Square-free graphs with no induced fork

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Abstract

The *claw* is the graph $K_{1,3}$, and the *fork* is the graph obtained from the claw $K_{1,3}$ by subdividing one of its edges once. In this paper, we prove a structure theorem for the class of (claw, C_4)-free graphs that are not quasi-line graphs, and a structure theorem for the class of (fork, C_4)-free graphs that uses the class of (claw, C_4)-free graphs as a basic class. Finally, we show that every (fork, C_4)-free graph G satisfies $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$ via these structure theorems with some additional work on coloring basic classes.

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1 Introduction

All graphs in this work are finite and simple. For a positive integer n, K_n will denote the complete graph on n vertices, and P_n will denote the path on n vertices. For integers $n > 2$, C_n will denote the cycle on n vertices; the graph C_4 is called a *square*. For positive integers m, n, $K_{m,n}$ will denote the complete bipartite graph with classes of size m and n. The *claw* is the graph $K_{1,3}$, and the *fork* is the tree obtained from the claw $K_{1,3}$ by subdividing one of its edges once. A *clique* (*stable set* or an *independent set*) is a set of vertices that are pairwise adjacent (nonadjacent). The *clique number* $\omega(G)$ (*independence number* $\alpha(G)$) of a graph G is the size of a largest clique (stable set) in G. A *triad* is a stable set of size 3. A *k-vertex coloring* of a graph G is a function $\phi : V(G) \to \{1, 2, \ldots, k\}$ such that for any adjacent vertices v and w, we have $\phi(v) \neq \phi(w)$. A *vertex coloring* of a graph G is a k-vertex coloring of G for some k. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G admits a k-vertex coloring. A graph is $(G_1, G_2, \ldots, G_k)$*-free* if it does not contain any graph in $\{G_1, G_2, \ldots, G_k\}$ as an induced subgraph.

Clearly, for every graph G, we have $\chi(G) \geqslant \omega(G)$. In 1955, Mycielski constructed an infinite sequence of graphs G_n with $\omega(G_n) = 2$ and $\chi(G) = n$ for every n [9]. Thus, in general, there is no function of $\omega(G)$ that gives an upper bound for $\chi(G)$; however, there do exist such upper bounding functions for some restricted classes of graphs. To be precise, if $\mathcal{G}$ is a class of graphs, and there exists a function f (called χ*-binding function*) such that $\chi(G) \leqslant f(\omega(G))$ for all $G \in \mathcal{G}$, then we say that $\mathcal{G}$ is χ*-bounded*; and is *linearly* χ*-bounded* if f is linear. The field of χ-boundedness is primarily concerned with determining which forbidden induced subgraphs $G_1, G_2, \ldots, G_k$ give χ-bounded classes, and finding the smallest χ-binding functions for these classes. It is known that if none of $G_1, G_2, \ldots, G_k$ is acyclic, then the class of $(G_1, G_2, \ldots, G_k)$-free graphs is not χ-bounded [11]. Gyárfás [6] and Sumner [12] both independently conjectured that for every tree T, the class of T-free graphs is χ-bounded. Gyárfás [6] showed that the class of $K_{1,t}$-free graphs is χ-bounded and its smallest χ-binding function f satisfies $\frac{R(t,\omega+1)-1}{t-1} \leqslant f(\omega) \leqslant R(t, \omega)$, where $R(m, n)$ denotes the classical Ramsey number. A famous result of Kim [8] shows that the Ramsey number $R(3, t)$ has order of magnitude $O(t^2/\log t)$. Thus for any claw-free graph G, we have $\chi(G) \leqslant O(\omega(G)^2/\log \omega(G))$. Further, it is known that there exists no linear χ-binding function for the class of claw-free graphs; see [11]. More precisely, for the class of claw-free graphs the smallest χ-binding function f satisfies $f(\omega) \in O(\omega^2/\log \omega)$. The first author and Seymour [4] studied the structure of claw-free graphs in detail, and they obtained the tight χ-bound for claw-free graphs containing a triad [5]. That is, if G is connected and claw-free with $\alpha(G) \geqslant 3$, then $\chi(G) \leqslant 2\omega(G)$.

The class of fork-free graphs generalizes the class of claw-free graphs. The class of fork-free graphs is comparatively less studied. Kierstead and Penrice showed that fork-free graphs are χ-bounded [7]. However, the best χ-binding function for fork-free graphs is not known, and an interesting question of Randerath and Schiermeyer [11] asks for the existence of a polynomial χ-binding function for the class of fork-free graphs. Randerath, in his thesis, obtained tight χ-bounds for several subclasses of fork-free graphs [10]. Here we are interested in linearly χ-bounded fork-free graphs. Recently the first author with Cook and Seymour [2] studied the structure of (fork, anti-fork)-free graphs and showed a linear χ-binding function for this class of graphs. Since the class of $(3K_1, 2K_2)$-free graphs does not admit a linear χ-binding function [1], if $\mathcal{G}$ is a linearly χ-bounded class of (fork, H)-free graphs with $|V(H)| = 4$, then $H \in \{P_4, C_4, K_4, K_4 - e, \overline{K_{1,3}}, \text{paw}\}$. When $H = P_4$, then every (fork, P_4)-free graph G is again P_4-free, and it is well known that every such G satisfies $\chi(G) = \omega(G)$; when $H \in \{K_4, K_4 - e, \text{paw}\}$, it follows from the results of [10] that every (fork, H)-free graph G satisfies $\chi(G) \leqslant \omega(G) + 1$, and from a result of [2] that every (fork, $\overline{K_{1,3}}$)-free graph G satisfies $\chi(G) \leqslant 2\omega(G)$. Thus the problem of obtaining a (best) linear χ-binding function for the class of (fork, C_4)-free graphs is open.

In this paper, we show that every (fork, C_4)-free graph G satisfies $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$. To do this, we need to achieve three major steps:

- First, we obtain a structure theorem for the class of (fork, C_4)-free graphs that uses the class of (claw, C_4)-free graphs as a basic class (Section 3).
- Next, we prove a new structure theorem for the class of (claw, C_4)-free graphs that are not quasi-line graphs (Section 4).
- Finally, we prove our $\left\lceil \frac{3\omega}{2} \right\rceil$-bound for the chromatic number via these structure theorems with additional work on coloring basic classes (Section 5).

2 Notation and terminology

Given a vertex $v \in V(G)$, we say the *neighborhood* of v, $N_G(v)$, is the set of neighbors of v; the *non-neighborhood* of v, $M_G(v)$, is the set of non-neighbors of v; and the *degree* of v, $d_G(v) = |N_G(v)|$; we may write $N(v)$, $M(v)$ and $d(v)$ when the relevant graph is unambiguous. We write $N[v]$ to denote the set $N(v) \cup \{v\}$, and $M[v]$ to denote the set $M(v) \cup \{v\}$. If $S \subseteq V(G)$, then $N(S)$ is the set $\cup_{v \in S} N(v) \setminus S$, and $M(S)$ is the set $\cup_{v \in S} M(v) \setminus S$.

