Answers to Two Questions on the DP Color Function

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Abstract

DP-coloring is a generalization of list coloring that was introduced in 2015 by Dvořák and Postle. The chromatic polynomial of a graph is a notion that has been extensively studied since the early 20th century. The chromatic polynomial of graph G is denoted $P(G, m)$, and it is equal to the number of proper m-colorings of G. In 2019, Kaul and Mudrock introduced an analogue of the chromatic polynomial for DP-coloring; specifically, the DP color function of graph G is denoted $P_{DP}(G, m)$. For vertex disjoint graphs G and H, suppose $G \vee H$ denotes the join of G and H. Two fundamental questions posed by Kaul and Mudrock are: (1) For any graph G with n vertices, is it the case that $P(G, m) - P_{DP}(G, m) = O(m^{n-3})$ as $m \to \infty$? and (2) For every graph G, does there exist $p, N \in \mathbb{N}$ such that $P_{DP}(K_p \vee G, m) = P(K_p \vee G, m)$ whenever $m \geqslant N$? We show that the answer to both these questions is yes. In fact, we show the answer to (2) is yes even if we require $p = 1$.

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1 Introduction

In this note all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [31] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \ldots, m\}$. Given a set A, $\mathcal{P}(A)$ is the power set of A. If G is a graph and $S, U \subseteq V(G)$, we use $G[S]$ for the subgraph of G induced by S, and we use $E_G(S, U)$ for the set consisting of all the edges in $E(G)$ that have at least one endpoint in S and at least one endpoint in U. If an edge in $E(G)$ connects the vertices u and v, the edge can be represented by uv or vu. If G and H are vertex disjoint graphs, we write $G \vee H$ for the join of G and H. The *cone of graph* G is $K_1 \vee G$.

1.1 List Coloring and DP-Coloring

In the classical vertex coloring problem we wish to color the vertices of a graph G with up to m colors from $[m]$ so that adjacent vertices receive different colors, a so-called *proper* m*-coloring*. The chromatic number of a graph G, denoted $\chi(G)$, is the smallest m such that G has a proper m-coloring. List coloring, a well-known variation on classical vertex coloring, was introduced independently by Vizing [29] and Erdős, Rubin, and Taylor [12] in the 1970s. For list coloring, we associate a *list assignment* L with a graph G such that each vertex $v \in V(G)$ is assigned a list of colors $L(v)$ (we say L is a list assignment for G). Then, G is L*-colorable* if there exists a proper coloring f of G such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to f as a *proper* L*-coloring* of G). A list assignment L is called a k*-assignment* for G if $|L(v)| = k$ for each $v \in V(G)$. The *list chromatic number* of a graph G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever L is a k-assignment for G. We say G is k*-choosable* if $k \geqslant \chi_\ell(G)$. Since G must be L-colorable whenever L is a $\chi_\ell(G)$-assignment for G that assigns the same list of colors to each element in $V(G)$, it is clear that $\chi(G) \leqslant \chi_\ell(G)$. This inequality may be strict since it is known that there are bipartite graphs with arbitrarily large list chromatic number (see [12]).

In 2015, Dvořák and Postle [11] introduced a generalization of list coloring called DP-coloring (they called it correspondence coloring) in order to prove that every planar graph without cycles of lengths 4 to 8 is 3-choosable. DP-coloring has been extensively studied over the past 5 years (see e.g., [3, 4, 5, 6, 7, 8, 16, 17, 18, 19, 22, 23, 25, 26]). Intuitively, DP-coloring is a variation on list coloring where each vertex in the graph still gets a list of colors, but identification of which colors are different can change from edge to edge. Following [7], we now give the formal definition. Suppose G is a graph. A *cover* of G is a pair $\mathcal{H} = (L, H)$ consisting of a graph H and a function $L : V(G) \to \mathcal{P}(V(H))$ satisfying the following four requirements:

(1) the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$ of size $|V(G)|$;

(2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;

(3) if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;

(4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (the matching may be empty).

Suppose $\mathcal{H} = (L, H)$ is a cover of G. We refer to the edges of H connecting distinct parts of the partition $\{L(v) : v \in V(G)\}$ as *cross-edges*. An $\mathcal{H}$*-coloring* of G is an independent set in H of size $|V(G)|$. It is immediately clear that an independent set $I \subseteq V(H)$ is an $\mathcal{H}$-coloring of G if and only if $|I \cap L(u)| = 1$ for each $u \in V(G)$. We say $\mathcal{H}$ is m*-fold* if $|L(u)| = m$ for each $u \in V(G)$. The *DP-chromatic number* of G, $\chi_{DP}(G)$, is the smallest $m \in \mathbb{N}$ such that G has an $\mathcal{H}$-coloring whenever $\mathcal{H}$ is an m-fold cover of G.

Suppose $\mathcal{H} = (L, H)$ is an m-fold cover of G. We say that $\mathcal{H}$ has a *canonical labeling* if it is possible to name the vertices of H so that $L(u) = \{(u, j) : j \in [m]\}$ and $(u, j)(v, j) \in E(H)$ for each $j \in [m]$ whenever $uv \in E(G)$. [1] Clearly, when $\mathcal{H}$ has a canonical labeling, G

[1]When $\mathcal{H} = (L, H)$ has a canonical labeling, we refer to the vertices of H using this naming scheme.

has an $\mathcal{H}$-coloring if and only if G has a proper m-coloring. Also, given an m-assignment, L, for a graph G, it is easy to construct an m-fold cover $\mathcal{H}'$ of G such that G has an $\mathcal{H}'$-coloring if and only if G has a proper L-coloring (see [7]). It follows that $\chi(G) \leqslant \chi_\ell(G) \leqslant \chi_{DP}(G)$. The second inequality may be strict since it is easy to prove that $\chi_{DP}(C_n) = 3$ whenever $n \geqslant 3$, but the list chromatic number of any even cycle is 2 (see [7] and [12]).

