Extending the Gyárfás-Sumner conjecture to digraphs

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Submitted: Sep, 28 2020; Accepted: Apr, 12 2021; Published: May 21, 2021

Abstract

The dichromatic number of a digraph D is the minimum number of colors needed to color its vertices in such a way that each color class induces an acyclic digraph. As it generalizes the notion of the chromatic number of graphs, it has become the focus of numerous works. In this work we look at possible extensions of the Gyárfás-Sumner conjecture. In particular, we conjecture a simple characterization of sets $\mathcal{F}$ of three digraphs such that every digraph with sufficiently large dichromatic number must contain a member of $\mathcal{F}$ as an induced subdigraph.

Among notable results, we prove that oriented K_4-free graphs without a directed path of length 3 have bounded dichromatic number where a bound of 414 is provided. We also show that an orientation of a complete multipartite graph with no directed triangle is 2-colorable. To prove these results we introduce the notion of *nice sets* that might be of independent interest.

Mathematics Subject Classifications: 05C15, 05C20

*Supported by ANR Project ALGORIDAM (ANR-19-CE48-0016)
†Supported by ANR Project DISTANCIA (ANR-17-CE40-0015)
‡Supported by ANR Project HOSIGRA (ANR-17-CE40-002)

1 Introduction

Despite the fact that the chromatic number of graphs is arguably the most studied invariant in graph theory, there are still many questions about chromatic number for which we do not have a satisfying answer. In particular, a lot of work has been done about the following question: what induced substructures are expected to be found inside a graph if we assume it has very large chromatic number? Or equivalently what are the minimal families $\mathcal{F}$ such that the class of graphs that do not contain any graph in $\mathcal{F}$ as an induced subgraph has bounded chromatic number? Since complete graphs have unbounded chromatic number and do not contain any induced subgraph other than complete graphs themselves, it is clear that such an $\mathcal{F}$ must contain a complete graph. Moreover, Erdős's celebrated result on the existence of graphs of high girth and high chromatic number [12] implies that if $\mathcal{F}$ is finite, then at least one member of $\mathcal{F}$ must be a forest. These two facts constitute the "only if" part of the following tantalizing and still widely open conjecture of Gyárfás and Sumner (see [28] for a survey on known results).

Conjecture 1 (Gyárfás-Sumner, [16, 32])**.** Given two graphs F_1 and F_2 the class of graphs with no induced F_1 or F_2 has bounded chromatic number if and only if one of F_1, F_2 is a complete graph and the other is a forest.

The goal of this paper is to study similar questions in the setting of digraphs.

Every graph and directed graph (*digraph* in short) in this paper is simple, loopless and finite, where simple means no multiedge (multiarc). A digraph in which there is at most one arc between each pair of vertices is called an *oriented graph.*

An oriented cycle in which the indegree and outdegree of each vertex is 1, is a *directed cycle.* The directed cycle of length k is denoted by $\overrightarrow{C_k}$ and the directed cycle on 2 vertices is sometimes referred to as a *digon.*

The *dichromatic number* of a digraph D, denoted $\vec{\chi}(D)$, is the minimum number of colors needed to color the vertices of D in such a way that no directed cycle is monochromatic. In other words, it is the minimum number of acyclic induced subdigraphs needed to partition $V(D)$. Clearly, if G is the *underlying graph* of D (uv is an edge in G if uv or vu is an arc in D), then $\vec{\chi}(D) \leqslant \chi(G)$, and equality holds if $D = \overleftrightarrow{G}$, where $\overleftrightarrow{G}$ is obtained from G by replacing each edge by a digon. The notion of dichromatic number has been introduced in 1982 by Neumann-Lara [27]. A surge of recent results on the subject suggests that this concept is accepted as the natural extension of the chromatic number of graphs to digraphs by most authors. For example see: [3, 7, 15, 20, 22, 25, 26, 30]. Note that the problem of deciding if a digraph has dichromatic number at most 2 is NP-complete (see [8]), even when restricted to tournaments (see [9]), which is in contrast to the undirected case, where 2-colorability is polynomial time to decide.

We note that some authors refer to the dichromatic number of a digraph as simply the *chromatic number of a digraph.* We choose to follow the notations of Neumann-Lara (also used by Erdős, see [11]) who introduced the notion. We then use the notation $\chi(D)$ for the chromatic number of the underlying graph of D. Moreover, an acyclic coloring of D will be called a *dicoloring* of D.

