda^* = 0$, and $r_\lambda^* = 3$. For μ , we have $a_\mu^* = 5$, $\varepsilon_\mu^* = 1$, and $r_\mu^* = 3$.

Definition 15. Let λ be a self-conjugate partition. The *BG-symbol* of λ is

$$\text{bg}_p(\lambda) := \begin{pmatrix} a_0^* & a_1^* & \cdots & a_l^* \\ r_0^* & r_1^* & \cdots & r_l^* \end{pmatrix}. \quad (3)$$

The *length* of the BG-symbol is l .

Example 16. If $p = 3$, the BG-symbol of the partition $\lambda = (6, 5^2, 3^2, 1)$ is



$$\text{bg}_3(\lambda) = \begin{pmatrix} 11 & 6 & 5 & 1 \\ 6 & 3 & 3 & 1 \end{pmatrix}.$$

In this diagram, each i -th 3-rim^* is shown in a different shade.

The following lemmas will allow us to prove that two different self-conjugate partitions correspond to different BG-symbols. Lemma 18 is an analogue of [Mul79a, 2.1]. Its proof is quite technical and the arguments are easier to understand with an example, see Example 19.

Lemma 17. *Let λ be a self-conjugate partition. If a_λ^* is an even number, then $p \mid a_\lambda^*$.*

Proof. From the definition (or see Remark 11), a_λ^* is even if and only if $U_\lambda \cap L_\lambda = \emptyset$. Then the $p\text{-rim}^*$ of λ does not contain diagonal nodes. From the definition of $\text{Rim}_p^*(\lambda)$, this means that the set U_λ only contains p -segments of length p . And then the same is true for L_λ . Therefore

$$p \mid \#(U_\lambda \cup L_\lambda) = a_\lambda^*. \quad \square$$

The converse is not true in general, for example, if $p = 3$ and $\lambda = (5, 3, 2, 1, 1)$, we have that $a_\lambda^* = 9$.

Lemma 18. *Let $\tilde{\lambda}$ be a self-conjugate partition, $\varepsilon \in \{0, 1\}$ and m , a residue modulo p , such that $m = 0$ if $\varepsilon = 0$. Then, there exists a unique self-conjugate partition λ such that*

- (i) $a_\lambda^* \equiv \varepsilon \pmod{2}$;
- (ii) $r_\lambda^* - \varepsilon_\lambda^* \equiv m \pmod{p}$ and
- (iii) $\lambda^{(1)*} = \tilde{\lambda}$.

Moreover, if $\tilde{\lambda} \in \text{BG}_p$, and $p \nmid 2m + 1$ when $\varepsilon = 1$, then $\lambda \in \text{BG}_p$.

Proof. Given $\varepsilon \in \{0, 1\}$ and m , a residue modulo p , let us see that there is a unique way to add nodes to $\tilde{\lambda}$ to obtain a self-conjugate partition λ such that the added nodes are the $p\text{-rim}^*$ of λ .

Let us study how nodes (i, j) over the diagonal ($i \leq j$) must be added. This will determine all nodes that must be added (if (i, j) is added to $\tilde{\lambda}$, then (j, i) is added as well).

First, the last row i over the diagonal that will contain new nodes (i, j) is uniquely fixed by $\tilde{\lambda}$ and ε . Indeed, let $d = k(\tilde{\lambda})$. If $\varepsilon = 0$, then $i = d$ and $(d, \tilde{\lambda}_d + 1)$ must be added to $\tilde{\lambda}$. If $\varepsilon = 1$, then $i = d + 1$ and $(d + 1, d + 1)$ must be added to $\tilde{\lambda}$.

Now, let (i, j) be the first node that we add (with i fixed as before by $\tilde{\lambda}$ and ε) and $j \in \{\tilde{\lambda}_d + 1, d + 1\}$ depending on $\tilde{\lambda}$ and ε . Starting from this node, it is clear that there is a unique way to add nodes such that (i), (ii), and (iii) hold: If the position $(i + 1, j)$ just above (i, j) is empty in $\tilde{\lambda}$, we add a node in that position, otherwise we add a node in $(i, j + 1)$. We repeat this procedure for adding nodes until we have added m nodes (including (i, j) if $\varepsilon = 0$, not including (i, j) if $\varepsilon = 1$). If the last added node is in row 1 we stop here. If it is added in row $k > 1$, we add a node in row $k - 1$ in position $(k - 1, \tilde{\lambda}_{k-1} + 1)$ and we restart the procedure to keep adding nodes until having added p nodes. We iterate this procedure, of adding groups of p nodes, until reaching the first row. This way we added nodes over the diagonal. Finally for each node (a, b) added, we add its reflection through the diagonal (b, a) . And we obtain a self-conjugate partition λ .

It remains to verify that $\lambda^{(1)*} = \tilde{\lambda}$. If $\varepsilon = 1$, it is straightforward that $\lambda^{(1)*} = \tilde{\lambda}$. Since when removing the nodes of the p -rim* of λ over the diagonal we eventually reach a diagonal node, and then just remove the reflection of the removed nodes. It is clear that in this case we obtain $\tilde{\lambda}$. If $\varepsilon = 0$, the condition $m = 0$ says that a p -segment of λ will eventually reach the row d and this p -segment has exactly p nodes, so that there is no ambiguity when removing p -segments and $\lambda^{(1)*} = \tilde{\lambda}$.

For the last part of the theorem, suppose that $\tilde{\lambda} \in \text{BG}_p$, and let us see that λ obtained as above is also in BG_p . In other words, we are assuming that $\tilde{\lambda}$ does not contain any diagonal (p) -hooks and we want to show that the same is true for λ .

Suppose that λ has a diagonal (p) -hook, say the $(i, i)_\lambda$ -th hook, that is $h_{ii}^\lambda = pk$ for some integer $k > 0$. For a partition μ , we set the convention $h_{ij}^\mu = 0$ if $(i, j) \notin [\mu]$.

Since $\tilde{\lambda} \in \text{BG}_p$, then the $(i, i)_\lambda$ -th hook is different from the $(i, i)_{\tilde{\lambda}}$ -th hook since if they were equal, $\tilde{\lambda}$ would have a (p) -hook, which is not possible. Therefore $(i, \tilde{\lambda}_i + 1) \in [\lambda]$. Since this node is not in $[\tilde{\lambda}]$, by definition, it is on the p -rim* of λ , in particular, it belongs to a p -segment of $\text{Rim}_p^*(\lambda)$ above the diagonal. Consider the two cases: this p -segment starts at row i , or this p -segment starts before row i , that is, this p -segment starts at a row j for $1 \leq j < i$.

- If the p -segment containing node $(i, \tilde{\lambda}_i + 1)$ starts at row i , let (i, j) be the first node of this p -segment and (a, b) its last node ($i \leq a$). Then $a \leq b$ because this segment is above the diagonal.

Let N be the number of nodes on this p -segment. Then we have:

$$\begin{aligned} N &= a - i + j - b + 1, \\ h_{ii}^\lambda &= 1 + 2(j - i), \\ h_{aa}^\lambda &= 1 + 2(b - a - 1), \end{aligned}$$

where the last identity holds if $b > a$ (since this implies that $(a, b - 1) \in \tilde{\lambda}$). Then we have

$$h_{ii}^\lambda = \begin{cases} 2N - 1 & \text{if } a = b, \\ 2N + h_{aa}^{\tilde{\lambda}} & \text{if } a < b. \end{cases}$$

If $a = b$, this p -segment is the last segment in U_λ and $\varepsilon_\lambda^* = 1$. So that $N = m + 1$ and we get $h_{ii}^\lambda = 2m + 1$. This contradicts $p \nmid 2m + 1$. If $a < b$, then the last node of this p -segment, (a, b) is not a diagonal node so that $N = p$ and we get $h_{ii}^\lambda = 2p + h_{aa}^\lambda$, which implies $p \mid h_{aa}^\lambda$, a contradiction.

- If the p -segment containing node $(i, \tilde{\lambda}_i + 1)$ starts at a row j with $j < i$, then it contains nodes on row $i - 1$, in particular $(i - 1, \tilde{\lambda}_{i-1} + 1)$. The next node on this p -segment is the node just below: $(i, \tilde{\lambda}_{i-1} + 1)$. Then $\lambda_i = \tilde{\lambda}_{i-1} + 1$. Let us see that $h_{(i-1, i-1)}^{\tilde{\lambda}} = h_{ii}^\lambda$. Since these are diagonal hooks contained in self-conjugate partitions, their lengths are calculated as follows

$$\begin{aligned} h_{(i-1, i-1)}^{\tilde{\lambda}} &= 2(\tilde{\lambda}_{i-1} - (i - 1)) + 1 \\ &= 2\tilde{\lambda}_{i-1} - 2i + 3, \end{aligned}$$

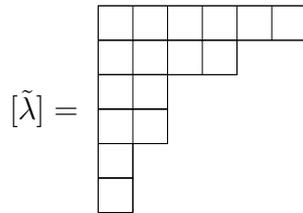
and

$$\begin{aligned} h_{(i, i)}^\lambda &= 2(\lambda_i - i) + 1 \\ &= 2(\tilde{\lambda}_{i-1} + 1 - i) + 1 \\ &= 2\tilde{\lambda}_{i-1} - 2i + 3. \end{aligned}$$

Since $p \mid h_{(i, i)}^\lambda$, then $p \mid h_{(i-1, i-1)}^{\tilde{\lambda}}$, a contradiction.