In some instances DP-coloring behaves similar to list coloring, but there are some interesting differences. Molloy [25] has shown that Kahn's [14] result that the list edge-chromatic number of a simple graph asymptotically equals the edge-chromatic number holds for DP-coloring as well. Thomassen [27] famously proved that every planar graph is 5-choosable, and Dvořák and Postle [11] observed that the DP-chromatic number of every planar graph is at most 5. Also, Molloy [24] recently improved a theorem of Johansson by showing that every triangle-free graph G with maximum degree $\Delta(G)$ satisfies $\chi_\ell(G) \leqslant (1 + o(1))\Delta(G)/\log(\Delta(G))$. Bernshteyn [5] subsequently showed that this bound also holds for the DP-chromatic number. On the other hand, Bernshteyn [4] showed that if the average degree of a graph G is d, then $\chi_{DP}(G) = \Omega(d/\log(d))$. This is in striking contrast to the celebrated result of Alon [1] that says $\chi_\ell(G) = \Omega(\log(d))$. It was also recently shown in [7] that there exist planar bipartite graphs with DP-chromatic number 4 even though the list chromatic number of any planar bipartite graph is at most 3 [2]. A famous result of Galvin [13] says that if G is a bipartite multigraph and $L(G)$ is the line graph of G, then $\chi_\ell(L(G)) = \chi(L(G)) = \Delta(G)$. However, it is also shown in [7] that every d-regular graph G satisfies $\chi_{DP}(L(G)) \geqslant d + 1$.

1.2 Counting Proper Colorings, List Colorings, and DP-Colorings

In 1912 Birkhoff introduced the notion of the chromatic polynomial in hopes of using it to make progress on the four color problem. For $m \in \mathbb{N}$, the *chromatic polynomial* of a graph G, $P(G, m)$, is the number of proper m-colorings of G. It can be shown that $P(G, m)$ is a polynomial in m of degree $|V(G)|$ (see [9]). For example, $P(K_n, m) = \prod_{i=0}^{n-1}(m - i)$, $P(C_n, m) = (m - 1)^n + (-1)^n(m - 1)$, $P(T, m) = m(m - 1)^{n-1}$ whenever T is a tree on n vertices, and $P(K_1 \vee G, m) = mP(G, m - 1)$ (see [31]).

The notion of chromatic polynomial was extended to list coloring in the 1990s [21]. In particular, if L is a list assignment for G, we use $P(G, L)$ to denote the number of proper L-colorings of G. The *list color function* $P_\ell(G, m)$ is the minimum value of $P(G, L)$ where the minimum is taken over all possible m-assignments L for G. It is clear that $P_\ell(G, m) \leqslant P(G, m)$ for each $m \in \mathbb{N}$ since we must consider the m-assignment that assigns the same m colors to all the vertices in G when considering all possible m-assignments for G. In general, the list color function can differ significantly from the chromatic polynomial for small values of m. However, for large values of m, Wang, Qian, and Yan [30] (improving upon results in [10] and [28]) showed the following in 2017.

Theorem 1 ([30])**.** *If G is a connected graph with l edges, then $P_\ell(G, m) = P(G, m)$ whenever $m > \frac{l-1}{\ln(1+\sqrt{2})}$.*

It is also known that $P_\ell(G, m) = P(G, m)$ for all $m \in \mathbb{N}$ when G is a cycle or chordal (see [20] and [21]). Moreover, if $P_\ell(G, m) = P(G, m)$ for all $m \in \mathbb{N}$, then $P_\ell(K_n \vee G, m) =$

$P(K_n \vee G, m)$ for each $n, m \in \mathbb{N}$ (see [15]). See [28] for a survey of known results and open questions on the list color function.

In 2019, Kaul and the first author introduced a DP-coloring analogue of the chromatic polynomial in hopes of gaining a better understanding of DP-coloring and using it as a tool for making progress on some open questions related to the list color function [16]. Specifically, suppose $\mathcal{H} = (L, H)$ is a cover of graph G. Let $P_{DP}(G, \mathcal{H})$ be the number of $\mathcal{H}$-colorings of G. Then, the *DP color function* of G, $P_{DP}(G, m)$, is the minimum value of $P_{DP}(G, \mathcal{H})$ where the minimum is taken over all possible m-fold covers $\mathcal{H}$ of G. [2] It is easy to show that for any graph G and $m \in \mathbb{N}$, $P_{DP}(G, m) \leqslant P_\ell(G, m) \leqslant P(G, m)$. [3] Note that if G is a disconnected graph with components: $H_1, H_2, \ldots, H_t$, then $P_{DP}(G, m) = \prod_{i=1}^{t} P_{DP}(H_i, m)$. So, we will only consider connected graphs from this point forward unless otherwise noted.

