A note on complex-4-colorability of signed planar graphs

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Abstract

A pair (G, σ) is called a *signed graph* if $\sigma : E(G) \longrightarrow \{1, -1\}$ is a mapping which assigns to each edge e of G a sign $\sigma(e) \in \{1, -1\}$. If (G, σ) is a signed graph, then a *complex-4-coloring* of (G, σ) is a mapping $f : V(G) \longrightarrow \{1, -1, i, -i\}$ with $i = \sqrt{-1}$ such that $f(u)f(v) \neq \sigma(e)$ for every edge $e = uv$ of G.

We prove that there are signed planar graphs that are not complex-4-colorable. This result completes investigations of Jin, Wong and Zhu as well as Jiang and Zhu on 4-colorings of generalized signed planar graphs disproving a conjecture of the latter authors.

Mathematics Subject Classifications: 05C10, 05C15, 05C22

1 Introduction and results

Let $G = (V, E)$ be a simple graph. A mapping $f : V(G) \longrightarrow [k] = \{1, 2, \ldots, k\}$ (k is a positive integer) is called a *k-coloring* of G if $f(u) \neq f(v)$ for every edge $e = uv$ of G. If such a coloring exists, then G is called *k-colorable*.

A pair (G, σ) is called a *signed graph* if $\sigma : E(G) \longrightarrow \{1, -1\}$ is a mapping which assigns to each edge e of G a sign $\sigma(e) \in \{1, -1\}$. The mapping σ is called a *signature* of G, and an edge e is positive or negative if $\sigma(e) = 1$ or $\sigma(e) = -1$, respectively.

Colorings of signed graphs have been introduced in the early 1980s by Zaslavsky [4] and were later the topic of various papers (see [1] for recent informations).

Let the set N_k be defined by $N_k = \{\pm 1, \pm 2, \ldots, \pm \frac{k}{2}\}$ if k is even or $N_k = \{0, \pm 1, \pm 2, \ldots, \pm \frac{k-1}{2}\}$ if k is odd, respectively.

Definition 1. If (G, σ) is a signed graph and k a positive integer, then a *k-coloring of* (G, σ) is a mapping $f : V(G) \longrightarrow N_k$ such that $f(u) \neq \sigma(e)f(v)$ for every edge $e = uv$ of G. A graph G is called *signed k-colorable* if (G, σ) has a k-coloring for every signature σ of G.

In [1] the concept of k-colorings of signed graphs was modified for the case $k = 4$ in the following way.

Definition 2. If (G, σ) is a signed graph, then a *complex-4-coloring* of (G, σ) is a mapping $f : V(G) \longrightarrow \{1, -1, i, -i\}$ with $i = \sqrt{-1}$ such that $f(u)f(v) \neq \sigma(e)$ for every edge $e = uv$ of G.

It is conjectured in [1] that every signed planar graph is complex-4-colorable. We will disprove the conjecture in this note (see Theorem 6). Moreover, our result completes the investigations of Jiang and Zhu [1] on S-k-colorability in that paper.

The concept of S-k-colorability generalizes colorings of signed graphs and was introduced in [1] and [2]. Discussions on the relationship to other coloring concepts and a list of corresponding references can be found in these references.

The *symmetric group* S_k consists of all permutations π of $1, 2, \ldots, k$. The identity permutation is written *id*. The permutation which interchanges exactly the elements i and j is written by (ij) and is called a cycle of length 2. All permutations can be written as product of cycles. A set S of permutations is called *inverse closed* if for each $\pi \in S$ also $\pi^{-1} \in S$ where π^{-1} is the inverse permutation of π.

Definition 3. If $G = (V, E)$ is a graph and $S \subseteq S_k$ an inverse closed set of permutations, then an S-signature of G is a pair (D, σ) where D is an orientation of G and $\sigma : E(D) \longrightarrow S$ a mapping which assigns to each arc $e = (u, v)$ a permutation $\sigma(e) \in S$. A k-coloring of an S-signature of G is a mapping $f : V(D) \longrightarrow [k]$ such that for each arc $e = (u, v)$ of D it holds $\sigma(e)(f(u)) \neq f(v)$. A graph G is called S-k-colorable if every S-signature (D, σ) of G has a k-coloring.

Moreover, an *involution* is a permutation π with $\pi^{-1} = \pi$. If all the permutations $\pi \in S$ are involutions, then the orientation of the edges is irrelevant. In our investigations we deal with involutions only. Therefore we give a further adapted definition.