Given a class $\mathcal{F}$ of digraphs we denote by $Forb_{ind}(\mathcal{F})$ the set of digraphs which have no member of $\mathcal{F}$ as an induced subdigraph. If $\mathcal{F}$ is explicitly given by a list of digraphs $F_1, F_2, \ldots, F_k$ we will write $Forb_{ind}(F_1, F_2, \ldots, F_k)$ instead of the heavier $Forb_{ind}(\{F_1, F_2, \ldots, F_k\})$. For example, if we denote by $\overrightarrow{K_2}$ the oriented graph on two vertices with one arc, $Forb_{ind}(\overrightarrow{K_2})$ is the class of *symmetric digraphs* (if uv is an arc, then vu is too) which is an equivalent representation of the class of graphs. A class $\mathcal{C}$ of digraphs is said to be *hereditary* if for every digraph $D \in \mathcal{C}$, every induced subdigraph D' of D is in $\mathcal{C}$. It is clear from the definition that $Forb_{ind}(\mathcal{F})$ is a hereditary class of digraph for any choice of $\mathcal{F}$. Moreover, every hereditary class $\mathcal{C}$ of digraphs can be presented as $Forb_{ind}(\mathcal{F})$ where $\mathcal{F}$ is the set of minimal digraphs not belonging to $\mathcal{C}$. This set $\mathcal{F}$ is not necessarily finite, but in this work we are only interested in the cases where $\mathcal{F}$ is finite.

Given a class $\mathcal{C}$ of digraphs, we define the *dichromatic number of* $\mathcal{C}$ as $\vec{\chi}(\mathcal{C}) = \max\{\vec{\chi}(D) \mid D \in \mathcal{C}\}$ with understanding that $\vec{\chi}(\mathcal{C}) = \infty$ when it is not bounded.

In this paper, we try to extend Gyárfás and Sumner conjecture to the world of digraphs, so we investigate the following problem:

Problem 2. What are the finite sets $\mathcal{F}$ of digraphs for which the class $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number?

Such sets are said to be *heroic*, so the goal of this paper is to investigate finite heroic sets. In Section 2 we recall the notion of *hero* introduced in [6] which forms the starting point of this work. In Section 3 we present four classes of digraphs each of which must intersect every heroic set. This permits us to prove that $\{\overleftrightarrow{K_2}, \overrightarrow{K_2}\}$ is the only heroic set of size two (Proposition 9), and that each heroic set of size three must intersect $\{\overleftrightarrow{K_2}, \overline{K_\alpha}, \overrightarrow{K_2}\}$, where $\overline{K_\alpha}$ is the graph with α vertices and no arc. Since a digraph with no $\overrightarrow{K_2}$ is symmetric, forbidding it puts us in the world of graphs (rather than digraphs) and thus in the framework of the well known Gyárfás-Sumner Conjecture. Hence, we study only heroic sets of size three that contains either $\overleftrightarrow{K_2}$, or $\overline{K_\alpha}$.

A characterization of heroic sets of size three containing $\overline{K_\alpha}$ is given in Section 4, and the proof in Section 5. Furthermore, we propose as a conjecture (Conjecture 11) a simple characterization of heroic sets of size three containing $\overleftrightarrow{K_2}$. Note that forbidding $\overleftrightarrow{K_2}$ means that we are in the world of oriented graphs, so the conjecture gives an oriented analogue to Gyárfás-Sumner Conjecture. As a modest support to it, we prove, in Section 6, that the sets $\{\overleftrightarrow{K_2}, \overrightarrow{C_3}, \overrightarrow{K_2} + K_1\}$ and $\{\overleftrightarrow{K_2}, K_4, P^+(3)\}$ are heroic, where $\overrightarrow{K_2} + K_1$ is the digraph on three vertices with a single arc and $P^+(3)$ is the directed path of length 3.

2 Tournaments, Heroes and Heroic Sets

A *tournament* is an oriented complete graph. Whereas complete graphs are somehow trivial objects regarding chromatic number, tournaments are already a complex and rich family regarding dichromatic number. Observe for example that the *transitive tournament* on n vertices (i.e. the unique up to isomorphism tournament on n vertices that contains no directed cycle), denoted by TT_n, has dichromatic number 1. On the other hand there exists tournaments of arbitrarily large dichromatic number, as we explain now.

Given two digraphs H_1 and H_2 on disjoint sets of vertices, we denote by $H_1 \Rightarrow H_2$ the digraph obtained from disjoint union of H_1 and H_2 by adding an arc from each vertex of H_1 to each vertex of H_2. Given a digraph D on k vertices labeled $\{1, \dots, k\}$ and k digraphs $D_1, D_2, \dots, D_k$ we denote by $\overrightarrow{D}(D_1, D_2, \dots, D_k)$ the digraph with vertex set $\cup_{i=1}^n V(D_i)$ and arc set $\cup_{i=1}^n A(D_i) \cup \{x_i x_j : x_i \in V(D_i),\, x_j \in V(D_j),\, ij \in A(D)\}$. When all D_i's are isomorphic to D_1 we may simply write $\overrightarrow{D}(D_1)$ in place of $\overrightarrow{D}(D_1, D_2, \dots, D_k)$.