As with list coloring and DP-coloring, the list color function and DP color function of certain graphs behave similarly. However, for some graphs there are surprising differences. For example, similar to the list color function, $P_{DP}(G, m) = P(G, m)$ for every $m \in \mathbb{N}$ whenever G is chordal or an odd cycle [16]. On the other hand, we have the following two results.

Theorem 2 ([16])**.** *If G is a graph with girth that is even, then there is an $N \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geqslant N$. Furthermore, for any integer $g \geqslant 3$ there exists a graph H with girth g and an $N \in \mathbb{N}$ such that $P_{DP}(H, m) < P(H, m)$ whenever $m \geqslant N$.*

This result is particularly surprising since Theorem 1 implies that the list color function of any graph eventually equals its chromatic polynomial. The following is also known.

Theorem 3 ([16])**.** *For any graph G with n vertices,*

$$P(G, m) - P_{DP}(G, m) = O(m^{n-2}) \text{ as } m \to \infty.$$

In studying the tightness of Theorem 3, the authors of [16] mentioned that if G is a unicyclic graph [4] on n vertices that contains a cycle of length 4, then $P(G, m) - P_{DP}(G, m) = \Theta(m^{n-3})$. However, they stated that "we do not have an example of a graph G such that $P(G, m) - P_{DP}(G, m) = \Theta(m^{n-2})$." Motivated by a result of Bernshteyn, Kostochka, and Zhu [8] that says for any graph G there exists an $N \leqslant 3|E(G)|$ such that $\chi_{DP}(K_p \vee G) = \chi(K_p \vee G)$ whenever $p \geqslant N$, the authors of [16] also studied $P_{DP}(K_p \vee G, m)$. Interestingly, it turns out that the question of whether there exist $p, N \in \mathbb{N}$ such that $P_{DP}(K_p \vee G, m) = P(K_p \vee G, m)$ whenever $m \geqslant N$ is related to the asymptotics of $P(G, m) - P_{DP}(G, m)$. In fact, the following two questions were both posed in [16]. These two questions are the focus of this note.

[2] We take $\mathbb{N}$ to be the domain of the DP color function of any graph.

[3] To prove this, recall that for any m-assignment L for G, an m-fold cover $\mathcal{H}'$ of G such that G has an $\mathcal{H}'$-coloring if and only if G has a proper L-coloring is constructed in [7]. It is easy to see from the construction in [7] that there is a bijection between the proper L-colorings of G and the $\mathcal{H}'$-colorings of G.

[4] A *unicyclic graph* is a connected graph containing exactly one cycle.

Question 4. For any graph G with n vertices, is it the case that $P(G,m) - P_{DP}(G,m) = O(m^{n-3})$ as $m \to \infty$?

Question 5. For every graph G, does there exist $p, N \in \mathbb{N}$ such that $P_{DP}(G \vee K_p, m) = P(G \vee K_p, m)$ whenever $m \geqslant N$?

In [16] it is shown that if the the answer to Question 4 is yes, then the answer to Question 5 must be yes. We will show that the answer to Question 4 is yes. This of course implies that the answer to Question 5 is yes, but we will show that its answer is yes even when p is fixed to 1.

1.3 Summary of Results

We begin by showing the following.

Theorem 6. *Suppose g is an odd integer with $g \geqslant 3$. If G is a graph on n vertices with girth g or $g+1$, then $P(G,m) - P_{DP}(G,m) = O(m^{n-g})$ as $m \to \infty$. Consequently, $P(M,m) - P_{DP}(M,m) = O(m^{|V(M)|-3})$ as $m \to \infty$ for any graph M.*

When considering the third sentence of Theorem 6, recall that if the girth of a graph is infinite, then the graph is acyclic and therefore chordal which means the DP color function of the graph is always equal to its chromatic polynomial. A result in [16] implies that if G is a unicyclic graph on n vertices with girth $2k+2$ where $k \in \mathbb{N}$, then $P(G,m) - P_{DP}(G,m) = (m-1)^n + (m-1)^{n-2k-1} - [(m-1)^n - (m-1)^{n-2k-2}] = \Theta(m^{n-2k-1})$. This together with a result in [3] implies that for any odd integer g with $g \geqslant 3$, if G consists of a cycle on g vertices and a cycle on $g+1$ vertices such that the cycles share exactly one vertex, then $P(G,m) - P_{DP}(G,m) = P(C_{g+1},m)P(C_g,m)/m - P_{DP}(C_{g+1},m)P_{DP}(C_g,m)/m = P(C_g,m)[P(C_{g+1},m) - P_{DP}(C_{g+1},m)]/m = P(C_g,m) = \Theta(m^g)$. This demonstrates the tightness of Theorem 6 for all possible girths.

We end this note by proving the following.

Theorem 7. *For any graph G, there is an $N \in \mathbb{N}$ such that $P_{DP}(K_1 \vee G, m) = P(K_1 \vee G, m)$ whenever $m \geqslant N$.*

Theorem 7 shows that the DP color function of $K_1 \vee G$ behaves like the list color function of $K_1 \vee G$ since the DP color function of $K_1 \vee G$ eventually equals the chromatic polynomial of $K_1 \vee G$. It is worth mentioning that in this note no attempt has been made to minimize the value of N in Theorem 7. It would be interesting to study the threshold at which $P_{DP}(K_1 \vee G, m) = P(K_1 \vee G, m)$ for a given graph G.

