Grundy domination of forests and the strong product conjecture

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Abstract

A maximum sequence S of vertices in a graph G, so that every vertex in S has a neighbor which is independent, or is itself independent, from all previous vertices in S, is called a Grundy dominating sequence. The Grundy domination number, $\gamma_{gr}(G)$, is the length of S. We show that for any forest F, $\gamma_{gr}(F) = |V(T)| - |\mathcal{P}|$ where $\mathcal{P}$ is a minimum partition of the non-isolate vertices of F into caterpillars in which if two caterpillars of $\mathcal{P}$ have an edge between them in F, then such an edge must be incident to a non-leaf vertex in at least one of the caterpillars. We use this result to show the strong product conjecture of B. Brešar, Cs. Bujtás, T. Gologranc, S. Klavžar, G. Košmrlj, B. Patkós, Zs. Tuza, and M. Vizer, *Dominating sequences in grid-like and toroidal graphs*, Electron. J. Combin. 23(4): P4.34 (2016), for all forests. Namely, we show that for any forest G and graph H, $\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H)$. We also show that every connected graph G has a spanning tree T so that $\gamma_{gr}(G) \leqslant \gamma_{gr}(T)$ and that every non-complete connected graph contains a Grundy dominating set S so that the induced subgraph of S contains no isolated vertices.

Mathematics Subject Classifications: 05C69, 05C76

1   Introduction

For any graph G, a sequence of vertices $S = (v_1, \ldots, v_k)$ is called a *legal sequence* if for every $i \in [k]$, $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$. A longest legal sequence is called a *Grundy dominating sequence* of G and the size of such a sequence is called the *Grundy domination number* and is denoted by $\gamma_{gr}(G)$.

Grundy domination was introduced in [6] several years ago, inspired by the much studied game domination number and the domination game. Since then, a multitude of papers have been published on aspects of this function and its variants, e.g. [7, 2, 3, 1].

A principal direction of inquiry in the original paper [6] was to understand $\gamma_{gr}(T)$ for any tree T. To describe the results of that paper, call $ES(T)$ the *end support vertices* of T which are support vertices (non-leaves adjacent to a leaf) each of which are adjacent to at most one non-leaf vertex of T. For a lower bound, the authors produced an algorithm to find a legal sequence of T of length at least $|V(T)| - |ES(T)| + 1$. To find an upper bound, the authors defined an equivalence relation between end support vertices with equivalence classes $\widetilde{T}$. They then proved that the Grundy domination number does not exceed $|V(T)| - |ES(T)| + |\widetilde{T}|$. To summarize, the authors showed that for any tree T,

$$|V(T)| - |ES(T)| + 1 \leqslant \gamma_{gr}(T) \leqslant |V(T)| - |ES(T)| + |\widetilde{T}|. \tag{1}$$

Another collection of natural questions for domination functions concern their behavior on graph products, inspired by the famous Vizing's conjecture [8], which states that for any graphs G and H, if $\square$ is the Cartesian product of graphs,

$$\gamma(G \square H) \geqslant \gamma(G)\gamma(H). \tag{2}$$

In [2], the authors investigated relations between the Grundy domination number of various graph products as they related to the Grundy domination numbers of the factor graphs. An outstanding conjecture concerned the strong product of graphs $\boxtimes$. For any graphs G and H, the strong product $G \boxtimes H$ is the graph on the vertices in (g, h) for every $g \in V(G)$ and $h \in V(H)$. Any pair of vertices (g_1, h_1) and (g_2, h_2) are adjacent if either

1. $g_1 = g_2$ and h_1 is adjacent to h_2 in H, or

2. g_1 is adjacent to g_2 in G and $h_1 = h_2$, or

3. g_1 is adjacent to g_2 in G and h_1 is adjacent to h_2 in H.

The authors easily showed that for any graphs G and H,

$$\gamma_{gr}(G \boxtimes H) \geqslant \gamma_{gr}(G)\gamma_{gr}(H) \tag{3}$$

and posed

Conjecture 1. For any graphs G and H,

$$\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H).$$

Among other results, the authors showed this conjecture holds if G is a caterpillar tree and H is any graph.

In this paper, we find an exact formula for the Grundy domination number of forests. In Section 3, we show that for any forest F, $\gamma_{gr}(F) = |V(F)| - |\mathcal{P}|$ where $\mathcal{P}$ is a minimum partition of the non-isolate vertices of F into caterpillars in which if two caterpillars of $\mathcal{P}$ have an edge between them in F, then such an edge must be incident to a non-leaf vertex in at least one of the caterpillars. In Section 4 we use our result for forests to show Conjecture 1 true when G is a forest. In section 5 we show that every connected graph G has a spanning tree with Grundy domination number at least as large as $\gamma_{gr}(G)$. We also show that every non-complete connected graph contains a Grundy dominating set S so that the induced subgraph of S contains no isolated vertices.

2   More definitions and known results

We follow established notation from [2] which can also be found in other papers on the subject. For any legal sequence $S = (v_1, \ldots, v_k)$ we call the set of vertices composed of the vertices from the sequence a *legal set* and write $\widehat{S} = \{v_1, \ldots, v_k\}$. For any $i \in [k]$, we say that v_i *footprints* the vertices $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j]$ and that the vertices of $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j]$ are *footprinted* by v_i or the *footprint* of v_i. Also, we say that v_i is the *footprinter* of $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j]$.

For notational convenience, when producing a legal sequence for a graph G, we identify *labels* on vertices of G with the indices of the legal sequence. That is, a label on a vertex will indicate the sequential position of that vertex.

Let G and H be arbitrary graphs. For any $h \in V(H)$, define the *G-fiber* of $G \boxtimes H$ with respect to h as the induced subgraph on $\{(g, h) \in G \boxtimes H : g \in V(G)\}$ and denote it by G^h. Similarly, for any $g \in V(G)$, define the *H-fiber* of $G \boxtimes H$ with respect to g as the induced subgraph on $\{(g, h) \in G \boxtimes H : h \in V(H)\}$ and denote it by H^g.

A tree C is called a *caterpillar* if for a maximum path P of C, every vertex is of distance at most 1 from P. A consequence of inequalities (1) from [6], is that if a tree C is a caterpillar, then $\gamma_{gr}(C) = |V(G)| - 1$.

A vertex is called *simplicial* if its neighbors form a clique. The following result was shown in [2] but we state the proof here for completeness.

Proposition 2. *[2] For any graphs G and H, if v is a simplicial vertex of G, then*

$$\gamma_{gr}(G \boxtimes H) \leqslant \gamma_{gr}(H) + \gamma_{gr}((G - v) \boxtimes H)$$

Proof. Among all Grundy dominating sequences of $G \boxtimes H$, let D be one that has the maximum number of vertices from H^v. Let D_1 be the subsequence of D consisting of the vertices in $\widehat{D} \cap H^v$ and $D_2 = D - D_1$. Notice that by projecting $\widehat{D}_1$ onto H we obtain vertices that form a legal sequence of H, taken in the same order as D_1. Call this projected sequence $p_H(D_1)$. To see this, assume that a vertex $(v, y) \in \widehat{D}_1$ footprints a

vertex $(v', y') \in G \boxtimes H$ and notice that in H, y must footprint y' with respect to the sequence $p_H(D_1)$. Thus, $|\widehat{D_1}| \leqslant \gamma_{gr}(H)$.