The following folklore theorem is straightforward to prove.

Theorem 3. *Given a digraph D and an integer $k \geqslant 3$, $\vec{\chi}(\overrightarrow{C_k}(D)) = \vec{\chi}(D) + 1$.*

Using this, we can give the aforementioned construction of tournaments of arbitrarily large dichromatic number. Consider a sequence of digraphs $(D_k)_{k \in \mathbb{N}}$ defined recursively as follows: $D_1 = K_1$, $D_k = \overrightarrow{C_3}(D_{k-1}, D_{k-1}, K_1)$. It satisfies $\vec{\chi}(D_k) = k$ (see Figure 1).

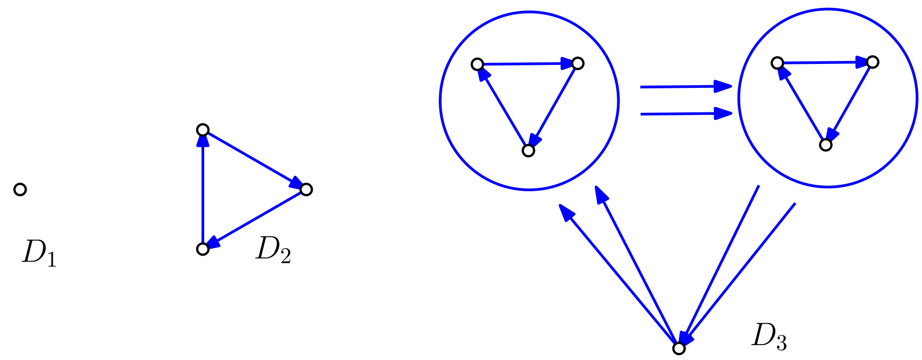

Figure 1: Digraphs D_1, D_2, D_3

Therefore, Problem 2 is already meaningful when restricted to the family of tournaments, and in a seminal paper [6], Berger et al. give a full characterization of tournaments H such that the class of tournaments not containing H has bounded dichromatic number. They call these tournaments *heroes.*

Theorem 4. *[6] A tournament is a hero if and only if it can be constructed by the following inductive rules:*

- *K_1 is a hero.*
- *If H_1 and H_2 are heroes, then $H_1 \Rightarrow H_2$ is also a hero.*
- *If H is a hero, then for every $k \geqslant 1$, $\overrightarrow{C_3}(H, TT_k, K_1)$ and $\overrightarrow{C_3}(TT_k, H, K_1)$ are both heroes.*

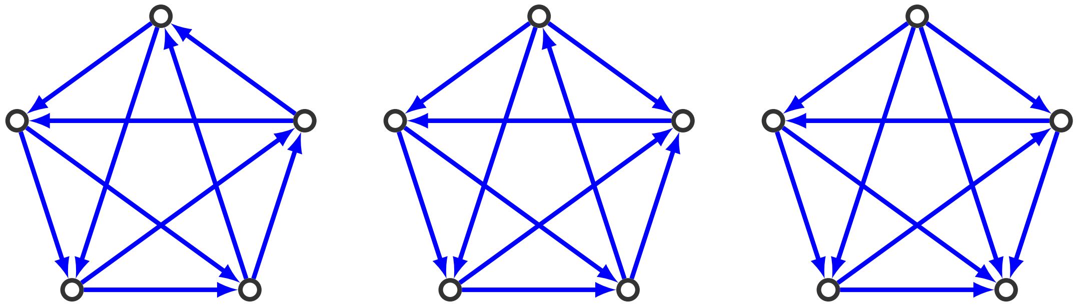

Figure 2: H_1, H_2 and H_3

Extending the notion of a hero, we say that a set $\mathcal{F}$ of digraphs is *heroic* if $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number.

Recall that $\overline{K_k}$ is the digraph on k vertices and no arc. Since $Forb_{ind}\{\overleftrightarrow{K_2}, \overline{K_2}\}$ is the set of all tournaments, heroes are precisely the digraphs H such that the set $\{\overleftrightarrow{K_2}, \overline{K_2}, H\}$ is heroic.

