Supplementary material for

The Graham–Knuth–Patashnik recurrence: Symmetries and continued fractions

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1 Introduction

We consider the Graham–Knuth–Patashnik recurrence, namely

$$T(n,k) \;=\; (\alpha n+\beta k+\gamma)\, T(n-1,k) \;+\; (\alpha' n+\beta' k+\gamma')\, T(n-1,k-1) \tag{1.1}$$

for $n \geqslant 1$, with initial condition $T(0,k) = \delta_{k0}$. For each choice of the parameters $\boldsymbol{\mu} = (\alpha,\beta,\gamma,\alpha',\beta',\gamma')$, we obtain a unique solution $T(n,k) = T(n,k;\boldsymbol{\mu})$, forming a triangular array $\boldsymbol{T}(\boldsymbol{\mu}) = \big(T(n,k;\boldsymbol{\mu})\big)_{0\leqslant k\leqslant n}$. Given a triangular array $\boldsymbol{T} = \big(T(n,k)\big)_{0\leqslant k\leqslant n}$, we define its row-generating polynomials

$$P_n(x) \;\stackrel{\rm def}{=}\; \sum_{k=0}^{n} T(n,k)\, x^k \tag{1.2}$$

and its ordinary generating function (ogf)

$$f(x,t) \;\stackrel{\rm def}{=}\; \sum_{n\geqslant 0} P_n(x)\, t^n\,. \tag{1.3}$$

We are looking for all the submanifolds in $\boldsymbol{\mu} \in \mathbb{C}^6$ such that the corresponding ogf (1.3) has an S-type continued fraction

$$f(x,t;\boldsymbol{\mu}) \;=\; \sum_{n\geqslant 0} P_n(x;\boldsymbol{\mu})\, t^n \;=\; \cfrac{c_0}{1-\cfrac{c_1 t}{1-\cfrac{c_2 t}{1-\cdots}}}\,, \tag{1.4}$$

where the coefficients c_i are *polynomials* (and not just rational functions) in x that moreover are not identically zero. The final result of this investigation is stated as Theorem 3.1 of the main paper. The first step in the proof of Theorem 3.1 is to make a computer-assisted search using MATHEMATICA to find a finite set of viable candidates for such an S-fraction. In this Supplementary Material we will give all the details of this search, which gives rise to the decision tree shown in Figure 1.

Each node in this decision tree will be labeled by a finite-length vector of the type $(\mathrm{b}_0\mathrm{x}_0,\, \mathrm{b}_1\mathrm{x}_1,\, \mathrm{b}_2\mathrm{x}_2,\, \ldots,\, \mathrm{b}_n\mathrm{x}_n)$, where in each entry $\mathrm{b}_i \in \{0,1\}$ and $\mathrm{x}_i \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots\}$. Here b_i denotes the degree of the denominator polynomial $R(x)$, as explained below, while the letter x_i labels the various cases that can occur at that stage. When there is only a single case, we can drop the letter x_i and write simply $\mathrm{b}_i\mathrm{x}_i = \mathrm{b}_i\mathrm{a} = \mathrm{b}_i$.

The reason behind this representation is that, as explained in Section 3.1 of the accompanying article, at the k-th level of this decision tree we always find that the corresponding coefficient c_k can be written as

$$c_k(x) \;=\; \frac{Q(x)}{R(x)} \tag{1.5}$$

where the denominator $R(x)$ is a polynomial of degree $\leqslant 1$ in x and the numerator $Q(x)$ is a polynomial of degree $\leqslant 2$ in x, and the coefficients in these polynomials are polynomial expressions in $\boldsymbol{\mu}$. Furthermore, the decisions taken at earlier stages will ensure that the denominator polynomial $R(x)$ is not identically zero; indeed, in all cases $R(x)$ turns out to be a nonzero multiple of the preceding coefficient $c_{k-1}(x)$. So $c_k(x) = Q(x)/R(x)$ is a polynomial in x if and only if either

(0) the denominator polynomial has degree 0 in x: that is, its leading coefficient $[x^1]R(x)$ is zero and its constant term $[x^0]R(x)$ is nonzero; or

(1) the denominator polynomial has degree 1 in x — that is, its leading coefficient $[x^1]R(x)$ is nonzero — and the remainder $\mathrm{rem}(Q(x), R(x))$ is zero.

Therefore, $\mathrm{b}_k = 0$ (resp. 1) if we are in the first (resp. second) case, and the letter x_k labels the different sub-cases that we encounter in each case (if more than one). In all these cases, it turns out that $c_k(x)$ is a polynomial in x of degree at most 1, i.e. it is of the form $c_k(x) = A + Bx$.

A branch in this decision tree will be terminated if

- We find a viable candidate family for a non-terminating S-fraction. We consider that a submanifold $\mathcal{M}$ of $\boldsymbol{\mu} \in \mathbb{C}^6$ gives rise to a viable candidate for a non-terminating

S-fraction if c_i is a polynomial in x that is generically nonvanishing on $\mathcal{M}$, for $0 \leqslant i \leqslant 10$. These are the red nodes in Figure 1.

In all other cases — i.e. those for which the most recent c_k is a polynomial in x that is generically nonvanishing on $\mathcal{M}$, but we have not yet obtained a viable candidate family — we proceed as follows. With $c_k(x) = A + Bx$, the manifold $\mathcal{M}$ divides into two parts:

(a) $A = B = 0$. Then $c_k(x)$ is the zero polynomial, and we have a terminating S-fraction. Although these cases are irrelevant for the proof of Theorem 3.1 (which concerns only non-terminating S-fractions), they are nevertheless of some interest in their own right; we will consider them in Section 4 below.

(b) $A \neq 0$ or $B \neq 0$. This disjunction of inequations corresponds to the statement that $c_k(x)$ is not the zero polynomial. Going forward, we will impose this disjunction of inequations on all children of the given node; that will ensure that their denominator polynomials $R(x)$ are not identically zero.

A branch in this decision tree will also be terminated if

- The given node has no children: that is, the set of solutions $\boldsymbol{\mu}$ is empty for both $\mathrm{b}_{k+1} = 0$ and $\mathrm{b}_{k+1} = 1$. These are the gray leaf nodes in Figure 1.

We will also encounter cases that are either (i) included in a previously found case, or (ii) have a nonempty intersection with a previously found case but without being included in it. In what follows, we will mention these cases explicitly where it seems relevant; in situation (i) we will discard them as redundant, while in situation (ii) we will sometimes (but not always) impose additional inequations to remove the overlap with the previously found case.

The full decision tree for this proof is shown in Figure 1.

2 Computer-assisted search

In this section we will describe in great detail the computer-assisted search using MATHEMATICA that led to the 10 candidate families presented in Theorem 3.1 of the main paper. Let us stress at the outset that our search now "computer-assisted" *only* in the sense that it used MATHEMATICA to perform elementary algebraic operations such as manipulation of polynomials and rational functions, series expansions, and `PolynomialRemainder`. We did *not* need to use any more advanced algebraic functions such as `Solve` or `Reduce`. If one assumes the correctness of MATHEMATICA's algebraic manipulations, the rest of the proof is easily human-verifiable (though tedious) and is explained in detail in Sections 2.1–2.8 herein.

2.1 Step 0: Coefficient c_0

This coefficient is always $c_0 = 1$, so it is a polynomial of degree 0 for any choice of $\boldsymbol{\mu} \in \mathbb{C}^6$. The denominator is thus $R(x) = 1$, a polynomial of degree 0. This is the root vertex of our decision tree (i.e., the top black node in Figure 1).

Case (0): $\boldsymbol{\mu} = (\alpha, \beta, \gamma, \alpha', \beta', \gamma')$.

2.2 Step 1: Coefficient c_1

The coefficient c_1 is also a polynomial in x,

$$c_1 \;=\; (\alpha + \gamma) + (\alpha' + \beta' + \gamma')\, x \;, \tag{2.1}$$

so the denominator is again $R(x) = 1$. This is equal to the value of c_0 for the Case (0). We thus have

Case (0, 0): $\boldsymbol{\mu} = (\alpha, \beta, \gamma, \alpha', \beta', \gamma')$. In this case, c_1 is the polynomial (2.1). Going forward, we impose the disjunction of inequations

$$\alpha + \gamma \;\neq\; 0 \quad \text{or} \quad \alpha' + \beta' + \gamma' \;\neq\; 0 \tag{2.2}$$

in order to ensure that $c_1 \neq 0$.

2.3 Step 2: Coefficient c_2

Imposing now the conditions (2.2) of the Case $(0, 0)$ — namely, $\alpha + \gamma \neq 0$ *or* $\alpha' + \beta' + \gamma' \neq 0$ — the coefficient c_2 is a rational function of x: $c_2 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) \;&=\; \alpha(\alpha + \gamma) \\ &\qquad + (2\alpha\alpha' + \alpha\beta' + \alpha\gamma' + \beta\beta' + \beta\gamma' + \beta\alpha' + \gamma\alpha')\, x \\ &\qquad + (\alpha' + \beta')(\alpha' + \beta' + \gamma')\, x^2 \end{aligned} \tag{2.3a}$$

$$R(x) \;=\; \alpha + \gamma \;+\; (\alpha' + \beta' + \gamma')\, x \tag{2.3b}$$

Note that the polynomial $R(x)$ is equal to the value (2.1) of c_1 for the Case $(0, 0)$.

The conditions (2.2) imply that the polynomial $R(x)$ is *not* identically zero. It follows that there are two cases: if $[x^1]\, R(x) = \alpha' + \beta' + \gamma' \neq 0$, then $\deg R = 1$ and

$$\operatorname{rem}(Q(x), R(x)) \;=\; \frac{(\alpha + \gamma)[\beta'(\alpha + \gamma) - \beta(\alpha' + \beta' + \gamma')]}{\alpha' + \beta' + \gamma'} \;; \tag{2.4}$$

while if $[x^1]\, R(x) = \alpha' + \beta' + \gamma' = 0$, then $\deg R = 0$ and $\operatorname{rem}(Q(x), R(x))$ is identically zero. [Henceforth, when we state a formula for $\operatorname{rem}(Q(x), R(x))$, it will be tacitly understood that it applies when and only when $[x^1]\, R(x) \neq 0$.]

We thus have:

- **$\deg R = 0$:** $[x^1]\, R(x) = \alpha' + \beta' + \gamma' = 0$ and $[x^0]\, R(x) = \alpha + \gamma \neq 0$. This gives:

  Case (0, 0, 0): $\boldsymbol{\mu} = (\alpha, \beta, \gamma, \alpha', \beta', -\alpha' - \beta')$ with $\alpha + \gamma \neq 0$. In this case, $c_2 = \alpha + \alpha' x$. Going forward, we impose the disjunction of inequations

  $$\alpha \;\neq\; 0 \quad \text{or} \quad \alpha' \;\neq\; 0 \tag{2.5}$$

  in order to ensure that $c_2 \neq 0$.

- **$\deg R = 1$:** $[x^1]\,R(x) = \alpha' + \beta' + \gamma' \neq 0$. We now need to impose the condition $\mathrm{rem}(Q(x), R(x)) = 0$. Since the numerator of (2.4) is explicitly factorized, each new case will correspond to one of its two factors being zero. Therefore we find two possibilities: either $\alpha + \gamma = 0$ or $\beta'(\alpha + \gamma) - \beta(\alpha' + \beta' + \gamma') = 0$. In the second case we can assume without loss of generality that we are not also in the first case, i.e. we can assume that $\alpha + \gamma \neq 0$.

Case $(0, 0, 1a)$: We have $\alpha + \gamma = 0$ and hence

$$\boldsymbol{\mu} \;=\; \big(\alpha, \beta, -\alpha, \alpha', \beta', \gamma'\big) \tag{2.6}$$

with $\alpha' + \beta' + \gamma' \neq 0$. In this case, $c_2 = (\alpha + \beta) + (\alpha' + \beta')x$. Going forward, we impose the disjunction of inequations

$$\alpha + \beta \;\neq\; 0 \quad \text{ or } \quad \alpha' + \beta' \;\neq\; 0 \tag{2.7}$$

in order to ensure that $c_2 \neq 0$.

Case $(0, 0, 1b)$: We have $\beta'(\alpha + \gamma) - \beta(\alpha' + \beta' + \gamma') = 0$ with $\alpha + \gamma \neq 0$ and $\alpha' + \beta' + \gamma' \neq 0$. Here it is convenient to express β as a rational function of the other variables in $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} \;=\; \left(\alpha, \frac{(\alpha + \gamma)\beta'}{\alpha' + \beta' + \gamma'}, \gamma, \alpha', \beta', \gamma'\right) \tag{2.8}$$

with $\alpha + \gamma \neq 0$ and $\alpha' + \beta' + \gamma' \neq 0$. In this case, $c_2 = \alpha + (\alpha' + \beta')x$. Going forward, we impose the disjunction of inequations

$$\alpha \;\neq\; 0 \quad \text{ or } \quad \alpha' + \beta' \;\neq\; 0 \tag{2.9}$$

in order to ensure that $c_2 \neq 0$.

Let us again stress that in Case $(0, 0, 1\mathrm{b})$ we have imposed the extra condition $\alpha + \gamma \neq 0$ in order to remove the overlap with Case $(0, 0, 1\mathrm{a})$.

2.4 Step 3: Coefficient c_3

We consider the three cases (i.e., vertices of the decision tree) obtained in Section 2.3; for each one, we compute the branches stemming from it by using the next coefficient c_3.

2.4.1 Branches starting from Case $(0, 0, 0)$

We have $c_3 = Q(x)/R(x)$ with

$$Q(x) \;=\; \alpha(2\alpha + \gamma) \;+\; \alpha'(3\alpha + \beta + \gamma)\,x \;+\; \alpha'(\alpha' + \beta')\,x^2 \tag{2.10a}$$

$$R(x) \;=\; \alpha \;+\; \alpha'\,x \tag{2.10b}$$

$$\mathrm{rem}(Q(x), R(x)) \;=\; \frac{\alpha(\alpha\beta' - \beta\alpha')}{\alpha'} \tag{2.10c}$$

and the conditions $\alpha + \gamma \neq 0$ and ($\alpha \neq 0$ or $\alpha' \neq 0$) of Case $(0,0,0)$. The polynomial $R(x)$ is equal to the coefficient c_2 for the Case $(0,0,0)$.