Given $S \subseteq V(G)$, we define $\alpha(S)$ to be $\alpha(G[S])$. A vertex v in G is *important* if for all $w \in V(G)$, $\alpha(N(v)) \geqslant \alpha(N(w))$. A vertex v in G is a *root of a claw* if v has neighbors a, b, c in G such that $\{v, a, b, c\}$ induces a claw in G. A vertex v in a graph G is *good* if $d_G(v) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil - 1$.

Given disjoint vertex sets S, T, we say that S is *complete* to T if every vertex in S is adjacent to every vertex in T; we say S is *anticomplete* to T if every vertex in S is nonadjacent to every vertex in T; and we say S is *mixed* on T if S is not complete or anticomplete to T. When $S = \{v\}$ is a single vertex, we can instead say that v is complete to, anticomplete to, or mixed on T. A vertex v is called *universal* if it is complete to $V(G) \setminus \{v\}$. A vertex set S in G is *homogeneous* if $1 < |S| < |V(G)|$ and for every $v \notin S$, v is complete to S or anticomplete to S. A *homogeneous clique* is a homogeneous set that is a clique. A *clique cutset* is a clique S in G such that $G[V(G) \setminus S]$ has more components than G.

We say that disjoint vertex sets Y, Z are *matched* (*antimatched*) if each vertex in Y has a unique neighbor (non-neighbor) in Z and vice versa. Note that if Y and Z are matched or antimatched, then $|Y| = |Z|$.

A graph H is called a *thin candelabrum* (with base Z) if its vertices can be partitioned into nontrivial disjoint sets Y, Z such that Y is a stable set, Z is a clique, and Y and Z are matched. Candelabra, which were introduced by Chudnovsky, Cook, and Seymour in [2], are a generalization of thin candelabra. In this work we deal only with thin candelabra, and henceforth use "candelabrum" to mean "thin candelabrum." One can add a candelabrum to a graph G via the following procedure: Let H be a candelabrum with base Z. Take the disjoint union of G and H, then add edges to make Z complete to $V(G)$. We refer to this construction procedure as *candling* the graph G. We say that a graph G is *candled* if it can be constructed by candling some induced subgraph $G_0 \subseteq G$.

An *anticandelabrum* with base Z is the complement of a candelabrum with base Z. We say that a graph G is *anticandled* if $\overline{G}$ is candled. We will refer to the analogous construction procedure as *anticandling*. Anticandling can also be thought of as adding an anticandelabrum H with base Z to a graph, so that Z is anticomplete to the graph and $V(H) \setminus Z$ is complete to the graph.

A graph G is a *quasi-line graph* if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques.

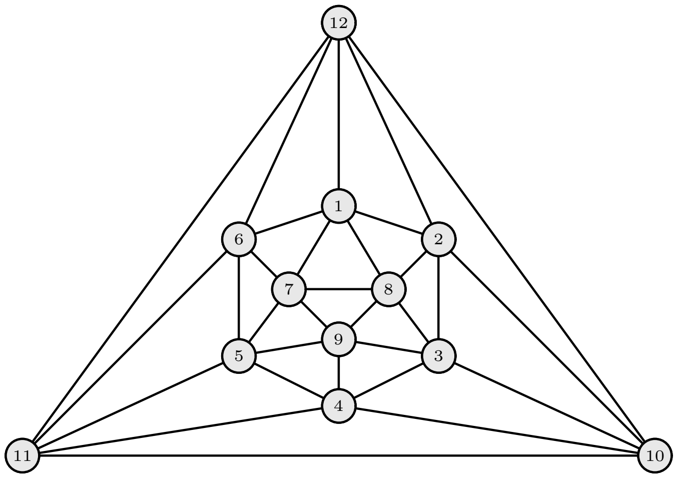

Figure 1: Icosahedron

The *icosahedron* is the unique planar graph with twelve vertices all of degree five; see Figure 1.

A *blowup* of a graph H is any graph G such that $V(G)$ can be partitioned into $|V(H)|$ (not necessarily non-empty) cliques Q_v, $v \in V(H)$, such that Q_u is complete to Q_v if $uv \in E(H)$, and Q_u is anticomplete to Q_v if $uv \notin E(H)$.

We say that a graph G is a *crown* (see Figure 2) if $V(G)$ can be partitioned into eleven sets $Q_1, \ldots, Q_{10}$ and M such that the following hold.

- Each Q_i is a clique.

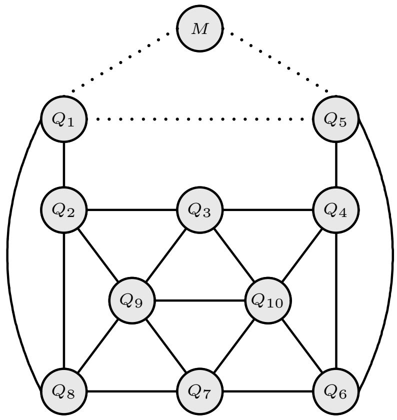

Figure 2: Schematic representation of a crown. Each circle represents a set. Each Q_i is a clique. A line between two sets means that the two sets are complete to each other, a dotted line between the two sets means that the edges between the two sets are arbitrary, and the absence of a line between two sets means that the two sets are anticomplete to each other.

- For $i \in \{1, 2, \ldots, 7\}$, Q_i is complete to Q_{i+1}; $Q_1 \cup Q_2$ is complete to Q_8; Q_4 is complete to Q_6; Q_9 is complete to $Q_2 \cup Q_3 \cup Q_7 \cup Q_8 \cup Q_{10}$; Q_{10} is complete to $Q_3 \cup Q_4 \cup Q_6 \cup Q_7$; the set of edges between Q_1 and Q_5 is arbitrary; and there are no other edges between Q_j and Q_k, where $j, k \in \{1, 2, \ldots, 10\}$ and $j \neq k$.
- The set M is anticomplete to $(\cup_{i=1}^{10} Q_i) \setminus (Q_1 \cup Q_5)$, and the set of edges between $Q_1 \cup Q_5$ and M is arbitrary.

3 Structure of (fork, C_4)-free graphs

In this section, we obtain a structure theorem for the class of (fork, C_4)-free graphs that uses the class of (claw, C_4)-free graphs as a basic class.