Notice that if u is a neighbor of v in G and for some h_1 in H, (u, h_1) footprints (v, h_2) for some h_2 in H, then since v is simplicial, (u, h_1) may be replaced by (v, h_1) in D to produce a legal sequence with more vertices from H^v than D. This contradicts the maximum choice of D. Hence, no vertex of $\widehat{D_2}$ footprints a vertex in H^v. This means that D_2 is a legal sequence of $(G - v) \boxtimes H$. Thus, we have the desired inequality

$$\gamma_{gr}(G \boxtimes H) = |\widehat{D_1}| + |\widehat{D_2}| \leqslant \gamma_{gr}(H) + \gamma_{gr}((G - v) \boxtimes H). \qquad \square$$

A vertex u is called a *twin vetex* if there exists another vertex v so that $N[u] = N[v]$. The next three results were shown in [5].

Theorem 3. *[5] If G is a graph and $e \in E(G)$, then*

$$\gamma_{gr}(G) - 1 \leqslant \gamma_{gr}(G - e) \leqslant \gamma_{gr}(G) + 1.$$

Moreover, there exist graphs G such that all values of $\gamma_{gr}(G - e)$ between $\gamma_{gr}(G) - 1$ and $\gamma_{gr}(G) + 1$ are realized for different edges $e \in E(G)$.

Theorem 4. *[5] If G is a graph and $v \in V(G)$, then*

$$\gamma_{gr}(G) - 2 \leqslant \gamma_{gr}(G - v) \leqslant \gamma_{gr}(G).$$

Moreover, there exist graphs G such that all values of $\gamma_{gr}(G - v)$ between $\gamma_{gr}(G) - 2$ and $\gamma_{gr}(G)$ are realized for different vertices $v \in V(G)$.

Proposition 5. *[5] Let G be a graph and $u \in V(G)$.*

1. *If u is a simplicial vertex, then $\gamma_{gr}(G - u) \geqslant \gamma_{gr}(G) - 1$.*

2. *If u is a twin vertex, then $\gamma_{gr}(G - u) = \gamma_{gr}(G)$.*

3 Grundy domination of forests

Let C be a caterpillar with more than one vertex. Choose a path $P = \{v_1, \ldots, v_k\}$ of maximum length with leaves v_1, v_k and the rest of the vertices of P non-leaf vertices. For $i \in \{2, \ldots, k - 1\}$, let L_i indicate the set of leaf neighbors of v_i. We now define a labeling on C which we will show produces a legal sequence of C.

Algorithm 1 (Caterpillar Labeling). Starting with v_2, label the vertices in L_2 by consecutive integers starting from 1. If $k = 2$, then stop. Otherwise, label v_2 by the next consecutive integer. Continue labeling L_3 by the next consecutive integers. If $k = 3$, then stop. Otherwise, label v_3 by the next consecutive integer. Repeat these steps for all non-leaf vertices in order along P, ending at v_{k-1} but in the last step label all but one vertex of L_{k-1} and then label v_{k-1}.

Proposition 6. *For any caterpillar C, the Caterpillar Labeling produces a length $|C| - 1$ legal sequence of C.*

Proof. Notice that every leaf vertex chosen in the Caterpillar Labeling footprints itself. For $i \in \{2, \ldots, k-2\}$, v_i footprints v_{i+1}. Finally, v_{k-1} footprints the unlabeled leaf in L_{k-1}. $\square$

For any forest F, a *minimum caterpillar partition* of F, $\mathcal{P}$, is a partition of the non-isolate vertices of F into sets $C_1, \ldots, C_\ell$ so that

1. For any i, the induced subgraph on C_i is a caterpillar.

2. If for some i and j, there is an edge between a vertex of C_i and C_j in T, then that edge must be adjacent to a non-leaf vertex of the induced subgraph on C_i or C_j.

3. $\mathcal{P}$ is chosen to have the minimum number of caterpillars.

We may refer to a minimum caterpillar partition of a forest F as $\mathcal{P}(F)$. For any minimum caterpillar partition $\mathcal{P}$ of a forest F we call the edges of F between caterpillars of $\mathcal{P}$, *branch edges* and the endpoints of branch edges, *branch vertices*. We say that a caterpillar $C \in \mathcal{P}$ is a *leaf caterpillar* if it contains only one branch vertex. Two caterpillars in $\mathcal{P}$ are *neighbors* if there is a branch edge between them.

We now introduce the useful concept of the structure produced by contracting caterpillars in a minimum caterpillar partition of a forest. For any forest F with minimum caterpillar partition $\mathcal{P}$, we define the *canopy graph* as the graph $CP(F, \mathcal{P})$, or just CP when clear from context, with vertex set corresponding to the caterpillars of $\mathcal{P}$ contracted to vertices. If $\mathcal{P} = \{C_1, \ldots, C_\ell\}$, then we write $V(CP) = \{c_1, \ldots, c_\ell\}$. Two vertices c_i and c_j of CP are adjacent if the corresponding caterpillars C_i and C_j are neighbors.

Observation 7. *For any caterpillar partition $\mathcal{P}$ of a forest F, $CP(F, \mathcal{P})$ is a forest.*

Proof. Notice that any cycle in CP can be extended to a cycle in F, yielding a contradiction. $\square$

Suppose $\mathcal{P} = \{C_1, \ldots, C_\ell\}$ and that for any $i \in [\ell]$, C_i contains $b(i)$ branch vertices. Let L be a maximum path of C_i represented from left to right. If v is a vertex of L, we say that the *position* of v is 1 when v is the left-most vertex on L. If v is some other vertex of L, then the position of v is one plus the distance from the vertex of position 1. If v is a vertex not on L, then the position of v is the distance of v from the vertex of position 1. Let us define the branch vertices of C_i as $v_{i_1}, \ldots, v_{i_{b(i)}}$ where $i_j < i_{j'}$ when the position of v_{i_j} is smaller than the position of $v_{i_{j'}}$. Define the *rank* of a branch vertex v_{i_j} as j.

In other words, the rank of a branch vertex on C_i is one plus the number of branch vertices that preceded it (with respect to position) on C_i, when counting from left to right. We note here that there may be more than one branch vertex with the same position.

We say that an integer label is *available* if it has not been used previously on a vertex.

Lemma 8. *For any minimum caterpillar partition $\mathcal{P}$ of a forest F, if F contains a component which is not a caterpillar, then there exist distinct integers i and j between 1 and ℓ such that C_i and C_j each contain vertices which are adjacent in F and have rank 1.*

Proof. Choose a caterpillar, say C_1, contained in a component which is not a caterpillar, and call the vertex on C_1 of rank 1, u_1. If u_1 is adjacent to a vertex of rank 1 on another caterpillar, the proof is complete. Otherwise, say u_1 is adjacent to a vertex on a caterpillar of $\mathcal{P} - C_1$, C_2'. Call the vertex of rank 1 on C_2', u_2. If u_2 is adjacent to a vertex of rank 1 on another caterpillar, the proof is complete. Otherwise, say u_2 is adjacent to a vertex on a caterpillar of $\mathcal{P} - C_1 - C_2'$, C_3'. Continue this process and notice that it must result in a vertex of rank 1 adjacent to another vertex of rank 1, since there are finitely many caterpillars in $\mathcal{P}$. $\qquad\square$

Algorithm 2 (Forest Labeling). Let F be a given non-trivial forest. For the forest F, choose a minimum caterpillar partition $\mathcal{P} = \{C_1, \ldots, C_\ell\}$. For every $i \in [\ell]$, define the branch vertices of C_i as $v_{i_1}, \ldots, v_{i_{b(i)}}$. In the labeling that follows, we label vertices by consecutive integers, starting with 1. Perform the following labeling for every $i \in [\ell]$.