Hence, Theorem 4 characterizes all heroic sets of size three containing $\overleftrightarrow{K_2}$ and $\overline{K_2}$. However, it should be noted that minimal heroic sets of size at least four containing $\overleftrightarrow{K_2}$ and $\overline{K_2}$ exist. An example of such a set is implicitly provided in [6]. Let H_1, H_2 and H_3 be the tournaments depicted in Figure 2. Furthermore, let H_4 be $\overrightarrow{C_3}(\overrightarrow{K_2}, \overrightarrow{K_2}, \overrightarrow{K_2})$ and let H_5 be $\overrightarrow{C_3}(\overrightarrow{C_3}, \overrightarrow{C_3}, K_1)$. In [6], it is proved that the set $\{H_1, H_2, H_3, H_4, H_5\}$ is the set of minimal non-hero tournaments, i.e., any tournament which is not a hero contains one of these five tournaments as a subdigraph. Using our terminology, it means that the set of heroes is precisely $Forb_{ind}\{\overleftrightarrow{K_2}, \overline{K_2}, H_1, H_2, H_3, H_4, H_5\}$. Moreover, it is easily observed that every hero is 2-colorable, so $\{\overleftrightarrow{K_2}, \overline{K_2}, H_1, H_2, H_3, H_4, H_5\}$ is a heroic set. We don't know if it is minimal or not but, since none of the H_i are heroes and by Theorem 4, a minimal heroic subset of it has at least four elements. An interesting direction of research is then the following:

Problem 5. Characterize all finite and minimal families $\{H_1, H_2, \dots, H_k\}$ of tournaments such that $\{\overleftrightarrow{K_2}, \overline{K_2}, H_1, H_2, \dots, H_k\}$ is a heroic set.

The following result of [19] is an extension of Theorem 4.

Theorem 6 ([19])**.** *For every integer $\alpha \geqslant 2$, the set $\{\overleftrightarrow{K_2}, \overline{K_\alpha}, H\}$ is heroic if and only if H is a hero.*

Further extension in this direction is given in Theorem 10.

3 Digraphs that must be contained in all heroic sets

One direction in the study of heroic sets is to find families of digraphs whose dichromatic number is not bounded. Given such a family $\mathcal{C}$, if $\mathcal{F}$ is a heroic set, then by definition there exists an integer k such that every digraph of dichromatic number at least k in $\mathcal{C}$ must contain some element of $\mathcal{F}$. Thus, finding well-suited such families $\mathcal{C}$ will allow us to force the existence of some particular types of digraphs in every heroic set.

In this section, we give four such families. They can be seen as analogues of the two sequences used to prove the *only if* part of Gyárfás-Sumner Conjecture (that is complete graphs and graphs with arbitrarily large girth and chromatic number). The situation is actually more complicated in the framework of digraphs, and two more such families will be given in Sections 5 and 6.1.

Theorem 7 (Erdős [12])**.** *Given positive integers g and k there exists a graph of girth at least g and chromatic number k.*

An analogue of this theorem for dichromatic number of oriented graphs is proved in [18].

Theorem 8. *[18] Given positive integers g and k there exists an oriented graph whose underlying graph has girth at least g and whose dichromatic number is at least k.*

Let us consider the four following families of digraphs:

1. The family $\overleftrightarrow{K_n}$ of complete symmetric digraphs (because $\vec{\chi}(\overleftrightarrow{K_n}) = n$).

2. The family $\overleftrightarrow{G_{g,k}}$, where for any positive integers g and k, $G_{g,k}$ is a graph of girth at least g and chromatic number at least k (whose existence is guaranteed by Theorem 7).

3. The family $H_{g,k}$, where for any positive integers g and k, $H_{g,k}$ is an oriented graph whose underlying graph is of girth at least g and whose dichromatic number is at least k (whose existence is guaranteed by Theorem 8).

4. The family of tournaments.

Thus for the dichromatic number of a class $Forb_{ind}(\mathcal{F})$ of digraphs to be finite, $\mathcal{F}$ must contain at least four *types* of elements:

- A complete symmetric digraph $\overleftrightarrow{K_k}$ for some integer k, because of the first family.
- A symmetric forest $\overleftrightarrow{F_1}$ because of the second family above. Indeed if $\mathcal{F}$ does not contain such a digraph, $Forb_{ind}(\mathcal{F})$ contains all graphs $\overleftrightarrow{G_{g,k}}$ for $g-1$ being the largest order of a member of $\mathcal{F}$.
- An oriented forest $\overrightarrow{F_2}$ because of the third family (same argument).
- A tournament T because of the fourth family.

We note that, as explained in the previous subsection, if a heroic set contains a single tournament, then that element must be a hero.

A digraph might be of several of the types above. In particular, the digraph made of a single vertex is of all four types. We describe below all other digraphs that are of several types (actually each of them is of exactly two distinct types):

- $\overleftrightarrow{K_2}$ is a complete symmetric digraph and a symmetric forest,
- for every integer $\alpha \geqslant 2$, $\overline{K_\alpha}$ is a symmetric forest, and an oriented forest,
- $\overrightarrow{K_2}$ is an oriented forest and a tournament.