Definition 4. If $G = (V, E)$ is a graph and $S \subseteq S_k$ a set of involutions, then an S-signature of G is a mapping $\sigma : E(G) \longrightarrow S$ which assigns to each edge $e = uv$ an involution $\sigma(e) \in S$. A k-coloring of an S-signature of G is a mapping $f : V(G) \longrightarrow [k]$ such that for each edge $e = uv$ of G it holds $\sigma(e)(f(u)) \neq f(v)$. A graph G is called S-k-colorable if every S-signature σ of G has a k-coloring.

Clearly if $S = \{id\}$, then S-k-colorability of G is equivalent to k-colorability of G. If $S = \{id, (12)(34) \ldots ((2\lfloor k/2 \rfloor - 1)(2\lfloor k/2 \rfloor))\}$, then S-k-colorability is equivalent to signed k-colorability. Moreover, if $S = \{(12), (34)\}$ then S-4-colorability is equivalent to complex-4-colorability as shown in the proof of Theorem 6.

In [1] the question is posed for which subsets S of S_4 every planar graph is S-4-colorable. We will prove in Theorem 1 that for $S = \{(12), (34)\}$ there exist planar graphs which are not S-4-colorable.

Theorem 5. *If $S = \{(12), (34)\}$, then there are planar graphs that are not S-4-colorable.*

Theorem 5 will be proved in Section 2.

Theorem 6. *There are signed planar graphs that are not complex-4-colorable.*

Proof. Let G be a planar graph, $S = \{(12), (34)\}$, and σ an S-signature of G such that G is not S-4-colorable. Such a configuration exists according to Theorem 5. Define a mapping $\sigma^* : E(G) \longrightarrow \{1, -1\}$ by

$$\sigma^*(e) = \begin{cases} -1 & if \quad \sigma(e) = (12), \\ 1 & if \quad \sigma(e) = (34). \end{cases}$$

Assume to the contrary that G has a complex-4-coloring $f^* : V(G) \longrightarrow \{1, -1, i, -i\}$ and define

$$f(v) = \begin{cases} 1 & if \quad f^*(v) = 1, \\ 2 & if \quad f^*(v) = -1, \\ 3 & if \quad f^*(v) = i, \\ 4 & if \quad f^*(v) = -i. \end{cases}$$

It is easy to check that f is a proper S-4-coloring of G contradicting the assumption. □

In [1] a subset S of S_4 is called *good* if every planar graph is S-4-colorable and the question was posed which subsets of S_4 are good. The authors mentioned that it is sufficient to restrict oneselves to subsets S of S_4 where for each $a \in \{1, 2, 3, 4\}$ there is a permutation $\pi \in S$ with $\pi(a) = a$. Such subsets are called *normal* subsets. If $S \in S_4$ is not normal then it is trivially good since there is an $a \in \{1, 2, 3, 4\}$ with $\pi(a) \neq a$ for all $\pi \in S$ and consequently the mapping $f(v) = a$ for all $v \in V(G)$ is an S-4-coloring for arbitrary G and arbitrary S-signature σ of G.

It is proved in [1] that all normal subsets $S \subseteq S_4$ not containing $\{id\}$, $S \neq S' = \{(12), (34)\}$ and $S \neq S'' = \{(12), (34), (12)(34)\}$ are not good. Moreover, it is proved in [2] that if S is a good subset of S_4 containing id then $S = \{id\}$. Note that $S = \{id\}$ is good because of the Four Color Theorem.

Summarizing these results the authors of [1] noticed that the only remaining open cases are whether S' and S'' are good subsets or not. Applying Theorem 5 and the results of [1] and [2] we obtain the following statement which is in some sense a strengthening of the Four Color Theorem.

Theorem 7. *$S = \{id\}$ is the only normal good subset of S_4.*

2 Proof of the main result

At first we will proof a claim and then our main result of Theorem 5.

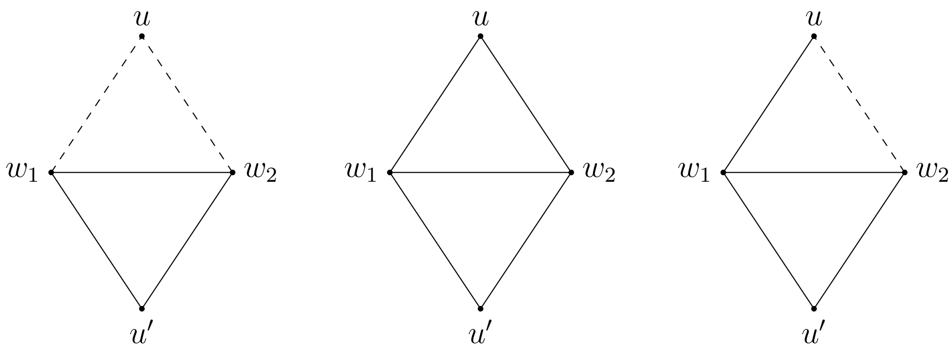
Claim 8. *If $S = \{(12), (34)\}$, then there is a planar graph G, a family of triangles $F_1, F_2, \ldots, F_q$ of G where each F_i is associated with a coloring ϕ_i such that the following holds:*

(1) ϕ_i colors the vertices of F_i with three distinct colors,

(2) for any S-4-coloring c of G, there is an index i such that c agrees with ϕ_i on F_i.