1. Set $j = 1$ and set $F_1 = F$.

2. For every i, perform the Caterpillar Labeling on C_i on all non-labeled vertices up to the vertices of the same position as the branch vertex of rank 1, except for the branch vertices on that position, if such a vertex exists. If no such vertex exists, perform the caterpillar labeling on the remaining non-labeled vertices of C_i.

3. For all branch vertices of rank 1 that do not have the largest position of all vertices on the caterpillar to which they belong, label the branch vertices of rank 1 which are adjacent to other branch vertices of rank 1 by consecutive integers, starting with the smallest available label. Furthermore, when labeling consecutive branch vertices such that one is a leaf, label the leaf branch vertex before the non-leaf branch vertex.

4. Label every caterpillar which does not contain an unlabeled branch vertex by the Caterpillar Labeling

5. Remove all caterpillars which have all but one vertex labeled and then remove all remaining labeled vertices and their incident edges.

6. Let $j = j + 1$ and call the remaining forest F_j, then repeat the labeling unless $F_j = \emptyset$.

7. Label all isolate vertices by the next consecutive available labels.

Definition 9. We say a forest F is *caterpillar-critical* if for any leaf edge e, $|\mathcal{P}(F - e)| < |\mathcal{P}(F)|$.

In other words, a forest is caterpillar-critical if the removal of any leaf edge produces a forest with fewer caterpillars in any minimum caterpillar partition.

Lemma 10. *Suppose F is a caterpillar-critical forest with minimum caterpillar partition $\mathcal{P}$. Then any leaf caterpillar $C \in \mathcal{P}$ is one of the following graphs*

1. *P_2 with one branch vertex*

2. *P_5 with the vertex of position 3 as the only branch vertex of C*

Proof. Suppose C is a leaf caterpillar of $\mathcal{P}$ which contains a path $L = \{v_1, \ldots, v_k\}$ of maximum length.

We first note that C can have no leaf edges of F other than $v_1 v_2$ and $v_{k-1} v_k$, since we can remove such an edge without reducing the number of caterpillars in a minimum caterpillar partition, contradicting the criticality of F. In other words, the removal of such a leaf edge does not allow for C to combine with another caterpillar to form a new caterpillar.

Next, we claim that only vertices of L may be branch vertices. Indeed, if v not on L is a branch vertex, then for any $k \geqslant 3$, v_1 and v_k are each of distance at least 2 to v. Now the removal of either edge $v_1 v_2$ or $v_{k-1} v_k$ does not reduce the number of caterpillars in a minimum caterpillar partition, contradicting the criticality of F. In other words, if we remove $v_1 v_2$ or $v_{k-1} v_k$ from C, then the resulting caterpillar cannot be combined with one of its neighbors to create a new caterpillar, which is a contradiction.

Together, these two observations imply that C is a path. Since C is a leaf caterpillar, it must have exactly one branch vertex. Notice that no branch vertex v of C can be of distance more than 2 from either v_1 (or v_k), since otherwise removing $v_1 v_2$ ($v_{k-1} v_k$) does not reduce the number of caterpillars in a minimum caterpillar partition. This means that $k \leqslant 5$. If k is 3 or 4, then there are two vertices of distance at least 2 to the neighboring caterpillar to C. In this case, the removal of a leaf edge from C does not allow us to combine it with its neighbor to form a caterpillar, since otherwise we could have combined C with its neighbor to form a caterpillar, contradicting the minimality of $\mathcal{P}$. Again, this contradicts the criticality of F. $\qquad\square$

Lemma 11. *If F is a caterpillar-critical forest so that every minimum caterpillar partition $\mathcal{P}$ does not contain a leaf caterpillar that is P_5, then every leaf caterpillar in $\mathcal{P}$ is a neighbor of a caterpillar that is P_5 with the central vertex as the only branch vertex.*

Proof. Let M be a path of maximum length in the canopy graph CP. If c_1 is an end vertex of M, then by Lemma 10, c_1 corresponds to a caterpillar of F which is either P_5 or P_2 and we must assume the latter. Suppose further that $M = \{c_1, c_2, \ldots, c_m\}$ with consecutive vertices as neighbors. Let C_2 be the caterpillar of F which corresponds to the vertex c_2 in CP. If C_2 does not contain a leaf in F, then the leaves in the induced subgraph of C_2 in F, $[C_2]_F$, must be branch vertices in F. Since all caterpillars have at least two leaves, there are at least two such branch vertices. Call one such branch vertex x and the other y, and call the caterpillar of $\mathcal{P}$ containing a branch vertex adjacent to x, C', and the one containing a branch vertex adjacent to y, C''. Notice that by maximality of M and the fact that c_1 is an end vertex of L, there exists at most one neighbor of c_2 in CP, which is not a leaf of CP, otherwise CP would have a longer path than M. This

means that either C' or C'' is a leaf caterpillar in $\mathcal{P}$. Without loss of generality suppose that it is C'. Notice that C' cannot be P_2 since in that case it could be combined with C_2 to produce a larger caterpillar, contradicting the minimality of $\mathcal{P}$. Thus, by Lemma 10, C' must be P_5, which leads to a contradiction to the preclusion of P_5 leaf caterpillars. Thus, C_2 must contain at least one leaf of F.

Next, let $L = \{v_1, \ldots, v_k\}$ be a path of maximum length of C_2 and notice as in the proof of Lemma 10, since F is caterpillar-critical, C_2 can have no leaf edges of F other than $v_1 v_2$ and $v_{k-1} v_k$. It is also easy to see that $k \geqslant 5$, since otherwise C_2 could be combined with C_1 to form a larger caterpillar, contradicting the minimality of the caterpillar partition $\mathcal{P}$. Furthermore, we may argue as in the proof of Lemma 10 that only vertices of L may be branch vertices since otherwise the removal of $v_1 v_2$ or $v_{k-1} v_k$ would not reduce the size of $\mathcal{P}$. This means that C_2 is a path.

By the maximality of M in CP, we note that there can be no vertex in CP of distance 2 to c_2 which is not on M. This means that any neighbors of c_2 other than c_1 or c_3 must be leaves, and by assumption, correspond to caterpillars of F which are P_2. We now note that v_3 and v_{k-3} in C_2 must be branch vertices since otherwise the removal of $v_1 v_2$ or $v_{k-1} v_k$ would not reduce the size of $\mathcal{P}$, contradicting the criticality of F.