As a direct consequence, we have a characterization of heroic set of size two. Note that $Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{K_2})$ is the class of digraphs with no arc, and is thus 1-colorable.

Proposition 9. *The set $\{\overleftrightarrow{K_2}, \overrightarrow{K_2}\}$ is the unique heroic set of size two.*

We now turn our attention to heroic sets of size three. If $\mathcal{F}$ is such a set, then it follows from the discussion above that $\mathcal{F}$ must contain at least one of $\overleftrightarrow{K_2}$, $\overline{K_\alpha}$, $\overrightarrow{K_2}$. As said in the introduction, if $\overrightarrow{K_2}$ belongs to $\mathcal{F}$, then we are in the world of graphs (rather than digraphs) and thus we are in the framework of the well known Gyárfás-Sumner conjecture. Hence, we study the two other cases. More precisely, we ask:

(**P1**) For which hero H and oriented forest F, is $\{\overleftrightarrow{K_2}, H, F\}$ heroic?

(**P2**) For which hero H and integers $k \geqslant 2$, $\alpha \geqslant 2$, is $\{\overleftrightarrow{K_k}, \overline{K_\alpha}, H\}$ heroic?

Observe that (**P1**) is about oriented graphs and it is, in our opinion, of particular interest.

4 Results and conjectures

In this section, we only state our results. Proofs are left to the following sections.

The first result here is a complete answer to (**P2**).

Theorem 10. *Let H be a hero and $k, \alpha \geqslant 2$ be integers. The set $\{\overleftrightarrow{K_k}, \overline{K_\alpha}, H\}$ is heroic if and only if H is a transitive tournament or $k = 2$*

For (**P1**), we venture to propose the following conjecture. A *star* is a tree with at most one vertex of degree more than 1. An *oriented star* is an orientation of a star. By *disjoint union of oriented stars* we mean an oriented forest whose connected components are oriented stars. If D_1 and D_2 are two digraphs, we denote by $D_1 + D_2$ the disjoint union of D_1 and D_2.

Conjecture 11. Let H be a hero and let F be an oriented forest. The set $\{\overleftrightarrow{K_2}, H, F\}$ is heroic if and only if: either

- F is the disjoint union of oriented stars,
  or

- H is a transitive tournament.

We prove the *only if* part of this conjecture in Section 6.1. Chudnovsky, Scott and Seymour [10] proved that, when both conditions hold, that is H is a transitive tournament and F is a disjoint union of oriented stars, the chromatic number of digraphs in $Forb_{ind}\{\overleftrightarrow{K_2}, H, F\}$ is bounded. As the chromatic number of a digraph is an upper bound for its dichromatic number, this indeed implies that $\{\overleftrightarrow{K_2}, H, F\}$ is a heroic set for such choices of H and F.

In Sections 6.2.1 and 6.2.2 we prove the following results in support of our conjecture. Before stating them, we need to extend our notation $Forb_{ind}(\mathcal{F})$ by allowing (non-oriented) graphs in $\mathcal{F}$. If $\mathcal{F}$ is a set of digraphs and graphs, we define $Forb_{ind}(\mathcal{F})$ to be the set of digraphs that does not contain as an induced subdigraph: any digraph of $\mathcal{F}$, and any orientation of any graph of $\mathcal{F}$. For example $Forb_{ind}(K_3, \overleftrightarrow{K_2})$ is the class of oriented triangle-free graphs.

Theorem 12. *The following sets are heroic:*

- $\{\overleftrightarrow{K_2}, \overrightarrow{C_3}, \overrightarrow{K_2} + K_1\}$,

- $\{\overleftrightarrow{K_2}, K_4, P^+(3)\}$, *where* $P^+(3)$ *is the directed path of length* 3.

The second claim of the theorem is equivalent to stating that the set of digraphs consisting of $\overleftrightarrow{K_2}$, $P^+(3)$ and all possible orientations of K_4 is a heroic set. As there are four non-isomorphic orientations on K_4, this is a set of order 6 and, at a first look, it may seem that this does not fit into our direction of research which is about characterizing heroic sets of order 3. However, using the fact that every tournament on at least 2^k vertices contains TT_k as an induced subdigraph, one easily observes that proving that "Given an oriented forest F and for every integer k, $\{\overleftrightarrow{K_2}, TT_k, F\}$ is a heroic set", is equivalent to proving that "Given an oriented forest F and for every integer k, $\{\overleftrightarrow{K_2}, K_k, F\}$ is heroic".