- **$\deg R = 0$:** If $[x^1]R(x) = \alpha' = 0$ and $[x^0]R(x) = \alpha \neq 0$, we get

  Case $(0,0,0,0)$: $\boldsymbol{\mu} = (\alpha, \beta, \gamma, 0, \beta', -\beta')$ with $\alpha + \gamma \neq 0$ and $\alpha \neq 0$. In this case, $c_3 = \gamma + 2\alpha$. Pursuing the computation of this family to higher order, we find that all the coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x that are generically nonzero:

$$c_{2k-1} \;=\; \gamma + k\alpha\,, \quad c_{2k} \;=\; k\alpha\,. \tag{2.11}$$

  So this is a viable candidate, corresponding to Family 2b in the main text, and this branch is terminated.

- **$\deg R = 1$:** If $[x^1]R(x) = \alpha' \neq 0$, the condition $\mathrm{rem}(Q(x), R(x)) = 0$ leads to two new cases: $\alpha = 0$ or $\alpha\beta' - \beta\alpha' = 0$.

  Case $(0,0,0,1\mathrm{a})$: We have $\alpha = 0$, so that

$$\boldsymbol{\mu} \;=\; \big(0, \beta, \gamma, \alpha', \beta', -\alpha' - \beta'\big) \tag{2.12}$$

  with $\gamma \neq 0$ and $\alpha' \neq 0$. In this case, $c_3 = (\beta + \gamma) + (\alpha' + \beta')\,x$. Going forward, we impose the disjunction of inequations

$$\beta + \gamma \;\neq\; 0 \quad \text{or} \quad \alpha' + \beta' \;\neq\; 0 \tag{2.13}$$

  in order to ensure that $c_3 \neq 0$.

  Case $(0,0,0,1\mathrm{b})$: We have $\alpha\beta' - \beta\alpha' = 0$ with $\alpha' \neq 0$. In this case it is convenient to express β as a rational function of the other variables in $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} \;=\; \left(\alpha, \frac{\alpha\beta'}{\alpha'}, \gamma, \alpha', \beta', -\alpha' - \beta'\right) \tag{2.14}$$

  with $\alpha + \gamma \neq 0$ and $\alpha' \neq 0$. In this case, $c_3 = (2\alpha + \gamma) + (\alpha' + \beta')\,x$. Going forward, we impose the disjunction of inequations

$$2\alpha + \gamma \;\neq\; 0 \quad \text{or} \quad \alpha' + \beta' \;\neq\; 0 \tag{2.15}$$

  in order to ensure that $c_3 \neq 0$.

  Remark. In Case $(0,0,0,1\mathrm{b})$ we could, if we wish, impose the extra condition $\alpha \neq 0$ in order to remove the overlap with Case $(0,0,0,1\mathrm{a})$. But we choose not to do so.

2.4.2 Branches starting from Case $(0, 0, 1a)$

We have $c_3 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) &= \alpha(\alpha+\beta) \;+\; (\alpha+\beta)(3\alpha'+2\beta'+\gamma')\,x \\ &\quad\;+\;(\alpha'+\beta')(2\alpha'+2\beta'+\gamma')\,x^2 \end{aligned} \tag{2.16a}$$

$$R(x) = \alpha+\beta \;+\; (\alpha'+\beta')\,x \tag{2.16b}$$

$$\mathrm{rem}(Q(x), R(x)) = \frac{(\alpha+\beta)(\alpha\beta'-\beta\alpha')}{\alpha'+\beta'} \tag{2.16c}$$

and the conditions $\alpha'+\beta'+\gamma' \neq 0$ and ($\alpha+\beta \neq 0$ or $\alpha'+\beta' \neq 0$) of the Case $(0, 0, 1a)$. The polynomial $R(x)$ is equal to the coefficient c_2 for the Case $(0, 0, 1a)$.

- **$\deg \boldsymbol{R} = 0$:** If $[x^1]R(x) = \alpha'+\beta' = 0$ and $[x^0]R(x) = \alpha+\beta \neq 0$, we get

  Case $(0, 0, 1a, 0)$: $\boldsymbol{\mu} = (\alpha, \beta, -\alpha, \alpha', -\alpha', \gamma')$ with $\gamma' \neq 0$ and $\alpha+\beta \neq 0$. In this case, $c_3 = \alpha + (\alpha'+\gamma')x$. Going forward, we impose the disjunction of inequations

  $$\alpha \neq 0 \quad \text{or} \quad \alpha'+\gamma' \neq 0 \tag{2.17}$$

  in order to ensure that $c_3 \neq 0$.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = \alpha'+\beta' \neq 0$, then $\mathrm{rem}(Q(x), R(x)) = 0$ gives two other cases: $\alpha+\beta = 0$ or $\alpha\beta'-\beta\alpha' = 0$. We divide this latter case into two, according as $\alpha' \neq 0$ or $\alpha' = 0$.

  Case $(0, 0, 1a, 1a)$: We have $\alpha+\beta = 0$, so that

  $$\boldsymbol{\mu} = \bigl(\alpha, -\alpha, -\alpha, \alpha', \beta', \gamma'\bigr) \tag{2.18}$$

  with $\alpha'+\beta' \neq 0$ and $\alpha'+\beta'+\gamma' \neq 0$. In this case, $c_3 = (2\alpha'+2\beta'+\gamma')x$. We find in fact that all coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x:

  $$c_{2k-1} = [\gamma' \;+\; k(\alpha'+\beta')]\,x\,, \quad c_{2k} = k(\alpha'+\beta')\,x\,. \tag{2.19}$$

  So this is a viable candidate, corresponding to Family 2a in the main text, and this branch is terminated.

  Case $(0, 0, 1a, 1b)$: We have $\alpha\beta'-\beta\alpha' = 0$ with $\alpha'+\beta' \neq 0$ and $\alpha' \neq 0$. The latter inequation allows us to express β as a rational function of the other variables in $\boldsymbol{\mu}$:

  $$\boldsymbol{\mu} = \left(\alpha, \frac{\alpha\beta'}{\alpha'}, -\alpha, \alpha', \beta', \gamma'\right) \tag{2.20}$$

with $\alpha' \neq 0$, $\alpha'+\beta' \neq 0$, and $\alpha'+\beta'+\gamma' \neq 0$. In this case, $c_3 = \alpha + (2\alpha'+2\beta'+\gamma')x$. Going forward, we impose the disjunction of inequations

$$\alpha \;\neq\; 0 \quad \text{or} \quad 2\alpha'+2\beta'+\gamma' \;\neq\; 0 \tag{2.21}$$

in order to ensure that $c_3 \neq 0$.

Case $(\mathbf{0,0,1a,1c})$: We have $\alpha\beta' - \beta\alpha' = 0$ and $\alpha' = 0$ with $\alpha'+\beta' \neq 0$. This implies (and is equivalent to) $\alpha = \alpha' = 0$ and $\beta' \neq 0$. Hence

$$\boldsymbol{\mu} \;=\; \big(0, \beta, 0, 0, \beta', \gamma'\big) \tag{2.22}$$

with $\beta' \neq 0$ and $\beta'+\gamma' \neq 0$. In this case, $c_3 = (2\beta'+\gamma')x$. We find in fact that all coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x:

$$c_{2k-1} \;=\; (\gamma' \,+\, k\beta')\, x\,, \quad c_{2k} \;=\; k(\beta+\beta' x)\;. \tag{2.23}$$

So this is a viable candidate, corresponding to Family 3a in the main text, and this branch is terminated.

2.4.3 Branches starting from Case $(\mathbf{0,0,1b})$

We have $c_3 = Q(x)/R(x)$ with

$$\begin{aligned}
Q(x) \;&=\; \alpha(2\alpha+\gamma)(\alpha'+\beta'+\gamma') \\
&\quad + \Big(\alpha\big(4(\alpha')^2 + 4(\beta')^2 + 4(\gamma')^2 + 8\alpha'\beta' + 5\alpha'\gamma' + 4\beta'\gamma'\big) \\
&\qquad + \gamma(\alpha'+\beta')(\alpha'+2\beta'+\gamma')\Big)\, x \\
&\quad + (\alpha'+\beta')(\alpha'+\beta'+\gamma')(2\alpha'+2\beta'+\gamma')\, x^2 && \text{(2.24a)} \\
R(x) \;&=\; (\alpha'+\beta'+\gamma')\,\big(\alpha + (\alpha'+\beta')\,x\big) && \text{(2.24b)} \\
\operatorname{rem}(Q(x), R(x)) \;&=\; \frac{\alpha\beta'(\alpha\gamma' - \gamma\alpha' - \gamma\beta')}{\alpha'+\beta'} && \text{(2.24c)}
\end{aligned}$$

and the conditions $\alpha+\gamma \neq 0$, $\alpha'+\beta'+\gamma' \neq 0$ and ($\alpha \neq 0$ or $\alpha'+\beta' \neq 0$) of the Case $(0,0,1b)$. The polynomial $R(x)$ is equal to the coefficient c_2 for the Case $(0,0,1b)$ multiplied by the factor $\alpha'+\beta'+\gamma' \neq 0$.

- **$\deg \boldsymbol{R} = 0$:** If $[x^1]R(x) = (\alpha'+\beta')(\alpha'+\beta'+\gamma') = 0$ and $[x^0]R(x) = \alpha(\alpha'+\beta'+\gamma') \neq 0$, these conditions reduce to $\alpha'+\beta' = 0$ (since $\alpha'+\beta'+\gamma' \neq 0$) and $\alpha \neq 0$. We get

  Case $(\mathbf{0,0,1b,0})$: We have

$$\boldsymbol{\mu} \;=\; \left(\alpha, -\frac{(\alpha+\gamma)\alpha'}{\gamma'}, \gamma, \alpha', -\alpha', \gamma'\right) \tag{2.25}$$

with $\alpha \neq 0$ and $\gamma' \neq 0$ and $\alpha + \gamma \neq 0$. In this case, $c_3 = (2\alpha + \gamma) + (\alpha' + \gamma')\, x$. Going forward, we impose the disjunction of inequations

$$2\alpha + \gamma \;\neq\; 0 \quad \text{or} \quad \alpha' + \gamma' \;\neq\; 0 \tag{2.26}$$

in order to ensure that $c_3 \neq 0$.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = (\alpha' + \beta')(\alpha' + \beta' + \gamma') \neq 0$, then $\operatorname{rem}(Q(x), R(x)) = 0$ gives three cases: $\alpha = 0$, $\beta' = 0$, and $\alpha\gamma' - \gamma(\alpha' + \beta') = 0$.

Case $(\mathbf{0}, \mathbf{0}, \mathbf{1b}, \mathbf{1a})$: We have $\alpha = 0$, so that

$$\boldsymbol{\mu} \;=\; \left(0, \frac{\gamma\beta'}{\alpha' + \beta' + \gamma'}, \gamma, \alpha', \beta', \gamma'\right) \tag{2.27}$$

with $\alpha' + \beta' \neq 0$, $\alpha' + \beta' + \gamma' \neq 0$ and $\gamma \neq 0$. In this case,

$$c_3 \;=\; \frac{\gamma(\alpha' + 2\beta' + \gamma')}{\alpha' + \beta' + \gamma'} \;+\; (2\alpha' + 2\beta' + \gamma')\, x\,. \tag{2.28}$$

Going forward, we impose the disjunction of inequations

$$\gamma(\alpha' + 2\beta' + \gamma') \;\neq\; 0 \quad \text{or} \quad 2\alpha' + 2\beta' + \gamma' \;\neq\; 0 \tag{2.29}$$

in order to ensure that $c_3 \neq 0$.

Case $(\mathbf{0}, \mathbf{0}, \mathbf{1b}, \mathbf{1b})$: We have $\beta' = 0$, so that

$$\boldsymbol{\mu} \;=\; \bigl(\alpha, 0, \gamma, \alpha', 0, \gamma'\bigr) \tag{2.30}$$

with $\alpha' + \gamma' \neq 0$, $\alpha' \neq 0$ and $\alpha + \gamma \neq 0$. In this case, $c_3 = (2\alpha + \gamma) + (2\alpha' + \gamma')\, x$. We find in fact that all coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x:

$$c_{2k-1} \;=\; (\gamma + \gamma' x) \;+\; k(\alpha + \alpha' x)\,, \quad c_{2k} \;=\; k(\alpha + \alpha' x)\,. \tag{2.31}$$

So this is a viable candidate, corresponding to Family 5 in the main text, and this branch is terminated.

Remark. Here we could impose the additional condition $\alpha \neq 0$ to avoid overlap with Case $(0, 0, 1\text{b}, 1\text{a})$, but we choose not to, since this is a perfectly good candidate irrespective of whether α is zero or nonzero.

Case $(\mathbf{0}, \mathbf{0}, \mathbf{1b}, \mathbf{1c})$: We have $\alpha\gamma' - \gamma(\alpha' + \beta') = 0$ with $\alpha' + \beta' \neq 0$, $\alpha' + \beta' + \gamma' \neq 0$, and $\alpha + \gamma \neq 0$. Here it is convenient to express γ as a rational function of the other variables in $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} \;=\; \left(\alpha, \frac{\alpha\beta'}{\alpha' + \beta'}, \frac{\alpha\gamma'}{\alpha' + \beta'}, \alpha', \beta', \gamma'\right) \tag{2.32}$$

with $\alpha'+\beta' \neq 0$, $\alpha'+\beta'+\gamma' \neq 0$, and $\alpha \neq 0$ (where this last inequation comes from $\alpha+\gamma \neq 0$). In this case,

$$c_3 \;=\; \bigl(2\alpha'+2\beta'+\gamma'\bigr)\left(\frac{\alpha}{\alpha'+\beta'}+x\right). \tag{2.33}$$

We find in fact that all coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x:

$$c_{2k-1} \;=\; \bigl[\gamma'+k(\alpha'+\beta')\bigr]\left(\frac{\alpha}{\alpha'+\beta'}+x\right) \qquad c_{2k} \;=\; k\bigl[\alpha+(\alpha'+\beta')x\bigr]. \tag{2.34}$$

The coefficients c_i (2.34) are *not* polynomials in the parameters $\boldsymbol{\mu}$, but if we perform the change of parameters $\alpha \mapsto \kappa(\alpha'+\beta')$, then (2.32) reduces to

$$\boldsymbol{\mu} \;=\; \bigl(\kappa(\alpha'+\beta'), \kappa\beta', \kappa\gamma', \alpha', \beta', \gamma'\bigr) \tag{2.35}$$

with $\alpha'+\beta' \neq 0$ and $\alpha'+\beta'+\gamma' \neq 0$, and $\kappa \neq 0$. The coefficients (2.34) then become

$$c_{2k-1} \;=\; \bigl[\gamma'+k(\alpha'+\beta')\bigr](\kappa+x), \quad c_{2k} \;=\; k(\alpha'+\beta')(\kappa+x), \tag{2.36}$$

which are polynomials jointly in $x, \alpha', \beta', \gamma', \kappa$. So this is a viable candidate, corresponding to Family 6 in the main text, and this branch is terminated.