If $k > 5$, then C_3 is either **not adjacent** to C_2 by a branch edge to v_3 or to v_{k-3}. Without loss of generality, we will assume that C_3 is not adjacent to C_2 by a branch edge to v_3. Let C be a caterpillar adjacent to C_2 by a branch edge to v_3. Then C is a leaf caterpillar $P_2 = \{u_1, u_2\}$ where u_2 is a branch vertex. Note now that we may produce a new minimum caterpillar partition $\mathcal{P}'$ from $\mathcal{P}$ by combining C_1 and C_2 into a leaf caterpillar P_5 with the remaining vertices of C_2 as its neighbor. That is, we form a leaf P_5 caterpillar from $\{v_1, v_2, v_3, u_2, u_1\}$, we define C_2' as the caterpillar $\{v_4, \ldots, v_k\}$, and we make no alterations to the rest of the caterpillars in $\mathcal{P}$. Since $|\mathcal{P}'| = |\mathcal{P}|$, this contradicts our assumptions about F. Thus, we conclude that $k = 5$ and notice that only v_3 can be a branch vertex, so the proof is complete. $\square$

Theorem 12. *For any forest F, $\gamma_{gr}(F) = |V(F)| - |\mathcal{P}|$.*

Proof. First, notice that by Lemma 8, every iteration of the Forest Labeling can be initiated. To prove that $\gamma_{gr}(T) \geqslant |V(T)| - |\mathcal{P}|$ we argue that the Forest Labeling produces a legal sequence for any forest F of length $|F| - |\mathcal{P}|$. First, notice that step (2) produces a legal sequence by the Caterpillar Labeling. In step (3), notice that a branch vertex which is a leaf footprints itself, and that any branch vertex which is not a leaf footprints a vertex of position one larger on the same caterpillar. Step (4) produces a legal sequence by the Caterpillar Labeling. Next, we note that while leaves may footprint themselves in the Caterpillar Labeling, non-leaf vertices footprint other vertices on the Caterpillar. This fact justifies the legality of iterating the Forest Labeling for increments of j, since if vertices that had been removed in the previous iteration footprinted vertices in a current iteration, the vertices in the current iteration do not only footprint themselves. Finally, notice that if u is a branch vertex with the largest position on the corresponding caterpillar C, then u is the only vertex of C which is not in the Grundy dominating sequence.

Next we show that

$$\gamma_{gr}(F) \leqslant |V(F)| - |\mathcal{P}|. \tag{4}$$

We induct on the size of F, $|E(F)|$. The statement is true for P_2 so we suppose that it holds for all forests with fewer than m edges. Let F be a forest of size m.

Claim 13. *For any forest F, if e is a leaf edge of F, then $\gamma_{gr}(F - e) \geqslant \gamma_{gr}(F)$.*

Proof. Let S be a Grundy dominating sequence of F and suppose that $e = xy$ with x a leaf vertex. Notice that if both x and y are elements of S, then x must be chosen in S previous to y, else it would not have a footprint in F. Thus, y must footprint some other vertex in F. This implies that we may choose S for $F - e$, where x footprints itself as an isolate.

If x is an element of S but y is not, then we may choose S for $F - e$ and, if y remains undominated, create the legal sequence S' by choosing y as a final vertex and adding it to S.

If y is an element of S but x is not, then let S'' be S with y replaced by x. If y remains undominated, create the legal sequence S''' by choosing y as a final vertex and adding it to S''.

Notice that in all of these instances, we produce a legal set of F of size at least $|S|$. This completes the proof of the claim. $\square$

Next, suppose that there exists a leaf edge e so that the size of a minimum caterpillar partition of F is the same as the size of a minimum caterpillar partition of $F - e$.

Let $\mathcal{P}'$ be a minimum caterpillar partition of $F - e$. By Claim 13, we write

$$\gamma_{gr}(F) \leqslant \gamma_{gr}(F - e) \leqslant |F - e| - |\mathcal{P}'| \leqslant |F| - |\mathcal{P}|.$$

We are now left to assume that F is caterpillar-critical. By Lemma 10, every leaf caterpillar of F is either P_2 or P_5. Let C be a leaf caterpillar.

Regardless of whether C is P_2 or P_5, by Lemma 10, it contains exactly one branch vertex which we will call v. Call the neighbor of v on the neighboring caterpillar, x and let $e = vx$.

We note that the size of any minimum caterpillar partition of F is the same as the size of any minimum caterpillar partition of $F - e$. By Theorem 3, we need only consider three possibilities, $\gamma_{gr}(F - e) = \gamma_{gr}(F) + 1$, $\gamma_{gr}(F - e) = \gamma_{gr}(F)$, and $\gamma_{gr}(F - e) = \gamma_{gr}(F) - 1$. Let S be a maximum legal sequence of F.

If there is a legal sequence S' of $F - e$, so that $|S| = |S'| - 1$, then by the induction hypothesis,

$$\gamma_{gr}(F) = |S| = |S'| - 1 \leqslant \gamma_{gr}(F - e) - 1 \leqslant |F - e| - |\mathcal{P}(F - e)| - 1 = |F| - |\mathcal{P}(F)| - 1$$

which which is a contradiction with the already proved direction of the theorem.

If there is a legal sequence S' of $F - e$, so that $|S| = |S'|$, then by the induction hypothesis,

$$\gamma_{gr}(F) = |S| = |S'| \leqslant \gamma_{gr}(F - e) \leqslant |F - e| - |\mathcal{P}(F - e)| = |F| - |\mathcal{P}(F)|$$

which again proves the theorem.

Thus, we restrict our attention to the case when $\gamma_{gr}(F) = \gamma_{gr}(F - e) + 1$. Since by the induction hypothesis together with the lower bound, $\gamma_{gr}(F - e) = |F - e| - |\mathcal{P}(F - e)| = |F| - |\mathcal{P}(F)|$, this implies that

$$\gamma_{gr}(F) = |F| - |\mathcal{P}(F)| + 1 \tag{5}$$

Notice that if either v is not in the footprint of x or x is not in the footprint v, then S is still a legal sequence of $F - e$. The same conclusion is attained if neither v nor x belong to $\widehat{S}$. Also, if $x \in \widehat{S}$ and the footprint of x contains v and some other vertex of F, then S is a legal sequence of $F - e$. The same is true if $v \in \widehat{S}$ and the footprint of v contains x and some other vertex of F.

Next we consider the two possibilies for C. Assume $C = P_5$ and that it is composed of the vertices v_1, v_2, v_3, v_4, v_5 with leaves v_1, v_5, where vertices with consecutive indices are adjacent. This means that the branch vertex $v = v_3$.

Suppose first that $\{x\}$ is the footprint of v_3. Applying the Pigeonhole Principle to equation (5), either C contains $|V(C)|$ vertices of $\widehat{S}$ or $F - C$ contains at least $|V(F)| - |V(C)| - |\mathcal{P}(F - C)| + 1 = |V(F)| - |V(C)| - |\mathcal{P}(F)| + 2$ vertices of $\widehat{S}$.

Suppose C contains $|V(C)|$ vertices of $\widehat{S}$. Notice that $v_1, v_5 \in \widehat{S}$ and that v_1 and v_5 must come earlier in S than v_2 and v_4. However, this leads to a contradiction since if v_2 came before v_4 in S, then v_4 cannot be in S, since it would not have a footprint, and if v_4 came before v_2, then v_2 could not be chosen in S. This means that C cannot contain $|V(C)|$ vertices.