Theorem 1. *Let G be a (fork, C_4)-free graph. Then at least one of the following hold:*

- *G is not connected.*
- *G contains a universal vertex.*
- *G contains a homogeneous clique.*
- *G is candled or anticandled.*
- *G is claw-free.*

Proof. Let G be a (fork, C_4)-free graph. Suppose that G is a connected graph which has no universal vertex, no homogeneous clique, and that G contains a claw. We show that G is either candled or anticandled. Let $v \in V(G)$ be an important vertex. Then since G is not claw-free, there is some claw rooted at v. Let $L(v) \subseteq N(v)$ be the leaves of claws rooted at v and let Q denote the set $N(v) \setminus L(v)$. So if S is a maximum stable set in $N(v)$, then $S \subseteq L(v)$. Since v is not a universal vertex, $M(v)$ is not empty. Then we have the following:

(1) $L(v)$ is anticomplete to $M(v)$.

Proof of (1): Suppose $x \in M(v)$ has a neighbor a in a triad $\{a, b, c\} \subseteq L(v)$. Since $\{v, a, x, b\}$ and $\{v, a, x, c\}$ do not induce C_4s, x is not adjacent to b or c. But then $\{x, a, v, b, c\}$ induces a fork, a contradiction. So (1) holds. $\lozenge$

Let $Q_1(v)$ be the maximal subset of Q that is anticomplete to $M(v)$, and let $Q_2(v) := N(M(v)) \cap Q = Q \setminus Q_1(v)$.

(2) If $t \in Q$ is complete to $L(v)$, then $t \in Q_1(v)$.

Proof of (2): Suppose $t \in Q$ is complete to $L(v)$. If t has a neighbor $x \in M(v)$, then, by (1), $\alpha(N(t)) > \alpha(N(v))$, a contradiction to the fact that v is an important vertex. So (2) holds. $\lozenge$

(3) $Q_2(v)$ is a clique, and $Q_1(v)$ is complete to $Q_2(v)$.

Proof of (3): Suppose to the contrary that there are nonadjacent vertices $t \in Q_2(v)$ and $t' \in Q_1(v) \cup Q_2(v)$. Let $x \in M(v)$ be a neighbor of t. Then since $\{v, t, x, t'\}$ does not induce a C_4, t' is not adjacent to x. By (2), t has a non-neighbor $a \in L(v)$. By (1), a is not adjacent to x. Then since $\{x, t, v, t', a\}$ does not induce a fork, t' is adjacent to a. Let $b, c \in L(v)$ be such that $\{v, a, b, c\}$ induces a claw. Again by (1), x is anticomplete to $\{b, c\}$. Now since $t, t' \notin L(v)$, we see that t and t' are each adjacent to at least two vertices in $\{a, b, c\}$. Thus t is adjacent to b and c, and we may assume that t' is adjacent to b. Then since $\{t, b, t', c\}$ does not induce a C_4, t' is not adjacent to c. But then $\{t', b, t, c, x\}$ induces a fork, a contradiction. So (3) holds. $\lozenge$

(4) Q is a clique.

Proof of (4): By (3), it is enough to show that $Q_1(v)$ is a clique. Suppose to the contrary that there are nonadjacent vertices in $Q_1(v)$, say t and t'. Since $M(v) \neq \varnothing$ and since G is connected, there exists a vertex $x \in M(v)$ which has a neighbor $w \in Q_2(v)$. By (3), w is complete to $\{t, t'\}$, and by the definition of $Q_1(v)$, x is anticomplete to $\{t, t'\}$. By (2), w has a non-neighbor $a \in L(v)$. Then by (1), x is not adjacent to a. Now since $\{a, t, t', w, x\}$ does not induce a fork and $\{a, t, w, t'\}$ does not induce a C_4, we see that a is anticomplete to $\{t, t'\}$. But then $\{v, a, t, t'\}$ induces a claw, contradicting $t, t' \notin L(v)$. So (4) holds. $\lozenge$

(5) If C is a connected component of $M(v)$, every $t \in N(v)$ is complete or anticomplete to C. In particular, C is a homogeneous set or a singleton.

Proof of (5): Suppose not. Then since G is connected, we may assume that there are adjacent vertices $x, y \in V(C)$, and there exists a vertex $t \in N(v)$ which is adjacent to x and not adjacent to y. By (1) and by our definition of $Q_1(v)$, $t \notin L(v) \cup Q_1(v)$. So $t \in Q_2(v)$. Then since $t \notin L(v)$, t is adjacent to at least two vertices in any given triad $\{a, b, c\} \subseteq L(v)$; we may assume $a, b \in N(t)$. Then $\{y, x, t, a, b\}$ induces a fork, a contradiction. So (5) holds. $\lozenge$

(6) If C is a connected component of $M(v)$, then $V(C)$ is a clique.

Proof of (6): Since G is connected, there is some $t \in N(V(C))$. As in (5), $t \in Q_2(v)$. So, by (2), t has a non-neighbor $a \in L(v)$. Now if there are nonadjacent vertices x and y in $V(C)$, then, by (5), we see that $\{a, v, t, x, y\}$ induces a fork. So any two vertices in $V(C)$ are adjacent, and hence $V(C)$ is a clique. $\Diamond$

(7) $M(v)$ is a stable set.

Proof of (7): Since G has no homogeneous cliques, the proof of (7) follows from (5) and (6). $\Diamond$

(8) Each vertex in $Q_2(v)$ has at most one neighbor in $M(v)$.

Proof of (8): Suppose to the contrary that $t \in Q_2(v)$ has two neighbors in C, say x and y. Then by (7), x and y are not adjacent. Since $t \in Q_2(v)$, by (2), t has a non-neighbor $a \in L(v)$. But then $\{a, v, t, x, y\}$ induces a fork, a contradiction. So (8) holds. $\Diamond$

(9) Every vertex in Q has a non-neighbor in $L(v)$.

Proof of (9): Suppose to the contrary that there exists a vertex $t \in Q$ which is complete to $L(v)$. Then by (2), $t \in Q_1(v)$. But then by (4), and by the definition of $Q_1(v)$, $\{v, t\}$ is a homogeneous clique in G, a contradiction to our assumption that G has no homogeneous cliques. So (9) holds. $\Diamond$

We now prove the theorem in two cases. Suppose that $|M(v)| > 1$. Then we have the following.

Claim 2. *Any $a \in L(v)$ is either complete to $Q_2(v)$ or anticomplete to $Q_2(v)$.*

Proof of Claim 2: Suppose to the contrary that there exists a vertex $a \in L(v)$ which is mixed on $Q_2(v)$. Then by using (3), there are adjacent vertices t and t' in $Q_2(v)$ such that a is adjacent to t and a is not adjacent to t'. Let $x \in M(v)$ be a neighbor of t and let $x' \in M(v)$ be a neighbor of t'. If $x \neq x'$, then by using (7) and (8), we see that $\{x', t', t, x, a\}$ induces a fork. So we may assume that $x = x'$. Then since $|M(v)| > 1$, there exists a vertex $y \in M(v)$ (which is distinct from x and x'), and so there exists a vertex $t'' \in Q_2(v)$ which is adjacent to y. Then by using (7), (8) and (3), we see that either $\{x, t', t'', y, a\}$ or $\{y, t'', t, x, a\}$ induces a fork, a contradiction. $\Diamond$

By Claim 2, we partition $L(v)$ into two sets as follows: Let $L_1(v)$ denote the set $\{a \in L(v) \mid a \text{ is complete to } Q_2(v)\}$ and let $L_0(v)$ denote the set $L(v) \setminus L_1(v) := \{a \in L(v) \mid a \text{ is anticomplete to } Q_2(v)\}$. Then by (9), $L_0(v) \neq \varnothing$. Fix a vertex $x \in M(v)$, and let $t \in Q_2(v)$ be a neighbor of x. Then we have the following.