Remark. Here we could impose the additional condition $\beta' \neq 0$ to avoid overlap with Case $(0, 0, 1\text{b}, \ 1\text{b})$; but we choose not to, since this is a perfectly good candidate irrespective of whether β' is zero or nonzero. Moreover, in obtaining this solution we have imposed the condition $\alpha \neq 0$; but this condition is actually superfluous, since this is a perfectly good candidate irrespective of whether α is zero or nonzero.

2.5 Step 4: Coefficient c_4

We consider the six cases obtained in Section 2.4 that require additional conditions to be terminated. For each of these vertices, we now compute the branches stemming from them by using the next coefficient c_4. To save space, from this section on, we will not make further remarks about the possibility of imposing additional conditions to avoid overlaps with other cases.

2.5.1 Branches starting from Case $(0, 0, 0, 1\text{a})$

We have $c_4 = Q(x)/R(x)$ with

$$Q(x) \;=\; (3\beta\alpha'+\beta\beta'+2\gamma\alpha')\,x \;+\; (\alpha'+\beta')(2\alpha'+\beta')\,x^2 \tag{2.37a}$$

$$R(x) \;=\; \beta+\gamma \;+\; (\alpha'+\beta')\,x \tag{2.37b}$$

$$\operatorname{rem}(Q(x), R(x)) \;=\; -\frac{(\beta+\gamma)(\beta\alpha'-\gamma\beta')}{\alpha'+\beta'} \tag{2.37c}$$

and the conditions $\gamma\alpha' \neq 0$ and ($\beta + \gamma \neq 0$ or $\alpha' + \beta' \neq 0$) of the Case $(0, 0, 0, 1\mathrm{a})$. The polynomial $R(x)$ is equal to the coefficient c_3 for the Case $(0, 0, 0, 1\mathrm{a})$.

- **deg $\boldsymbol{R} = 0$:** If $[x^1]R(x) = \alpha' + \beta' = 0$ and $[x^0]R(x) = \beta + \gamma \neq 0$, we get

  Case $(\mathbf{0, 0, 0, 1a, 0})$: $\boldsymbol{\mu} = \big(0, \beta, \gamma, \alpha', -\alpha', 0\big)$ with $\gamma\alpha' \neq 0$ and $\beta + \gamma \neq 0$. In this case, $c_4 = 2\alpha' x$. Pursuing the computation of this family to higher order, we find that all the coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x that are generically nonzero:

  $$c_{2k-1} \;=\; \gamma \;+\; (k-1)\beta\,, \quad c_{2k} \;=\; k\alpha'\,x\,. \tag{2.38}$$

  So this is a viable candidate, corresponding to Family 1b in the main text, and this branch is terminated.

- **deg $\boldsymbol{R} = 1$:** If $[x^1]R(x) = \alpha' + \beta' \neq 0$, then $\operatorname{rem}(Q(x), R(x)) = 0$ gives two cases: $\beta + \gamma = 0$, and $\beta\alpha' - \gamma\beta' = 0$.

  Case $(\mathbf{0, 0, 0, 1a, 1a})$: We have $\beta + \gamma = 0$, so that

  $$\boldsymbol{\mu} \;=\; \big(0, \beta, -\beta, \alpha', \beta', -\alpha' - \beta'\big) \tag{2.39}$$

  with $\alpha' + \beta' \neq 0$ and $\beta\alpha' \neq 0$. In this case, $c_4 = \beta + (2\alpha' + \beta')\,x \neq 0$, as $\beta \neq 0$.

  Case $(\mathbf{0, 0, 0, 1a, 1b})$: We have $\beta\alpha' - \gamma\beta'$. As $\gamma \neq 0$, the solution is $\beta' = \beta\alpha'/\gamma$, so that

  $$\boldsymbol{\mu} \;=\; \left(0, \beta, \gamma, \alpha', \frac{\beta\alpha'}{\gamma}, -\frac{(\beta+\gamma)\alpha'}{\gamma}\right) \tag{2.40}$$

  with $\gamma\alpha' \neq 0$ and $\beta + \gamma \neq 0$. In this case,

  $$c_4 \;=\; \frac{\alpha'}{\gamma}\,(\beta + 2\gamma)\,x\,. \tag{2.41}$$

  Going forward, we impose the inequation $\beta + 2\gamma \neq 0$ in order to ensure that $c_4 \neq 0$.

2.5.2 Branches starting from Case $(0, 0, 0, 1\mathrm{b})$

We have that $c_4 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) \;&=\; 2\alpha\alpha'(2\alpha + \gamma) \\ &\qquad + \big(6\alpha(\alpha')^2 + 3\alpha\alpha'\beta' + \alpha(\beta')^2 + 2\gamma(\alpha')^2\big)\,x \\ &\qquad + \alpha'(\alpha' + \beta')(2\alpha' + \beta')\,x^2 \end{aligned} \tag{2.42a}$$

$$R(x) \;=\; \alpha'\,\big(2\alpha + \gamma \;+\; (\alpha' + \beta')\,x\big) \tag{2.42b}$$

$$\operatorname{rem}(Q(x), R(x)) \;=\; \frac{\beta'(2\alpha + \gamma)(\alpha\alpha' - \alpha\beta' + \gamma\alpha')}{\alpha' + \beta'} \tag{2.42c}$$

and the conditions $\alpha' \neq 0$, $\alpha+\gamma \neq 0$, and ($2\alpha+\beta \neq 0$ or $\alpha'+\eta' \neq 0$) of the Case $(0,0,0,1\mathrm{b})$. The polynomial $R(x)$ is equal to the coefficient c_3 for the Case $(0,0,0,1\mathrm{b})$ multiplied by the factor $\alpha' \neq 0$.

- **deg $\boldsymbol{R} = 0$:** If $[x^1]R(x) = \alpha'(\alpha'+\beta') = 0$ and $[x^0]R(x) = \alpha'(2\alpha+\gamma) \neq 0$, these conditions reduce to $\alpha'+\beta' = 0$ and $2\alpha+\gamma \neq 0$, as $\alpha' \neq 0$. We get

  Case $(\mathbf{0,0,0,1b,0})$: $\boldsymbol{\mu} = \big(\alpha,-\alpha,\gamma,\alpha',-\alpha',0\big)$ with $\alpha' \neq 0$, $\alpha+\gamma \neq 0$, and $2\alpha+\gamma \neq 0$. In this case, $c_4 = 2(\alpha+\alpha' x)$. Pursuing the computation of this family to higher order, we find that all the coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x that are generically nonzero:

  $$c_{2k-1} \;=\; \gamma \,+\, k\alpha\,, \quad c_{2k} \;=\; k(\alpha+\alpha' x)\,. \tag{2.43}$$

  So this is a viable candidate, corresponding to Family 3b in the main text, and this branch is terminated.

- **deg $\boldsymbol{R} = 1$:** If $[x^1]R(x) = \alpha'(\alpha'+\beta') \neq 0$, then $\alpha'+\beta' \neq 0$, as $\alpha' \neq 0$. Then $\mathrm{rem}(Q(x),R(x)) = 0$ gives three cases: $2\alpha+\gamma = 0$, $\alpha\alpha' - \alpha\beta' + \gamma\alpha' = 0$, and $\beta' = 0$. Notice that the latter one $\beta' = 0$, leads to $\boldsymbol{\mu} = (\alpha,0,\gamma,\alpha',0,-\alpha')$, which is the specialization of Family 5 [i.e., Case $(0,0,1\mathrm{b},1\mathrm{b})$] when $\gamma' = -\alpha'$. Therefore, it should be discarded. The other two solutions leads to:

  Case $(\mathbf{0,0,0,1b,1a})$: We have $2\alpha+\gamma = 0$, so that

  $$\boldsymbol{\mu} \;=\; \left(\alpha, \frac{\alpha\beta'}{\alpha'}, -2\alpha, \alpha', \beta', -\alpha'-\beta'\right) \tag{2.44}$$

  with $\alpha'+\beta' \neq 0$ and $\alpha\alpha' \neq 0$. In this case,

  $$c_4 \;=\; \frac{1}{\alpha'}\,(2\alpha'+\beta')\,(\alpha+\alpha'\,x)\,. \tag{2.45}$$

  Going forward, we impose the inequation $2\alpha'+\beta' \neq 0$ in order to ensure that $c_4 \neq 0$.

  Case $(\mathbf{0,0,0,1b,1b})$: We have $\alpha\alpha' - \alpha\beta' + \gamma\alpha' = 0$. As $\alpha' \neq 0$, the solution is $\gamma = -\alpha(\alpha'-\beta')/\alpha'$, so that

  $$\boldsymbol{\mu} \;=\; \left(\alpha, \frac{\alpha\beta'}{\alpha'}, -\frac{\alpha(\alpha'-\beta')}{\alpha'}, \alpha', \beta', -\alpha'-\beta'\right) \tag{2.46}$$

  with $\alpha'+\beta' \neq 0$ and $\alpha\alpha'\beta' \neq 0$. In this case, $c_4 = 2\alpha + (2\alpha'+\beta')x \neq 0$, as $\alpha \neq 0$.

2.5.3 Branches starting from Case $(0, 0, 1\mathrm{a}, 0)$

We have $c_4 = Q(x)/R(x)$ with

$$Q(x) = \alpha(2\alpha+\beta) + \big(3\alpha\alpha' + 2\beta\alpha' + 2\alpha\gamma' + 2\beta\gamma'\big)\, x \tag{2.47a}$$

$$R(x) = \alpha + (\alpha'+\gamma')\, x \tag{2.47b}$$

$$\mathrm{rem}(Q(x), R(x)) = -\frac{\alpha(\alpha\alpha' + \beta\alpha' + \beta\gamma')}{\alpha'+\gamma'} \tag{2.47c}$$

and the conditions $\gamma' \neq 0$, $\alpha+\beta \neq 0$, and ($\alpha \neq 0$ or $\alpha'+\gamma' \neq 0$) of the Case $(0, 0, 1\mathrm{a}, 0)$. The polynomial $R(x)$ is equal to the coefficient c_3 for the Case $(0, 0, 1\mathrm{a}, 0)$.

- $\mathbf{deg}\, \boldsymbol{R} = \mathbf{0}$**:** If $[x^1]R(x) = \alpha'+\gamma' = 0$ and $[x^0]R(x) = \alpha \neq 0$, we get

  Case $(0, 0, 1\mathrm{a}, 0, 0)$: $\boldsymbol{\mu} = \big(\alpha, \beta, -\alpha, \alpha', -\alpha', -\alpha'\big)$ with $\alpha+\beta \neq 0$ and $\alpha\alpha' \neq 0$. In this case, $c_4 = (2\alpha+\beta) + \alpha' x \neq 0$, as $\alpha' \neq 0$.

- $\mathbf{deg}\, \boldsymbol{R} = \mathbf{1}$**:** If $[x^1]R(x) = \alpha'+\gamma' \neq 0$, then $\mathrm{rem}(Q(x), R(x)) = 0$ gives two cases: $\alpha = 0$ and $\alpha\alpha' + \beta\alpha' + \beta\gamma' = 0$.

  Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{a})$: We have $\alpha = 0$, so that

  $$\boldsymbol{\mu} = \big(0, \beta, 0, \alpha', -\alpha', \gamma'\big) \tag{2.48}$$

  with $\alpha'+\gamma' \neq 0$ and $\beta\gamma' \neq 0$. In this case, $c_4 = 2\beta$. Pursuing the computation of this family to higher order, we find that all the coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x that are generically nonzero:

  $$c_{2k-1} = [\gamma' + (k-1)\alpha']\, x\,, \quad c_{2k} = k\beta\,. \tag{2.49}$$

  So this is a viable candidate, corresponding to Family 1a in the main text, and this branch is terminated.

  Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{b})$: We have $\alpha\alpha' + \beta\alpha' + \beta\gamma' = 0$. As $\alpha'+\gamma' \neq 0$, the solution is given by

  $$\beta = -\frac{\alpha\alpha'}{\alpha'+\gamma'}\,, \tag{2.50}$$

  so that

  $$\boldsymbol{\mu} = \left(\alpha, -\frac{\alpha\alpha'}{\alpha'+\gamma'}, -\alpha, \alpha', -\alpha', \gamma'\right) \tag{2.51}$$

  with $\alpha'+\gamma' \neq 0$ and $\alpha\gamma' \neq 0$. In this case,

  $$c_4 = \frac{\alpha}{\alpha'+\gamma'}\big(\alpha' + 2\gamma'\big)\,. \tag{2.52}$$

  Going forward, we impose the inequation $\alpha'+2\gamma' \neq 0$ in order to ensure that $c_4 \neq 0$.

2.5.4 Branches starting from Case $(0, 0, 1a, 1b)$

We find that $c_4 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) &= \alpha^2(2\alpha' + \beta') \\ &\quad + \alpha\big(6(\alpha')^2 + 9\alpha'\beta' + 4(\beta')^2 + 2\alpha'\gamma' + 2\beta'\gamma'\big)\, x \\ &\quad + 2\alpha'\big(\alpha' + \beta'\big)\,\big(2\alpha' + 2\beta' + \gamma'\big)\, x^2 \end{aligned} \tag{2.53a}$$

$$R(x) = \alpha'\,\big(\alpha \;+\; (2\alpha' + 2\beta' + \gamma')\, x\big) \tag{2.53b}$$

$$\operatorname{rem}(Q(x), R(x)) = -\frac{\alpha^2\beta'(\alpha' + 2\beta' + \gamma')}{2\alpha' + 2\beta' + \gamma'} \tag{2.53c}$$

and the conditions $\alpha' \neq 0$, $\alpha' + \beta' \neq 0$, $\alpha' + \beta' + \gamma' \neq 0$, and ($\alpha \neq 0$ or $2\alpha' + 2\beta' \neq 0$) of the Case $(0, 0, 1a, 1b)$. The polynomial $R(x)$ is equal to the coefficient c_3 for the Case $(0, 0, 1a, 1b)$ multiplied by $\alpha' \neq 0$.