Suppose $F - C$ contains at least $|V(F)| - |V(C)| - |\mathcal{P}(F)| + 2$ vertices of $\widehat{S}$. Since the footprint of v_3 is $\{x\}$, we note that $\widehat{S} \cap (F - C)$ is a legal set for $F - C$. Since $|\mathcal{P}(F - C)| = |\mathcal{P}(F)| - 1$, by the induction hypothesis we have that any legal set of $F - C$ must be of size at most $|V(F)| - |V(C)| - |\mathcal{P}(F - C)| = |V(F)| - |V(C)| - |\mathcal{P}(F)| + 1$, which contradicts the assumption on the number of vertices of $\widehat{S}$ in $F - C$.

Next, we suppose that $\{v_3\}$ is the footprint of x. Applying the Pigeonhole Principle, as previously, if we assume that C contains $|V(C)|$ vertices of $\widehat{S}$, then v_1 and v_5 must be in $\widehat{S}$ and appear in S before v_2 and v_4. Again, the appearance of v_2 in S precludes v_4 and the appearance of v_4 precludes v_2.

This means that we may assume that $F - C$ contains at least $|V(F)| - |V(C)| - |\mathcal{P}(F)| + 2$ vertices of $\widehat{S}$. First, let us assume that $F - C$ contains exactly $|V(F)| - |V(C)| - |\mathcal{P}(F)| + 2$ vertices of $\widehat{S}$. Notice that since $\{v_3\}$ is the footprint of x, if $v_2 \in \widehat{S}$, then v_2 footprints v_1 and so $v_1 \notin \widehat{S}$. Similarly, if $v_4 \in \widehat{S}$, then v_4 footprints v_5 and so $v_5 \notin \widehat{S}$. This means that some two vertices of C are not in $\widehat{S}$. However, now we have that

$$|S| \leqslant |V(F)| - |V(C)| - |\mathcal{P}(F)| + 2 + |V(C)| - 2 = |V(F)| - |\mathcal{P}(F)|.$$

Next, assume that $F-C$ contains more than $|V(F)|-|V(C)|-|\mathcal{P}(F)|+2$ vertices of $\widehat{S}$. We first consider the forest $F' = F - \{v_1, v_2, v_4, v_5\}$ and note that v_3 is a simplicial vertex in F'. Applying Proposition 5, we note that $\gamma_{gr}(F-C) \geqslant \gamma_{gr}(F') - 1$. Furthermore, note that $\gamma_{gr}(F') \geqslant |\widehat{S} \cap (F-C)|$ since $\widehat{S} \cap (F-C)$ is a legal set in F'. However, this leads to

$$\gamma_{gr}(F-C) \geqslant |\widehat{S} \cap (F-C)| - 1 > |V(F)| - |V(C)| - |\mathcal{P}(F)| + 1,$$

which contradicts the induction hypothesis that

$$\gamma_{gr}(F-C) \leqslant |V(F)| - |V(C)| - |\mathcal{P}(F-C)| \leqslant |V(F) - |V(C)| - |\mathcal{P}(F)| + 1.$$

Finally, we may assume that F contains no leaf caterpillars which are P_5 so that $C = P_2$ and that it is composed of the vertices u_1 and u_2 where u_2 is a branch vertex. By Lemma 11, C is a neighbor of a caterpillar $C_1 = P_5$ which is composed of the vertices v_1, v_2, v_3, v_4, v_5 with leaves v_1, v_5 and branch vertex v_3. If v_3 does not have a neighbor which is not on C or C_1, then we say we are in case $(*)$, which implies that that the component of the canopy graph of F contains only two vertices, caterpillars P_2 and P_5, and so P_5 is a leaf of the canopy graph, which has already been considered. Otherwise, if we are not in case $(*)$, let x be the neighbor of v_3 which is not on C or C_1 and let $e = v_3 x$.

As in the previous case, we suppose first that $\{x\}$ is the footprint of v_3. Applying the Pigeonhole Principle to equation (5), either $C \cup C_1$ contains at least $|V(C)| + |V(C_1)| - 1$ vertices of $\widehat{S}$ or $F - C - C_1$ contains at least $|V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F-C-C_1)| + 1 = |V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 3$ vertices of $\widehat{S}$.

Suppose $C \cup C_1$ contains $|V(C)| + |V(C_1)| - 1$ vertices of $\widehat{S}$. Notice that the footprinter of v_3 is either v_2, v_4, or u_2. If u_2 footprints v_3, then only one of v_1 or v_2 may belong to $\widehat{S}$, and similarly, only one of v_4 or v_5 may belong to $\widehat{S}$. This implies that $C \cup C_1$ may contain at most $|V(C)| + |V(C_1)| - 2$ vertices of $\widehat{S}$. If v_2 footprints v_3, then only one of v_4 or v_5 may belong to $\widehat{S}$, and only one of u_1 or u_2 may belong to $\widehat{S}$. Again, this implies that $C \cup C_1$ may contain at most $|V(C)| + |V(C_1)| - 2$ vertices of $\widehat{S}$. The assumption that v_4 footprints v_3 proceeds in the same way.

Suppose $F - C - C_1$ contains $|V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 3$ vertices of $\widehat{S}$. Since the footprint of v_3 is $\{x\}$, we note that $\widehat{S} \cap (F-C-C_1)$ is a legal set for $F - C - C_1$. Since $|\mathcal{P}(F-C-C_1)| = |\mathcal{P}(F)| - 2$, by the induction hypothesis we have that any legal set of $F - C - C_1$ must be of size at most $|V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F-C-C_1)| = |V(F)| - |V(C)| - |V(C_1)| + 2$, which contradicts the assumption on the number of vertices of $\widehat{S}$ in $F - C - C_1$.

Next, we suppose that $\{v_3\}$ is the footprint of x. Again, we apply the Pigeonhole Principle, assuming first that $C \cup C_1$ contains at least $|V(C)| + |V(C_1)| - 1$ vertices of $\widehat{S}$. Since $\{v_3\}$ is the footprint of x, and not of v_2 or v_4, if v_2 is in $\widehat{S}$, then v_2 footprints v_1, and v_1 is not in $\widehat{S}$. Likewise, if v_4 is in $\widehat{S}$, then v_4 footprints v_5, and v_5 is not in $\widehat{S}$. However, this implies that $C \cup C_1$ may contain at most $|V(C)| + |V(C_1)| - 2$ vertices of $\widehat{S}$. Note that this argument also holds if we are in case $(*)$.