In conclusion, λ does not have any diagonal (p)-hooks, that is, $\lambda \in \text{BG}_p$. □

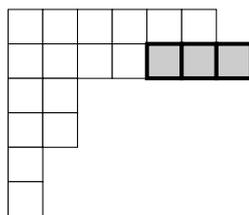
Example 19. Let $p = 3$. We use the notations of Lemma 18. Consider the self-conjugate partition $\tilde{\lambda} = (6, 4, 2^2, 1^2)$.



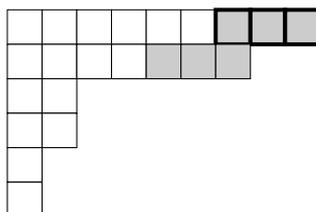
- Let $\varepsilon = 0$, then $m = 0$. Let us see that there is only one self-conjugate partition λ satisfying: a_λ^* is even, $r_\lambda^* - \varepsilon_\lambda^* = \# U_\lambda - \varepsilon_\lambda^* \equiv 0 \pmod{3}$ and $\lambda^{(1)*} = \tilde{\lambda}$. We add to $[\tilde{\lambda}]$ the nodes of $\text{Rim}_3^*(\lambda) = U_\lambda \cup L_\lambda$.

In this case, since $a_\lambda^* \equiv 0 \pmod{2}$, then U_λ does not contain diagonal nodes. That is, U_λ consists only on nodes strictly over the diagonal, so that the last row containing nodes from U_λ is row 2, since $k(\tilde{\lambda}) = 2$.

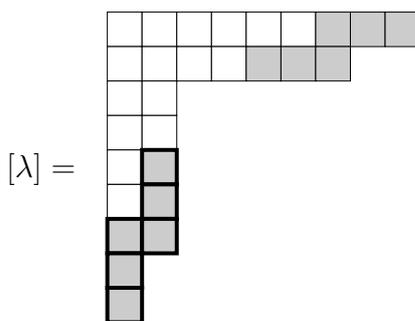
Since $m = 0$, then every 3-segment of U_λ has 3 nodes. The bottom 3-segment of U_λ is shown in shaded nodes in the following diagram



Since we have not reached the top of $[\tilde{\lambda}]$, there is at least another 3-segment, which starts at the following upper row :



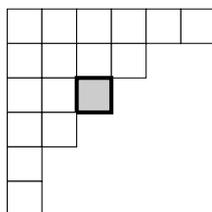
Now, λ is self-conjugate, then for each upper node (i, j) that we added, we add the node (j, i) (or also because $(i, j) \in U_\lambda$ if and only if $(j, i) \in L$)



So that $\lambda = (9, 7, 2^5, 1^2)$ is the only possibility for λ self-conjugate such that a_λ^* is even, $r_\lambda^* - \varepsilon_\lambda^* = \#U_\lambda \equiv 0 \pmod{3}$ and $\lambda^{(1)*} = \tilde{\lambda}$.

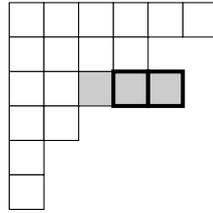
- Let $\varepsilon = 1$, and $m = 2$. Let us see that there is only one possible partition λ satisfying: a_λ^* is odd, $r_\lambda^* - \varepsilon_\lambda^* = \#U_\lambda - \varepsilon_\lambda^* \equiv 2 \pmod{3}$ and $\lambda^{(1)*} = \tilde{\lambda}$. We add to $[\tilde{\lambda}]$ the nodes of $\text{Rim}_3^*(\lambda) = U_\lambda \cup L_\lambda$.

In this case, $a_\lambda^* \equiv 1 \pmod{2}$, then there is a diagonal node in U_λ :

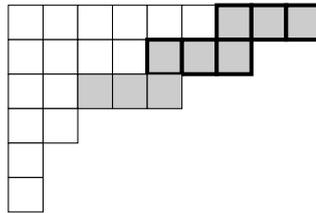


We add now the rest of the nodes in U_λ . Here $r_\lambda^* - \varepsilon_\lambda^* = \#(U_\lambda \setminus \{(3, 3)\}) \equiv 2 \pmod{3}$. That means that the rest of the nodes in U_λ contain one 3-segment of 2 nodes, we

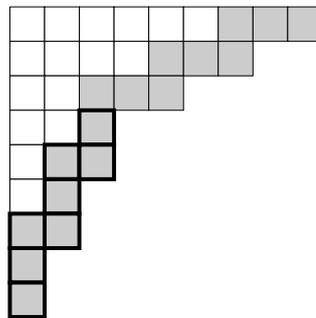
add this 3-segment



and the rest are 3-segments of 3 nodes:



and finally, for making λ self-conjugate:



And we see that $\lambda = (9, 7, 5, 3^2, 2^2, 1^2)$ is the only possibility for having a_λ^* odd, $r_\lambda^* - \varepsilon_\lambda^* = \#U_\lambda - 1 \equiv 2 \pmod{3}$, and $\lambda^{(1)*} = \tilde{\lambda}$.

Proposition 20. *Let p be an odd prime. Two different self-conjugate partitions have different BG-symbols. In other words, the BG-symbol gives rise to an injective map from self-conjugate partitions to the set of two-row positive integer symbols.*

Proof. We proceed by induction on l , the length of the BG-symbol. Let $l = 0$. Let λ be a self-conjugate partition with BG-symbol

$$\text{bg}_p(\lambda) = \begin{pmatrix} a_0^* \\ r_0^* \end{pmatrix}.$$

Notice that having a BG-symbol of length 0 means that $\lambda = \lambda^{(0)*} = \lambda^{(l)*}$ is a hook and its size is a_0^* , that is, there are positive integers u, v such that $\lambda = (u, 1^v)$ and $u + v = a_0^*$. But λ is self-conjugate, then $u - 1 = v$, so that $a_0^* = 2u - 1$, and $\varepsilon_0^* = 1$. Then

$$r_0^* = \frac{a_0^* + \varepsilon_0^*}{2} = u.$$

Therefore $\lambda = (r_0^*, 1^{r_0^*-1})$. This way, λ is determined from its BG-symbol and, from this reasoning, we see that any self-conjugate partition with BG-symbol $\text{bg}_p(\lambda)$ is completely determined by it and is then equal to λ .

Now, fix $l > 0 \in \mathbb{N}$, let λ be a self-conjugate partition with BG-symbol

$$\text{bg}_p(\lambda) = \begin{pmatrix} a_0^* & a_1^* & \cdots & a_l^* \\ r_0^* & r_1^* & \cdots & r_l^* \end{pmatrix},$$

and let μ be a self-conjugate partition such that $\text{bg}_p(\mu) = \text{bg}_p(\lambda)$.

Then, by definition, the BG-symbol of $\lambda^{(1)*}$ (and also of $\mu^{(1)*}$) is the BG-symbol of λ after removing its first column

$$\text{bg}_p(\lambda^{(1)*}) = \text{bg}_p(\mu^{(1)*}) = \begin{pmatrix} a_1^* & \cdots & a_l^* \\ r_1^* & \cdots & r_l^* \end{pmatrix}.$$

By induction, there exists a unique self-conjugate partition $\tilde{\tau}$ such that

$$\text{bg}_p(\tilde{\tau}) = \begin{pmatrix} a_1^* & \cdots & a_l^* \\ r_1^* & \cdots & r_l^* \end{pmatrix}.$$

Then $\tilde{\tau} = \lambda^{(1)*} = \mu^{(1)*}$. Let us see, from Lemma 18, that $\lambda = \mu$.

Let $\varepsilon = a_0^* \bmod 2$, and $m = r_0^* - \varepsilon_0^* \bmod p$. By Lemma 17, if $\varepsilon = 0$ then $m = 0$. Therefore, by Lemma 18 there exists a unique self-conjugate partition τ such that

- (i) $a_\tau^* \equiv \varepsilon \pmod{2}$;
- (ii) $r_\tau^* - \varepsilon_\tau^* \equiv m \pmod{p}$ and
- (iii) $\tau^{(1)*} = \tilde{\tau}$.

But partitions λ and μ are self-conjugate and they satisfy (i) and (ii) since $a_0^* = a_\lambda^* = a_\mu^*$. Moreover $\lambda^{(1)*} = \mu^{(1)*} = \tilde{\tau}$, then by the uniqueness of τ we have that $\tau = \lambda = \mu$. \square

As it turns out, the BG-symbol of a BG-partition is a Mullineux symbol of some self-Mullineux partition. Denote by \mathcal{M}_p the set of Mullineux symbols of the self-Mullineux partitions M_p .

Proposition 21. *Let p be an odd prime and λ a BG-partition. The BG-symbol of λ , $\text{bg}_p(\lambda)$ is the Mullineux symbol of some self-Mullineux partition. That is*

$$\text{bg}_p(\text{BG}_p) \subseteq \mathcal{M}_p.$$

We postpone the proof of this proposition to Section 3.2, since some technical lemmas are necessary for this proof.