- **$\deg \boldsymbol{R} = 0$:** If $[x^1]R(x) = \alpha'(2\alpha' + 2\beta' + \gamma') = 0$ and $[x^0]R(x) = \alpha\alpha' \neq 0$, these conditions reduce to $2\alpha' + 2\beta' + \gamma' = 0$ and $\alpha \neq 0$, as $\alpha' \neq 0$. We get

  Case $(0, 0, 1a, 1b, 0)$: we have the family

  $$\boldsymbol{\mu} = \left(\alpha, \frac{\alpha\beta'}{\alpha'}, -\alpha, \alpha', \beta', -2\alpha' - 2\beta'\right) \tag{2.54}$$

  with $\alpha' + \beta' \neq 0$ and $\alpha\alpha' \neq 0$. In this case,

  $$c_4 = \frac{1}{\alpha'}\,(2\alpha' + \beta')\,(\alpha + \alpha'\, x)\,. \tag{2.55}$$

  Going forward, we impose the inequation $2\alpha' + \beta' \neq 0$ in order to ensure that $c_4 \neq 0$.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = \alpha'(2\alpha' + 2\beta' + \gamma') \neq 0$, this condition reduces to $2\alpha' + 2\beta' + \gamma' \neq 0$, as $\alpha' \neq 0$. Then $\operatorname{rem}(Q(x), R(x)) = 0$ gives three cases: $\alpha = 0$, $\beta' = 0$, and $\alpha' + 2\beta' + \gamma' = 0$. Notice that the first solution implies $\boldsymbol{\mu} = (0, 0, 0, \alpha', \beta', \gamma')$, which is the specialization of Family 2a [i.e., Case $(0, 0, 1a, 1a)$] when $\alpha = 0$. Moreover, the second one leads to $\boldsymbol{\mu} = (\alpha, 0, -\alpha, \alpha', 0, \gamma')$, which is the specialization of Family 5 [i.e., Case $(0, 0, 1b, 1b)$] when $\gamma = -\alpha$. Therefore, we have only one case left:

  Case $(0, 0, 1a, 1b, 1a)$: We have $\alpha' + 2\beta' + \gamma' = 0$:

  $$\boldsymbol{\mu} = \left(\alpha, \frac{\alpha\beta'}{\alpha'}, -\alpha, \alpha', -\beta', -\alpha' - 2\beta'\right) \tag{2.56}$$

  with $\alpha' + \beta' \neq 0$ and $\alpha'\beta' \neq 0$. In this case,

  $$c_4 = \frac{\alpha(2\alpha' + \beta')}{\alpha'} \;+\; 2\big(\alpha' + \beta'\big)\, x \;\neq\; 0\,, \tag{2.57}$$

  as $\alpha' + \beta' \neq 0$.

2.5.5 Branches starting from Case $(0, 0, 1b, 0)$

We find that $c_4 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) &= 2\alpha(2\alpha+\gamma)\gamma' \\ &\qquad + \big(3\alpha\alpha'\gamma' + 2\alpha(\gamma')^2 - \alpha(\alpha')^2 - \gamma(\alpha')^2\big)\, x && \text{(2.58a)} \\ R(x) &= \gamma'\big(2\alpha+\gamma \;+\; (\alpha'+\gamma')\, x\big) && \text{(2.58b)} \\ \operatorname{rem}(Q(x), R(x)) &= \frac{\alpha'(2\alpha+\gamma)(\alpha\alpha'+\gamma\alpha'-\alpha\gamma')}{\alpha'+\gamma'} && \text{(2.58c)} \end{aligned}$$

and the conditions $\alpha+\gamma \neq 0$, $\alpha\gamma' \neq 0$, and ($2\alpha+\gamma \neq 0$ or $\alpha'+\gamma' \neq 0$) of the Case $(0, 0, 1b, 0)$. The polynomial $R(x)$ is equal to the coefficient c_3 for the Case $(0, 0, 1b, 0)$ multiplied by $\gamma' \neq 0$.

- $\mathbf{deg}\, \boldsymbol{R} = \mathbf{0}$**:** If $[x^1]R(x) = \gamma'(\alpha'+\gamma') = 0$ and $[x^0]R(x) = \gamma'(2\alpha+\gamma) \neq 0$, these conditions reduce to $\alpha'+\gamma' = 0$ and $2\alpha+\gamma \neq 0$, as $\gamma' \neq 0$. We get

  Case $(\mathbf{0, 0, 1b, 0, 0})$: $\boldsymbol{\mu} = \big(\alpha, \alpha+\gamma, \gamma, \alpha', -\alpha', -\alpha'\big)$ with $\alpha+\gamma \neq 0$, $2\alpha+\gamma \neq 0$, and $\alpha\alpha' \neq 0$. In this case, $c_4 = 2\alpha + \alpha' x \neq 0$, as $\alpha\alpha' \neq 0$.

- $\mathbf{deg}\, \boldsymbol{R} = \mathbf{1}$**:** If $[x^1]R(x) = \gamma'(\alpha'+\gamma') \neq 0$, this condition reduces to $\alpha'+\gamma' \neq 0$, as $\gamma' \neq 0$. Then $\operatorname{rem}(Q(x), R(x)) = 0$ gives three cases: $\alpha' = 0$, $2\alpha+\gamma = 0$, and $\alpha\alpha'+\gamma\alpha'-\alpha\gamma' = 0$. Notice that the first solution implies $\boldsymbol{\mu} = (\alpha, 0, \gamma, 0, 0, \gamma')$, which is the specialization of Family 5 [i.e., Case $(0, 0, 1b, 1b)$] when $\alpha' = 0$. The other two cases give:

  Case $(\mathbf{0, 0, 1b, 0, 1a})$: We have $2\alpha+\gamma = 0$, so that

  $$\boldsymbol{\mu} \;=\; \left(\alpha, \frac{\alpha\alpha'}{\gamma'}, -2\alpha, \alpha', -\alpha', \gamma'\right) \tag{2.59}$$

  with $\alpha'+\gamma' \neq 0$ and $\alpha\gamma' \neq 0$. In this case,

  $$c_4 \;=\; \frac{\alpha}{\gamma'}\,(\alpha'+2\gamma')\,. \tag{2.60}$$

  Going forward, we impose the inequation $\alpha'+2\gamma' \neq 0$ in order to ensure that $c_4 \neq 0$.

  Case $(\mathbf{0, 0, 1b, 0, 1b})$: We have $\alpha\alpha'+\gamma\alpha'-\alpha\gamma' = 0$. As $\alpha+\gamma \neq 0$, the solution is

  $$\alpha' \;=\; \frac{\alpha\gamma'}{\alpha+\gamma}\,, \tag{2.61}$$

so that

$$\boldsymbol{\mu} \;=\; \left(\alpha, -\alpha, \gamma, \frac{\alpha\gamma'}{\alpha+\gamma}, -\frac{\alpha\gamma'}{\alpha+\gamma}, \gamma'\right) \tag{2.62}$$

with $\alpha+\gamma \neq 0$, $2\alpha+\gamma \neq 0$, and $\alpha\gamma' \neq 0$. In this case, $c_4 = 2\alpha$. Pursuing the computation of this family to higher order, we find that all the coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x that are generically nonzero:

$$c_{2k-1} \;=\; (\gamma \,+\, k\alpha)\left(1 + \frac{\gamma'}{\alpha+\gamma}\, x\right), \quad c_{2k} \;=\; k\alpha\,. \tag{2.63}$$

The coefficients c_i (2.63) are *not* polynomials in α or γ, but if we perform the change of parameters $\gamma' \mapsto \kappa(\alpha+\gamma)$, then $\boldsymbol{\mu}$ (2.62) reduces to

$$\boldsymbol{\mu} \;=\; \big(\alpha, -\alpha, \gamma, \kappa\alpha, -\kappa\alpha, \kappa(\alpha+\gamma)\big) \tag{2.64}$$

with $\alpha+\gamma \neq 0$, $2\alpha+\gamma \neq 0$, and $\alpha\kappa \neq 0$. The coefficients c_i (2.63) transform to

$$c_{2k-1} \;=\; (\gamma \,+\, k\alpha)\,(1+\kappa x)\,, \quad c_{2k} \;=\; k\alpha\,. \tag{2.65}$$

where these coefficients are polynomials jointly in x, α, γ, and κ. So this is a viable candidate, corresponding to Family 4b in the main text, and this branch is terminated.

2.5.6 Branches starting from Case $(\mathbf{0}, \mathbf{0}, \mathbf{1b}, \mathbf{1a})$

We have $c_4 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) \;&=\; \gamma\big(2(\alpha')^2 + 7\alpha'\beta' + 4(\beta')^2 + 2\alpha'\gamma' + 2\beta'\gamma'\big)\, x \\ &\qquad + 2(\alpha'+\beta')(\alpha'+\beta'+\gamma')(2\alpha'+2\beta'+\gamma')\, x^2 \end{aligned} \tag{2.66a}$$

$$R(x) \;=\; \gamma(\alpha'+2\beta'+\gamma') \;+\; (\alpha'+\beta'+\gamma')(2\alpha'+2\beta'+\gamma')\, x \tag{2.66b}$$

$$\operatorname{rem}(Q(x), R(x)) \;=\; -\frac{\alpha'\beta'\gamma^2(\alpha'+2\beta'+\gamma')}{(\alpha'+\beta'+\gamma')(2\alpha'+2\beta'+\gamma')} \tag{2.66c}$$

and the conditions $\alpha'+\beta'+\gamma' \neq 0$, $\alpha'+\beta' \neq 0$, and $\gamma \neq 0$ of the Case $(0,0,1\mathrm{b},1\mathrm{a})$. The polynomial $R(x)$ is the coefficient c_3 of the Case $(0,0,1\mathrm{b},1\mathrm{a})$ multiplied by the factor $\alpha'+2\beta'+\gamma' \neq 0$.

- $\mathbf{deg}\,\boldsymbol{R} = \mathbf{0}$**:** If $[x^1]R(x) = (\alpha'+\beta'+\gamma')(2\alpha'+2\beta'+\gamma') = 0$ and $[x^0]R(x) = \gamma(\alpha'+2\beta'+\gamma') \neq 0$, these conditions reduce to $2\alpha'+2\beta'+\gamma' = 0$ and $\alpha'+2\beta'+\gamma' \neq 0$, as $\gamma \neq 0$ and $\alpha'+\beta'+\gamma' \neq 0$. We get

  Case $(\mathbf{0}, \mathbf{0}, \mathbf{1b}, \mathbf{1a}, \mathbf{0})$: we have the family

  $$\boldsymbol{\mu} \;=\; \left(0, -\frac{\gamma\beta'}{\alpha'+\beta'}, \gamma, \alpha', \beta', -2\alpha'-2\beta'\right) \tag{2.67}$$

  with $\alpha'+\beta' \neq 0$, $\alpha'+2\beta'+\gamma' \neq 0$, and $\gamma\alpha' \neq 0$. In this case, $c_4 = (2\alpha'+\beta')\,x$. Going forward, we impose the inequation $2\alpha'+\beta' \neq 0$ in order to ensure that $c_4 \neq 0$.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = (\alpha' + \beta' + \gamma')(2\alpha' + 2\beta' + \gamma') \neq 0$, it reduces to $2\alpha' + 2\beta' + \gamma' \neq 0$, as $\alpha' + \beta' + \gamma' \neq 0$. Then $\mathrm{rem}(Q(x), R(x)) = 0$ gives three cases: $\alpha' = 0$, $\beta' = 0$, and $\alpha' + 2\beta' + \gamma' = 0$. Notice that the first solution implies $\boldsymbol{\mu} = (0, 0, \gamma, \alpha', 0, \gamma')$, which is the specialization of Family 5 [i.e., Case $(0, 0, 1\mathrm{b}, 1\mathrm{b})$] when $\alpha = 0$. The other two cases give:

  Case $(\mathbf{0}, \mathbf{0}, \mathbf{1b}, \mathbf{1a}, \mathbf{1a})$: We have $\alpha' = 0$, so that

  $$\boldsymbol{\mu} \;=\; \left(0, \frac{\gamma\beta'}{\beta' + \gamma'}, \gamma, 0, \beta', \gamma'\right) \tag{2.68}$$

  with $\beta' + \gamma' \neq 0$, $2\beta' + \gamma' \neq 0$, and $\gamma\beta' \neq 0$. In this case, $c_4 = 2\beta' x$. We find in fact that all coefficients c_i for $1 \leqslant i \leqslant 10$ are polynomials in x:

  $$c_{2k-1} \;=\; (\beta' k + \gamma') \left(\frac{\gamma}{\beta' + \gamma'} + x\right), \quad c_{2k} \;=\; k\beta' x\,. \tag{2.69}$$

  The coefficients c_i (2.69) are *not* polynomials in β' or γ', but if we perform the change of parameters $\gamma \mapsto \kappa(\beta' + \gamma')$, then $\boldsymbol{\mu}$ (2.68) reduces to

  $$\boldsymbol{\mu} \;=\; \big(0, \kappa\beta', \kappa(\beta' + \gamma'), 0, \beta', \gamma'\big) \tag{2.70}$$

  with $\beta' + \gamma' \neq 0$, $2\beta' + \gamma' \neq 0$, and $\kappa\beta' \neq 0$. The coefficients c_i (2.69) transform to

  $$c_{2k-1} \;=\; (\beta' k + \gamma')\,(\kappa \;+\; x)\,, \quad c_{2k} \;=\; k\;\beta' x\,. \tag{2.71}$$

  where these coefficients are polynomials jointly in x, β', γ', and κ. So this is a viable candidate, corresponding to Family 4a in the main text, and this branch is terminated.

  Case $(\mathbf{0}, \mathbf{0}, \mathbf{1b}, \mathbf{1a}, \mathbf{1b})$: We have $\alpha' + 2\beta' + \gamma' = 0$, so that

  $$\boldsymbol{\mu} \;=\; \big(0, \beta, -\beta, \alpha', \beta', -\alpha' - 2\beta'\big) \tag{2.72}$$

  with $\alpha' + \beta' \neq 0$ and $\beta\alpha'\beta' \neq 0$. In this case, $c_4 = \beta + 2(\alpha' + \beta')\,x \neq 0$, as $\beta(\alpha' + \beta') \neq 0$.