This means that we are left with the case where $F - C - C_1$ contains at least $|V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 3$ vertices of $\widehat{S}$. First, assume that $F - C - C_1$ contains exactly

$|V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 3$ vertices of $\widehat{S}$. Since v_3 is in the footprint of x it is not footprinted by v_2, v_4, or u_2. This means that if v_2 is in $\widehat{S}$, then it footprints v_1, if v_4 is in $\widehat{S}$, then it footprints v_5, and if u_2 is in $\widehat{S}$, then it footprints u_1. In each of these cases, the footprinted vertex is not in $\widehat{S}$. However, now we have $|\widehat{S} \cap (C \cup C_1)| \leqslant |V(C)| + |V(C_1)| - 3$, which leads to

$$|S| \leqslant |V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 3 + |V(C)| + |V(C_1)| - 3 = |V(F)| - |\mathcal{P}(F)|.$$

Finally, we assume that $F - C - C_1$ contains more than $|V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 3$ vertices of $\widehat{S}$. We first consider forest $F' = F - \{v_1, v_2, v_4, v_5, u_1, u_2\}$ and note that v_3 is a simplicial vertex in F'. Applying Proposition 5, we note that $\gamma_{gr}(F - C - C_1) \geqslant \gamma_{gr}(F') - 1$. Furthermore, note that $\gamma_{gr}(F') \geqslant |\widehat{S} \cap (F - C - C_1)|$ since $\widehat{S} \cap (F - C - C_1)$ is a legal set in F'. However, this leads to

$$\gamma_{gr}(F - C - C_1) \geqslant |\widehat{S} \cap (F - C - C_1)| - 1 > |V(F) - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 2,$$

which contradicts the induction hypothesis that

$$\begin{aligned} \gamma_{gr}(F - C - C_1) &\leqslant |V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F - C - C_1)| \\ &\leqslant |V(F)| - |V(C)| - |V(C_1)| - |\mathcal{P}(F)| + 2. \qquad \square \end{aligned}$$

We now show two examples of the Forest Labeling. In the first example we assume all caterpillars are paths so as to focus on running the algorithm through several iterations. In the second example, we allow for arbitrary caterpillars to illustrate the labeling of branch vertices which are leaves in a caterpillar. The marked edges in each of the following forests are the edges which can be removed to produce the caterpillars in the decomposition. The rank of vertices is shown inside branch vertices. The enclosed vertices are those which are removed at the end of the iteration for the given value of j.
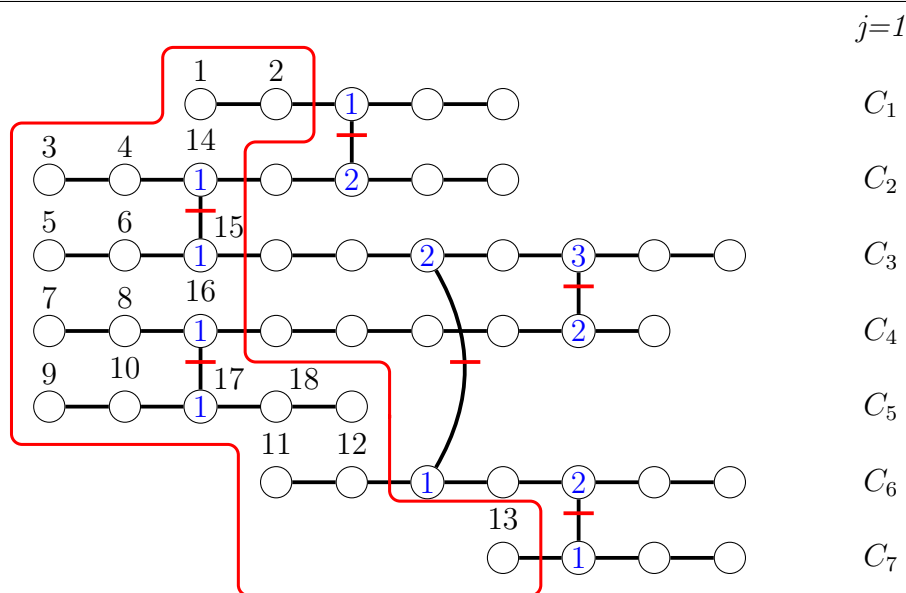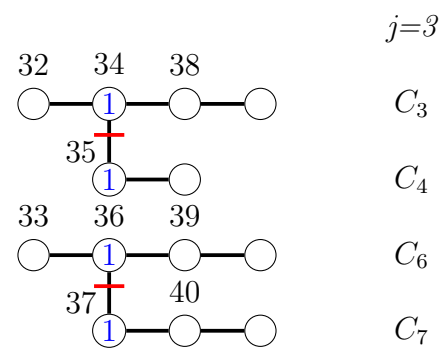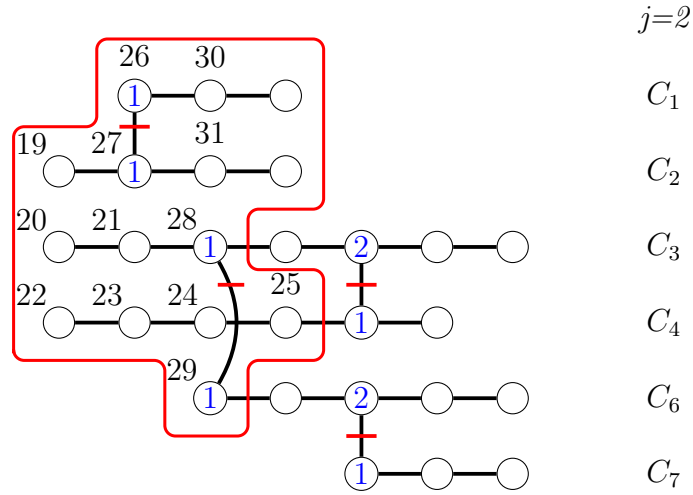
Forest Labeling Algorithm Example 1
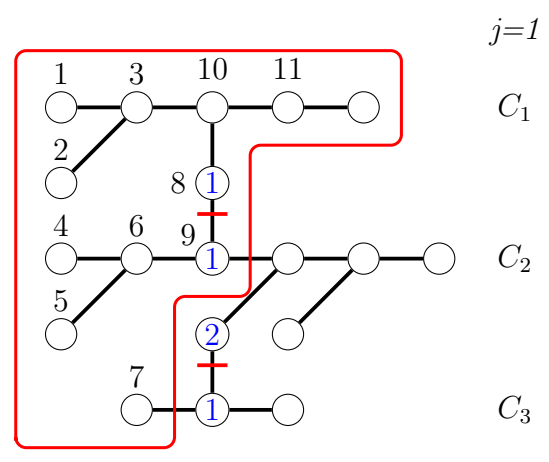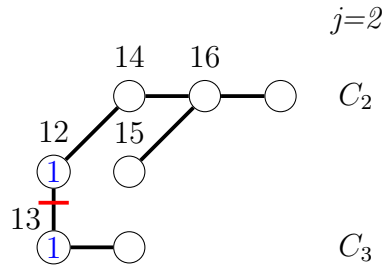
Forest Labeling Algorithm Example 2

4 Grundy domination of strong products of graphs

We first note that to solve Conjecture 1, we may assume that G and H are connected graphs, since strong products of graphs have components corresponding to the components of the factor graphs.

To prove our main result, we need to introduce some notation and an essential lemma.

For any graphs G and H, choose a maximum legal sequence D of $G \boxtimes H$. For some $v \in V(G)$ let $D_v = \widehat{D} \cap (\{v\} \times H)$ and F_v be the set of vertices of $\widehat{D}$ which have a footprint in H^v. More specifically, for any $u \in N[v]$, let $D_u(F_v)$ be the set of vertices in D_u which have a footprint in H^v.