2 Proofs of Results

The key to proving our results is generalizing the proof technique of the following classical result to the context of DP-coloring.

Theorem 8 ([32]). *Suppose G is a graph. Then,*

$$P(G,m) = \sum_{A \subseteq E(G)} (-1)^{|A|} m^{k_A}$$

where k_A is the number of components of the spanning subgraph of G with edge set A.

The next four results will also be useful tools to keep in mind.

Proposition 9 ([32]). *Suppose G is a graph on n vertices. Then there are nonnegative integers $a_0, \dots, a_n$ such that $P(G,m) = \sum_{i=0}^{n} (-1)^i a_i m^{n-i}$. Furthermore, if G has c components, then $a_0, \dots, a_{n-c}$ are all positive integers, and $a_{n-c+1} = \cdots = a_n = 0$.*

Proposition 10 ([32]). *Suppose G is a graph with s edges and n vertices having girth $g \in \mathbb{N}$. Suppose $P(G,m) = \sum_{i=0}^{n} (-1)^i a_i m^{n-i}$. Then, for $i = 0, 1, \dots, g-2$*

$$a_i = \binom{s}{i} \text{ and } a_{g-1} = \binom{s}{g-1} - t$$

where t is the number of cycles of length g contained in G.

Proposition 11 ([16]). *Suppose T is a tree and $\mathcal{H} = (L, H)$ is an m-fold cover of T such that $E_H(L(u), L(v))$ is a perfect matching whenever $uv \in E(T)$. Then, $\mathcal{H}$ has a canonical labeling.*

Proposition 12. *Suppose that $\mathcal{H} = (L, H)$ is an m-fold cover of graph G and $\mathcal{H}$ has a canonical labeling. Let $B_i = \{(v, i) : v \in V(G)\}$ for each $i \in [m]$. Then, $I \subset V(H)$ satisfies: $|I \cap L(v)| = 1$ for each $v \in V(G)$ and $H[I]$ is isomorphic to G if and only if $I = B_j$ for some $j \in [m]$.*

Proof. For each $i \in [m]$, it is clear that $|B_i \cap L(v)| = 1$ for each $v \in V(G)$ and $H[B_i]$ is isomorphic to G. Conversely, suppose that $I \notin \{B_1, \dots, B_m\}$. Since $|I \cap L(v)| = 1$ for each $v \in V(G)$, $H[I]$ has fewer edges than G contradicting the fact that $H[I]$ is isomorphic to G. □

2.1 Proof of Theorem 6

We will now introduce some notation that will be used for the remainder of this note. Suppose that G is a graph on $n \geqslant 3$ vertices with $|E(G)| \geqslant 3$. Let $s = |E(G)|$, and $E(G) = \{e_1, \dots, e_s\}$. Also, for some $m \in \mathbb{N}$ suppose that $\mathcal{H} = (L, H)$ is an m-fold cover of G satisfying $|E_H(L(u), L(v))| = m$ whenever $uv \in E(G)$.

Let $\mathcal{U} = \{I \subseteq V(H) : |L(v) \cap I| = 1 \text{ for each } v \in V(G)\}$. Clearly, $|\mathcal{U}| = m^n$. Now, for each $i \in [s]$, suppose $e_i = u_i v_i$, and let S_i be the set consisting of each $I \in \mathcal{U}$ with the property that $H[I]$ contains an edge in $E_H(L(u_i), L(v_i))$. Also, for each $i \in [s]$ let $C_i = \mathcal{U} - S_i$. Clearly,

$$P_{DP}(G, \mathcal{H}) = \left| \bigcap_{i=1}^{s} C_i \right|.$$

So, by the Inclusion-Exclusion Principle, we see that

$$P_{DP}(G,\mathcal{H}) = |\mathcal{U}| - \left|\bigcup_{i=1}^{s} S_i\right| = m^n - \sum_{k=1}^{s}(-1)^{k-1}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant s}\left|\bigcap_{j=1}^{k} S_{i_j}\right|\right).$$

The following Lemma is the key to our proof of Theorem 6.

Lemma 13. *Assuming the set-up established above, suppose that G is a graph of girth $g \in \mathbb{N}$. Then, the following three statements hold.*
(i) For any $k \in [g-1]$ and $i_1, \ldots, i_k \in [s]$ satisfying $i_1 < \cdots < i_k$, $\left|\bigcap_{j=1}^{k} S_{i_j}\right| = m^{n-k}$.
(ii) If $e_{i_1}, \ldots, e_{i_g}$ are distinct edges in G, then $\left|\bigcap_{j=1}^{g} S_{i_j}\right| \leqslant m^{n-g+1}$. Moreover, $\left|\bigcap_{j=1}^{g} S_{i_j}\right| = m^{n-g}$ when $e_{i_1}, \ldots, e_{i_g}$ are not the edges of a g-cycle in G.
(iii) For any $k \geqslant g+1$ and $i_1, \ldots, i_k \in [s]$ satisfying $i_1 < \cdots < i_k$, $\left|\bigcap_{j=1}^{k} S_{i_j}\right| \leqslant m^{n-g}$.