2.6 Step 5: Coefficient c_5

We consider the 12 cases obtained in Section 2.5 that require additional conditions to be terminated. For each of these vertices, we now compute the branches stemming from them by using the next coefficient c_5.

2.6.1 Branches starting from Case $(0,0,0,1\mathrm{a},1\mathrm{a})$

We have $c_5 = Q(x)/R(x)$ with

$$Q(x) \;=\; 2\beta(2\alpha'+\beta')\,x \;+\; 2(\alpha'+\beta')(2\alpha'+\beta')\,x^2 \tag{2.73a}$$

$$R(x) \;=\; \beta \;+\; (2\alpha'+\beta')\,x \tag{2.73b}$$

$$\mathrm{rem}(Q(x),R(x)) \;=\; -\frac{2\beta^2\alpha'}{2\alpha'+\beta'} \tag{2.73c}$$

and the conditions $\alpha'+\beta'\neq 0$ and $\beta\alpha'\neq 0$ of the Case $(0,0,0,1\mathrm{a},1\mathrm{a})$. The polynomial $R(x)$ is equal to the coefficient c_4 for the Case $(0,0,0,1\mathrm{a},1\mathrm{a})$.

- **$\deg \boldsymbol{R} = 0$:** If $[x^1]R(x) = 2\alpha'+\beta' = 0$ and $[x^0]R(x) = \beta \neq 0$, we get $\boldsymbol{\mu} = (0,\beta,-\beta,\alpha',-2\alpha',\alpha')$ with $\beta\alpha'\neq 0$. In this case, $c_5 = 0$, so this solution should not be taken into account.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = 2\alpha'+\beta' \neq 0$, then $\mathrm{rem}(Q(x),R(x)) \neq 0$ as $\beta\alpha'\neq 0$. Therefore, there is no solution of this type.

This branch does not have any viable solution. Therefore, Case $(0,0,0,1\mathrm{a},1\mathrm{a})$ is a leaf of the decision tree.

2.6.2 Branches starting from Case $(0,0,0,1\mathrm{a},1\mathrm{b})$

We find that

$$c_5 \;=\; 2\beta+\gamma \;+\; \frac{2(\beta+\gamma)\alpha'}{\gamma}\,x \tag{2.74}$$

is a nonzero polynomial in x, due to the inequations coming from Case $(0,0,0,1\mathrm{a},1\mathrm{b})$: $\beta+\gamma\neq 0$ and $\gamma\alpha'\neq 0$. Then $R(x)=1$, and we have only one case:

- **$\deg \boldsymbol{R} = 0$: Case $(0,0,0,1\mathrm{a},1\mathrm{b},0)$**: we have

$$\boldsymbol{\mu} \;=\; \left(0,\beta,\gamma,\alpha',\frac{\beta\alpha'}{\gamma},-\frac{(\beta+\gamma)\alpha'}{\gamma}\right) \tag{2.75}$$

  with $\gamma\alpha'\neq 0$ and $\beta+\gamma\neq 0$.

2.6.3 Branches starting from Case $(0,0,0,1\mathrm{b},1\mathrm{a})$

We find that $c_5 = \alpha + 2(\alpha'+\beta')\,x$ is a nonzero polynomial in x, due to the inequations $\alpha'+\beta'\neq 0$, $2\alpha'+\beta'\neq 0$, and $\alpha\alpha'\neq 0$ of the Case $(0,0,0,1\mathrm{b},1\mathrm{a})$. Then $R(x)=1$, and we have only one case:

- **$\deg \boldsymbol{R} = 0$: Case $(0,0,0,1\mathrm{b},1\mathrm{a},0)$**: we have

$$\boldsymbol{\mu} \;=\; \left(\alpha,\frac{\alpha\beta'}{\alpha'},-2\alpha,\alpha',\beta',-\alpha'-\beta'\right) \tag{2.76}$$

  with $\alpha'+\beta'\neq 0$, $2\alpha'+\beta'\neq 0$, and $\alpha\alpha'\neq 0$.

2.6.4 Branches starting from Case $(0, 0, 0, 1b, 1b)$

We have $c_5 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) &= 2\alpha^2(2\alpha' + \beta') \ + \ 2\alpha(2\alpha' + \beta')^2\, x \\ &\qquad + 2\alpha'(\alpha' + \beta')(2\alpha' + \beta')\, x^2 \end{aligned} \tag{2.77a}$$

$$R(x) = \alpha' \big(2\alpha \ + \ (2\alpha' + \beta')\, x\big) \tag{2.77b}$$

$$\mathrm{rem}(Q(x), R(x)) = -\frac{2\alpha^2(\beta')^2}{2\alpha' + \beta'} \tag{2.77c}$$

and the conditions $\alpha' + \beta' \neq 0$ and $\alpha\alpha'\beta' \neq 0$ of the Case $(0, 0, 0, 1b, 1b)$. The polynomial $R(x)$ is equal to the coefficient c_4 of the Case $(0, 0, 0, 1b, 1b)$ multiplied by $\alpha' \neq 0$.

- **$\deg \boldsymbol{R} = \mathbf{0}$:** If $[x^1]R(x) = \alpha'(2\alpha' + \beta') = 0$ and $[x^0]R(x) = 2\alpha\alpha' \neq 0$, they reduce to $2\alpha' + \beta' = 0$, as $\alpha\alpha' \neq 0$. Then there is a single solution when $2\alpha' + \beta' = 0$: $\boldsymbol{\mu} = \big(\alpha, -2\alpha, -3\alpha, \alpha', -2\alpha', \alpha'\big)$ with $\alpha\alpha' \neq 0$. In this case, $c_5 = 0$, so this solution should not be taken into account.
- **$\deg \boldsymbol{R} = \mathbf{1}$:** If $[x^1]R(x) = 2\alpha' + \beta' \neq 0$, then $\mathrm{rem}(Q(x), R(x)) \neq 0$ as $\alpha\beta' \neq 0$. So there is no solution of this type.

This branch does not have any viable solution. Therefore, Case $(0, 0, 0, 1b, 1b)$ is a leaf of the decision tree.

2.6.5 Branches starting from Case $(0, 0, 1a, 0, 0)$

We have $c_5 = Q(x)/R(x)$ with

$$Q(x) = 2\alpha(2\alpha + \beta) \ + \ 2\alpha'(2\alpha + \beta)^2\, x \tag{2.78a}$$

$$R(x) = 2\alpha + \beta \ + \ \alpha'\, x \tag{2.78b}$$

$$\mathrm{rem}(Q(x), R(x)) = -2(\alpha + \beta)(2\alpha + \beta) \tag{2.78c}$$

and the conditions $\alpha + \beta \neq 0$ and $\alpha\alpha' \neq 0$ of the Case $(0, 0, 1a, 0, 0)$. The polynomial $R(x)$ is the coefficient c_4 of the Case $(0, 0, 1a, 0, 0)$.

- **$\deg \boldsymbol{R} = \mathbf{0}$:** As $[x^1]R(x) = \alpha \neq 0$, there is no solution of this type.
- **$\deg \boldsymbol{R} = \mathbf{1}$:** If $[x^1]R(x) = \alpha \neq 0$, then $\mathrm{rem}(Q(x), R(x)) = 0$ gives the case $2\alpha + \beta = 0$: $\boldsymbol{\mu} = \big(\alpha, -2\alpha, -\alpha, \alpha', -\alpha', -\alpha'\big)$ with $\alpha\alpha' \neq 0$. In this case, $c_5 = 0$, so this solution should not be taken into account.

This branch does not have any viable solution. Therefore, Case $(0, 0, 1a, 0, 0)$ is a leaf of the decision tree.

2.6.6 Branches starting from Case $(0, 0, 1a, 0, 1b)$

We find that $c_5 = 2\alpha + (2\alpha' + \gamma')\, x$ is a nonzero polynomial in x, due to the inequations $\alpha\gamma' \neq 0$, $\alpha' + \gamma' \neq 0$, and $\alpha' + 2\gamma' \neq 0$ of the Case $(0, 0, 1a, 0, 1b)$. Then $R(x) = 1$, and we have only one case:

- **$\deg \boldsymbol{R} = \mathbf{0}$: Case $(\mathbf{0}, \mathbf{0}, \mathbf{1a}, \mathbf{0}, \mathbf{1b}, \mathbf{0})$**: we have

$$\boldsymbol{\mu} \;=\; \left(\alpha, -\frac{\alpha\alpha'}{\alpha' + \gamma'}, -\alpha, \alpha', -\alpha', \gamma'\right) \tag{2.79}$$

  with $\alpha\gamma' \neq 0$, $\alpha' + \gamma' \neq 0$, and $\alpha' + 2\gamma' \neq 0$.

2.6.7 Branches starting from Case $(0, 0, 1a, 1b, 0)$

We find that $c_5 = 2\alpha + (\alpha' + \beta')\, x$ is a nonzero polynomial in x, due to the inequations $\alpha\alpha' \neq 0$, $\alpha' + \beta' \neq 0$, and $2\alpha' + \beta' \neq 0$ of the Case $(0, 0, 1a, 1b, 0)$. Then $R(x) = 1$, and we have only one case:

- **$\deg \boldsymbol{R} = \mathbf{0}$: Case $(\mathbf{0}, \mathbf{0}, \mathbf{1a}, \mathbf{1b}, \mathbf{0}, \mathbf{0})$**: we have

$$\boldsymbol{\mu} \;=\; \left(\alpha, \frac{\alpha\beta'}{\alpha'}, -\alpha, \alpha', \beta', -2\alpha' - 2\beta'\right) \tag{2.80}$$

  with $\alpha' + \beta' \neq 0$, $2\alpha' + \beta' \neq 0$, and $\alpha\alpha' \neq 0$.

2.6.8 Branches starting from Case $(0, 0, 1a, 1b, 1a)$

We have $c_5 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) \;&=\; 2\alpha^2(2\alpha' + \beta') \;+\; 2\alpha(2\alpha' + \beta')^2\, x \\ &\qquad + 2\alpha'(\alpha' + \beta')(2\alpha' + \beta')\, x^2 \end{aligned} \tag{2.81a}$$

$$R(x) \;=\; \alpha(2\alpha' + \beta') \;+\; 2\alpha'(\alpha' + \beta')\, x \tag{2.81b}$$

$$\operatorname{rem}(Q(x), R(x)) \;=\; -\frac{\alpha^2(\beta')^2(2\alpha' + \beta')}{2\alpha'(\alpha' + \beta')} \tag{2.81c}$$

and the conditions $\alpha' + \beta' \neq 0$ and $\alpha'\beta' \neq 0$ of the Case $(0, 0, 1a, 1b, 1a)$. The polynomial $R(x)$ is equal to the coefficient c_4 of the Case $(0, 0, 1a, 1b, 1a)$ multiplied by $\alpha' \neq 0$.

- **$\deg \boldsymbol{R} = \mathbf{0}$:** As $[x^1]R(x) = 2\alpha'(\alpha' + \beta') \neq 0$, there is no solution of this type.
- **$\deg \boldsymbol{R} = \mathbf{0}$:** If $[x^1]R(x) = 2\alpha'(\alpha' + \beta') \neq 0$, then $\operatorname{rem}(Q(x), R(x)) = 0$ gives two cases $\alpha = 0$, and $2\alpha' + \beta' = 0$. The first one gives $\boldsymbol{\mu} = (0, 0, 0, \alpha', \beta', -\alpha' - 2\beta')$. But this is the specialization of Family 2a [i.e., Case $(0, 0, 1a, 1a)$] when $\alpha = 0$ and $\gamma' = -\alpha' - 2\beta'$. Therefore, it should be discarded. The latter case gives: $\boldsymbol{\mu} = (\alpha, -2\alpha, -\alpha, \alpha', -2\alpha', -3\alpha')$ with $\alpha'\beta' \neq 0$. In this case, $c_5 = 0$, so this solution should not be taken into account.

This branch does not have any viable solution. Therefore, Case $(0, 0, 1a, 1b, 1a)$ is a leaf of the decision tree.

2.6.9 Branches starting from Case $(0, 0, 1b, 0, 0)$

We have $c_5 = Q(x)/R(x)$ with

$$Q(x) = 2\alpha(3\alpha + \gamma) + 2\alpha'(3\alpha + \gamma)\, x \tag{2.82a}$$

$$R(x) = 2\alpha + \alpha'\, x \tag{2.82b}$$

$$\operatorname{rem}(Q(x), R(x)) = -2\alpha(3\alpha + \gamma) \tag{2.82c}$$

and the conditions $\alpha + \gamma \neq 0$, $2\alpha + \gamma \neq 0$, and $\alpha\alpha' \neq 0$ of the Case $(0, 0, 1b, 0, 0)$. The polynomial $R(x)$ is equal to the coefficient c_4 of the Case $(0, 0, 1b, 0, 0)$.

- **deg $\boldsymbol{R} = 0$:** As $[x^1]R(x) = \alpha' \neq 0$, there is no solution of this type.
- If $[x^1]R(x) = \alpha' \neq 0$, then $\operatorname{rem}(Q(x), R(x)) = 0$ gives the case $3\alpha + \gamma = 0$: $\boldsymbol{\mu} = (\alpha, -2\alpha, -3\alpha, \alpha', -\alpha', -\alpha')$ with $\alpha\alpha' \neq 0$. In this case, $c_5 = 0$, so this solution should not be taken into account.

This branch does not have any viable solution. Therefore, Case $(0, 0, 1b, 0, 0)$ is a leaf of the decision tree.