Recall, from Proposition 4 and Remark 5, that to a Mullineux symbol corresponds a unique p -regular partition. So that the Mullineux symbol determines a bijection between p -regular partitions and their Mullineux symbols. In particular, to a symbol in \mathcal{M}_p corresponds a unique self-Mullineux partition in M_p . As a corollary from Proposition 20, Proposition 21 and from the fact that the sets BG_p^n and M_p^n have the same number of elements, we obtain the following result.

Theorem 22. *We have that*

$$\text{bg}_p(\text{BG}_p) = \mathcal{M}_p,$$

and the BG-symbol provides an explicit bijection between BG-partitions and self-Mullineux partitions. This bijection is given by associating to a BG-partition λ its BG-symbol $\text{bg}_p(\lambda)$, which corresponds to a unique self-Mullineux partition. This bijection restricts to bijections between BG_p^n and M_p^n for every $n \in \mathbb{N}$.

Remark 23. If we consider the Mullineux symbol G_p as a bijection from the set of p -regular partitions into its image in the set of two-row arrays of integers, then the bijection in Theorem 22, from BG_p to M_p is given precisely by $G_p^{-1} \circ \text{bg}_p$.

Remark 24. In [AO91], G. Andrews and J. Olsson prove a general partition identity, which depends on some integer parameters. A special case of this identity is the fact that the number of self-Mullineux partitions of a non-negative integer n equals the number of partitions of n with different odd parts, none of them divisible by p , which is in turn equal to the number of BG-partitions.

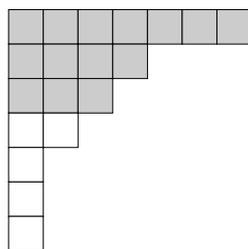
Now, in [Bes91], C. Bessenrodt shows a combinatorial proof of the Andrews–Olsson identity, which provides, by choosing the right parameters, an explicit bijection between BG_p^n and M_p^n . The bijection from Theorem 22 is obtained in a more direct way and it is different from Bessenrodt’s bijection. In particular, for $p = 5$ and $n = 20$, the partition $(7, 6, 3, 2^2) \in M_5^{20}$ is mapped to partition $(9, 3, 2, 1^6) \in \text{BG}_5^{20}$ under Bessenrodt’s bijection, and it is mapped to $(7, 5, 2^3, 1^2) \in \text{BG}_5^{20}$ under bijection from Theorem 22.

3.2 Proof of the main result

In this subsection we prove Proposition 21. For this proof we need some technical lemmas.

Lemma 25. *Let p be an odd prime and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ a partition in BG_p . Let $k = k(\lambda)$ as defined in Section 2. Then the partition $\mu = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is p -regular.*

Example 26. Let $p = 3$ and $\lambda = (7, 4, 3, 2, 1^3)$. This partition is self-conjugate and does not have diagonal (3)-hooks, that is, $\lambda \in \text{BG}_3$. Here $k = 3$, so that $l(\mu) = 3$, and indeed, the partition $\mu = (7, 4, 3)$ is 3-regular. The following diagram illustrates both the partitions λ and μ (in shaded boxes).



Proof of the lemma. Suppose that μ is not p -regular. So that there exists $1 \leq i \leq k$ such that

$$\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+p-1}.$$

Since $i + p - 1 \leq k$, then $\lambda_{i+p-1} \geq i + p - 1$. Then $(i + p - 1, i + p - 1) \in [\mu] \subseteq [\lambda]$. Let a be the length of the $(i + p - 1, i + p - 1)$ -th hook of λ , that is $a = h_{(i+p-1, i+p-1)}^\lambda$.

Then, the length of the $(i + p - 2, i + p - 2)$ -th hook of λ , is $h_{(i+p-2, i+p-2)} = a + 2$, since $\lambda_{i+p-1} = \lambda_{i+p-2}$ and λ is self-conjugate. And $h_{(i+p-3, i+p-3)} = a + 4$. In general $h_{(i+p-1-j, i+p-1-j)} = a + 2j$ for $j = 0, \dots, p - 1$. That is, the lengths of these hooks are:

$$a, a + 2, a + 4, \dots, a + 2j, \dots, a + 2(p - 1).$$

But since $p \neq 2$, this list, modulo p , forms a complete collection of residues. Then, there exists $j \in \{0, \dots, p - 1\}$ such that $p \mid a + 2j = h_{(i+p-1-j, i+p-1-j)}$, and this contradicts the fact that $\lambda \in \text{BG}_p$. \square

In the set of BG-partitions, the implication in Lemma 17 becomes an equivalence:

Lemma 27. *Let $\lambda \in \text{BG}_p$. Then a_λ^* is even if and only if $p \mid a_\lambda^*$.*

Proof. As already noted, the fact that a_λ^* implies $p \mid a_\lambda^*$ is proved in Lemma 17.

Suppose that $p \mid a_\lambda^*$. If a_λ^* is odd, then $\text{Rim}_p^*(\lambda)$ contains a diagonal node. Then U_λ is formed by p -segments of length p and one last p -segment of length possibly less than p , which, in this case contains the diagonal node. Let us name B the set of nodes in this last p -segment, and let A be the set

$$A = B \cup \{(j, i) \in [\lambda] \mid (i, j) \in B\} \subseteq \text{Rim}_p^*(\lambda).$$

The set A is formed by a symmetrical segment along the rim of $[\lambda]$. See Figure 1a.

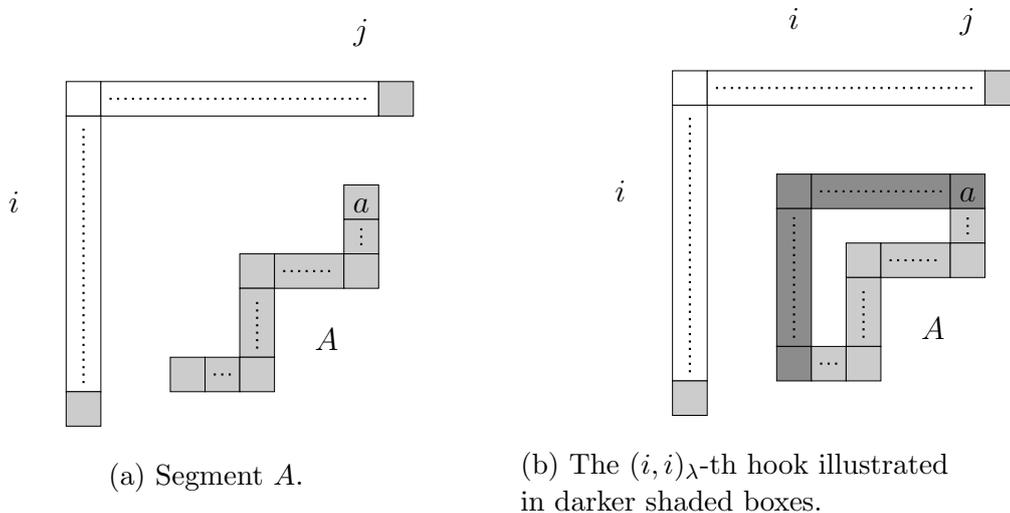


Figure 1

The set $\text{Rim}_p^*(\lambda)$ is formed by the disjoint union of A and some p -segments. Therefore, since $p \mid a_\lambda^*$, we have that $|A| = p$.

Now, let $a = (i, j)$ be the first node of the segment B , that is $i = \min\{r \mid (r, s) \in B\}$ and $j = \max\{s \mid (r, s) \in B\}$. We have that the (i, i) -th hook contains exactly $|A| = p$ nodes. See Figure 1b.

This means that λ has a diagonal (p) -hook, which is a contradiction because $\lambda \in \text{BG}_p$. \square

We obtain the following corollary from Remark 11 and Lemma 27.

Corollary 28. *Let $\lambda \in \text{BG}_p$. The following statements are equivalent*

1. $\varepsilon_\lambda^* = 0$.
2. a_λ^* is an even integer.
3. $\text{Rim}_p^*(\lambda)$ has no diagonal nodes.
4. $p \mid a_\lambda^*$.

Consider a partition $\lambda \in M_p$, that is, a fixed point of the Mullineux map. This is a condition that depends only on the columns of the Mullineux symbol of λ . Therefore, the partition $\lambda^{(1)}$ obtained by removing the p -rim of λ is also a fixed point of the Mullineux map, since its Mullineux symbol is obtained by removing the first column of the Mullineux symbol of λ . The following lemma is an analogue property for partitions in BG_p .

Lemma 29. *If $\lambda \in \text{BG}_p$ then $\lambda^{(1)*} \in \text{BG}_p$. In other words, if λ is a BG-partition, then, removing its p -rim^{*} results in a BG-partition.*

Proof. Recall (Remark 12) that if λ is self-conjugate, then so it is for $\lambda^{(1)*}$. In particular, if $\lambda \in \text{BG}_p$, then $\lambda^{(1)*}$ is self-conjugate. It remains to prove that $\lambda^{(1)*}$ does not have any diagonal (p) -hooks.

For simplicity of notations let $\mu = \lambda^{(1)*}$. Suppose that μ has a diagonal (p) -hook, say the $(i, i)_\mu$ -th hook, with $h_{i,i}^\mu = pk$ for some $k \in \mathbb{N}$.