Proof. For Statement (i), suppose that G' is the spanning subgraph of G with $E(G') = \{e_{i_1}, \ldots, e_{i_k}\}$. Let $H' = H - \bigcup_{xy \in E(G)-E(G')} E_H(L(x), L(y))$. Since G has girth g and $k \in [g-1]$, G' is an acyclic graph with $n-k$ components. Suppose the components of G' are $W_1, \ldots, W_{n-k}$ (each component is a tree). Note that we can construct each element I of $\bigcap_{j=1}^{k} S_{i_j}$ in $(n-k)$ steps as follows. For each $i \in [n-k]$ consider the component W_i. Suppose $V(W_i) = \{w_1, \ldots, w_l\}$. Choose one element from each of $L(w_1), \ldots, L(w_l)$ so that the subgraph of H' induced by the set containing these chosen elements is isomorphic to W_i. Then, place these chosen elements in I. By Propositions 11 and 12, this step can be done in m ways (regardless of the choices made in previous steps). So, $\left|\bigcap_{j=1}^{k} S_{i_j}\right| = m^{n-k}$.

For Statement (ii), the first part follows from Statement (i) since $\left|\bigcap_{j=1}^{g} S_{i_j}\right| \leqslant \left|\bigcap_{j=1}^{g-1} S_{i_j}\right| = m^{n-g+1}$. So, suppose that $e_{i_1}, \ldots, e_{i_g}$ are not the edges of a g-cycle in G. Let G'' be the spanning subgraph of G with $E(G'') = \{e_{i_1}, \ldots, e_{i_g}\}$. Clearly, G'' is an acyclic graph with $n-g$ components. We can obtain $\left|\bigcap_{j=1}^{g} S_{i_j}\right| = m^{n-g}$ by using an argument similar to the argument used for the proof of Statement (i).

For Statement (iii), notice that we can assume without loss of generality that $e_{i_1}, \ldots, e_{i_g}$ are not the edges of a g-cycle in G. So, by Statement (ii), we see that $\left|\bigcap_{j=1}^{k} S_{i_j}\right| \leqslant \left|\bigcap_{j=1}^{g} S_{i_j}\right| = m^{n-g}$. □

We are now ready to prove Theorem 6.

Proof. Suppose $s = |E(G)|$ and t is the number of g-cycles in G (note that $t = 0$ in the case that G has girth $g+1$). Since g is odd, Propositions 9 and 10 tell us that there is an $N \in \mathbb{N}$ such that

$$P(G,m) \leqslant \left(\binom{s}{g-1} - t\right) m^{n-g+1} + \sum_{i=0}^{g-2}(-1)^i \binom{s}{i} m^{n-i}$$

whenever $m \geqslant N$. Suppose that m is a fixed natural number satisfying $m \geqslant N$.

Suppose that $\mathcal{H} = (L, H)$ is an m-fold cover of G satisfying $P_{DP}(G, \mathcal{H}) = P_{DP}(G, m)$. Clearly, we may assume that $|E_H(L(u), L(v))| = m$ whenever $uv \in E(G)$. Now, assume we use the same notation described at the start of this Subsection. By Statement (i) of Lemma 13, we have that

$$\begin{aligned}
&P_{DP}(G, m) \\
&= m^n + \sum_{k=1}^{s} (-1)^k \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant s} \left| \bigcap_{j=1}^{k} S_{i_j} \right| \right) \\
&= \sum_{i=0}^{g-1} (-1)^i \binom{s}{i} m^{n-i} - \sum_{1 \leqslant i_1 < \cdots < i_g \leqslant s} \left| \bigcap_{j=1}^{g} S_{i_j} \right| + \sum_{k=g+1}^{s} (-1)^k \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant s} \left| \bigcap_{j=1}^{k} S_{i_j} \right| \right).
\end{aligned}$$

We see that Statement (ii) of Lemma 13 implies that

$$\sum_{1 \leqslant i_1 < \cdots < i_g \leqslant s} \left| \bigcap_{j=1}^{g} S_{i_j} \right| \leqslant t m^{n-g+1} + \left(\binom{s}{g} - t \right) m^{n-g}.$$

Furthermore, Statement (iii) of Lemma 13 implies that

$$\sum_{k=g+1}^{s} (-1)^k \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant s} \left| \bigcap_{j=1}^{k} S_{i_j} \right| \right) \geqslant -2^s m^{n-g}.$$

These facts imply that

$$P_{DP}(G, m) \geqslant \left(\binom{s}{g-1} - t \right) m^{n-g+1} + \sum_{i=0}^{g-2} (-1)^i \binom{s}{i} m^{n-i} - \left(\binom{s}{g} - t \right) m^{n-g} - 2^s m^{n-g}.$$

So, we see that

$$P(G, m) - P_{DP}(G, m) \leqslant \left(\binom{s}{g} - t + 2^s \right) m^{n-g}.$$

The desired result immediately follows. □

2.2 Proof of Theorem 7

Notice that the result of Theorem 7 is obvious when G is acyclic since the cone of such a graph is chordal. So, throughout this Subsection suppose that G is a graph with $n-1$ vertices where $n \geqslant 4$ and $s \geqslant 3$ edges. Suppose that $E(G) = \{e_1, \ldots, e_s\}$. Also, suppose that $M = K_1 \vee G$, and w is the vertex corresponding to the copy of K_1 used to form M. We use $e_{s+1}, \ldots, e_{s+n-1}$ to denote the edges in $E(M)$ that have w as an endpoint. We want to show that $P_{DP}(M, m) = P(M, m)$ for sufficiently large m, or equivalently, $P_{DP}(M, m) \geqslant P(M, m)$ for sufficiently large m.