Proof. First we construct a graph G. Then we describe the assignment of permutations to its edges to identify the triangles F_i and its corresponding colorings depending on the possible S-4-colorings c of G.

Let T be a graph consisting of two triangles $D = w_1w_2u$ and $D' = w_1w_2u'$ sharing the edge w_1w_2. If we want to color the vertices u and u' with colors from $\{1, 2, 3, 4\}$, then we have 16 possibilities: $(1, 1), (1, 2), \ldots, (4, 4)$. For each of these cases $(c(u), c(u')) = (a, b)$ ($a = b$ is possible) we take one copy of T, denoted by $T_{a,b}$, and construct G by identifying the 16 vertices u to one vertex, also denoted by u, and the 16 vertices u' to one vertex, also denoted by u'. Therefore, $|V(G)| = 34$ and $|E(G)| = 80$.



(a) $c(u) = 1, c(u') = 1$, (b) $c(u) = 1, c(u') = 2$, (c) $c(u) = 1, c(u') = 3$

Figure 1: Copies of $T_{a,b}$, a solid edge e has assigned permutation $\sigma(e) = (12)$, a dashed edge e has $\sigma(e) = (34)$.

For each of the copies $T_{a,b}$ we assign permutations to the edges depending from a and b. We consider the possible colorings of D and D' which can be created by these permutations provided that $c(u) = a$ and $c(u') = b$. Based on these considerations we specify colorings of the vertices of D and D' given below. Note that each of these colorings uses three distinct colors and either the coloring of D or the coloring of D' must occur if $c(u) = a$ and $c(u') = b$.

Because of symmetry it is sufficient to consider the cases $(a, b) \in \{(1, 1), (1, 2), (1, 3)\}$ which are depicted in Figure 1.

At first assume that $c(u) = 1$, $c(u') = 1$ and assign $\sigma(e)$ as indicated in Figure 1(a). Then either $D = w_1w_2u$ with $c(w_1) = 3$, $c(w_2) = 4$, and $c(u) = 1$ or $D' = w_1w_2u'$ with $c(w_1) = 4$, $c(w_2) = 3$, and $c(u') = 1$.

Next assume that $c(u) = 1$, $c(u') = 2$ and permutations assigned to the edges according to Figure 1(b). Analogously, we have either $D = w_1w_2u$ with $c(w_1) = 3$, $c(w_2) = 4$, and $c(u) = 1$ or $D' = w_1w_2u'$ with $c(w_1) = 4$, $c(w_2) = 3$, and $c(u') = 2$.

Finally, we assume that $c(u) = 1$, $c(u') = 3$ and $\sigma(e)$ as shown in Figure 1(c). Then either $D = w_2uw_1$ with $c(w_2) = 2$, $c(u) = 1$, and $c(w_1) = 4$ or $D' = u'w_2w_1$ with $c(u') = 3$, $c(w_2) = 4$, and $c(w_1) = 1$.

The family $F_1, F_2, \ldots, F_q$ consists of all triangles D and D' where the colorings ϕ_i are given by the specified colorings indicated above. Thus Claim 8 is proved. □

Proof of Theorem 5. In every triangle $F_i = xyz$ of G we insert a structure $H_{\alpha,\beta,\gamma}$ as described below which depends on the coloring ϕ_i as indicated in the above proof. Let G^* be the resulting graph.