We claim that the node $(i, \mu_i + 1)$ is in the p -rim^{*} of λ . Indeed, $(i, \mu_i + 1) \in [\lambda]$ since if $(i, \mu_i + 1) \notin [\lambda]$, then $\mu_i = \lambda_i$ and $h_{i,i}^\lambda = h_{i,i}^\mu = pk$ so that λ has a diagonal (p) -hook, which contradicts the fact that $\lambda \in \text{BG}_p$. Now, since $(i, \mu_i + 1) \in [\lambda] \setminus [\mu]$, then $(i, \mu_i + 1) \in \text{Rim}_p^*(\lambda)$. See Figure 2a.

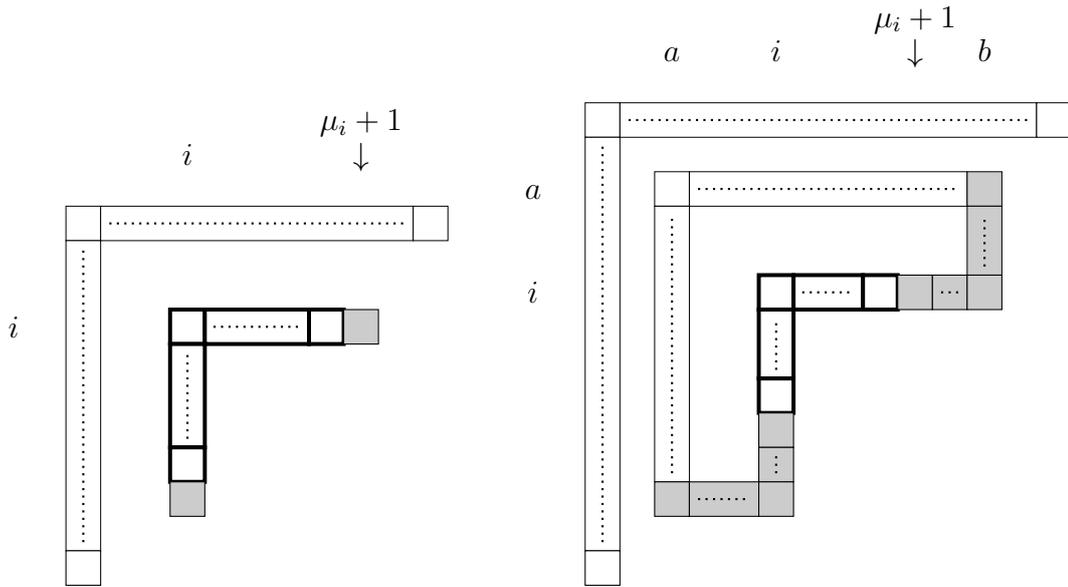
There are now two possible cases: either $(i, \mu_i + 1)$ is the last node of a p -segment of U_λ (the nodes on the p -rim^{*} of λ over the diagonal), or it is not the last node of the p -segment to which it belongs. Let us examine these two cases.

Suppose $(i, \mu_i + 1)$ is the last node of a p -segment of U_λ , and this p -segment starts on a node (a, b) . See Figure 2b.

Then, the $(a, a)_\lambda$ -th hook has length equal to the length of the $(i, i)_\mu$ -th hook plus twice the length of the p -segment of $\text{Rim}_p^*(\lambda)$ containing the node $(i, \mu_i + 1)$, that is

$$h_{a,a}^\lambda = p + h_{i,i}^\mu + p = p + pk + p = p(k + 2),$$

so that λ contains diagonal (p) -hook, which is impossible.



(a) Shaded boxes are in $\text{Rim}_p^*(\lambda)$. (b) Shaded boxes are p -segments in $\text{Rim}_p^*(\lambda)$.

Figure 2

Suppose now that $(i, \mu_i + 1)$ is not the last node of a p -segment of U_λ . First, notice that the node $(i + 1, \mu_i + 2) \notin [\lambda]$. This is true because $(i, \mu_i + 1)$ is in the p -rim* of λ . We claim that $(i + 1, \mu_i + 1) \in \text{Rim}_p^*(\lambda) \subseteq [\lambda]$. In Figure 3, node $(i + 1, \mu_i + 2)$ is illustrated as a cross (meaning it is not in $[\lambda]$) and node $(i + 1, \mu_i + 1)$ as a shaded box (as are their opposites with respect to the diagonal). Indeed, $(i + 1, \mu_i + 1) \in \text{Rim}_p^*(\lambda)$ because, since $(i, \mu_i + 1)$ is not the last node of a p -segment, then the next node of its p -segment is either to the left or down. But the node to the left of $(i, \mu_i + 1)$, that is, (i, μ_i) is not in the p -rim* of λ since it is in μ , so that the next node of this p -segment is $(i + 1, \mu_i + 1)$, which is then in $\text{Rim}_p^*(\lambda)$.

The fact that $(i + 1, \mu_i + 1) \in \text{Rim}_p^*(\lambda) \subseteq [\lambda]$ and $(i + 1, \mu_i + 2) \notin [\lambda]$ implies that $\lambda_{i+1} = \mu_i + 1$ and therefore the $(i + 1, i + 1)_\lambda$ -th hook has length

$$h_{(i+1, i+1)}^\lambda = h_{(i, i)}^\mu = pk,$$

that is, λ has a diagonal (p)-hook, a contradiction.

We conclude that μ does not have any diagonal (p)-hooks and then, $\mu = \lambda^{(1)*} \in \text{BG}_p$. \square

Proof of Proposition 21. Let us first state which properties characterize elements in \mathcal{M}_p . That is, if $\lambda \in M_p^n$ which conditions characterize its Mullineux symbol

$$G_p(\lambda) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}.$$

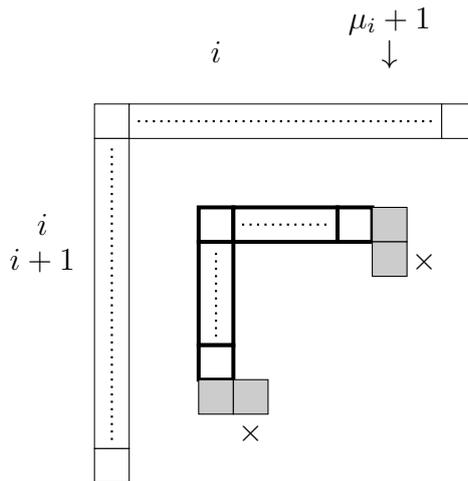


Figure 3

Let ε_i be as in Proposition 4 and $s_i = a_i + \varepsilon_i - r_i$. The partition λ is a fixed point of the Mullineux map if and only if $r_i = s_i$, that is

$$a_i = 2r_i - \varepsilon_i.$$

We also know that λ is the only p -regular partition whose Mullineux symbol satisfies properties (1)–(5) from Proposition 4. This way, the properties that characterize Mullineux symbols of partitions in M_p are equivalent to the following properties

1. $\varepsilon_i \leq r_i - r_{i+1} < p + \varepsilon_i$ for $0 \leq i < l$,
2. $1 \leq r_l < p + \varepsilon_l$,
3. $\sum_{i=0}^l a_i = n$, and
4. $a_i = 2r_i - \varepsilon_i$.

On the other hand, from the definition of ε_i^* and Corollary 28, we have that

$$\varepsilon_i^* = \begin{cases} 0 & \text{if } p \mid a_i^* \\ 1 & \text{if } p \nmid a_i^* \end{cases}$$

Let $\lambda \in \text{BG}_p^n$. Let us see that its BG-symbol

$$\text{bg}_p(\lambda) = \begin{pmatrix} a_0^* & a_1^* & \cdots & a_l^* \\ r_0^* & r_1^* & \cdots & r_l^* \end{pmatrix}$$

is in \mathcal{M}_p by verifying properties (1)–(4) for a_i^* , ε_i^* and r_i^* :

From the definition of the sequence a_0^*, \dots, a_l^* , it is clear that (3) holds. We have that (4) is satisfied from Remark 11. Let us first show that (2) holds. Since $\lambda^{(l)*}$ is

not the empty partition, $r_l^* \geq 1$. On the other hand, the partition $\lambda^{(l)*}$ is a hook and it is self-conjugate; more precisely $\lambda^{(l)*} = (r_l^*, 1^{a_l^* - r_l^*})$. Then $a_l^* = |\lambda^{(l)*}|$ is odd, so that $\varepsilon_l^* = a_l^* \bmod 2 = 1$. Suppose that $r_l^* \geq p + \varepsilon_l^* = p + 1$. This means that the first p -segment over the diagonal of $\lambda^{(l)*}$ consists of p nodes. But then, there are more nodes remaining in the first row of $[\lambda^{(l)*}]$ that are not in the p -rim * , but this contradicts the maximality of l .

It remains to prove (1). A key element for this task is Lemma 25, which roughly says that truncating a BG-partition to some particular row results in a p -regular partition. The idea is to use the fact that this truncated partition, being p -regular, satisfies properties from Proposition 4, which uses numbers from the Mullineux symbol, and these will give us information about r_i^* and ε_i^* , which are numbers appearing in the BG-symbol.

Let us see that λ satisfies

$$\varepsilon_i^* \leq r_i^* - r_{i+1}^* < p + \varepsilon_i^* \quad \text{for } 0 \leq i < l.$$

It suffices to prove this for $i = 0$ and then, the property follows recursively by Lemma 29.