Claim 3. *$L_0(v)$ is anticomplete to $L_1(v)$.*

Proof of Claim 3: Suppose to the contrary that there are adjacent vertices $c \in L_1(v)$ and $d \in L_0(v)$. Then by definitions of $L_0(v)$ and $L_1(v)$, we have c is adjacent to t, and d is not adjacent to t. Let $\{a, b\} \subset L(v)$ be such that $\{a, b, c\}$ is a triad in $L(v)$. Since $t \notin L(v)$, we may assume that t is adjacent to a. By (1), x is anticomplete to $\{a, b, c, d\}$. Then since

$\{a,t,c,d\}$ does not induce a C_4, a is not adjacent to d. But then $\{d,c,t,x,a\}$ induces a fork, a contradiction. $\lozenge$

Claim 4. *$L_0(v)$ is a clique.*

Proof of Claim 4: If there are nonadjacent vertices a and b in $L_0(v)$, then $\{x,t,v,a,b\}$ induces a fork, a contradiction. $\lozenge$

Consider a maximum stable set $S \subseteq N(v)$; then $S \subseteq L(v)$. We have $|S \cap L_0(v)| = 1$, because $L_0(v)$ is a clique component of $L(v)$ (by Claim 3 and Claim 4). So $|S \cap L_1(v)| = |S| - 1$. A maximum stable set in $N(t)$ is $(S \cap L_1(v)) \cup \{x\}$, which has size $|S| = \alpha(N(v))$. Therefore, $\alpha(N(t)) = \alpha(N(v))$, so t is also an important vertex. So $M(t)$ is a stable set, by (7). Since $L_0(v)$ is a nonempty component of $M(t)$, it is a singleton, say $L_0(v) := \{l\}$. Then we have the following claim.

Claim 5. *$L_0(v) = \{l\}$ is anticomplete to $Q_1(v)$.*

Proof of Claim 5: Suppose that there exists a vertex $q \in Q_1(v)$ which is adjacent to l. Then by (3), t and q are adjacent, and by the definition of $L_0(v)$, l and t are not adjacent. Now by (9), q has a non-neighbor, say $a \in L(v)$. Then $a \in L_1(v)$, and hence a is adjacent to t. Also by Claim 3 and (1), a is anticomplete to $\{l,x\}$. But then $\{l,q,t,x,a\}$ induces a fork, a contradiction. $\lozenge$

Claim 6. *No two vertices in $Q_2(v)$ share a common neighbor in $M(v)$.*

Proof of Claim 6: Suppose that there are vertices t' and t'' in $Q_2(v)$ which have a common neighbor $x' \in M(v)$. Then by (4) and (8), since $\{t',t''\}$ is complete to $(Q \setminus \{t',t''\}) \cup L_1(v) \cup \{v,x'\}$, and is anticomplete to $L_0(v) \cup (M(v) \setminus \{x'\})$, $\{t',t''\}$ is a homogenous clique, a contradiction to our assumption that G has no homogenous cliques. $\lozenge$

Now let Z denote the set $\{v\} \cup Q_2(v)$. Since $M(Q_2) \subseteq M(v) \cup \{l\}$, we have $M(Z) = M(v) \cup \{l\}$. Then by (4), we see that Z is a clique. By (1) and (7), $M(Z)$ is a stable set which is anticomplete to $V(G) \setminus (Z \cup M(Z))$. By Claim 6 and (8), Z and $M(Z)$ are matched. Thus we conclude that G is candled.

So we may assume that every important vertex in G has exactly one non-neighbor. In this case, we claim that G is anticandled. Let $Y = Q_2(v) \cup \{v\}$. Then by (3), Y is a clique. Let m be the unique vertex in $M(v)$. Then there exists a vertex $t \in Q_2(v)$ such that t is adjacent to m. If S is a maximum stable set in $N(v)$, then by (1), $S \cup \{m\}$ is a stable set of size $\alpha(N(v)) + 1$. Since $t \notin L(v)$, t is adjacent to at least $|S| - 1$ of the vertices in S, so $\alpha(N(t)) = |S| = \alpha(N(v))$. So every vertex $t \in Q_2(v)$ is important and hence by assumption has a unique non-neighbor.

Since $\{t,t'\}$ is not a homogeneous clique, for any $t,t' \in Y$, they do not share a non-neighbor. Therefore, each vertex in $M(Y)$ has a distinct non-neighbor in Y, so in particular $M(Y)$ and Y are antimatched.

Consider distinct $m,m' \in M(Y)$ with respective non-neighbors $t,t' \in Y$. Then since $\{m',m,t,t'\}$ does not induce a C_4, m and m' are not adjacent. Thus $M(Y)$ is stable.

Suppose $m \in M(Y)$ has a neighbor u. Let $t \in Y$ be a non-neighbor of m; then since u is adjacent to the unique non-neighbor of t, we have by (1) that $u \in Q_2(t)$. Then u is important, so by assumption u has a unique non-neighbor. Thus $u \notin L(v)$, since it is not part of a triad. Moreover, by (9), every vertex in $Q_1(v)$ has a non-neighbor in $L(v)$, so has at least two non-neighbors. Then $u \notin Q_1(v)$. Thus, $u \in Y$. So $M(Y)$ is anticomplete to $V(G) \setminus (Y \cup M(Y))$.

Hence we conclude that $Y \cup M(Y)$ induces an anticandelabrum with base $M(Y)$, with $G \setminus (Y \cup M(Y))$ complete to Y and anticomplete to $M(Y)$.

This completes the proof of the theorem. □

Corollary 7. *Let G be a connected (fork, C_4)-free graph. Then G is claw-free or G has a universal vertex or G has a clique cutset.*

Proof. Let G be a (fork, C_4)-free graph. Suppose that G has no universal vertex, and no clique cutset. We show that G is claw-free. Suppose to the contrary that G contains a claw. Let $v \in V(G)$ be an important vertex. Let $L(v) \subseteq N(v)$ be the leaves of claws rooted at v and let Q denote the set $N(v) \setminus L(v)$. So if S is a maximum stable set in $N(v)$, then $S \subseteq L(v)$. Since v is not a universal vertex, $M(v)$ is not empty. Let Q_1 be the maximal subset of Q that is anticomplete to $M(v)$, and let $Q_2 := N(M(v)) \cap Q = Q \setminus Q_1$. Then it follows from Theorem 1 (See item (3) and note that items (1)–(3) hold regardless of whether G has a homogeneous clique or not.) that Q_2 is a clique. But then we see that Q_2 is a clique cutset separating $\{v\}$ and $M(v)$ which is a contradiction. This completes the proof. □

4 Structure of (claw, C_4)-free graphs

In this section, we obtain a structure theorem for the class of (claw, C_4)-free graphs that are not quasi-line graphs. A graph is *chordal* if it does not contain any induced cycle of length at least four.