2.6.10 Branches starting from Case $(0, 0, 1b, 0, 1a)$

We find that $c_5 = \alpha + (2\alpha' + \gamma')\, x$ is a non-zero polynomial in x due to the inequations $\alpha\gamma' \neq 0$, $\alpha' + \gamma' \neq 0$, and $\alpha' + 2\gamma' \neq 0$ of the Case $(0, 0, 1b, 0, 1a)$. Then $R(x) = 1$, and we have only one case:

- **deg $\boldsymbol{R} = 0$: Case $(0, 0, 1b, 0, 1a, 0)$**: we have

$$\boldsymbol{\mu} = \left(\alpha, \frac{\alpha\alpha'}{\gamma'}, -2\alpha, \alpha', -\alpha', \gamma'\right) \tag{2.83}$$

  with $\alpha\gamma' \neq 0$, $\alpha' + \gamma' \neq 0$, and $\alpha' + 2\gamma' \neq 0$

2.6.11 Branches starting from Case $(0, 0, 1b, 1a, 0)$

We find that

$$c_5 = \frac{\gamma(\alpha - \beta')}{\alpha' + \beta'} + (\alpha' + \beta')\, x \tag{2.84}$$

is a nonzero polynomial in x, due to the conditions $\gamma\alpha' \neq 0$, $\alpha' + \beta' \neq 0$, and $2\alpha' + \beta' \neq 0$ of the Case $(0, 0, 1b, 1a, 0)$. Then $R(x) = 1$, and we have only one case:

- **deg $\boldsymbol{R} = 0$: Case $(0, 0, 1b, 1a, 0, 0)$**: we have

$$\boldsymbol{\mu} = \left(0, -\frac{\gamma\beta'}{\alpha' + \beta'}, \gamma, \alpha', \beta', -2\alpha' - 2\beta'\right) \tag{2.85}$$

  with $\gamma\alpha' \neq 0$, $\alpha' + \beta' \neq 0$, and $2\alpha' + \beta' \neq 0$.

2.6.12 Branches starting from Case $(0,0,1\mathrm{b},1\mathrm{a},1\mathrm{b})$

We have $c_5 = Q(x)/R(x)$ with

$$Q(x) \;=\; 2\beta(2\alpha'+\beta')\,x \;+\; 2(\alpha'+\beta')(2\alpha'+\beta')\,x^2 \tag{2.86a}$$

$$R(x) \;=\; \beta \;+\; 2(\alpha'+\beta')\,x \tag{2.86b}$$

$$\operatorname{rem}(Q(x),R(x)) \;=\; -\frac{\beta^2(2\alpha'+\beta')}{2(\alpha'+\beta')} \tag{2.86c}$$

and the conditions $\alpha'+\beta' \neq 0$ and $\beta\alpha'\beta' \neq 0$ of the Case $(0,0,1\mathrm{b},1\mathrm{a},1\mathrm{b})$. The polynomial $R(x)$ is equal to the coefficient c_4 of the Case $(0,0,1\mathrm{b},1\mathrm{a},1\mathrm{b})$.

- **deg $\boldsymbol{R} = 0$:** As $[x^1]R(x) = 2(\alpha'+\gamma') \neq 0$, there are no solution of this type.
- **deg $\boldsymbol{R} = 1$:** If $[x^1]R(x) = 2(\alpha'+\gamma') \neq 0$, then $\operatorname{rem}(Q(x),R(x)) = 0$ gives the case $2\alpha'+\beta' = 0$: $\boldsymbol{\mu} = (0,\beta,-\beta,\alpha',-2\alpha',3\alpha')$ with $\beta\alpha' \neq 0$. In this case, $c_5 = 0$, so this solution should not be taken into account.

This branch does not have any viable solution. Therefore, Case $(0,0,1\mathrm{b},1\mathrm{a},1\mathrm{b})$ is a leaf of the decision tree.

2.7 Step 6: Coefficient c_6

We consider the six cases obtained in Section 2.6 that require additional conditions to be terminated. For each of these vertices, we now compute the branches stemming from them by using the next coefficient c_6.

2.7.1 Branches starting from Case $(0,0,0,1\mathrm{a},1\mathrm{b},0)$

We have $c_6 = Q(x)/R(x)$ with

$$Q(x) \;=\; \gamma\alpha'(4\beta^2+9\beta\gamma+3\gamma^2)\,x \;+\; 2(\alpha')^2(\beta+\gamma)(2\beta+3\gamma)\,x^2 \tag{2.87a}$$

$$R(x) \;=\; \gamma^2(2\beta+\gamma) \;+\; 2\gamma\alpha'(\beta+\gamma)\,x \tag{2.87b}$$

$$\operatorname{rem}(Q(x),R(x)) \;=\; -\frac{\beta\gamma^3(2\beta+\gamma)}{2(\beta+\gamma)} \tag{2.87c}$$

with the conditions $\beta+\gamma \neq 0$ and $\gamma\alpha' \neq 0$ of the Case $(0,0,0,1\mathrm{a},1\mathrm{b},0)$. Note that $R(x)$ is equal to the coefficient c_5 of Case $(0,0,0,1\mathrm{a},1\mathrm{b},0)$ multiplied by $\gamma^2 \neq 0$.

- **deg $\boldsymbol{R} = 0$:** As $[x^1]R(x) = \gamma\alpha'(\beta+\gamma) \neq 0$, there is no solution of this type.
- **deg $\boldsymbol{R} = 1$:** If $[x^1]R(x) = \gamma\alpha'(\beta+\gamma) \neq 0$, then $\operatorname{rem}(Q(x),R(x)) = 0$ gives two cases $\beta = 0$, or $2\beta+\gamma = 0$. In the former one, we obtain $\boldsymbol{\mu} = (0,0,\gamma,\alpha',0,-\alpha')$ which is the specialization of Family 5 [i.e., Case $(0,0,1\mathrm{b},1\mathrm{b})$] when $\alpha = 0$ and $\gamma' = -\alpha'$. The latter case give:

  Case $(0,0,0,1\mathrm{a},1\mathrm{b},0,1\mathrm{a})$: we have (after some trivial scale transformation) the family $\boldsymbol{\mu} = (0,\beta,-2\beta,-2\beta',\beta',\beta')$ with $\beta\beta' \neq 0$. In this case, $c_6 = \beta - 4\beta'\,x \neq 0$.

2.7.2 Branches starting from Case $(0,0,0,1\text{b},1\text{a},0)$

We have $c_6 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) &= \alpha^2(3\alpha'+\beta') + \alpha(9(\alpha')^2+11\alpha'\beta'+4(\beta')^2)\,x \\ &\qquad + 2(\alpha')^2(\alpha'+\beta')(3\alpha'+2\beta')\,x^2 \end{aligned} \tag{2.88a}$$

$$R(x) = \alpha'\bigl(\alpha + 2(\alpha'+\beta')\,x\bigr) \tag{2.88b}$$

$$\operatorname{rem}(Q(x),R(x)) = -\frac{\alpha^2\beta'(\alpha'+2\beta')}{2(\alpha'+\beta')} \tag{2.88c}$$

and the conditions $\alpha'+\beta' \neq 0$, $2\alpha'+\beta' \neq 0$, and $\alpha\alpha' \neq 0$ of the Case $(0,0,0,1\text{b},1\text{a},0)$. Note that $R(x)$ is equal to the coefficient c_5 of the Case $(0,0,0,1\text{b},1\text{a},0)$ multiplied by $\alpha' \neq 0$.

- **$\deg \boldsymbol{R} = 0$:** As $[x^1]R(x) = 2\alpha'(\alpha'+\beta') \neq 0$, there is no solution of this type.
- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = 2\alpha'(\alpha'+\beta') \neq 0$, then $\operatorname{rem}(Q(x),R(x)) = 0$ gives two cases: $\beta' = 0$, and $\alpha'+2\beta' = 0$. The former one leads to $\boldsymbol{\mu} = (\alpha,0,-2\alpha,\alpha',0,-\alpha')$, which is a specialization of Family 5 [i.e., Case $(0,0,1\text{b},\ 1\text{b})$] when $\gamma = -2\alpha$ and $\gamma' = -\alpha'$; so it must be discarded. The other one gives:

  Case $(0,0,0,1\text{b},1\text{a},0,1\text{a})$: we have (after some trivial scale transformations) the family $\boldsymbol{\mu} = (-2\beta,\beta,4\beta,-2\beta',\beta',\beta')$ with $\beta\beta' \neq 0$. In this case, $c_6 = -5\beta - 4\beta'\,x \neq 0$.

2.7.3 Branches starting from Case $(0,0,1\text{a},0,1\text{b},0)$

We have $c_6 = Q(x)/R(x)$ with

$$Q(x) = 2\alpha^2(2\alpha'+3\gamma') + \alpha\bigl(4(\alpha')^2+9\alpha'\gamma'+3(\gamma')^2\bigr)\,x \tag{2.89a}$$

$$R(x) = (\alpha'+\gamma')\bigl(2\alpha + (2\alpha'+\gamma')\,x\bigr) \tag{2.89b}$$

$$\operatorname{rem}(Q(x),R(x)) = -\frac{2\alpha^2\alpha'\gamma'}{2\alpha'+\gamma'} \tag{2.89c}$$

with $\alpha\gamma' \neq 0$, $\alpha'+\gamma' \neq 0$, and $\alpha'+2\gamma' \neq 0$. Note that $R(x)$ is equal to the coefficient of Case $(0,0,1\text{a},0,1\text{b},0)$ multiplied by $\alpha'+\gamma' \neq 0$.

- **$\deg \boldsymbol{R} = 0$:** If $[x^1]R(x) = (\alpha'+\gamma')(2\alpha'+\gamma') = 0$, and $[x^0]R(x) = 2\alpha(\alpha'+\gamma') \neq 0$, they reduce to $2\alpha'+\gamma' = 0$ if we take into account the above inequations. This solution leads to:

  Case $(0,0,1\text{a},0,1\text{b},0,0)$: we have the family $\boldsymbol{\mu} = (\alpha,\alpha,-\alpha,\alpha',-\alpha',-2\alpha')$ with $\alpha\alpha' \neq 0$. In this case, $c_6 = 4\alpha + \alpha'\,x \neq 0$.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = (\alpha'+\gamma')(2\alpha'+\gamma') \neq 0$, this condition reduces to $2\alpha'+\gamma' \neq 0$. Then $\mathrm{rem}(Q(x), R(x)) = 0$ gives the case $\alpha' = 0$. We obtain the family $\boldsymbol{\mu} = (\alpha, 0, -\alpha, 0, 0, \gamma')$; but this is the specialization of Family 5 [i.e., Case $(0, 0, 1\mathrm{b}, 1\mathrm{b})$] when $\gamma = -\alpha$ and $\alpha' = 0$; so it must be discarded.

2.7.4 Branches starting from Case $(0, 0, 1\mathrm{a}, 1\mathrm{b}, 0, 0)$

We have $c_6 = Q(x)/R(x)$ with

$$\begin{aligned} Q(x) &= 2\alpha^2(3\alpha' + \beta') + \alpha\big(9(\alpha')^2 + 7\alpha'\beta' + 2(\beta')^2\big)\, x \\ &\quad + \alpha'(\alpha' + \beta')(3\alpha' + 2\beta')\, x^2 \end{aligned} \tag{2.90a}$$

$$R(x) = \alpha' \left(2\alpha + (\alpha' + \beta')\, x\right) \tag{2.90b}$$

$$\mathrm{rem}(Q(x), R(x)) = \frac{2\alpha^2\beta'(\alpha' - \beta')}{\alpha' + \beta'} \tag{2.90c}$$

and the conditions $\alpha' + \beta' \neq 0$, $2\alpha' + \beta' \neq 0$, and $\alpha\alpha' \neq 0$ of Case $(0, 0, 1\mathrm{a}, 1\mathrm{b}, 0, 0)$. Note that $R(x)$ is equal to the coefficient c_5 of the Case $(0, 0, 1\mathrm{a}, 1\mathrm{b}, 0, 0)$ multiplied by $\alpha' \neq 0$.

- **$\deg \boldsymbol{R} = 0$:** As $[x^1]R(x) = \alpha'(\alpha' + \beta') \neq 0$, there are no solution of this type.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = \alpha'(\alpha'+\beta') \neq 0$, then $\mathrm{rem}(Q(x), R(x)) = 0$ gives two cases $\beta' = 0$, and $\alpha' - \beta' = 0$. The former one corresponds to $\boldsymbol{\mu} = (\alpha, 0, -\alpha, \alpha', 0, -2\alpha')$; but this is the specialization of Family 5 [i.e., Case $(0, 0,\ 1\mathrm{b}, 1\mathrm{b})$] when $\gamma = -\alpha$ and $\gamma' = -2\alpha'$; so it must be discarded. The latter case gives:

  Case $(0, 0, 1\mathrm{a}, 1\mathrm{b}, 0, 0, 1\mathrm{a})$: we have the family $\boldsymbol{\mu} = (\alpha, \alpha, -\alpha, \alpha', \alpha', -4\alpha')$ with $\alpha\alpha' \neq 0$. In this case, $c_6 = 4\alpha + 5\alpha'\, x \neq 0$.

2.7.5 Branches starting from Case $(0, 0, 1\mathrm{b}, 0, 1\mathrm{a}, 0)$

We have $c_6 = Q(x)/R(x)$ with

$$Q(x) = \alpha^2(\alpha' + 3\gamma') + \alpha\big(4(\alpha')^2 + 9\alpha'\gamma' + 2(\gamma')^2\big)\, x \tag{2.91a}$$

$$R(x) = \gamma' \left(\alpha + (2\alpha' + \gamma')\, x\right) \tag{2.91b}$$

$$\mathrm{rem}(Q(x), R(x)) = -\frac{2\alpha^2\alpha'(\alpha' + \beta')}{2\alpha' + \gamma'} \tag{2.91c}$$

with the conditions $\alpha\gamma' \neq 0$, $\alpha' + \gamma' \neq 0$, and $\alpha' + 2\gamma' \neq 0$ of Case $(0, 0, 1\mathrm{b}, 0, 1\mathrm{a}, 0)$. Note that $R(x)$ is equal to the coefficient c_5 of the Case $(0, 0, 1\mathrm{b}, 0, 1\mathrm{a}, 0)$ multiplied by $\gamma' \neq 0$.

- **$\deg \boldsymbol{R} = 0$:** We have $[x^1]R(x) = \gamma'(2\alpha' + \gamma') = 0$ and $[x^0]R(x) = \alpha\gamma' \neq 0$, which reduce to $2\alpha' + \gamma' = 0$. The single solution leads to:

  Case $(0, 0, 1\mathrm{b}, 0, 1\mathrm{a}, 0, 0)$: we have (after a trivial change of parameters) the family $\boldsymbol{\mu} = (-2\beta, \beta, 4\beta, \alpha', -\alpha', -2\alpha')$ with $\beta\alpha' \neq 0$. In this case, $c_6 = -5\beta + \alpha'\, x \neq 0$.