Lemma 14. *For any graphs G and H, and any $v \in V(G)$, $|F_v| \leqslant \gamma_{gr}(H)$.*

Proof. Let D be a maximum legal sequence of $G \boxtimes H$. Say $N_G(v) = \{u_1, \ldots, u_k\}$. Notice that $|F_v| = |D_{u_1}(F_v) \cup \cdots \cup D_{u_k}(F_v) \cup D_v(F_v)|$. Choose any vertex $x \in F_v$ and without loss of generality, suppose $x = (u_1, h)$ for $h \in V(H)$. Notice that x has the same footprint in H^v as (v, h). This means that we may project $D_{u_1}(F_v) \cup \cdots \cup D_{u_k}(F_v) \cup D_v(F_v)$ onto H and produce a legal set of H. Such a legal set has size at most $\gamma_{gr}(H)$ and the lemma is proven. $\square$

Theorem 15. *For any tree G and graph H,*

$$\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H).$$

Proof. By inequality (3), it suffices to show that

$$\gamma_{gr}(G \boxtimes H) \leqslant \gamma_{gr}(G)\gamma_{gr}(H).$$

We induct on the order n of G. The statement is trivial for $n = 1$. Since every tree contains at least two leaf vertices, and leaves are simplicial, we may choose a leaf v and consider the tree $T = G - \{v\}$. Notice that if $\mathcal{P}$ is a minimum caterpillar partition for G

and $\mathcal{P}'$ is a minimum caterpillar partition for T, then $|\mathcal{P}| - 1 \leqslant |\mathcal{P}'| \leqslant |\mathcal{P}|$. If $|\mathcal{P}'| = |\mathcal{P}|$, we may apply Proposition 2, the inductive hypothesis, and Theorem 12 to show

$$\begin{aligned}
\gamma_{gr}(G \boxtimes H) &\leqslant \gamma_{gr}(H) + \gamma_{gr}\Big((G - \{v\}) \boxtimes H\Big) \\
&\leqslant \gamma_{gr}(H) + \gamma_{gr}(G - \{v\})\gamma_{gr}(H) \\
&\leqslant \gamma_{gr}(H) + \Big((n-1) - |\mathcal{P}'|\Big)\gamma_{gr}(H) \\
&\leqslant (n - |\mathcal{P}'|)\gamma_{gr}(H) = (n - |\mathcal{P}|)\gamma_{gr}(H) \\
&= \gamma_{gr}(G)\gamma_{gr}(H).
\end{aligned}$$

We may now suppose that for any leaf v, $|\mathcal{P}'| = |\mathcal{P}| - 1$, which means that G is caterpillar-critical. By Lemma 10, we may have chosen $\mathcal{P}$ as a minimum caterpillar partition of G so that $P_5 = v_1 - v_2 - v_3 - v_4 - v_5$ is a leaf caterpillar in $\mathcal{P}$ or so that $P_2 = u_1 - u_2$ is a leaf caterpillar. First suppose we chose such a minimum partition $\mathcal{P}$ with leaf caterpillar $C = P_5$. Let v_6 be the neighbor of v_3 which is not on C.

Let D' be a maximum legal sequence of $G \boxtimes H$ with the maximum number of entries from $\{v_1\} \times H$. For all such sequences D', let D be one with the maximum number of entries from $\{v_5\} \times H$.

Claim 16. *Every vertex $v \in D \cap (\{v_2\} \times H)$ must only footprint vertices of $\{v_3\} \times H$ and every vertex $v' \in D \cap (\{v_4\} \times H)$ must only footprint vertices of $\{v_3\} \times H$.*

Proof. If v footprints $x \in \{v_1\} \times H$, then say $v = (v_2, h)$, notice that (v_1, h) footprints x and may replace v, and we still retain a maximum legal sequence of $G \boxtimes H$. However, now the resulting sequence contains more vertices in $\{v_1\} \times H$ than D which contradicts the choice of D.

Next, suppose (v_2, h) footprints (v_2, h') for some h and h' in $V(H)$. Since $\widehat{D}$ is a legal set and dominates $\{v_1\} \times H$, some vertex y of $\widehat{D}$ must dominate (v_1, h'), and thus, y dominates (v_2, h'). This means that either y or (v_2, h) footprint both (v_2, h') and (v_1, h').

The symmetric argument applies to v'. $\qquad\square$

Using our notation from this section, we note that by Lemma 14,

$$|D_{v_2}(F_{v_3})| + |D_{v_3}(F_{v_3})| + |D_{v_4}(F_{v_3})| + |D_{v_6}(F_{v_3})| = |F_{v_3}| \leqslant \gamma_{gr}(H). \tag{6}$$

By Claim 16,

$$D_{v_1} = F_{v_1}, \; D_{v_5} = F_{v_5}, \; D_{v_2} = D_{v_2}(F_{v_3}), \; D_{v_4} = D_{v_4}(F_{v_3}). \tag{7}$$

We apply inequality (6) and identities (7) to show

$$\begin{aligned}
|\widehat{D}| \;=\; & |D_{v_1} \cup D_{v_2} \cup D_{v_3} \cup D_{v_4} \cup D_{v_5}| + |D \cap ((G - C) \boxtimes H)| \\
\leqslant \;& |D_{v_1}| + |D_{v_5}| + |D_{v_2}(F_{v_3})| + |D_{v_4}(F_{v_3})| + |D_{v_3}| + |D_{v_6}(F_{v_3})| \\
& + |D \cap ((G - C) \boxtimes H)| \\
=\; & |D_{v_1}| + |D_{v_5}| + \Big[|D_{v_2}(F_{v_3})| + |D_{v_4}(F_{v_3})| + |D_{v_6}(F_{v_3})| \\
& + |D_{v_3}(F_{v_3})|\Big] + |D_{v_3} - D_{v_3}(F_{v_3})| + |D \cap ((G - C) \boxtimes H)|
\end{aligned}$$

$$\leqslant \quad 4\gamma_{gr}(H) + \gamma_{gr}((G - C) \boxtimes H)$$

(which we bound by applying the inductive hypothesis to $G - C$)

$$\leqslant \quad 4\gamma_{gr}(H) + \gamma_{gr}((G - C))\gamma_{gr}(H)$$

(and applying Theorem 12)

$$= \quad 4\gamma_{gr}(H) + (|V(G)| - 5 - |\mathcal{P}| + 1)\gamma_{gr}(H)$$

$$= \quad (|V(G)| - |\mathcal{P}|)\gamma_{gr}(H) = \gamma_{gr}(G)\gamma_{gr}(H)$$

We must now consider the case where no P_5 leaf caterpillar exists and $\mathcal{P}$ is chosen as a minimum caterpillar partition of G so that $C = P_2 = u_1 - u_2$ is a leaf caterpillar. By Lemma 11, we may assume that $C_1 = P_5 = v_1 - v_2 - v_3 - v_4 - v_5$ is a neighbor of C with u_2 and v_3 as branch vertices. Let v_6 be the neighbor of v_3 which is not on C or C_1.

Let D'' be a maximum legal sequence of $G \boxtimes H$ with the maximum number of entries from $\{v_1\} \times H$. For all such sequences D'', let D' be one with the maximum number of entries in $\{v_5\} \times H$, and for all such sequences D', let D be the one with the maximum number of entries in $\{u_1\} \times H$.