Since M is a graph with n vertices and $n+s-1$ edges, Propositions 9 and 10 tell us that

$$P(M,m) = m^n - (n+s-1)m^{n-1} + \left(\binom{n+s-1}{2} - t\right)m^{n-2} - a_3 m^{n-3} + O(m^{n-4}) \text{ as } m \to \infty$$

where t is the number of 3-cycles contained in M (note that $t \geqslant s$). We now give a formula for a_3. Let $A = \{A_1, \ldots, A_q\}$ be the set of spanning subgraphs of M with $(n-3)$ components. For each $i \in [q]$, it is straightforward to verify that $3 \leqslant |E(A_i)| \leqslant 6$. For $i \in \{3,4,5,6\}$, let $P_i = \{E(A_j) : A_j \in A, |E(A_j)| = i\}$. By Theorem 8, $a_3 = |P_3| - |P_4| + |P_5| - |P_6|$.

In this Subsection we are interested in finding a lower bound for $P_{DP}(M,m)$. So, whenever $\mathcal{H} = (L,H)$ is an m-fold cover for M, we will assume that $|E_H(L(u),L(v))| = m$ for each $uv \in E(M)$. We will also suppose without loss of generality that $L(u) = \{(u,j) : j \in [m]\}$ for each $u \in V(M)$, and $(w,j)(v,j) \in E(H)$ for each $v \in V(G)$ and $j \in [m]$ (this is permissible by Proposition 11 since the spanning subgraph of M with edge set $\{e_{s+1}, \ldots, e_{s+n-1}\}$ is a tree).

Suppose $\mathcal{H} = (L,H)$ is an m-fold cover for M. For each $e_i \in E(G)$ we suppose that $e_i = u_i v_i$, and we let $x_{i,\mathcal{H}}$ [5] be the number of edges in $E_H(L(u_i),L(v_i))$ that connect endpoints with differing second coordinates. Finally, we let $x_{\mathcal{H}} = \sum_{i=1}^{s} x_{i,\mathcal{H}}$. Clearly, if $x_{\mathcal{H}} = 0$, then $\mathcal{H}$ has a canonical labeling and $P_{DP}(M,\mathcal{H}) = P(M,m)$. Also, $x_{\mathcal{H}}$ is the number of cross edges in H that connect vertices with differing second coordinates.

For the next Lemma assume that $\mathcal{H} = (L,H)$ is an m-fold cover for M, and assume we are using the same notation as the beginning of Subsection 2.1 (with M playing the role of G).

Lemma 14. *The following statments hold.*
(i) $\sum_{1 \leqslant i_1 < \cdots < i_3 \leqslant n+s-1} \left| \bigcap_{j=1}^{3} S_{i_j} \right| \leqslant tm^{n-2} - x_{\mathcal{H}} m^{n-3} + |P_3| m^{n-3}$,
(ii) $\sum_{1 \leqslant i_1 < \cdots < i_4 \leqslant n+s-1} \left| \bigcap_{j=1}^{4} S_{i_j} \right| \geqslant |P_4| m^{n-3} - 2|P_4| x_{\mathcal{H}} m^{n-4}$,
(iii) $\sum_{1 \leqslant i_1 < \cdots < i_5 \leqslant n+s-1} \left| \bigcap_{j=1}^{5} S_{i_j} \right| \leqslant |P_5| m^{n-3} + \left(\binom{n+s-1}{5} - |P_5|\right) m^{n-4}$,
(iv) $\sum_{1 \leqslant i_1 < \cdots < i_6 \leqslant n+s-1} \left| \bigcap_{j=1}^{6} S_{i_j} \right| \geqslant |P_6| m^{n-3} - 2|P_6| x_{\mathcal{H}} m^{n-4}$, *and*
(v) For $k \geqslant 7$, $\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n+s-1} \left| \bigcap_{j=1}^{k} S_{i_j} \right| \leqslant \binom{n+s-1}{k} m^{n-4}$.

Proof. For Statement (i), suppose x, y, and z are distinct edges in $E(M)$. Let M' be the spanning subgraph of M with $E(M') = \{x,y,z\}$. If x, y, and z form a 3-cycle in M containing w, then M' consists of this 3-cycle and $n-3$ isolated vertices. Notice that each 3-cycle in M containing w contains exactly one edge in $E(G)$. So, we suppose that $z = e_i$ for some $i \in [s]$, then it is clear that $|S_x \cap S_y \cap S_z| = m^{n-3}(m - x_i) = m^{n-2} - x_i m^{n-3}$. In the case that x, y, and z form a 3-cycle in M not containing w, then Lemma 13 implies that $|S_x \cap S_y \cap S_z| \leqslant m^{n-2}$. Finally, in the case that x, y, and z do not form a 3-cycle in M

[5] We will just write x_i when $\mathcal{H}$ is clear from context.

(note that there are $|P_3|$ such sets of three edges), Lemma 13 implies $|S_x \cap S_y \cap S_z| = m^{n-3}$. Statement (i) now follows immediately from these facts.