Theorem 8. *Let G be a connected (claw, C_4)-free graph. Then at least one of the following hold:*

- *G has a clique cutset.*
- *G has a good vertex.*
- *G is a quasi-line graph.*
- *G is a blowup of the icosahedron graph.*
- *G is a crown with $|M \cup Q_1 \cup Q_5| < |V(G)|$.*

Proof. Let G be a connected (claw, C_4)-free graph. We may assume that G has no clique cutset. Let $v \in V(G)$. First suppose that $G[N(v)]$ is chordal. Then since G is claw-free, $G[N(v)]$ is a chordal graph with no triad. Since the complement graph of a chordal graph with no triad is a bipartite graph, we see that $N(v)$ can be expressed as the union of two cliques. Since v is arbitrary, G is a quasi-line graph. So we may assume that $G[N(v)]$ is not chordal and hence $G[N(v)]$ contains an induced C_k for some $k \geqslant 5$. Since $\alpha(C_k) \geqslant 3$

for $k \geqslant 6$, and since G is (claw, C_4)-free, we have $k = 5$. That is, $G[N(v)]$ contains an induced C_5, say $C{:=}\ v_1\text{-}v_2\text{-}v_3\text{-}v_4\text{-}v_5\text{-}v_1$. Let T denote the set $\{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = V(C)\}$, let R denote the set $\{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \varnothing\}$, and for each $i \in \{1, 2, \ldots, 5\}$, i mod 5, let:

$$\begin{aligned} A_i &:= \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \{v_i, v_{i+1}\}\}, \\ B_i &:= \{x \in V(G) \setminus V(C) \mid N(x) \cap V(C) = \{v_{i-1}, v_i, v_{i+1}\}\} \cup \{v_i\}. \end{aligned}$$

Let $A := A_1 \cup \cdots \cup A_5$ and $B := B_1 \cup \cdots \cup B_5$. Note that $v \in T$, and so $T \neq \varnothing$. Then the following properties hold for each $i \in \{1, 2, \ldots, 5\}$, i mod 5:

(1) $V(G) = A \cup B \cup T \cup R$.

Proof of (1): Suppose that there is a vertex $x \in V(G) \setminus (A \cup B \cup T \cup R)$. Then for some i, either $N(x) \cap V(C) = \{v_i\}$ or $\{v_{i-1}, v_{i+1}\} \subseteq N(x) \cap V(C)$ with $v_i \notin N(x)$. But then $\{v_i, v_{i-1}, x, v_{i+1}\}$ induces either a claw or a C_4. $\diamondsuit$

(2) A_i and $B_i \cup T$ are cliques.

Proof of (2): Let $i = 1$ and suppose that there are nonadjacent vertices x and y in one of the listed sets. If $x, y \in A_1$, then $\{v_1, x, y, v_5\}$ induces a claw, and if $x, y \in B_1 \cup T$, then $\{x, v_5, y, v_2\}$ induces a C_4. $\diamondsuit$

(3) A_i is anticomplete to T.

Proof of (3): If there are adjacent vertices $a \in A_i$ and $t \in T$, then $\{t, v_{i-1}, v_{i+2}, a\}$ induces a claw. $\diamondsuit$

(4) A_i is complete to $A_{i-1} \cup A_{i+1} \cup B_i \cup B_{i+1}$.

Proof of (4): By symmetry, it suffices to show that A_i is complete to $A_{i+1} \cup B_{i+1}$. Suppose that there are nonadjacent vertices $x \in A_i$ and $y \in A_{i+1} \cup B_{i+1}$. If $y \in A_{i+1}$, then $\{x, y\}$ is anticomplete to v (by (3)), and then $\{v_{i+1}, v, x, y\}$ induces a claw. So $y \in B_{i+1}$. Then $\{v_i, v_{i-1}, x, y\}$ induces a claw. $\diamondsuit$

(5) A_i is anticomplete to $A_{i+2} \cup A_{i-2} \cup B_{i+2} \cup B_{i-1} \cup B_{i-2}$.

Proof of (5): By symmetry, it suffices to show that A_i is anticomplete to $A_{i+2} \cup B_{i+2} \cup B_{i-2}$. Suppose that there are adjacent vertices $x \in A_i$ and $y \in A_{i+2} \cup B_{i+2} \cup B_{i-2}$. If $y \in A_{i+2} \cup B_{i-2}$, then $\{x, v_{i+1}, v_{i+2}, y\}$ induces a C_4. So $y \in B_{i+2}$. Now since x is not adjacent to v (by (3)), and y is adjacent to v (by (2)), we see that $\{x, v_i, v, y\}$ induces a C_4. $\diamondsuit$

(6) B_i is complete to $B_{i+1} \cup B_{i-1}$.

Proof of (6): By symmetry, it suffices to show that B_i is complete to B_{i+1}. If there are nonadjacent vertices $x \in B_i$ and $y \in B_{i+1}$, then $\{x, y\}$ is complete to v (by (2)), and then $\{v, v_{i-2}, x, y\}$ induces a claw. $\diamondsuit$

(7) B_i is anticomplete to $B_{i+2} \cup B_{i-2}$.

Proof of (7): If there are adjacent vertices $x \in B_i$ and $y \in B_{i+2} \cup B_{i-2}$, then either $\{x, v_{i-1}, v_{i-2}, y\}$ or $\{x, v_{i+1}, v_{i+2}, y\}$ induces a C_4. $\Diamond$

(8) If $r \in R$, then $N(r) \cap (B \cup T) = \varnothing$.

Proof of (8): If there is a vertex $x \in N(r) \cap (B \cup T)$, then for some i, $\{v_{i-1}, v_{i+1}\} \subset N(x) \cap V(C)$, and then $\{x, v_{i-1}, v_{i+1}, r\}$ induces a claw. $\Diamond$

(9) Any $r \in R$ which has a neighbor in A_i is complete to $A_{i+1} \cup A_{i-1}$.