- **$\deg \boldsymbol{R} = 1$:** If $[x^1]R(x) = \gamma'(2\alpha' + \gamma') \neq 0$, then $\mathrm{rem}(Q(x), R(x)) = 0$ gives the case $\alpha' = 0$: $\boldsymbol{\mu} = (\alpha, 0, -2\alpha, 0, 0, \gamma')$, but this is the specialization of Family 5 [i.e., Case $(0, 0, 1\mathrm{b}, 1\mathrm{b})$] when $\gamma = -2\alpha$ and $\alpha' = 0$; so it must be discarded.

2.7.6 Branches starting from Case $(0, 0, 1b, 1a, 0, 0)$

We have $c_6 = Q(x)/R(x)$ with

$$Q(x) = \gamma\big(3(\alpha')^2 - 3\alpha'\beta' - 2(\beta')^2\big)\, x \ + \ (\alpha' + \gamma')^2(3\alpha' + 2\beta')\, x^2 \quad (2.92a)$$

$$R(x) = \gamma(\alpha' - \beta') \ + \ (\alpha' + \beta')^2\, x \quad (2.92b)$$

$$\mathrm{rem}(Q(x), R(x)) = \frac{2\gamma^2\alpha'\beta'(\alpha' - \beta')}{(\alpha' + \gamma')^2} \quad (2.92c)$$

with the conditions $\gamma\alpha' \neq 0$, $\alpha' + \beta' \neq 0$, and $2\alpha' + \beta' \neq 0$ of Case $(0, 0, 1b, 1a, 0, 0)$. Note that $R(x)$ is equal to the coefficient c_5 of the Case $(0, 0, 1b, 1a, 0, 0)$ multiplied by $\alpha' + \beta' \neq 0$.

- **deg $\boldsymbol{R} = 0$:** As $[x^1]R(x) = (\alpha' + \gamma')^2 \neq 0$, there is no solution of this type.
- **deg $\boldsymbol{R} = 1$:** If $[x^1]R(x) = (\alpha' + \gamma')^2 \neq 0$, then $\mathrm{rem}(Q(x), R(x)) = 0$ gives two cases $\beta' = 0$ and $\alpha' - \beta' = 0$. The former one leads to $\boldsymbol{\mu} = (0, 0, \gamma, \alpha', 0, -2\alpha')$, which corresponds to the specialization of Family 5 [i.e., Case $(0, 0, 1b, 1b)$] when $\alpha = 0$ and $\gamma' = -2\alpha'$. Therefore, this solution should be discarded. The latter case gives:

  Case $(0, 0, 1b, 1a, 0, 0, 1a)$: we have (after a trivial change of parameters) the family $\boldsymbol{\mu} = (0, \beta, -2\beta, \alpha', \alpha', -4\alpha')$ with $\beta\alpha' \neq 0$. In this case, $c_6 = \beta + 5\alpha'\, x \neq 0$.

2.8 Step 7: Coefficient c_7

We consider the six cases obtained in Section 2.7 that require additional conditions to be terminated. For each of these vertices, we now compute the branches stemming from them by using the next coefficient c_7. It will turn out that none of the six cases have any viable solutions; therefore, all six correspond to leaves of the decision tree, and this is the end of the computation.

2.8.1 Branches starting from Case $(0, 0, 0, 1a, 1b, 0, 1a)$

We find that $c_7 = Q(x)/R(x)$ with

$$Q(x) = -8\beta\beta'\, x \ + \ 12(\beta')^2\, x^2 \quad (2.93a)$$

$$R(x) = \beta \ - \ 4\beta'\, x \quad (2.93b)$$

$$\mathrm{rem}(Q(x), R(x)) = -\frac{5\beta^2}{4} \quad (2.93c)$$

and the conditions $\beta\beta' \neq 0$ of the Case $(0, 0, 0, 1a, 1b, 0, 1a)$. Note that $R(x)$ is equal to the coefficient c_6 of the Case $(0, 0, 0, 1a, 1b, 0, 1a)$.

- **deg $\boldsymbol{R} = 0$:** As $[x^1]R(x) = -4\beta' \neq 0$, there is no solution of this type.

- **deg $\boldsymbol{R}$ = 1:** In this case, $\mathrm{rem}(Q(x), R(x)) \neq 0$, so there is no solution of this type.

This branch does not have any viable solution, so Case $(0, 0, 0, 1\mathrm{a}, 1\mathrm{b}, 0, 1\mathrm{a})$ is a leaf of the decision tree.

2.8.2 Branches starting from Case $(0, 0, 0, 1\mathrm{b}, 1\mathrm{a}, 0, 1\mathrm{a})$

We find that $c_7 = Q(x)/R(x)$ with

$$Q(x) = -20\beta^2 - 32\beta\beta' x - 12(\beta')^2 x^2 \tag{2.94a}$$

$$R(x) = 5\beta + 4\beta' x \tag{2.94b}$$

$$\mathrm{rem}(Q(x), R(x)) = -\frac{5\beta^2}{4} \tag{2.94c}$$

and the conditions $\beta\beta' \neq 0$ of the Case $(0, 0, 0, 1\mathrm{b}, 1\mathrm{a}, 0, 1\mathrm{a})$. Note that $-R(x)$ is equal to the coefficient c_6 of the Case $(0, 0, 0, 1\mathrm{b}, 1\mathrm{a}, 0, 1\mathrm{a})$.

- **deg $\boldsymbol{R}$ = 0:** As $[x^1]R(x) = -4\beta' \neq 0$, there is no solution of this type.
- **deg $\boldsymbol{R}$ = 1:** In this case, $\mathrm{rem}(Q(x), R(x)) \neq 0$, so there is no solution of this type.

This branch does not have any viable solution, so Case $(0, 0, 0, 1\mathrm{b}, 1\mathrm{a}, 0, 1\mathrm{a})$ is a leaf of the decision tree.

2.8.3 Branches starting from Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{b}, 0, 0)$

We find that $c_7 = Q(x)/R(x)$ with

$$Q(x) = 12\alpha^2 + 8\alpha\alpha' x \tag{2.95a}$$

$$R(x) = 4\alpha + \alpha' x \tag{2.95b}$$

$$\mathrm{rem}(Q(x), R(x)) = -20\alpha^2 \tag{2.95c}$$

and the conditions $\alpha\alpha' \neq 0$ of the Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{a}, 0, 0)$. Note that $R(x)$ is equal to the coefficient c_6 of Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{b}, 0, 0)$.

- **deg $\boldsymbol{R}$ = 0:** As $[x^1]R(x) = \alpha' \neq 0$, there is no solution of this type.
- **deg $\boldsymbol{R}$ = 1:** In this case, $\mathrm{rem}(Q(x), R(x)) \neq 0$, so there is no solution of this type.

This branch does not have any viable solution, so Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{b}, 0, 0)$ is a leaf of the decision tree.

2.8.4 Branches starting from Case $(\mathbf{0,0,1a,1b,0,0,1a})$

We find that $c_7 = Q(x)/R(x)$ with

$$Q(x) = 12\alpha^2 + 32\alpha\alpha' x + 20(\alpha')^2 x^2 \tag{2.96a}$$

$$R(x) = 4\alpha + 5\alpha' x \tag{2.96b}$$

$$\mathrm{rem}(Q(x), R(x)) = -\frac{4\alpha^2}{5} \tag{2.96c}$$

and the conditions $\alpha\alpha' \neq 0$ of the Case $(0, 0, 1a, 1b, 0, 0, 1a)$. Note that $R(x)$ is equal to the coefficient c_6 of Case $(0, 0, 1a, 1b, 0, 0, 1a)$.

- $\mathbf{deg}\,\boldsymbol{R} = \mathbf{0}$**:** As $[x^1]R(x) = 5\alpha' \neq 0$, there is no solution of this type.
- $\mathbf{deg}\,\boldsymbol{R} = \mathbf{1}$**:** In this case, $\mathrm{rem}(Q(x), R(x)) \neq 0$, so there is no solution of this type.

This branch does not have any viable solution, so Case $(0, 0, 1a, 1b, 0, 0, 1a)$ is a leaf of the decision tree.

2.8.5 Branches starting from Case $(\mathbf{0,0,1b,0,1a,0,0})$

We find that $c_7 = Q(x)/R(x)$ with

$$Q(x) = -20\beta^2 + 8\beta\alpha' x \tag{2.97a}$$

$$R(x) = 5\beta - \alpha' x \tag{2.97b}$$

$$\mathrm{rem}(Q(x), R(x)) = 20\beta^2 \tag{2.97c}$$

and the conditions $\beta\alpha' \neq 0$ of the Case $(0, 0, 1b, 0, 1a, 0, 0)$. Note that $-R(x)$ is equal to the coefficient c_6 of Case $(0, 0, 1b, 0, 1a, 0, 0)$.

- $\mathbf{deg}\,\boldsymbol{R} = \mathbf{0}$**:** As $[x^1]R(x) = -\alpha' \neq 0$, there is no solution of this type.
- $\mathbf{deg}\,\boldsymbol{R} = \mathbf{1}$**:** In this case, $\mathrm{rem}(Q(x), R(x)) \neq 0$, so there is no solution of this type.

This branch does not have any viable solution, so Case $(0, 0, 1b, 0, 1a, 0, 0)$ is a leaf of the decision tree.

2.8.6 Branches starting from Case $(\mathbf{0,0,1b,1a,0,0,1a})$

We find that $c_7 = Q(x)/R(x)$ with

$$Q(x) = 8\beta\alpha' x + 20(\alpha')^2 x^2 \tag{2.98a}$$

$$R(x) = \beta + 5\alpha' x \tag{2.98b}$$

$$\mathrm{rem}(Q(x), R(x)) = -\frac{4\beta^2}{5} \tag{2.98c}$$

and the conditions $\beta\alpha' \neq 0$ of the Case $(0, 0, 1b, 1a, 0, 0, 1a)$. Note that $R(x)$ is equal to the coefficient c_6 of Case $(0, 0, 1b, 1a, 0, 0, 1a)$.

- **$\deg R = 0$:** As $[x^1]R(x) = 5\alpha' \neq 0$, there is no solution of this type.
- **$\deg R = 1$:** In this case, $\mathrm{rem}(Q(x), R(x)) \neq 0$, so there is no solution of this type.

This branch does not have any viable solution, so Case $(0, 0, 1\mathrm{b}, 1\mathrm{a}, 0, 0, 1\mathrm{a})$ is a leaf of the decision tree.

3 Non-terminating S-fractions

The final result of Section 2 is that there are ten families that are viable candidates for having an ogf with a non-terminating S-fraction representation. These families are:

1a. Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{a})$ given by $\boldsymbol{\mu} = (0, \beta, 0, \alpha', -\alpha', \gamma')$ with $\alpha' + \gamma' \neq 0$ and $\beta\gamma' \neq 0$. The conjectured coefficients are $c_{2k-1} = [\gamma' + (k-1)\alpha']x$ and $c_{2k} = k\beta$.

1b. Case $(0, 0, 0, 1\mathrm{a}, 1\mathrm{b})$ given by $\boldsymbol{\mu} = (0, \beta, \gamma, \alpha', -\alpha', 0)$ with $\beta + \gamma \neq 0$ and $\gamma\alpha' \neq 0$. The conjectured coefficients are $c_{2k-1} = \gamma + (k-1)\beta$ and $c_{2k} = k\alpha' x$.

2a. Case $(0, 0, 1\mathrm{a}, 1\mathrm{a})$ given by $\boldsymbol{\mu} = (\alpha, -\alpha, -\alpha, \alpha', \beta', \gamma')$ with $\alpha'+\beta'+\gamma' \neq 0$, $\alpha'+\beta' \neq 0$, and $2\alpha' + 2\beta' + \gamma' \neq 0$. The conjectured coefficients are $c_{2k-1} = [\gamma' + k(\alpha' + \beta')]x$ and $c_{2k} = k(\alpha' + \beta')x$.

2b. Case $(0, 0, 0, 0)$ given by $\boldsymbol{\mu} = (\alpha, \beta, \gamma, 0, \beta', -\beta')$ with $\alpha + \gamma \neq 0$, $2\alpha + \gamma \neq 0$, and $\alpha \neq 0$. The conjectured coefficients are $c_{2k-1} = k\alpha + \gamma$ and $c_{2k} = k\alpha$.

3a. Case $(0, 0, 1\mathrm{a}, 1\mathrm{a})$ given by $\boldsymbol{\mu} = (0, \beta, 0, 0, \beta', \gamma')$ with $\beta' + \gamma' \neq 0$, $2\beta' + \gamma' \neq 0$, and $\beta' \neq 0$. The conjectured coefficients are $c_{2k-1} = (\gamma' + k\beta')x$ and $c_{2k} = k(\beta + \beta' x)$.

3b. Case $(0, 0, 0, 1\mathrm{b}, 0)$ given by $\boldsymbol{\mu} = (\alpha, -\alpha, \gamma, \alpha', -\alpha', 0)$ with $\alpha + \gamma \neq 0$, $\alpha + 2\gamma \neq 0$, and $\alpha' \neq 0$. The conjectured coefficients are $c_{2k-1} = k\alpha + \gamma$ and $c_{2k} = k(\alpha + \alpha' x)$.

4a. Case $(0, 0, 1\mathrm{b}, 1\mathrm{a}, 1\mathrm{a})$ given by $\boldsymbol{\mu} = (0, \kappa\beta', \kappa(\beta' + \gamma'), 0, \beta', \gamma')$ with $\beta' + \gamma' \neq 0$, $2\beta' + \gamma' \neq 0$, and $\kappa\beta' \neq 0$. The conjectured coefficients are $c_{2k-1} = (k\beta' + \gamma')(\kappa + x)$ and $c_{2k} = k\beta' x$.

4b. Case $(0, 0, 1\mathrm{b}, 0, 1\mathrm{b})$ given by $\boldsymbol{\mu} = (\alpha, -\alpha, \gamma, \kappa\alpha, -\kappa\alpha, \kappa(\alpha + \gamma))$ with $\alpha + \gamma \neq 0$, $2\alpha + \gamma \neq 0$, and $\alpha\kappa \neq 0$. The conjectured coefficients are $c_{2k-1} = (\gamma + k\alpha)(1 + \kappa x)$ and $c_{2k} = k\alpha$.