Claim 17. *Every vertex $v \in D \cap (\{v_2\} \times H)$ must only footprint vertices of $\{v_3\} \times H$, $v' \in D \cap (\{v_4\} \times H)$ must only footprint vertices of $\{v_3\} \times H$, and $v'' \in D \cap (\{u_2\} \times H)$ must only footprint vertices of $\{v_3\} \times H$.*

The proof of Claim 17 is identical to that of Claim 16 with the argument for v repeated for u_2.

By Lemma 14,

$$|D_{v_2}(F_{v_3})| + |D_{v_3}(F_{v_3})| + |D_{v_4}(F_{v_3})| + |D_{u_2}(F_{v_3}| + |D_{v_6}(F_{v_3})| \tag{8}$$
$$= |F_{v_3}| \leqslant \gamma_{gr}(H).$$

By Claim 17

$$D_{v_1} = F_{v_1}, \; D_{v_5} = F_{v_5}, \; D_{v_2} = D_{v_2}(F_{v_3}), \; D_{v_4} = D_{v_4}(F_{v_3}), \; D_{u_2} = D_{u_2}(F_{v_3}). \tag{9}$$

We apply inequality (8) and identities (9) to show

$$|\widehat{D}| = |D_{u_1} \cup D_{u_2} \cup D_{v_1} \cup D_{v_2} \cup D_{v_3} \cup D_{v_4} \cup D_{v_5}| + |D \cap ((G - C - C_1) \boxtimes H)|$$

$$\leqslant |D_{u_1}| + |D_{v_1}| + |D_{v_5}| + |D_{u_2}(F_{v_3})| + |D_{v_2}(F_{v_3})| + |D_{v_4}(F_{v_3})| + |D_{v_3}|$$
$$\quad + |D_{v_6}(F_{v_3})| + |D \cap ((G - C - C_1) \boxtimes H)|$$

$$= |D_{u_1}| + |D_{v_1}| + |D_{v_5}| + \Big[|D_{u_2}(F_{v_3})| + |D_{v_2}(F_{v_3})| + |D_{v_4}(F_{v_3})| + |D_{v_6}(F_{v_3})|$$
$$\quad + |D_{v_3}(F_{v_3})| \Big] + |D_{v_3} - D_{v_3}(F_{v_3})| + |D \cap ((G - C - C_1) \boxtimes H)|$$

$$\leqslant 5\gamma_{gr}(H) + \gamma_{gr}((G - C - C_1) \boxtimes H)$$

(which we bound by applying the inductive hypothesis to $G - C - C_1$)

$$\leqslant 5\gamma_{gr}(H) + \gamma_{gr}((G - C - C_1))\gamma_{gr}(H)$$

(and applying Theorem 12)

$$= 5\gamma_{gr}(H) + (|V(G)| - 7 - |\mathcal{P}| + 2)\gamma_{gr}(H)$$

$$= (|V(G)| - |\mathcal{P}|)\gamma_{gr}(H) = \gamma_{gr}(G)\gamma_{gr}(H) \qquad \square$$

Corollary 18. *For any forest G and graph H,*

$$\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H).$$

5 Some results about Grundy dominating sets

Theorem 19. *For any connected graph G, there exists a spanning tree T of G so that $\gamma_{gr}(G) \leqslant \gamma_{gr}(T)$.*

Proof. We induct on the number of cycles in G. Notice that the theorem is trivially true if G is a tree. Assume that G contains m cycles and that the theorem holds for every graph with fewer cycles. Let C be a cycle on vertices $x_1, \ldots, x_k$ so that consecutive vertices in the list are adjacent as are x_1 and x_k. Call the edge of C from x_i to x_{i+1}, e_i, for $i \in [k-1]$, and the edge from x_k to x_1, e_k.

We will now show that for some $i \in [k]$, $\gamma_{gr}(G) \leqslant \gamma_{gr}(G - e_i)$, completing the proof.

Let S be a maximal legal set for G. Notice that for any edge $e = xy$ of C, $\gamma_{gr}(G-e) < \gamma_{gr}(G)$ only if either x footprints only y, or y footprints only x, in S. Indeed, if x does not footprint y and y does not footprint x, then S is a legal sequence in $G - e$ which means that $\gamma_{gr}(G) \leqslant \gamma_{gr}(G - e)$. Furthermore, if x footprints y as well as some other vertex z, then y cannot footprint x and notice that removing e allows the selection of every vertex of S for a legal sequence S' of $G - e$ to which we may add y if it is not dominated. Again, this means that $\gamma_{gr}(G) \leqslant \gamma_{gr}(G - e)$. The argument is identical if y footprints x and another vertex.

Note further that two vertices may not footprint the same vertex, by definition of a legal sequence. We now consider whether the removal of any e_i results in the statement of the claim. Suppose the claim is false. Without loss of generality, suppose x_1 footprints only x_2. This means that x_2 must footprint only x_3, and following this chain of reasoning, x_i must footprint only x_{i+1} for every $i \in [k-1]$. Finally, x_k must footprint only x_1. This leads to a contradiction, because some vertex from the cycle must be chosen first from the vertices of C for S, and thus cannot be footprinted by any other vertex in the cycle. $\square$

Originally, we hoped that Theorem 19, together with Theorem 15, would allow us to induct on the number of cycles in G to solve (or at least make progress on) the strong product conjecture. However, at this time, we do not know the right incantation to conjure that spell.

Theorem 20. *For any connected non-complete graph G with at least one edge, there exists a Grundy dominating set $\widehat{S}$ of G, so that the induced subgraph of $\widehat{S}$ in G contains no isolated vertices.*

Proof. We show that if v is an isolated vertex in the induced subgraph of $\widehat{S}$ in G, then we can remove v from S and add another vertex to S, possibly at a different point in the sequence, so that the resulting sequence is legal and the new vertex is a neighbor of another vertex of $\widehat{S}$.

To this end, we choose such a vertex v and note that no vertex of $N(v)$ belongs to $\widehat{S}$. We consider two cases.

Case 1. Suppose there exists a vertex $x \in \widehat{S}$ at distance 2 from v.

Notice that v must footprint itself. Let u be a common neighbor of v and x. We form the desired legal sequence S' by removing v from S and adding u as the last vertex in S', so that u footprints v.

Case 2. Suppose no vertex of $\widehat{S}$ is at distance 2 from v.

Since $\widehat{S}$ is a dominating set, any vertex u' which is at distance 2 from v, must be a neighbor of at least one vertex of $\widehat{S}$. Let w be the footprinter of u' and let u be a common neighbor of v and u'. Notice that v must footprint u since it is the only neighbor of u which is in $\widehat{S}$. Form the legal sequence S' by removing v from S and adding u' as the penultimates vertex and u as the ultimate vertex in S'. Notice that u' footprints u and u footprints v in S'. However, we now have $|\widehat{S'}| = |\widehat{S}| + 1$ which is impossible, so this case cannot occur. $\qquad\square$

Theorem 20 implies that every connected non-complete graph has a Grundy dominating set which is a total dominating set.

Acknowledgements

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