To simplify notation, set:

values associated to λ	values associated to $\lambda^{(1)*}$
$a := a_0^*$	$a' := a_1^*$
$r := r_0^*$	$r' := r_1^*$
$\varepsilon := \varepsilon_0^*$	$\varepsilon' := \varepsilon_1^*$

Let us prove that

$$\varepsilon \leq r - r' < p + \varepsilon.$$

We study the four possible cases for the values of ε and ε' , namely

	ε	ε'
(i)	0	0
(ii)	0	1
(iii)	1	0
(iv)	1	1

In each case we will consider some diagram

$$[\underline{\lambda}] := \{(i, j) \in [\lambda] \mid i \leq k(\lambda) \text{ and } j \geq k(\lambda) - x + 1\},$$

for a certain $1 \leq x \leq k(\lambda)$ (which will be chosen depending on the case). That is, $[\underline{\lambda}]$ is obtained from $[\lambda]$ by removing rows below row $k(\lambda)$ and columns up to column $k(\lambda) - x$. In an abuse of notation we will call $\underline{\lambda}$ the partition with Young diagram obtained by shifting the diagram $[\underline{\lambda}]$ to the left by $k(\lambda) - x$ columns. This will allow us to identify nodes of $\underline{\lambda}$ and nodes of λ (for example (i, λ_i) , and not $(i, \lambda_i - k(\lambda) + x)$, will be the last node on row i of $\underline{\lambda}$ for $1 \leq i \leq k(\lambda)$).

partition. In particular, the diagonal node $(k(\lambda), k(\lambda))$ is in the j -th p -rim of λ for some $j > 1$ since $\text{Rim}_p(\lambda)$ and $\text{Rim}_p(\lambda^{(1)})$ do not contain diagonal nodes. On the other hand the p -rim of any partition contains nodes in the last row of the partition and since $k(\lambda)$ is the last row of both λ and $\lambda^{(j)}$, then it is also the last row of $\lambda^{(1)}$. So that both λ and $\lambda^{(1)}$ contain nodes in row $k(\lambda)$.

Now, the fact that $\text{Rim}_p(\lambda)$ and $\text{Rim}_p(\lambda^{(1)})$ end at the same row means that $l(\lambda) = k(\lambda) = l(\lambda^{(1)})$, that is $\underline{r} - \underline{r}' = 0$.

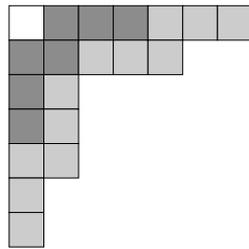
On the other hand, in this case, we have that $\varepsilon' = 0$, which means that $p \mid a'$. But since $a' = 2r' - \varepsilon'$, then $p \mid r' = \underline{a}'$ ($p \neq 2$), which means that $\underline{\varepsilon}' = 0$.

The fact that $\underline{r} - \underline{r}' = 0$, together with the fact that $\underline{a} = r$, $\underline{a}' = r'$ and $\underline{\varepsilon} = 0$, make Equation 4 become

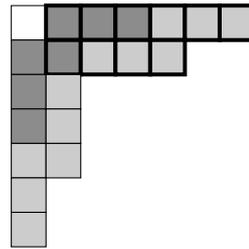
$$0 \leq r - r' < p + 0.$$

So that $\varepsilon \leq r - r' < p + \varepsilon$, as we wanted to show, since in this case $\varepsilon = 0$.

- (ii) Suppose that $\varepsilon = 0$ and $\varepsilon' = 1$. For example as in the partition $(7, 5, 2^3, 1^2)$ with $p = 3$. In Figure 5a we represent the 3-rims* of $(7, 5, 2^3, 1^2)$ and $(7, 5, 2^3, 1^2)^{(1)*}$ in different shades.



(a) $\text{Rim}_3^*((7, 5, 2^3, 1^2))$ and $\text{Rim}_3^*((7, 5, 2^3, 1^2)^{(1)*})$.



(b) Partition $(7, 5, 2^3, 1^2)$ in thicker lines.

Figure 5

As in the previous case, let $x = 1$. We illustrate $(7, 5, 2^3, 1^2)$ by thicker lines in Figure 5b.

Let us see that also in this case we have that $\underline{r} - \underline{r}' = 0$. As before, the nodes in $\text{Rim}_p^*(\lambda)$ over the diagonal of λ , or U_λ , are exactly the nodes of $\text{Rim}_p(\lambda)$. And we also have that $U_{\lambda^{(1)*}} = \text{Rim}_p(\lambda^{(1)})$. So that $r = \underline{a}$ and $r' = \underline{a}'$. On the other hand, since in this case $(k(\lambda), k(\lambda)) \in \text{Rim}_p^*(\lambda^{(1)*})$, then $(k(\lambda), k(\lambda)) \in \text{Rim}_p(\lambda^{(1)})$. Furthermore, the fact that $\text{Rim}_p^*(\lambda^{(1)*})$ has a node on row $k(\lambda)$, implies that $\lambda^{(1)}$ has a node on row $k(\lambda)$, and then so it is for λ . Therefore $l(\lambda) = k(\lambda) = l(\lambda^{(1)})$, then $\underline{r} - \underline{r}' = 0$.

Now, consider the two possible cases for $\underline{\varepsilon}'$. If $\underline{\varepsilon}' = 0$, we obtain, as in the previous case

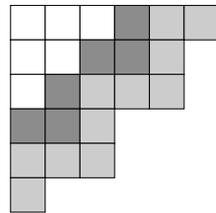
$$0 \leq r - r' < p + 0,$$

which is what we wanted to show. If $\underline{\varepsilon}' = 1$, Equation 4 becomes

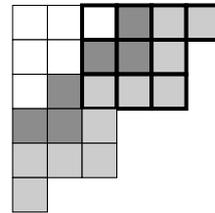
$$1 \leq r - r' < p + 1,$$

In particular $0 \leq r - r' \leq p$. But actually, $r - r' < p$. Indeed, if $r - r' = p$, since $p \mid a = 2r$, then $p \mid r$ and therefore $p \mid r' = \underline{a}'$, which contradicts the fact that $\underline{\varepsilon}' = 1$. In conclusion $0 \leq r - r' < p$, which ends this case.

- (iii) Suppose that $\varepsilon = 1$ and $\varepsilon' = 0$. For example as in the partition $(6, 5^2, 3^2, 1)$ with $p = 3$. In Figure 6a we represent the 3-rims* of $(6, 5^2, 3^2, 1)$ and $(6, 5^2, 3^2, 1)^{(1)*}$ in different shades.



(a) $\text{Rim}_3^*((6, 5^2, 3^2, 1))$ and $\text{Rim}_3^*((6, 5^2, 3^2, 1)^{(1)*})$.



(b) Partition $(6, 5^2, 3^2, 1)$ in thicker lines.

Figure 6

As before, let $x = 1$. We illustrate $(6, 5^2, 3^2, 1)$ by thicker lines in Figure 6b. Let us see that in this case $\underline{r} - \underline{r}' = 1$.

As in the preceding cases, the nodes in $\text{Rim}_p^*(\lambda)$ over the diagonal of λ are exactly the nodes of $\text{Rim}_p(\underline{\lambda})$. This fact implies that $\underline{a} = r$, and since $\varepsilon' = 0$, by the same argument that in case (i), we have that $\underline{a}' = r'$. Now, since $\varepsilon = 1$, the last diagonal node of λ , that is, the node $(k(\lambda), k(\lambda))$ is in $\text{Rim}_p^*(\lambda)$. In particular $(k(\lambda), k(\lambda)) \in \text{Rim}_p(\underline{\lambda})$, and since it is the first node of the last row of $\underline{\lambda}$, that means that all nodes on this last row are in the p -rim of $\underline{\lambda}$. So that this last row $k(\lambda) = \underline{r}$ of $\underline{\lambda}$ does not have any nodes from $\text{Rim}_p^*(\lambda^{(1)*})$ (or $\text{Rim}_p(\underline{\lambda}^{(1)})$). We claim that row $k(\lambda) - 1$ in $\underline{\lambda}$ contains at least one node in $\text{Rim}_p^*(\lambda^{(1)*})$ (or $\text{Rim}_p(\underline{\lambda}^{(1)})$). Indeed, node $(k(\lambda) - 1, k(\lambda) - 1)$ is in $\text{Rim}_p^*(\lambda^{(j)*})$ for a $j > 1$, since it is not in $\text{Rim}_p^*(\lambda^{(1)*})$ (because $\varepsilon' = 0$). If we suppose that row $k(\lambda) - 1$ does not have node in $\text{Rim}_p^*(\lambda^{(1)*})$, we are supposing that to the left of node $(k(\lambda) - 1, k(\lambda) - 1)$ there are only nodes from $\text{Rim}_p^*(\lambda)$. If this is the case, the last node in row $k(\lambda) - 1$ in $\lambda^{(1)*}$ is $(k(\lambda) - 1, k(\lambda) - 1)$, that is: $\lambda_{k(\lambda)-1}^{(1)*} = k(\lambda) - 1$. But the last node on every row (over

or on the diagonal) belongs to the p -rim*. In this case, node $(k(\lambda) - 1, k(\lambda) - 1)$ belongs to the p -rim* of $\lambda^{(1)*}$, a contradiction since $\lambda^{(1)*}$ does not have any diagonal nodes on its p -rim*. In conclusion, row $k(\lambda) - 1$ in $\underline{\lambda}$ contains at least one node in $\text{Rim}_p^*(\lambda^{(1)*})$, in particular, row $k(\lambda) - 1$ in $\underline{\lambda}$ contains at least one node in $\underline{\lambda}^{(1)}$, so that $l(\underline{\lambda}^{(1)}) = k(\lambda)$. Therefore $\underline{r} - \underline{r}' = k(\lambda) - l(\underline{\lambda}^{(1)}) = 1$.