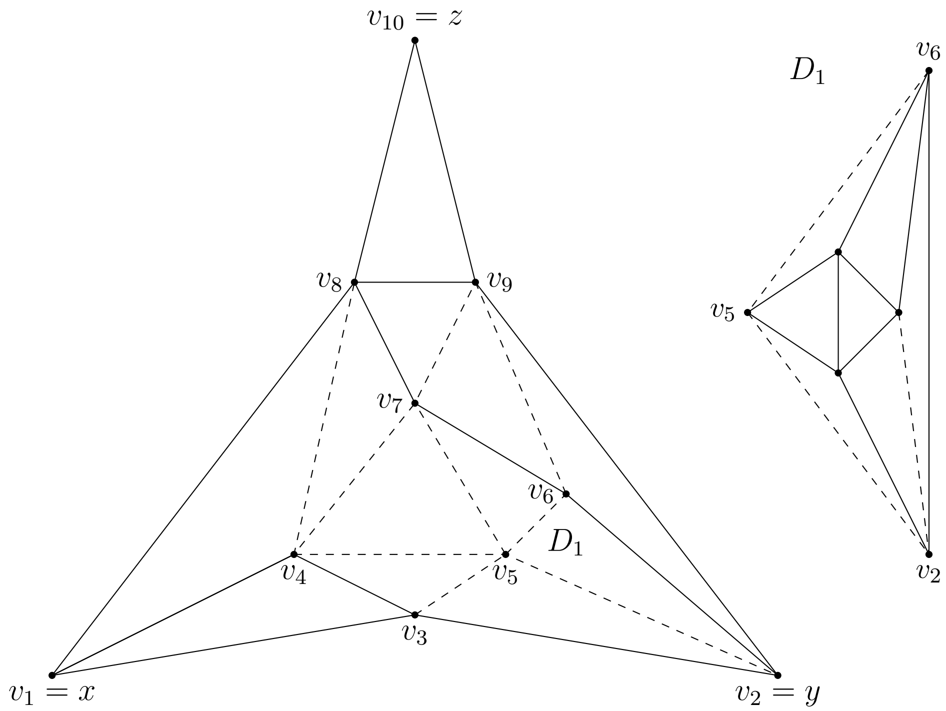


Figure 2: Subgraph $H_{1,2,3}$: $c(v_1) = 1, c(v_2) = 2, c(v_{10}) = 3$, a solid edge e has assigned permutation $\sigma(e) = (12)$, a dashed edge e has $\sigma(e) = (34)$.

Consider the triangle $F_i = xyz$ with associated coloring ϕ_i where $c(x) = \alpha, c(y) = \beta$, and $c(z) = \gamma$. We will show that the precoloring of x, y, z is not extendable to the inserted $H_{\alpha,\beta,\gamma}$. Note that by suitable assignment of the vertices of F_i we may assume that $\{c(x), c(y)\} = \{1, 2\}$ and $c(z) \in \{3, 4\}$ or $\{c(x), c(y)\} = \{3, 4\}$ and $c(z) \in \{1, 2\}$. Thus, because of symmetry it is sufficient to consider

$$c(x) = 1, c(y) = 2, c(z) = 3.$$

Consider the subgraph $H_{1,2,3}$ as shown in Figure 2 with the specified permutations on the edges and assume that $v_1 = x$, $v_2 = y$, and $v_{10} = z$ with $c(v_1) = 1$, $c(v_2) = 2$, and $c(v_{10}) = 3$.

According to the assigned permutations (12) to the edges v_1v_4 and v_2v_5 color 2 is not possible for the vertices v_4 and v_5. If both v_4 and v_5 would not be colored by color 1 then v_4 and v_5 must be colored by the same color 3 or 4 because of $\sigma(v_4v_5) = (34)$. Then v_3 would not be colorable. It follows that one of the vertices v_4 and v_5 must be colored by 1. We consider two cases.

Case 1: $c(v_4) = 1$

Because of $c(v_5) \in \{3, 4\}$ and $\{c(v_6), c(v_7)\} \subset \{2, 3, 4\}$ it follows that either v_6 or v_7 is colored by color 2. Hence $c(v_9) = 4$ and v_8 is not colorable.

Case 2: $c(v_5) = 1$

Clearly, we have $c(v_9) \in \{2, 4\}$

- $c(v_9) = 2$ It follows that $c(v_8) = 4$ and $c(v_7) = 3$ which implies that v_4 is not colorable.

- $c(v_9) = 4$ It follows that $c(v_8) = 1$, $c(v_7) = 4$, and $c(v_6) = 2$.

  Now all three vertices of the triangle inside D_1 must be colored by 3 or 4 which is not possible.

Therefore, the precoloring of v_1, v_2, and v_{10} cannot be extended to $H_{1,2,3}$.

By Claim 8, one of the specified triangles of G must be colored by the corresponding colors α, β, γ. However, the coloring is not extendable to the inserted subgraph $H_{\alpha,\beta,\gamma}$. Thus G^* is not S-4-colorable and the proof of Theorem 5 is complete. □

Added in proof. Our main result is also contained in [3] which was presubmitted during the reviewing process of this paper.

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