5 Proof of Theorem 10

As mentioned earlier, each tournament on at least 2^k vertices contains TT_k. In the next proposition we use Ramsey number $R(a, b, c)$: that is the smallest integer such that for $n \geqslant R(a, b, c)$, if edges of K_n are 3-colored, then it either contains a K_a all whose edges are colored by the first color, or a K_b all whose edges are colored by the second color or a K_c all whose edges are colored by the third color.

Proposition 13. *Given positive integers k, α, t, for any $D \in Forb_{ind}(\overleftrightarrow{K_k}, \overline{K_\alpha}, TT_t)$, we have $|V(D)| < R(k, \alpha, 2^{t-1})$. In particular $\{\overleftrightarrow{K_k}, \overline{K_\alpha}, TT_t\}$ is a heroic set.*

Proof. Given a digraph D on k vertices we associate with it a 3-edge-colored complete graph D' whose vertices are the vertices of D, and whose edges are colored as follows: for each pair $\{x, y\}$ of vertices that induce no arc in D, xy is a red edge in D', if the pair induces a $\overrightarrow{K_2}$, then xy is a blues edge of D' and if it induces a $\overleftrightarrow{K_2}$, then xy is a green edge. The claim of the proposition now follows from the definition of $R(k, \alpha, 2^{t-1})$ and the fact that each tournament on 2^{t-1} contains a transitive tournament of order t. □

We now give a construction of digraphs with arbitrarily large dichromatic number proving that if H is not a transitive tournament, then the set $\{\overleftrightarrow{K_3}, \overline{K_2}, H\}$ is not heroic. Consider a symmetric digraph $\overleftrightarrow{G}$ where G is a graph of arbitrarily large girth and large chromatic number. We fix an arbitrary enumeration $v_1, \ldots, v_n$ of the vertices of $\overleftrightarrow{G}$ and create a digraph $D \in Forb_{ind}(\overline{K_2})$ as follows: if v_iv_j, $i < j$, is a non-edge of $\overleftrightarrow{G}$, then v_iv_j is an arc of D. Such a construction has arbitrarily large dichromatic number, and belongs to $Forb_{ind}(\overleftrightarrow{K_3}, \overline{K_2}, \overrightarrow{C_3})$. Moreover, Theorem 4 easily implies that a hero is either a transitive tournament, or contains a $\overrightarrow{C_3}$, so if H is not a transitive tournament, then the set $\{\overleftrightarrow{K_3}, \overline{K_2}, H\}$ is not heroic.

Finally, since for all heroes H the set $\{\overleftrightarrow{K_2}, \overline{K_\alpha}, H\}$ is heroic (Theorem 6), Theorem 10 follows.

6 Supports for Conjecture 11

In what follows, first of all we prove "the only if" part of Conjecture 11. Then, toward support for the main direction of this conjecture, we first recall that the case where both conditions are satisfied (i.e., H is a transitive tournament and F is a union of oriented stars) is already settled in [10]. We then prove the conjecture for specific choices of H and F where only one of the two conditions is satisfied.

6.1 The "only if" part

Let D_1 be the digraph on one vertex and let $D_{i+1} = \overrightarrow{C_4}(D_i)$. From Theorem 3 it follows that $\vec{\chi}(D_i) = i$. Furthermore, it is easily verified that $D_i \in Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, P_4)$ for every i.

By Theorem 4, a hero with no $\overrightarrow{C_3}$ is a transitive tournament. Observe that an oriented forest with no induced P_4 in its underlying graph is a disjoint unions of oriented stars. This implies that if H is a hero which is not a transitive tournament, and F is a forest which is not a union of oriented stars, then $Forb_{ind}(\overleftrightarrow{K_2}, F, H)$ contains $Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, P_4)$ whose chromatic number is not bounded. This proves the necessary part of Conjecture 11.

6.2 The "if" part

We first discuss the case where H is $\overrightarrow{C_3}$ and F has at most 3 vertices, then we discuss the case where H is a transitive tournament and F is the directed path on 4 vertices.

6.2.1 $H = \overrightarrow{C_3}$ and F has at most 3 vertices

Since it is proved that the set is heroic when both conditions hold, we may assume that only one of the two conditions holds. In this subsection we are interested in the case when H is not transitive, which is equivalent to say that it contains a $\overrightarrow{C_3}$, so the first step is then $H = \overrightarrow{C_3}$ itself.

If F has only two vertices the question is trivial. On three vertices the possibilities for F are:

- $\overline{K_3}$,
- $P^+(2)$ (the directed path of length 2)
- $\overrightarrow{K_2} + K_1$,
- S_2^+ (the oriented star with two outgoing arcs from the center)
- S_2^- (the oriented star with two ingoing arcs to the center)

By Theorem 6, $F = \overline{K_3}$ gives a heroic set. The second item is simple: the digraphs in $Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, P^+(2))$ have dichromatic number 1, as one can prove easily by induction that they consists of disjoint unions of transitive tournaments.