On the other hand, since we have that $\varepsilon' = 0$, by the same argument as in case (i), we have that $\underline{\varepsilon}' = 0$.

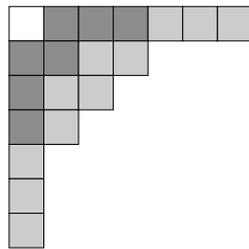
Putting all together in Equation 4, we obtain

$$1 \leq r - r' < p + 1.$$

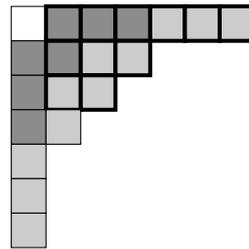
That is, $\varepsilon \leq r - r' < p + \varepsilon$.

- (iv) Suppose finally that $\varepsilon = \varepsilon' = 1$. An example is given by partition $(7, 4, 3, 2, 1^3)$ for $p = 3$. In Figure 7a we represent the 3-rims* of $(7, 4, 3, 2, 1^3)$ and $(7, 4, 3, 2, 1^3)^{(1)*}$ in different shades.

Let $x = 2$. Lemma 25 still assures that $\underline{\lambda}$ is p -regular. And from the way that it is defined, $\underline{\lambda}$ contains both diagonal nodes in $\text{Rim}_p^*(\lambda)$ and $\text{Rim}_p^*(\lambda^{(1)*})$. We illustrate $(7, 4, 3, 2, 1^3)$ by thicker lines in Figure 7b.



(a) $\text{Rim}_3^*((7, 4, 3, 2, 1^3))$ and $\text{Rim}_3^*((7, 4, 3, 2, 1^3)^{(1)*})$.



(b) Partition $(7, 4, 3, 2, 1^3)$ in thicker lines.

Figure 7

Notice that in this case it is not necessarily true that $\underline{a} = r$ and $\underline{a}' = r'$. Since $\underline{\lambda}$ contains the node $(k(\lambda), k(\lambda) - 1)$ which is under the diagonal of λ , where the p -rim* does not behave as the p -rim. For the partition $(7, 4, 3, 2, 1^3)$, this node is the node $(3, 2)$, which in this case is in the p -rim* of $(7, 4, 3, 2, 1^3)$. But it could be the case that the node $(k(\lambda), k(\lambda) - 1)$ is not in the p -rim* of λ but in the p -rim* of $\lambda^{(1)*}$. This depends on the divisibility of r by p .

Recall that $r = \#U_\lambda$ is the number of nodes in the p -rim* of λ that are above (or on) the diagonal of λ . Let us consider the two cases: $p \mid r$ and $p \nmid r$.

- Suppose that $p \mid r$. As in $\lambda = (7, 4, 3, 2, 1^3)$ with $p = 3$, see Figure 7a. In this case every p -segment of the p -rim* of λ contains exactly p nodes. In particular the segment which contains the node $(k(\lambda), k(\lambda))$. And since this node is the last (and p -th) node of this p -segment, then the node to its left $(k(\lambda), k(\lambda) - 1)$ is not in the p -rim of $\underline{\lambda}$. And we have $\underline{a} = r$ and $\underline{r} - \underline{r}' = 0$. Moreover, the node $(k(\lambda), k(\lambda) - 1)$ is then in the p -rim of $\underline{\lambda}^{(1)}$. So that $\underline{a}' = r' + 1$ (the r' nodes of $\text{Rim}_p^*(\underline{\lambda}^{(1)*})$ above the diagonal, together with node $(k(\lambda), k(\lambda) - 1)$, form $\text{Rim}_p(\underline{\lambda}^{(1)})$). Therefore we have $\underline{a} = r$, $\underline{a}' = r' + 1$ and $\underline{r} - \underline{r}' = 0$. Putting this together in Equation 4 we get

$$\underline{\varepsilon}' \leq r - (r' + 1) < p + \underline{\varepsilon}',$$

or

$$\underline{\varepsilon}' + 1 \leq r - r' < p + \underline{\varepsilon}' + 1.$$

But $\underline{\varepsilon}' + 1 \geq 1 = \varepsilon$. Therefore we have

$$1 \leq r - r' < p + \underline{\varepsilon}' + 1.$$

Let us see that $r - r' < p + 1 = p + \varepsilon$. There are two possibilities for $\underline{\varepsilon}'$. Either $\underline{\varepsilon}' = 0$, in which case $r - r' < p + 1$, or $\underline{\varepsilon}' = 1$. If $\underline{\varepsilon}' = 1$, we have $r - r' < p + 2$, so that $r - r' \leq p + 1$. But in fact $r - r' < p + 1$, since if $r - r' = p + 1$, then $r - (r' + 1) = p$. But in this case $p \mid r$, therefore, $p \mid r' + 1 = \underline{a}'$, and this contradicts the fact that $\underline{\varepsilon}' = 1$.

- Suppose that $p \nmid r$. As in $\lambda = (6, 2, 1^4)$ with $p = 3$, see Figure 8a. In this case, the p -segment of $\text{Rim}_p^*(\lambda)$ which contains the node $(k(\lambda), k(\lambda))$ has less than p nodes. This implies that the node to the left of this diagonal node, namely $(k(\lambda), k(\lambda) - 1)$, is on the p -rim of $\underline{\lambda}$. Then $\underline{a} = r + 1$ (the r nodes of $\text{Rim}_p^*(\lambda)$ above the diagonal, together with node $(k(\lambda), k(\lambda) - 1)$, form $\text{Rim}_p(\underline{\lambda})$), and we also have that $\underline{r} - \underline{r}' = 1$ and $\underline{a}' = r'$. Equation 4 gives

$$1 + \underline{\varepsilon}' \leq (r + 1) - r' < p + 1 + \underline{\varepsilon}',$$

or

$$\underline{\varepsilon}' \leq r - r' < p + \underline{\varepsilon}'.$$

But $p + \underline{\varepsilon}' \leq p + 1 = p + \varepsilon$. Then $\underline{\varepsilon}' \leq r - r' < p + \varepsilon$. Let us show that $r - r' \geq 1 = \varepsilon$. There are two possibilities for $\underline{\varepsilon}'$. Either $\underline{\varepsilon}' = 1$, in which case $r - r' \geq 1 = \varepsilon$ or $\underline{\varepsilon}' = 0$. If $\underline{\varepsilon}' = 0$, then $r - r' \geq 0$. But actually $r - r' \geq 1$, since if $r - r' = 0$, from the fact that $p \mid \underline{a}' = r'$ we would have that $p \mid r$, a contradiction. \square



(a) $\text{Rim}_3^*((6, 2, 1^4))$ and $\text{Rim}_3^*((6, 2, 1^4)^{(1)*})$. (b) Partition $(6, 2, 1^4)$ in thicker lines.

Figure 8

4 From self-Mullineux partitions to BG-partitions

From Theorem 22, we know that the notion of BG-symbol induces an algorithm for the correspondence between BG-partitions and self-Mullineux partition, since it defines an injective mapping between sets of the same cardinality. Then we know that to each Mullineux symbol of a self-Mullineux partition, corresponds a unique BG-partition. Moreover, from the definition of the BG-symbol, and Lemma 18, we know how to find the BG-partition associated to such a Mullineux symbol under this correspondence. In this section we prove that this inverse algorithm is well defined, that is, we prove that applying it to a Mullineux symbol of a self-Mullineux partition results in a BG-partition. This confirms Theorem 22 in a combinatorial way without using the fact that $\#\text{BG}_p^n = \#M_p^n$.

Proposition 30. *Let p be an odd prime and λ a self-Mullineux partition. The Mullineux symbol of λ , $G_p(\lambda)$ is the BG-symbol of some BG-partition. That is*

$$\mathcal{M}_p \subseteq \text{bg}_p(\text{BG}_p).$$

Proof. We give a combinatorial proof of this fact, although it follows also directly from Proposition 20 and Proposition 21.

We proceed by induction on l , the length of the Mullineux symbol.

Let $l = 0$ and $S = \begin{pmatrix} a_l \\ r_l \end{pmatrix} \in \mathcal{M}_p$, that is, $S = G_p(\lambda)$ for some $\lambda \in M_p$. Let $\varepsilon_l = 0$ if $p \mid a_l$ and $\varepsilon_l = 1$ otherwise. Since S has exactly one column, then $\lambda = \lambda^{(l)}$ is a hook, that is λ is of the form $\lambda = (x, 1^y)$, with $x \leq p$. On the other hand, since λ is fixed by the Mullineux map, we know that

$$a_l = 2r_l - \varepsilon_l.$$

We claim that $\varepsilon_l = 1$. If $\varepsilon_l = 0$, that is, if $p \mid a_l$, then the p -segments that form $\text{Rim}_p(\lambda) = [\lambda]$ are all of length exactly p . We know that λ is a p -regular hook, this means that λ is formed by exactly one p -segment. If there was more than one p -segment, then $a_l > p$ (so that $a_l = 2kp$ for some $k \geq 1$) it follows that $y \geq p$, and then λ would not be p -regular. Thus $a_l = p = 2r_l$. But this is not possible since p is odd. Then $\varepsilon_l = 1$ and $a_l = 2r_l - 1$.