5. Case $(0, 0, 1\mathrm{b}, 1\mathrm{b})$ given by $\boldsymbol{\mu} = (\alpha, 0, \gamma, \alpha', 0, \gamma')$ with $\alpha' + \gamma' \neq 0$, $\alpha + \gamma \neq 0$, $\alpha' \neq 0$, and ($2\alpha + \gamma \neq 0$ or $2\alpha' + \gamma' \neq 0$). The conjectured coefficients are $c_{2k-1} = (\gamma + \gamma' x) + k(\alpha + \alpha' x)$ and $c_{2k} = k(\alpha + \alpha' x)$.

6. Case $(0, 0, 1\mathrm{b}, 1\mathrm{c})$ given by $\boldsymbol{\mu} = (\kappa(\alpha' + \beta'), \kappa\beta', \kappa\gamma', \alpha', \beta', \gamma')$ with $\alpha' + \beta' + \gamma' \neq 0$, $\alpha' + \beta' \neq 0$, $2\alpha' + 2\beta' + \gamma' \neq 0$, and $\alpha' \neq 0$. The conjectured coefficients are $c_{2k-1} = [\gamma' + k(\alpha' + \beta')](\kappa + x)$ and $c_{2k} = k(\alpha' + \beta')(\kappa + x)$.

Please note that the inequations that accompany each family are needed only to ensure that the first few coefficients c_i are nonvanishing. Thus, the families obtained in Section 2.4 (i.e., 2a, 2b, 3a, 5, and 6) have inequations that imply that $c_1, c_2, c_3 \neq 0$, while the families obtained in Section 2.5 (i.e., 1a, 1b, 3b, 4a, and 4b) have inequations that imply that $c_1, c_2, c_3, c_4 \neq 0$. Of course, an infinite set of inequations (which are easily deducible from the given formulae) will be needed to ensure that *all* $c_i \neq 0$. But since Theorem 3.1 of the main text explicitly allows that, in each family, the continued fraction might be terminating in some degenerate cases, these inequations can be discarded. This is why they do not appear in the main text.

4 Rational ordinary generating functions

In Section 2 we found the ten families 1a–6 of non-terminating S-fractions compiled in Section 3. A side effect of this computation now to find also several non-trivial rational ogfs (that is, families $\boldsymbol{\mu} \in \mathbb{C}^6$ that lead to terminating S-fractions). By non-trivial, we mean that they are *not* particular cases of any of the ten families 1a–6.

In Section 2.2, we find the first non-trivial family with a rational ogf. The Case $(0,0)$ provides the coefficient $c_1 = \alpha + \gamma + (\alpha' + \beta' + \gamma')x$, which vanishes on the submanifold $\alpha + \gamma = \alpha' + \beta' + \gamma' = 0$. This leads to the self-dual family:

s0. $\boldsymbol{\mu} = (\alpha, \beta, -\alpha, \alpha', \beta', -\alpha' - \beta')$ corresponding to Case $(0,0)$. The coefficients are $c_0 = 1$, and $c_1 = 0$.

Of course, this case corresponds to the trivial matrix $T(n,k) = \delta_{n0}\delta_{k0}$. (We remark that although the parameters $\boldsymbol{\mu}$ corresponding to Case $(0,0)$ are not a special case of any of the families 1a–6, this trivial matrix *is* a special case of *all* the families 1a–6: it suffices to specialize the parameters so that $c_1 = 0$. For instance, in the family 1a we take $\gamma' = 0$. This illustrates the parametric ambiguities discussed in [56, section 2.4] [2, section 3], corresponding to the non-injectivity of the map $\boldsymbol{\mu} \mapsto \boldsymbol{T}(\boldsymbol{\mu})$.)

In Section 2.3, we find that c_2 vanishes in three submanifolds corresponding to the families $\boldsymbol{\mu} = (0, \beta, \gamma, 0, \beta', -\beta')$ [Case $(0,0,0)$], $\boldsymbol{\mu} = (\alpha, -\alpha, -\alpha, \alpha', -\alpha', \gamma')$ [Case $(0,0,1a)$], and $\boldsymbol{\mu} = (0, -\gamma\alpha', \gamma\gamma', \alpha', -\alpha', \gamma')$ [Case $(0,0,1b)$]. These families are actually specializations of Family 2b (when $\alpha = 0$), Family 2a (when $\beta' = -\alpha'$), and Family 6 (when $\beta' = -\alpha'$ and $\kappa = \gamma$), respectively. Therefore, they all are trivial families leading to a rational ogf.

In Section 2.4, we have six possible submanifolds where $c_3 = 0$. We find that all of them correspond to specializations of families 1a–6, so they correspond to trivial rational ogf.

On the other hand, in Section 2.5 we find six non-trivial rational ogfs. For instance, in Case $(0,0,0,\ 1a,1b)$, we found that $c_4 = [\alpha'(\beta + 2\gamma)/\gamma]\, x$. This coefficient can be made equal to zero if $\beta + 2\gamma = 0$. This leads to $\boldsymbol{\mu} = (0, -2\gamma, \gamma, \alpha', -2\alpha', \alpha')$, which corresponds to the family s1a below. We can work out the other five cases in a similar fashion. The results can be summarized in the following list:

s1a. $\boldsymbol{\mu} = (0, -2\gamma, \gamma, \alpha', -2\alpha', \alpha')$ corresponding to Case $(0, 0, 0, 1\mathrm{a}, 1\mathrm{b})$. The coefficients are $c_1 = \gamma$, $c_2 = \alpha' x$, $c_3 = -\gamma - \alpha' x$, and $c_4 = 0$.

s1b. $\boldsymbol{\mu} = (\alpha, -2\alpha, -\alpha, -2\gamma', 2\gamma', \gamma')$ corresponding to Case $(0, 0, 1\mathrm{a}, 0, 1\mathrm{b})$. The coefficients are $c_1 = \gamma' x$, $c_2 = -\alpha$, $c_3 = \alpha - \gamma' x$, and $c_4 = 0$.

s2a. $\boldsymbol{\mu} = (0, -2\gamma, \gamma, \alpha', -2\alpha', 2\alpha')$ corresponding to Case $(0, 0, 1\mathrm{b}, 1\mathrm{a}, 0)$. The coefficients are $c_1 = \gamma + \alpha' x$, $c_2 = -\alpha' x$, $c_3 = -\gamma$, and $c_4 = 0$.

s2b. $\boldsymbol{\mu} = (\alpha, -2\alpha, -2\alpha, -2\gamma', 2\gamma', \gamma')$ corresponding to Case $(0, 0, 1\mathrm{b}, 0, 1\mathrm{a})$. The coefficients are $c_1 = -\alpha + \gamma' x$, $c_2 = \alpha$, $c_3 = -\gamma' x$, and $c_4 = 0$.

s3a. $\boldsymbol{\mu} = (\alpha, -2\alpha, -2\alpha, \alpha', -2\alpha', \alpha')$ corresponding to Case $(0, 0, 0, 1\mathrm{b}, 1\mathrm{a})$. The coefficients are $c_1 = -\alpha$, $c_2 = \alpha + \alpha' x$, $c_3 = -\alpha' x$, and $c_4 = 0$.

s3b. $\boldsymbol{\mu} = (\alpha, -2\alpha, -\alpha, \alpha', -2\alpha', 2\alpha')$ corresponding to Case $(0, 0, 1\mathrm{a}, 1\mathrm{b}, 0)$. The coefficients are $c_1 = \alpha' x$, $c_2 = -\alpha - \alpha' x$, $c_3 = \alpha$, and $c_4 = 0$.

It is clear that under duality s1a $\leftrightarrow$ s1b, s2a $\leftrightarrow$ s2b, s3a $\leftrightarrow$ s3b.

In Section 2.6, we find explicitly another six non-trivial rational ogf's with $c_5 = 0$:

s4a. $\boldsymbol{\mu} = (0, \beta, -\beta, \alpha', -2\alpha', \alpha')$ corresponding to Case $(0, 0, 0, 1\mathrm{a}, 1\mathrm{a}, 0)$. The coefficients are $c_1 = -\beta$, $c_2 = \alpha' x$, $c_3 = -\alpha' x$, $c_4 = \beta$, and $c_5 = 0$.

s4b. $\boldsymbol{\mu} = (\alpha, -2\alpha, -\alpha, \alpha', -\alpha', -\alpha')$ corresponding to Case $(0, 0, 1\mathrm{a}, 0, 0, 1\mathrm{a})$. The coefficients are $c_1 = -\alpha' x$, $c_2 = -\alpha$, $c_3 = \alpha$, $c_4 = \alpha' x$, and $c_5 = 0$.

s5a. $\boldsymbol{\mu} = (\alpha, -2\alpha, -3\alpha, \alpha', -2\alpha', \alpha')$ corresponding to Case $(0, 0, 0, 1\mathrm{b}, 1\mathrm{b}, 0)$. The coefficients are $c_1 = -2\alpha$, $c_2 = \alpha + \alpha' x$, $c_3 = -\alpha - \alpha' x$, $c_4 = 2\alpha$, and $c_5 = 0$.

s5b. $\boldsymbol{\mu} = (\alpha, -2\alpha, -\alpha, \alpha', -2\alpha', 3\alpha')$ corresponding to Case $(0, 0, 1\mathrm{a}, 1\mathrm{b}, 1\mathrm{a}, 1\mathrm{a})$. The coefficients are $c_1 = 2\alpha' x$, $c_2 = -\alpha - \alpha' x$, $c_3 = \alpha + \alpha' x$, $c_4 = -2\alpha' x$, and $c_5 = 0$.

s6a. $\boldsymbol{\mu} = (\alpha, -2\alpha, -3\alpha, \alpha', -\alpha', -\alpha')$ corresponding to Case $(0, 0, 1\mathrm{b}, 0, 0, 1\mathrm{a})$. The coefficients are $c_1 = -2\alpha - \alpha' x$, $c_2 = \alpha$, $c_3 = -\alpha$, $c_4 = 2\alpha + \alpha' x$, and $c_5 = 0$.

s6b. $\boldsymbol{\mu} = (0, \beta, -\beta, \alpha', -2\alpha', 3\alpha')$ corresponding to Case $(0, 0, 1\mathrm{b}, 1\mathrm{a}, 1\mathrm{b}, 1\mathrm{a})$. The coefficients are $c_1 = -\beta + 2\alpha' x$, $c_2 = -\alpha' x$, $c_3 = \alpha' x$, $c_4 = \beta - 2\alpha' x$, and $c_5 = 0$.

It is clear that under duality s4a $\leftrightarrow$ s4b, s5a $\leftrightarrow$ s5b, s6a $\leftrightarrow$ s6b.

Finally, we have not found any non-trivial rational ogf coming from $c_6 = 0$ or $c_7 = 0$. Therefore, we have found *all* the non-trivial families $\boldsymbol{\mu} \in \mathbb{C}^6$ whose ogf is a rational function in t that can be expressed by a (terminating) S-fraction in which all the coefficients c_i are polynomials in x.

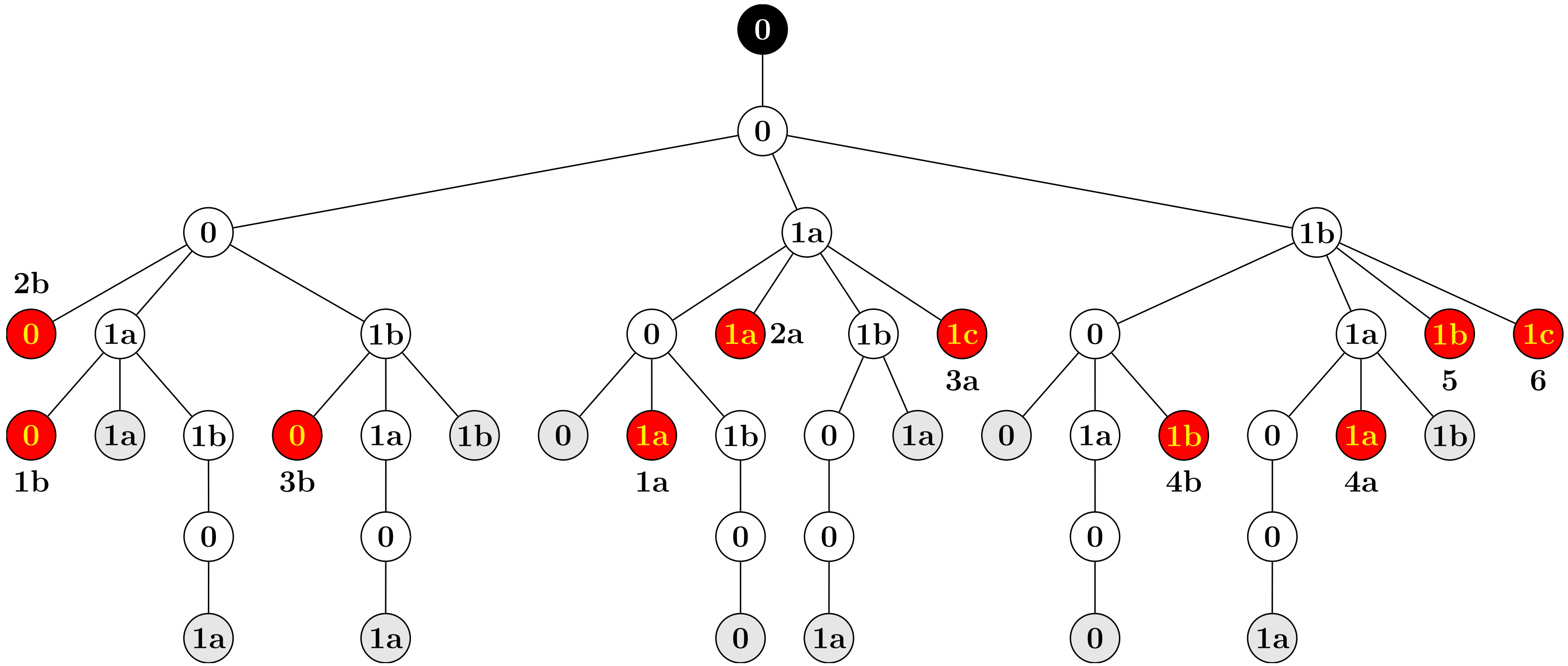

Figure 1: Decision tree for the computation performed in Section 2. The black node is the root vertex, corresponding to Case (0). The successive levels are based on analyzing the coefficients $c_1, c_2, c_3, c_4, c_5, c_6, c_7$. The 10 red nodes correspond to the candidate families found in Section 3; they are labeled as in Theorem 3.1 of the main article. Finally, the 12 gray leaves are the nodes with no children, and the white nodes are the internal vertices of the tree.