Proof of (9): Let $r \in R$ be such that r has a neighbor $a \in A_i$. If r is not adjacent to a vertex $b \in A_{i+1} \cup A_{i-1}$, then since a is adjacent to b (by (4)), we see that either $\{a, r, v_i, b\}$ or $\{a, r, v_{i+1}, b\}$ induces a claw. $\Diamond$

(10) If A_i and A_{i+1} are not empty, for some i, then any $r \in R$ which has a neighbor in $A_i \cup A_{i+1} \cup A_{i-1}$ is complete to $A_i \cup A_{i+1} \cup A_{i-1}$.

Proof of (10): This follows from (4) and (9). $\Diamond$

If R is empty, then by above properties we see that G is a blowup of the icosahedron graph, where we set $Q_1 := B_1$, $Q_8 := B_2$, $Q_9 := B_3$, $Q_5 := B_4$, $Q_6 := B_5$, $Q_7 := T$, $Q_2 := A_1$, $Q_3 := A_2$, $Q_4 := A_3$, $Q_{11} := A_4$ and $Q_{12} := A_5$.

So we may assume that $R \neq \varnothing$. Then by (8), $A \neq \varnothing$. Since G has no clique cutset, using (4) and (10), we may assume that there exists an index i such that A_i and A_{i+2} are not empty, say $i = 1$. Now if $A_2 \neq \varnothing$, then by (10) and since G is claw-free, any $r \in R$ is complete to A. Moreover, since G is C_4-free, R is a clique. So again G is a blowup of the icosahedron graph, where we set $Q_1 := B_1$, $Q_8 := B_2$, $Q_9 := B_3$, $Q_5 := B_4$, $Q_6 := B_5$, $Q_7 := T$, $Q_2 := A_1$, $Q_3 := A_2$, $Q_4 := A_3$, $Q_{11} := A_4$, $Q_{12} := A_5$ and $Q_{10} := R$. So we may assume that $A_2 = \varnothing$.

Next suppose that $A_4 \cup A_5 = \varnothing$. In this case, we show that one of the vertices v_2 or v_5 is good. Suppose not. Then since $T \cup B_1 \cup B_5$ and $T \cup B_4 \cup B_5$ are cliques, we see that $|B_1| > \frac{\omega(G)}{2}$ and $|B_4| > \frac{\omega(G)}{2}$. Since v_2 is not a good vertex and since $A_1 \cup B_1 \cup B_2$ is a clique, we have $|T \cup B_3| > \frac{\omega(G)}{2}$. Then we see that $T \cup B_3 \cup B_4$ is a clique of size $> \omega(G)$ which is a contradiction. Thus one of the vertices v_2 or v_5 is good.

So we may assume that $A_5 \neq \varnothing$ and $A_4 = \varnothing$. Let R' be the set $\{r \in R \mid r$ has a neighbor in $A_1 \cup A_5\}$, and let R'' be the set $R \setminus R'$. Then by (10), R' is complete to $A_1 \cup A_5$. Also if there are nonadjacent vertices $r_1, r_2 \in R'$, then for any $a \in A_1$, $\{r_1, r_2, v_2, a\}$ induces a claw, and so R' is a clique. Now by above properties we see that G is a crown, where we set $Q_{10} := B_1$, $Q_7 := B_2$, $Q_8 := B_3$, $Q_2 := B_4$, $Q_3 := B_5$, $Q_9 := T$, $Q_6 := A_1$, $Q_1 := A_3$, $Q_4 := A_5$, $Q_5 := R'$ and $M := R''$. Since $A_5 \neq \varnothing$, it follows that $|M \cup Q_1 \cup Q_5| < |V(G)|$.

This completes the proof of the theorem. $\square$

5 Coloring (claw/fork, C_4)-free graphs

In this section, we show that every (fork, C_4)-free graph satisfies $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$. We will use the following known result.

Theorem 9 ([3])**.** *If G is a quasi-line graph, then $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$.*

Let $[k]$ denote the set $\{1, 2, \ldots, k\}$. A *k-list assignment* of a graph G is a function $L : V(G) \to 2^{[k]}$. The set $L(v)$, for a vertex v in G, is called the *list* of v. In the *list k-coloring* problem, we are given a graph G with a k-list assignment L and asked whether G has an *L-coloring*, i.e., a k-coloring of G such that every vertex is assigned a color from its list. We say that G is *L-colorable* if G has an L-coloring. We say that a graph F with list assignment L is *L-degenerate* if there exists a vertex ordering $v_1, \ldots, v_n$ of $V(F)$ such that each v_i has at most $|L(v)| - 1$ neighbors in $\{v_1, \ldots, v_{i-1}\}$ for $1 \leqslant i \leqslant n$. Clearly, if a graph is L-degenerate, then it is L-colorable.

Lemma 10. *Suppose that G is a crown and $k \geqslant 1$ be an integer. If $\phi : M \cup Q_1 \cup Q_5 \to S$ with $|S| = \left\lceil \frac{3k}{2} \right\rceil$ is a vertex coloring of $G[M \cup Q_1 \cup Q_5]$ and $\omega(G - M) \leqslant k$, then $\chi(G) \leqslant \left\lceil \frac{3k}{2} \right\rceil$.*

Proof. We prove the lemma by induction on k. If $k = 1$, then any non-trivial component of G is an induced subgraph of $G[M \cup Q_1 \cup Q_5]$ and the lemma holds. We now assume that $k \geqslant 2$ and the lemma holds for any positive integer smaller than k. Let $H = G - (M \cup Q_1 \cup Q_5)$. Note that, for each $i \in \{1, 2, \ldots 10\} \setminus \{1, 5\}$, any two vertices in Q_i have the same degree in H. Let L be the list assignment of H such that

$$L(v) = \begin{cases} S \setminus \phi(Q_1) & \text{if } v \in Q_2 \cup Q_8, \\ S \setminus \phi(Q_5) & \text{if } v \in Q_4 \cup Q_6, \\ S & \text{if } v \in Q_3 \cup Q_7 \cup Q_9 \cup Q_{10}. \end{cases}$$

Note that if H is L-colorable, then $\chi(G) \leqslant \left\lceil \frac{3k}{2} \right\rceil$. Since $|Q_1|+|Q_2|+|Q_8| \leqslant \omega(G-M) \leqslant k$, it follows that for any $v \in Q_2 \cup Q_8$, $|L(v)| = |S| - |Q_1| = \left\lceil \frac{3k}{2} \right\rceil - |Q_1| \geqslant |Q_2| + |Q_8| + \left\lceil \frac{k}{2} \right\rceil$. Similarly, for any $v \in Q_4 \cup Q_6$, $|L(v)| \geqslant |Q_4| + |Q_6| + \left\lceil \frac{k}{2} \right\rceil$. Next, we claim that:

$$\text{We may assume that: } Q_9 \neq \varnothing. \text{ Likewise, } Q_{10} \neq \varnothing. \tag{1}$$