For Statement (ii), suppose a, x, y, and z are distinct edges in $E(M)$. Let M' be the spanning subgraph of M with $E(M') = \{a, x, y, z\}$. Let $H' = H - \bigcup_{xy \in E(M)-E(M')} E_H(L(x), L(y))$. If M' contains a cycle, then M' contains one cycle and consists of $n-3$ components (note that there are $|P_4|$ sets of four edges for which this happens). Suppose that the components of M' are $W_1, \ldots, W_{n-3}$, and assume that W_1 is the component of M' containing the cycle. Also, suppose that $V(W_1) = \{w_1, \ldots, w_l\}$. Now, let $B_i = \{(w_j, i) : j \in [l]\}$ for each $i \in [m]$. If $H'[B_i]$ is not isomorphic to W_1, then an element of B_i must be the endpoint of a cross edge in H that connects vertices with differing second coordinates. Let $\mathcal{B}$ consist of each $B_i \in \{B_1, \ldots, B_m\}$ with the property that $H'[B_i]$ is not isomorphic to W_1. Notice this means that for each $B_j \in \{B_1, \ldots, B_m\} - \mathcal{B}$, $H'[B_j]$ is isomorphic to W_1, and there are are least $|\{B_1, \ldots, B_m\} - \mathcal{B}|$ ways to select one element from each of $L(w_1), \ldots, L(w_l)$ so that the subgraph of H' induced by the set containing these chosen elements is isomorphic to W_1. Let $\mathcal{E}$ be the set of cross edges in H that connect vertices with differing second coordinates (note that $|\mathcal{E}| = x_{\mathcal{H}}$). We can construct a function $\eta : \mathcal{B} \to \mathcal{E}$ that maps each $B_i \in \mathcal{B}$ to one of the edges in $\mathcal{E}$ that has an endpoint in B_i. Furthermore, if B_i, B_j, and B_t are distinct elements of $\mathcal{B}$, then it is not possible for $\eta(B_i) = \eta(B_j) = \eta(B_t)$ since an edge only has two endpoints. Consequently, $|\{B_1, \ldots, B_m\} - \mathcal{B}| \geqslant (m - 2x_{\mathcal{H}})$. So, $|S_a \cap S_x \cap S_y \cap S_z| \geqslant m^{n-4}(m - 2x_{\mathcal{H}}) = m^{n-3} - 2x_{\mathcal{H}} m^{n-4}$. Statement (ii) now immediately follows.

For Statement (iii), suppose a, b, x, y, and z are distinct edges in $E(M)$. Let M' be the spanning subgraph of M with $E(M') = \{a, b, x, y, z\}$. Let $H' = H - \bigcup_{xy \in E(M)-E(M')} E_H(L(x), L(y))$. Suppose M' consists of $n-3$ components (note that there are $|P_5|$ sets of five edges for which this happens). Suppose that the components of M' are $W_1, \ldots, W_{n-3}$. Note that we can construct each element I of $(S_a \cap S_b \cap S_x \cap S_y \cap S_z)$ in $(n-3)$ steps as follows. For each $i \in [n-3]$ consider the component W_i. If $\{a, b, x, y, z\} \cap E(W_i) \neq \emptyset$, then $V(W_i)$ has at least 2 elements, say $V(W_i) = \{w_1, \ldots, w_l\}$, choose one element from each of $L(w_1), \ldots, L(w_l)$ so that the subgraph of H' induced by these chosen elements is isomorphic to W_i (this can be done in at most m ways [6]). Then, place these chosen elements in I. If $\{a, b, x, y, z\} \cap E(W_i) = \emptyset$, then W_i is a single vertex, say $V(W_i) = \{q\}$, and we choose an element of $L(q)$ to place in I. Notice that in either case there are at most m ways to complete the step. Consequently, $|S_a \cap S_b \cap S_x \cap S_y \cap S_z| \leqslant m^{n-3}$. A similar argument shows that when M' has fewer than $n-3$ components, $|S_a \cap S_b \cap S_x \cap S_y \cap S_z| \leqslant m^{n-4}$. Statement (iii) now follows from the fact that $\sum_{1 \leqslant i_1 < \cdots < i_5 \leqslant n+s-1} \left| \bigcap_{j=1}^{5} S_{i_j} \right|$ has $\binom{n+s-1}{5}$ terms.

For Statement (iv), suppose a, b, c, x, y, and z are distinct edges in $E(M)$. Let M' be the spanning subgraph of M with $E(M') = \{a, b, c, x, y, z\}$. Suppose M' consists of $n-3$ components (note that there are $|P_6|$ sets of six edges for which this happens). It is easy to see that M' must consist of a complete graph on four vertices and $n-4$

[6] To see why this is so, consider a spanning tree of W_i and apply Propositions 11 and 12.

isolated vertices. Suppose that the components of M' are $W_1, \dots, W_{n-3}$, and assume that $W_1 = K_4$. Using an argument similar to the argument used to prove Statement (ii), we obtain $|S_a \cap S_b \cap S_c \cap S_x \cap S_y \cap S_z| \geqslant m^{n-4}(m - 2x_{\mathcal{H}}) = m^{n-3} - 2x_{\mathcal{H}}m^{n-4}$. Statement (iv) now immediately follows.

For Statement (v), suppose $k \geqslant 7$ and $1 \leqslant i_1 < \cdots < i_k \leqslant n+s-1$. Let M' be the spanning subgraph of M with $E(M') = \{e_{i_1}, \dots, e_{i_k}\}$. Then, M' must consist of at most $n-4$ components. An argument similar to the argument used to prove Statement (iii) then yields $\left|\bigcap_{j=1}^{k} S_{i_j}\right| \leqslant m^{n-4}$. Statement (v) now immediately follows. □

We need one more Lemma before proving Theorem 7.