The partition $\mu = (r_l, 1^{r_l-1})$ is self-conjugate, and is a hook of length $2r_l - 1 = a_l$. Since $p \nmid a_l$, then $\mu \in \text{BG}_p$. Its BG-symbol is

$$\text{bg}_p(\mu) = \begin{pmatrix} 2r_l - 1 \\ r_l \end{pmatrix} = \begin{pmatrix} a_l \\ r_l \end{pmatrix} = S.$$

In fact $\mu = \lambda$.

Consider now $l > 0$. Let

$$S = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}$$

be a symbol in \mathcal{M}_p corresponding to a partition λ in M_p . Consider the array

$$\tilde{S} = \begin{pmatrix} a_1 & \cdots & a_l \\ r_1 & \cdots & r_l \end{pmatrix}.$$

By definition, \tilde{S} is the Mullineux symbol of the partition $\lambda^{(1)}$, obtained from λ by removing the nodes on the p -rim. We know that $\lambda^{(1)}$ is fixed by the Mullineux map, given that this only depends on the columns of the symbol. Then $\lambda^{(1)} \in M_p$, and $\tilde{S} \in \mathcal{M}_p$. By induction, there exists a partition $\tilde{\mu} \in \text{BG}_p$ such that

$$\text{bg}_p(\tilde{\mu}) = \tilde{S}.$$

We will apply Lemma 18. Let $\varepsilon_0 = 0$ if $p \mid a_0$, or $\varepsilon_0 = 1$ otherwise. Let $m = (r_0 - \varepsilon_0) \bmod p$.

Suppose that $\varepsilon_0 = 0$ and let us see that in this case $m = 0$. Since $\varepsilon_0 = 0$, then $p \mid a_0$. But $a_0 = 2r_0 - \varepsilon_0 = 2r_0$, since λ is a fixed point of the Mullineux map. Now, since p is odd, then $p \mid r_0$ so that $m = (r_0 - \varepsilon_0) \bmod p = r_0 \bmod p = 0$.

If $\varepsilon_0 = 1$, we have that $p \nmid a_0$. Therefore $p \nmid 2m + 1$ since $2m + 1 \equiv 2(r_0 - \varepsilon_0) + 1 \pmod{p}$ and $2(r_0 - \varepsilon_0) + 1 = 2r_0 - \varepsilon_0 = a_0$.

Lemma 18 implies that there exists a unique self-conjugate partition $\mu \in \text{BG}_p$ such that

- (i) $a_\mu^* \equiv \varepsilon_0 \pmod{2}$;
- (ii) $r_\mu^* - \varepsilon_\mu^* \equiv m \pmod{p}$ and
- (iii) $\mu^{(1)*} = \tilde{\mu}$.

The condition $\mu^{(1)*} = \tilde{\mu}$ implies that

$$\text{bg}_p(\mu) = \begin{pmatrix} a_\mu^* & & & \\ & \text{bg}_p(\tilde{\mu}) & & \\ r_\mu^* & & & \end{pmatrix} = \begin{pmatrix} a_\mu^* & & & \\ & \tilde{S} & & \\ r_\mu^* & & & \end{pmatrix} = \begin{pmatrix} a_\mu^* & a_1 & \cdots & a_l \\ r_\mu^* & r_1 & \cdots & r_l \end{pmatrix}.$$

Let us see that in fact $\text{bg}_p(\mu) = S$, that is, $a_\mu^* = a_0$ and $r_\mu^* = r_0$. Indeed, from (i), a_μ^* is even if and only if $\varepsilon_0 = 0$, if and only if $p \mid a_\lambda$. But a_μ^* is even if and only if $p \mid a_\mu^*$, by

Corollary 28. This sequence of equivalences says that $\varepsilon_0 = \varepsilon_\mu^*$. Then, by (ii) we have that $r_\mu^* \equiv r_0 \pmod{p}$.

Since $S \in \mathcal{M}_p$, then, from Proposition 4, we have, in particular

$$\varepsilon_0 \leq r_0 - r_1 < p + \varepsilon_0. \tag{5}$$

On the other hand, since $\mu \in \text{BG}_p$, then $\text{bg}_p(\mu) \in \mathcal{M}_p$, by Proposition 21, so that, in particular we have

$$\varepsilon_\mu^* \leq r_\mu^* - r_1 < p + \varepsilon_\mu^*. \tag{6}$$

Subtracting Equation 5 from Equation 6, we get

$$-p < r_\mu^* - r_0 < p,$$

but since $p \mid r_\mu^* - r_0$ we can conclude that $r_\mu^* - r_0 = 0$, so that $r_\mu^* = r_0$. Therefore $a_\mu^* = 2r_\mu^* - \varepsilon_\mu^* = 2r_0 - \varepsilon_0 = a_0$, and

$$\text{bg}_p(\mu) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix} = S. \quad \square$$

A Appendix

Let $n \in \mathbb{N}$ and p an odd prime. Consider the two following sets of partitions of n

$$\text{BG}_p^n = \{\lambda \mid \lambda \vdash n; \lambda = \lambda' \text{ and } \lambda \text{ has no diagonal } (p)\text{-hooks}\}, \text{ and}$$

$$M_p^n = \{\lambda \mid \lambda \in \text{Reg}_p(n) \text{ and } m(\lambda) = \lambda\},$$

where m is the Mullineux map.

Example 31. Let $n = 18$ and $p = 3$. There are, in total, 385 partitions of 18. The partitions λ of 18 such that $\lambda = \lambda'$ are

$$(5, 4^3, 1), (6, 5, 2^3, 1), (7, 4, 2^2, 1^3), (8, 3, 2, 1^5), (9, 2, 1^7)$$

Among them, those with no diagonal (3)-hooks are

$$\text{BG}_3^{18} = \{(6, 5, 2^3, 1), (7, 4, 2^2, 1^3), (9, 2, 1^7)\}$$

There are 135 3-regular partitions of 18. Those which are fixpoints of the Mullineux map are

$$M_3^{18} = \{(7, 5, 2^2, 1^2), (9, 4^2, 1), (10, 4^2)\}$$

In general the sets M_p^n of partitions of n fixed by the Mullineux map and BG_p^n of BG-partitions of n , have the same number of elements. Indeed, it is easy to see that the number of self-Mullineux partitions is equal to the number of p -regular conjugacy classes of the symmetric group \mathfrak{S}_n , contained in A_n , which split into two different conjugacy classes of A_n . Here, a p -regular conjugacy class means that the order of its elements is not divisible

by p . This follows by [AO91, Proposition 2]. Now, the number of p -regular splitting conjugacy classes of \mathfrak{S}_n contained in A_n is equal to the number of self-conjugate partitions with diagonal hook-lengths not divisible by p . This is straightforward by [JK81, 1.2.10] and the standard bijection between partitions in odd distinct parts and self-conjugate partitions (explained in detail later in this appendix).

This appendix contains an alternative proof of the fact that $\#M_p^n = \#\text{BG}_p^n$.

Remark 32. The generating function for the cardinality of BG_p^n is

$$\prod_{\substack{i \geq 0 \\ p \nmid 2i+1}} (1 + t^{2i+1}).$$

For proving the mentioned identity, we need make some remarks about the intersection of conjugacy classes in \mathfrak{S}_n with A_n and about Brauer characters of A_n .

Splitting of conjugacy classes of \mathfrak{S}_n . Let C be a conjugacy class of \mathfrak{S}_n of even permutations. That is $C \subseteq A_n$. Then one of the two following possibilities holds:

- C is a conjugacy class in A_n , or
- C splits into two conjugacy classes in A_n .

In the second possibility, say $C = C_1 \sqcup C_2$, these two conjugacy classes have the same size. Moreover, conjugating such classes by any element of $\mathfrak{S}_n \setminus A_n$ permutes them, that is, if $\sigma \in \mathfrak{S}_n \setminus A_n$, then $\sigma C_1 \sigma^{-1} = C_2$ and $\sigma C_2 \sigma^{-1} = C_1$. Furthermore, the conjugacy class C splits if and only if the cycle type of elements in C consists of different odd integers ([JK81, 1.2.10]). We call a conjugacy class p -regular when the order of its elements is not divisible by p .