Proof of (1): Suppose that $Q_9 = \varnothing$. Since $|Q_2| + |Q_8| \leqslant k$, one of Q_2 and Q_8 has size at most $\frac{k}{2}$, say Q_2 by symmetry. Then for any $v \in Q_3$, it follows that $d_H(v) = |Q_3 \cup Q_4 \cup Q_{10}| - 1 + |Q_2| \leqslant (k-1) + |Q_2| \leqslant \frac{3k}{2} - 1$. If $|Q_8| \leqslant \frac{k}{2}$, then $d_H(v) \leqslant \frac{3k}{2} - 1$ for any $v \in Q_7$. This implies that H is L-degenerate with the ordering of the vertices $Q_2, Q_4, Q_6, Q_8,\ Q_{10},\ Q_3, Q_7$. (It does not matter which vertex comes first in Q_i.) So $|Q_8| > \frac{k}{2}$. This implies that $|Q_7| < \frac{k}{2}$. Then for any $v \in Q_8$, it follows that $d_H(v) = |Q_2| + |Q_8| - 1 + |Q_7| < |Q_2| + |Q_8| - 1 + \frac{k}{2} < |L(v)|$. So H is L-degenerate with the ordering $Q_2, Q_4, Q_6, Q_{10}, Q_7, Q_8, Q_3$. This proves (1). $\lozenge$

Next:

$$\text{We may assume that } Q_3 \neq \varnothing\text{. Likewise, } Q_7 \neq \varnothing. \tag{2}$$

Proof of (2): Suppose $Q_3 = \varnothing$. If $|Q_7 \cup Q_9| \leqslant \frac{k}{2}$, then for any $v \in Q_2 \cup Q_8$, it follows that $d_H(v) \leqslant |Q_2| + |Q_8| - 1 + \frac{k}{2} < |L(v)|$. Then H is L-degenerate with the ordering $Q_4, Q_6, Q_{10}, Q_7, Q_9, Q_8, Q_2$. So we assume that $|Q_7 \cup Q_9| > \frac{k}{2}$. By symmetry, $|Q_7 \cup Q_{10}| > \frac{k}{2}$. This implies that each of Q_6, Q_8, Q_9, Q_{10} has size less than $\frac{k}{2}$. Since $Q_3 = \varnothing$, for any $v \in Q_2$, $d_H(v) = |Q_2| + |Q_8| + |Q_9| - 1 \leqslant |Q_2| + |Q_8| + \frac{k}{2} - 1 < |L(v)|$. By symmetry, for any $v \in Q_4$, $d_H(v) < |L(v)|$. Moreover, each vertex in $Q_9 \cup Q_{10}$ has degree at most $\frac{3k}{2} - 1$ in $H - (Q_2 \cup Q_4)$. So H is L-degenerate with the ordering $Q_8, Q_6, Q_7, Q_{10}, Q_9, Q_4, Q_2$. This proves (2). $\Diamond$

Moreover:

$$\text{We may assume that } Q_2 \neq \varnothing\text{. Likewise, } Q_4,\ Q_6,\ Q_8 \text{ are nonempty.} \tag{3}$$

Proof of (3): Suppose $Q_2 = \varnothing$. If $|Q_9| \leqslant \frac{k}{2}$, then for any $v \in Q_3$ it follows that $d_H(v) \leqslant \frac{3k}{2} - 1$. Then as in the proof of (2), $H - Q_3$ is L-degenerate and thus H is L-degenerate. So $|Q_9| > \frac{k}{2}$. This implies that $|Q_i \cup Q_{10}| < \frac{k}{2}$ for $i \in \{3, 7\}$. This implies that, for any $v \in Q_4 \cup Q_6$, $d_H(v) < |Q_4| + |Q_6| - 1 + \frac{k}{2} < |L(v)|$. So H is L-degenerate with the ordering $Q_8, Q_7, Q_9, Q_{10}, Q_3, Q_4, Q_6$. This proves (3). $\Diamond$

By (1), (2) and (3), we conclude that for each $i \in \{1, 2, \ldots, 10\} \setminus \{1, 5\}$, Q_i contains at least one vertex, say q_i. In particular, this implies that $k \geqslant 3$, $|\phi(Q_1)| \leqslant k - 2$ and $|\phi(Q_5)| \leqslant k - 2$. Next we claim that:

There are three distinct colors $c_1, c_2, c_3 \in S$ such that $c_1 \notin \phi(Q_1)$, $c_2 \notin \phi(Q_5)$, either $Q_1 = \varnothing$ or $|\{c_2, c_3\} \cap \phi(Q_1)| \geqslant 1$, and either $Q_5 = \varnothing$ or $|\{c_1, c_3\} \cap \phi(Q_5)| \geqslant 1$. (4)

Proof of (4): Since $|\phi(Q_1)| \leqslant k - 2$ and $|S| = \lceil \frac{3k}{2} \rceil$, there are at least $\lceil \frac{3k}{2} \rceil - k + 2 \geqslant 4$ colors in S that are not in $\phi(Q_1)$. Similarly, there are at least 4 colors in S that are not in $\phi(Q_5)$.

First suppose that $\phi(Q_1) \cap \phi(Q_5) \neq \varnothing$. Let $c_3 \in \phi(Q_1) \cap \phi(Q_5)$. Now we choose a color $c_1 \in S \setminus \phi(Q_1)$, and then a color $c_2 \in S \setminus \phi(Q_5)$ with $c_2 \neq c_1$. Clearly, c_1, c_2 and c_3 are the desired colors. So we may assume that $\phi(Q_1) \cap \phi(Q_5) = \varnothing$.

If $Q_1 = Q_5 = \varnothing$, then any three colors $c_1, c_2, c_3 \in S$ are the desired colors. If $Q_1 = \varnothing$ and $Q_5 \neq \varnothing$, then we choose $c_3 \in \phi(Q_5)$, and then choose a color $c_2 \in S \setminus \phi(Q_5)$, and finally choose a color $c_1 \in S \setminus \{c_2, c_3\}$. Clearly, c_1, c_2 and c_3 are desired colors. If $Q_1 \neq \varnothing$ and $Q_5 = \varnothing$, we can choose the three colors in a similar way. Finally, we assume that $Q_1, Q_5 \neq \varnothing$. Then it is possible to pick a color $c_3 \in \phi(Q_1)$ and a color $c_1 \in \phi(Q_5)$. Since $\phi(Q_1) \cap \phi(Q_5) = \varnothing$, it follows that $c_1 \neq c_3$ and $c_1 \notin \phi(Q_1)$. Moreover, $|\{c_2, c_3\} \cap \phi(Q_1)| \geqslant 1$ and $|\{c_1, c_3\} \cap \phi(Q_5)| \geqslant 1$ by the choice of c_1 and c_3. Since there are at least 4 colors in S that are not in $\phi(Q_5)$, we can choose such a color $c_2 \notin \{c_1, c_3\}$.