Lemma 15. *Suppose that $m \geqslant 2(|P_4| + |P_6|)$, and $\mathcal{H} = (L, H)$ is an m-fold cover for M with $x_{\mathcal{H}} > 0$. Then,*

$$\begin{aligned}
&P_{DP}(M, \mathcal{H}) \\
&\geqslant m^n - (n+s-1)m^{n-1} + \left(\binom{n+s-1}{2} - t\right) m^{n-2} - a_3 m^{n-3} \\
&+ m^{n-3} - (2(|P_4| + |P_6| + 2^{n+s-2}))m^{n-4}.
\end{aligned}$$

Proof. Using the notation established in Subsection 2.1 (with M playing the role of G) along with Lemma 13, we know that

$$\begin{aligned}
&P_{DP}(M, \mathcal{H}) \\
&= m^n + \sum_{k=1}^{n+s-1} (-1)^k \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n+s-1} \left| \bigcap_{j=1}^{k} S_{i_j} \right| \right) \\
&= m^n - (n+s-1)m^{n-1} + \binom{n+s-1}{2} m^{n-2} - \sum_{1 \leqslant i_1 < \cdots < i_3 \leqslant n+s-1} \left| \bigcap_{j=1}^{3} S_{i_j} \right| \\
&+ \sum_{1 \leqslant i_1 < \cdots < i_4 \leqslant n+s-1} \left| \bigcap_{j=1}^{4} S_{i_j} \right| - \sum_{1 \leqslant i_1 < \cdots < i_5 \leqslant n+s-1} \left| \bigcap_{j=1}^{5} S_{i_j} \right| + \sum_{1 \leqslant i_1 < \cdots < i_6 \leqslant n+s-1} \left| \bigcap_{j=1}^{6} S_{i_j} \right| \\
&+ \sum_{k=7}^{n+s-1} (-1)^k \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n+s-1} \left| \bigcap_{j=1}^{k} S_{i_j} \right| \right).
\end{aligned}$$

Then, Lemma 14 yields:

$$\begin{aligned}
&P_{DP}(M, \mathcal{H}) \\
&\geqslant m^n - (n+s-1)m^{n-1} + \binom{n+s-1}{2} m^{n-2} - \left(t m^{n-2} - x_{\mathcal{H}} m^{n-3} + |P_3| m^{n-3}\right) \\
&+ |P_4| m^{n-3} - 2|P_4| x_{\mathcal{H}} m^{n-4} - \left(|P_5| m^{n-3} + \left(\binom{n+s-1}{5} - |P_5| \right) m^{n-4} \right) \\
&+ |P_6| m^{n-3} - 2|P_6| x_{\mathcal{H}} m^{n-4} - \sum_{k=7}^{n+s-1} \binom{n+s-1}{k} m^{n-4}
\end{aligned}$$

$$\geqslant m^n - (n+s-1)m^{n-1} + \left(\binom{n+s-1}{2} - t\right) m^{n-2} - a_3 m^{n-3}$$
$$+ (m - 2|P_4| - 2|P_6|)x_{\mathcal{H}} m^{n-4} - 2^{n+s-1} m^{n-4}$$
$$\geqslant m^n - (n+s-1)m^{n-1} + \left(\binom{n+s-1}{2} - t\right) m^{n-2} - a_3 m^{n-3}$$
$$+ m^{n-3} - (2(|P_4| + |P_6| + 2^{n+s-2}))m^{n-4}. \qquad \square$$

We now prove Theorem 7.

Proof. By Propositions 9 and 10 there are $C, N_1 \in \mathbb{N}$ such that $P(M,m) \leqslant m^n - (n+s-1)m^{n-1} + \left(\binom{n+s-1}{2} - t\right) m^{n-2} - a_3 m^{n-3} + Cm^{n-4}$ whenever $m \geqslant N_1$. Also, when $m \geqslant 2(|P_4| + |P_6|)$ and $\mathcal{H} = (L, H)$ is an m-fold cover for M with $x_{\mathcal{H}} > 0$, Lemma 15 tells us $P_{DP}(M, \mathcal{H}) \geqslant m^n - (n+s-1)m^{n-1} + \left(\binom{n+s-1}{2} - t\right) m^{n-2} - a_3 m^{n-3} + m^{n-3} - (2(|P_4| + |P_6| + 2^{n+s-2}))m^{n-4}$. Finally, there must be an $N_2 \in \mathbb{N}$ such that $m^{n-3} - (2(|P_4| + |P_6| + 2^{n+s-2}) + C)m^{n-4} \geqslant 0$ whenever $m \geqslant N_2$.

Let $N = \max\{N_1, N_2, 2(|P_4| + |P_6|)\}$. If $m \geqslant N$ and $\mathcal{H} = (L, H)$ is an m-fold cover for M with $x_{\mathcal{H}} > 0$, then $P_{DP}(M, \mathcal{H}) - P(M,m) \geqslant m^{n-3} - (2(|P_4| + |P_6| + 2^{n+s-2}) + C)m^{n-4} \geqslant 0$. Since we know that when $\mathcal{H} = (L, H)$ is an m-fold cover for M with $x_{\mathcal{H}} = 0$, $P_{DP}(M, \mathcal{H}) = P(M,m)$, we may conclude that $P_{DP}(M,m) = P(M,m)$ whenever $m \geqslant N$. $\square$

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