The set of p -regular conjugacy classes of \mathfrak{S}_n contained in A_n is then formed by two types of conjugacy classes:

$$A \sqcup B,$$

where A is set of p -regular conjugacy classes of \mathfrak{S}_n of even permutations which are also conjugacy classes in A_n and B is the set conjugacy classes of \mathfrak{S}_n which split into two conjugacy classes in A_n . Hence, the set of p -regular conjugacy classes of A_n is

$$A \sqcup \overline{B},$$

where \overline{B} consists of conjugacy classes coming for restriction of those conjugacy classes in B . These conjugacy classes in \overline{B} come by pairs, in the sense that if $\sigma \in \mathfrak{S}_n \setminus A_n$ and $\overline{C} \in \overline{B}$, then $\sigma \overline{C} \sigma^{-1} \in \overline{B}$ and $C = \overline{C} \cup \sigma \overline{C} \sigma^{-1}$ is a conjugacy class of \mathfrak{S}_n in B . Hence, a basis for the space of \mathbb{C} -valued functions defined on p -regular elements of A_n and constant on conjugacy classes is

$$\{\mathbb{1}_C \mid C \in A\} \sqcup \{\mathbb{1}_{\overline{C}} \mid \overline{C} \in \overline{B}\},$$

or

$$\{\mathbb{1}_C \mid C \in A\} \sqcup \{\mathbb{1}_{\overline{C}}, \mathbb{1}_{\sigma \overline{C} \sigma^{-1}} \mid \overline{C} \cup \sigma \overline{C} \sigma^{-1} = C \in B\}.$$

We claim that the set BG_p^n is in bijection with the set of p -regular conjugacy classes of S_n , contained in A_n which split in two different conjugacy classes of A_n . Notice that the set of conjugacy classes of \mathfrak{S}_n with cycle type consisting of different odd integers is in bijection with self-conjugate partitions of size n . Indeed, a conjugacy class whose cycle type consists of different odd integers is associated to a unique partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n with $\lambda_1, \dots, \lambda_r$ different odd integers (the lengths of the cycles in the cycle decomposition, in decreasing order). Consider the self-conjugate partition μ defined by the lengths of its diagonal hooks as follows: $h_{11}^\mu = \lambda_1, h_{22}^\mu = \lambda_2, \dots, h_{rr}^\mu = \lambda_r$. The condition of λ_i 's being different, and then strictly decreasing, ensures that μ is a well defined partition. Conversely, any self-conjugate partition of n corresponds to a unique finite sequence of different odd integers; the lengths of its hooks ([JK81, 2.5.11]).

Conjugacy classes of \mathfrak{S}_n with cycle type consisting of different odd integers are in particular contained in A_n . If we consider those conjugacy classes with the additional condition of being p -regular, which form in fact the set B , they are therefore in bijection with self-conjugate partitions such that p does not divide the length of any diagonal hook, that is, the set BG_p^n . Hence B is in bijection with BG_p^n .

Brauer characters of A_n . Let D be an irreducible FA_n -module. To D we can associate a function χ_D which is called the (*irreducible*) Brauer character of A_n afforded by D . This function χ_D is a complex-valued function defined on the set of p -regular elements of A_n and it is constant on conjugacy classes. Furthermore, isomorphic FA_n -modules are associated to equal Brauer characters. See [Isa06, §15] for the precise definition of Brauer character and for further information.

In particular [Isa06, Theorem 15.10] says that the set of irreducible Brauer characters of A_n form a basis of the space of \mathbb{C} -valued functions defined on p -regular elements of A_n and constant on conjugacy classes. This implies that there are as many irreducible Brauer characters of A_n as p -regular conjugacy classes of A_n , and by [Isa06, Corollary 15.11], this is also the number of isomorphism classes of FA_n -modules. Therefore to each element μ of the set

$$\{\lambda \mid \lambda \vdash n, \lambda \text{ } p\text{-regular and } \lambda \neq m(\lambda)\} \sqcup \{\lambda^+, \lambda^- \mid \lambda \vdash n, \lambda \text{ } p\text{-regular and } \lambda = m(\lambda)\},$$

which parametrizes irreducible FA_n -modules (see 1 in the introduction), we can associate an irreducible Brauer character $\chi_{[\mu]}$. That way, a basis of the space of \mathbb{C} -valued functions defined on p -regular elements of A_n and constant on conjugacy classes is

$$\{\chi_{[\lambda]} \mid \lambda \in \text{Reg}_p(n) \text{ and } \lambda \neq m(\lambda)\} \sqcup \{\chi_{[\lambda^+]}, \chi_{[\lambda^-]} \mid \lambda \in \text{Reg}_p(n) \text{ and } \lambda = m(\lambda)\},$$

considering only one partition λ for each couple $\{\lambda, m(\lambda)\}$ with $\lambda \neq m(\lambda)$.

Proposition 33. *The sets M_p^n and BG_p^n have the same number of elements.*

Proof. To prove this, we will give two bases of a same space of functions, and the equality of the cardinality of these bases will give the result.

Denote by E the space of \mathbb{C} -valued functions defined on p -regular elements of A_n and constant on conjugacy classes

$$E = \{f : \{p\text{-regular elements of } A_n\} \longrightarrow \mathbb{C} \mid f \text{ is a class function of } A_n\}.$$

Define an action of \mathfrak{S}_n on E by conjugation as follows: for $\sigma \in \mathfrak{S}_n$ and $f \in E$, f^σ is the class function

$$f^\sigma(\tau) := f(\sigma\tau\sigma^{-1}).$$

For $\sigma \in \mathfrak{S}_n \setminus A_n$, let E^σ be the set of class functions fixed by conjugation by σ :

$$E^\sigma = \{f \in E \mid f^\sigma = f\}.$$

This is a subspace of E . From the above discussion about splitting of conjugacy classes and how conjugation permutes some conjugacy classes, a basis for E^σ is

$$\{\mathbb{1}_C \mid C \in A\} \sqcup \{\mathbb{1}_{\bar{C}} + \mathbb{1}_{\sigma\bar{C}\sigma^{-1}} \mid \bar{C} \cup \sigma\bar{C}\sigma^{-1} = C \in B\}.$$

We claim that a basis for E^σ is

$$\{\chi_{[\lambda]} \mid \lambda \in \text{Reg}_p(n) \text{ and } \lambda \neq m(\lambda)\} \sqcup \{\chi_{[\lambda^+]} + \chi_{[\lambda^-]} \mid \lambda \in \text{Reg}_p(n) \text{ and } \lambda = m(\lambda)\}.$$

Indeed, this comes from the fact that, as with usual characters of representations in characteristic zero, conjugation of the character of a representation is the character of conjugation of the representation, here with Brauer characters. And also from the fact that conjugation by σ permutes the modules associated to λ^+ and λ^- above.

Now, we have two bases for E and two bases for E^σ . On one hand, from the characteristic function basis, the dimension of E is

$$\#A + \#\bar{B} = \#A + 2(\#B),$$

and from the Brauer character basis, the dimension of E is

$$\#\{\{\lambda, m(\lambda)\} \mid \lambda \in \text{Reg}_p(n) \text{ and } \lambda \neq m(\lambda)\} + \#\{\lambda^+, \lambda^- \mid \lambda \in \text{Reg}_p(n) \text{ and } \lambda = m(\lambda)\}.$$

That is

$$\#D + 2(\#M_p^n),$$

where $D = \{\lambda, m(\lambda)\} \mid \lambda \in \text{Reg}_p(n) \text{ and } \lambda \neq m(\lambda)\}$. Hence,

$$\#A + 2(\#B) = \#D + 2(\#M_p^n).$$

Counting the elements on the two bases for E^σ , we obtain that the dimension of E^σ is

$$\#A + \#B = \#D + \#M_p^n.$$

These two identities imply that $\#B = \#M_p^n$. Since B is in bijection with BG_p^n we obtain the result. \square

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References

- [AO91] George E. Andrews and Jørn B. Olsson. Partition identities with an application to group representation theory. *J. Reine Angew. Math.*, 413:198–212, 1991.
- [Bes91] Christine Bessenrodt. A combinatorial proof of a refinement of the Andrews-Olsson partition identity. *European J. Combin.*, 12(4):271–276, 1991.
- [BG10] Olivier Brunat and Jean-Baptiste Gramain. A basic set for the alternating group. *J. Reine Angew. Math.*, 641:177–202, 2010.
- [BO94] Christine Bessenrodt and Jørn B. Olsson. On Mullineux symbols. *J. Combin. Theory Ser. A*, 68(2):340–360, 1994.
- [BO98] Christine Bessenrodt and Jørn B. Olsson. On residue symbols and the Mullineux conjecture. *J. Algebraic Combin.*, 7(3):227–251, 1998.
- [BOX99] Christine Bessenrodt, Jørn B. Olsson, and Maozhi Xu. On properties of the Mullineux map with an application to Schur modules. *Math. Proc. Cambridge Philos. Soc.*, 126(3):443–459, 1999.
- [FK97] Ben Ford and Alexander S. Kleshchev. A proof of the Mullineux conjecture. *Math. Z.*, 226(2):267–308, 1997.
- [For97] Ben Ford. Irreducible representations of the alternating group in odd characteristic. *Proc. Amer. Math. Soc.*, 125(2):375–380, 1997.
- [Isa06] I. Martin Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423].
- [JK81] Gordon James and Adalbert Kerber. The representation theory of the symmetric group. *Encyclopedia of Mathematics and its Applications*, 16, 1981.
- [Kle96] Alexander S. Kleshchev. Branching rules for modular representations of symmetric groups. III. Some corollaries and a problem of Mullineux. *J. London Math. Soc. (2)*, 54(1):25–38, 1996.
- [Mul79a] Glen Mullineux. Bijections of p -regular partitions and p -modular irreducibles of the symmetric groups. *J. London Math. Soc. (2)*, 20(1):60–66, 1979.
- [Mul79b] Glen Mullineux. On the p -cores of p -regular diagrams. *J. London Math. Soc. (2)*, 20(2):222–226, 1979.