Thus, in all the cases, we have found the required colors, and the proof of (4) is complete. $\Diamond$

Now for each $j \in \{1,2,3\}$, let $T_j = \{x \in M \cup Q_1 \cup Q_5 \mid \phi(x) = c_j\}$. Let $I_1 := T_1 \cup \{q_2, q_{10}\}$, $I_2 := T_2 \cup \{q_6, q_9\}$, and $I_3 := T_3 \cup \{q_3, q_7\}$. It follows from (4) that I_1, I_2 and I_3 are three pairwise disjoint independent sets. Moreover, ϕ restricted to $(M \cup Q_1 \cup Q_5) \setminus (T_1 \cup T_2 \cup T_3)$ maps to $S \setminus \{c_1, c_2, c_3\}$ with

$$|S \setminus \{c_1, c_2, c_3\}| = |S| - 3 = \left\lceil \frac{3k}{2} \right\rceil - 3 = \left\lceil \frac{3(k-2)}{2} \right\rceil.$$

Let $G' = G - (I_1 \cup I_2 \cup I_3)$, and $M' = M - (T_1 \cup T_2 \cup T_3)$. By (4), it follows that $G' - M'$ is obtained from $G - M$ by deleting $\{q_2, q_3, q_6, q_7, q_9, q_{10}\}$ and at least one vertex in Q_j if $Q_j \neq \varnothing$ for each $j \in \{1, 5\}$. Therefore,

$$\omega(G' - M') \leqslant \omega(G - M) - 2 \leqslant k - 2.$$

Now by the inductive hypothesis,

$$\chi(G) \leqslant \chi(G') + 3 \leqslant \left\lceil \frac{3(k-2)}{2} \right\rceil + 3 = \left\lceil \frac{3k}{2} \right\rceil.$$

This proves Lemma 10. □

Lemma 11. *Let G be a blowup of the icosahedron. Then $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$.*

Proof. Let I be the icosahedron graph with vertex labels as in Figure 1. Let G be a blowup of the icosahedron I. We prove the lemma by induction on $|V(G)|$. Let $\omega = \omega(G)$. We may assume that $\omega \geqslant 2$. Let Q_i be the clique corresponding to the vertex $i \in V(I)$. Let X be a subset of $V(G)$ obtained by taking $\min\{1, |Q_i|\}$ vertices from Q_i for each $i \in \{1, 2, \ldots, 12\}$. Clearly, $G[X]$ is an induced subgraph of the icosahedron, and so $\chi(G[X]) \leqslant 4$. First suppose that $\omega(G - X) \geqslant \omega - 2$. Then there are two indices $i, j \in \{1, 2, \ldots, 12\}$ such that $Q_i \cup Q_j$ is a clique of size ω. Since the icosahedron is edge-transitive, we may assume that $i = 10$ and $j = 11$. Since $Q_4 \cup Q_{10} \cup Q_{11}$ and $Q_{10} \cup Q_{11} \cup Q_{12}$ are cliques, we conclude that Q_4 and Q_{12} are empty. Then we see that G is a crown (with $M = \varnothing$, and Q_{10} and Q_{11} being Q_1 and Q_5 in the definition of the crown), and the lemma follows from Lemma 10. So suppose that $\omega(G - X) \leqslant \omega - 3$. Then by induction, we have $\chi(G - X) \leqslant \left\lceil \frac{3\omega(G-X)}{2} \right\rceil \leqslant \left\lceil \frac{3(\omega-3)}{2} \right\rceil = \left\lceil \frac{3\omega}{2} - \frac{9}{2} \right\rceil$. Since $\chi(G) \leqslant \chi(G - X) + \chi(G[X])$, we have $\chi(G) \leqslant \left\lceil \frac{3\omega}{2} \right\rceil$. This proves Lemma 11. □

Theorem 12. *Let G be a (claw, C_4)-free graph. Then $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$.*

Proof. Let G be a (claw, C_4)-free graph. By Theorem 9, we may assume that G is not a quasi-line graph. We prove the theorem by induction on $|V(G)|$, and we apply Theorem 8.

If G has a clique cutset K, let A, B be a partition of $V(G) \setminus K$ such that both A, B are non-empty, and A is anticomplete to B. Clearly $\chi(G) = \max\{\chi(G[K \cup A]), \chi(G[K \cup B])\}$, so the desired result follows from the induction hypothesis on $G[K \cup A]$ and $G[K \cup B]$.

If G has a good vertex u, then by induction, $\chi(G - \{u\}) \leqslant \left\lceil \frac{3\omega(G-\{u\})}{2} \right\rceil$. Now consider any $\chi(G - \{u\})$-coloring of $G - \{u\}$ and extend it to a $\left\lceil \frac{3\omega(G)}{2} \right\rceil$-coloring of G, using for u a (possibly new) color that does not appear in its neighborhood.

If G is a blowup of the icosahedron graph, then the theorem follows from Lemma 11.

Finally, suppose that G is a crown with $|M \cup Q_1 \cup Q_5| < |V(G)|$. By the inductive hypothesis, let $\phi : M \cup Q_1 \cup Q_5 \to S$ with $|S| = \left\lceil \frac{3\omega(G)}{2} \right\rceil$ be a vertex coloring of $G[M \cup Q_1 \cup Q_5]$. It then follows from Lemma 10 that $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$. □

Theorem 13. *Let G be a (fork, C_4)-free graph. Then* $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$.

Proof. Let G be any (fork, C_4)-free graph. We prove the theorem by induction on $|V(G)|$.

If G has a universal vertex u, then $\omega(G) = \omega(G - \{u\}) + 1$, and by the induction hypothesis, we have $\chi(G) = \chi(G - \{u\}) + 1 \leqslant \left\lceil \frac{3\omega(G-\{u\})}{2} \right\rceil + 1$, which implies $\chi(G) \leqslant \left\lceil \frac{3\omega(G)}{2} \right\rceil$.

If G has a clique cutset K, let A, B be a partition of $V(G) \setminus K$ such that both A, B are non-empty, and A is anticomplete to B. Clearly $\chi(G) = \max\{\chi(G[K \cup A]), \chi(G[K \cup B])\}$, so the desired result follows from the induction hypothesis on $G[K \cup A]$ and $G[K \cup B]$.

Finally, if G has no universal vertex and no clique cutset, then the result follows from Corollary 7 and Theorem 12. □

We remark that we do not have any example of a (claw/fork, C_4)-free graph G such that $\chi(G) = \left\lceil \frac{3}{2}\omega(G) \right\rceil$ except C_5. However, for an integer $m \geqslant 1$, consider the blowup G of the icosahedron graph I where $|Q_v| = m$, for each vertex v in I. Then clearly $\omega(G) = 3m$, and since $\alpha(G) = 3$, we have $\chi(G) \geqslant \frac{|V(G)|}{\alpha(G)} = \frac{12m}{3} = 4m = \frac{4\omega(G)}{3}$.

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