Let us know prove the case $F = \overrightarrow{K_2} + K_1$ (the second item of Theorem 12).

Theorem 14. $\vec{\chi}(Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, \overrightarrow{K_2} + K_1)) = 2.$

Proof. Let $D \in Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, \overrightarrow{K_2}+K_1)$. Observe that an oriented graph with no induced $\overrightarrow{K_2} + K_1$, and, in particular, our digraph D, is an orientation of a complete multipartite graph. We denote by $X_1, \dots X_n$ its parts. Furthermore, since D has neither $\overleftrightarrow{K_2}$ nor $\overrightarrow{C_3}$ as induced subdigraph, all its induced directed cycles have length 4 and all its $\overrightarrow{C_4}$'s are induced. Hence, an acyclic coloring of D is the same as a $\overrightarrow{C_4}$-free coloring, that is a coloring of the vertices such that no $\overrightarrow{C_4}$ is monochromatic.

Hence, we can look for a $\overrightarrow{C_4}$-free coloring with two colors. The proof is based on the following claim that says that all $\overrightarrow{C_4}$'s containing a fixed vertex are included in the union of two parts of D.

Claim: *Let a_1 be a vertex of D. There exists i, j, $1 \leqslant i \neq j \leqslant n$ such that all $\overrightarrow{C_4}$'s containing a_1 are included in $X_i \cup X_j$.*

Let $a_1 \in X_i$ and assume a_1 is contained in a $\overrightarrow{C_4}$: $a_1 \to b_1 \to a_2 \to b_2 \to a_1$. Since this $\overrightarrow{C_4}$ is induced, and as the underlying graph of D is a complete multipartite graph, we may assume that $a_2 \in X_i$ and b_1, b_2 belong to a same part, say X_j, $j \neq i$. Assume now that a_1 belongs to another $\overrightarrow{C_4}$, say $a_1 \to c_1 \to a_3 \to c_2$. Similarly, $a_3 \in X_i$ and c_1, c_2 are in a same part. Assume for a contradiction that $c_1, c_2 \in X_k$ with $k \neq j$. In particular b_1 and b_2 are connected with both c_1 and c_2.

If $a_2 = a_3$, then either $c_1 \to b_2$ and $\{c_1, b_2, a_1\}$ induces a $\overrightarrow{C_3}$, or $b_2 \to c_1$ and $\{b_2, c_1, a_2\}$ induces a $\overrightarrow{C_3}$, a contradiction in both cases.

So $a_2 \neq a_3$. We then have:

- $b_2 \to c_1$, as otherwise $\{b_2, a_1, c_1\}$ induces a $\overrightarrow{C_3}$.
- $a_2 \to c_1$, as otherwise $\{a_2, b_2, c_1\}$ induces a $\overrightarrow{C_3}$.
- $c_2 \to b_1$, as otherwise $\{c_2, a_1, b_1\}$ induces a $\overrightarrow{C_3}$.
- $a_3 \to b_1$, as otherwise $\{a_3, c_2, b_1\}$ induces a $\overrightarrow{C_3}$.

Now, if $b_1 \to c_1$, then $\{b_1, c_1, a_3\}$ induces a $\overrightarrow{C_3}$ and if $c_1 \to b_1$, then $\{b_1, c_1, a_2\}$ is a $\overrightarrow{C_3}$, a contradiction in both cases. This complete the proof of the claim.

We can now partition each X_i into n subparts as follow. For every i and j, $1 \leqslant i \neq j \leqslant n$, define X_i^j as the set of vertices in X_i that are involved in $\overrightarrow{C_4}$ only with some vertices of $X_i \cup X_j$ and let X_i^i be the remaining set of vertices of X_i (those not in any $\overrightarrow{C_4}$).

Build an auxiliary graph with vertices x_i^j (representing the set X_i^j) and, for every i and j, $1 \leqslant i \neq j \leqslant n$, connect x_i^j and x_j^i. This graph is a disjoint union of K_1's and K_2's and, therefore, can be properly 2-colored. Take a 2 coloring of it. Then, giving to all vertices of X_i^j the color of x_i^j, we obtain a $\overrightarrow{C_4}$-free coloring of D with 2 colors. □

To conclude the discussion started at the beginning of this section, we note that the next case where $H = \overrightarrow{C_3}$ and $F = S_2^+$ is already unsettled and we do not know if $\{\overleftrightarrow{K_2}, \overrightarrow{C_3}, S_2^+\}$ is a heroic set, however we conjecture that:

Conjecture 15. $\vec{\chi}(Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, S_2^+)) = 2$

To support this, we point out that $\vec{\chi}(Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, S_2^+, S_2^-)) = 2$. Indeed, in [5] (Chapter 6), it is proven that the strongly connected elements of $Forb_{ind}(\overleftrightarrow{K_2}, \overrightarrow{C_3}, S_2^+, S_2^-)$ is the class of so called *round digraphs*: a digraph is round if its vertices can be ordered cyclically $(v_1, v_2, \dots v_n)$ such that whenever v_iv_j is an arc, then for any k, $i < k < j$, v_iv_k and v_kv_j are both arcs (indices are taken modulo n). It is easy to see that round digraphs have dichromatic number 2: consider the longest arc on the cyclic order, assume w.l.o.g. that it is v_1v_k, and then observe that $(\{v_i,\, i \leqslant k\}, \{v_i,\, i > k\})$ is a partition of $V(D)$ into two acyclic digraphs.

6.2.2 $H = TT_k$ and F is an orientation of the path of length 3

In this subsection we consider the case where H *is* a transitive tournament and F is an oriented forest *but not* a disjoint union of oriented stars. The smallest non trivial case is thus when F is an orientation of a path on 4 vertices.

Given the path P on vertices $v_1, v_2, \dots, v_n$ where v_iv_{i+1} are the edges, an orientation of P can be coded by starting with a sign (+ or −) which decides the orientation of the first edge (v_1v_2) followed by a sequence of numbers, first of which tells the number of consecutive arcs in the same direction starting at v_1, then number of consecutive arcs at the opposite direction and so on. For example $P^+(3,4)$ is an orientation of a path of length 7 (8 vertices) where first 3 arcs are directed away from v_1 and last four are directed toward v_1. Using this terminology, the four orientations of the path on 4 vertices are represented on Figure 3.

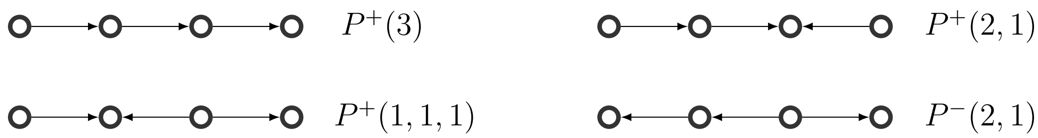

Figure 3: The four orientations of P_4

Recall that in this paper the chromatic number of an oriented graph is the chromatic number of its underlying graph.

The chromatic number of the classes of oriented graphs where an orientation of P_4 is forbidden has been already studied. On the positive side, Chudnovsky et al [10] proved that $Forb_{ind}(\overleftrightarrow{K_2}, TT_k, P^+(2,1))$ has bounded chromatic number for every k, which implies that $\{\overleftrightarrow{K_2}, TT_k, P^+(2,1)\}$ is heroic. Reversing the arcs trivially implies the same for $\{\overleftrightarrow{K_2}, TT_k, P^-(2,1)\}$.

Another interesting result is that of Galeana and Sánchez who proved in [14] that $Forb_{ind}(\overleftrightarrow{K_2}, K_3, P^+(3))$ has chromatic number at most 2. (Recall that forbidding K_3 means forbidding all orientation of K_3, that is $\vec{C}_3$ and TT_3). Interestingly, it is proved in [24] (see also Section 4.1 of [2]) that $Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3))$ has unbounded chromatic number.

In contrast to the chromatic number, we will prove that $Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3))$ has bounded dichromatic number, and more precisely dichromatic number at most 414. The rest of the subsection is dedicated to the proof of this result. We first introduce the notion of nice sets that will be the key tool of our proofs.

Definition 16. Let D be a digraph. A nonempty set of vertices S of D is said to be *nice* if each vertex in S either has no out-neighbor in $V(D) \setminus S$ or has no in-neighbor in $V(D) \setminus S$. The set of vertices in S with no out-neighbor in $V(D) \setminus S$ is called the *in-part of* S, and the set of vertices in S with no in-neighbor in $V(D) \setminus S$ is the *out-part of* S.

The next lemma gives a sufficient condition for a class of digraphs to have bounded dichromatic number.

Lemma 17. *Let* $\mathcal{C}$ *be a hereditary class of digraphs. Assume that there exists two integers* c_1 *and* c_2 *such that every digraph in* $\mathcal{C}$ *contains a nice set* S *such that the in-part of* S *has dichromatic number at most* c_1 *and its out-part has dichromatic number at most* c_2.

Then $\vec{\chi}(\mathcal{C}) \leqslant c_1 + c_2$. In particular, if there exists c such that every digraph in $\mathcal{C}$ admits a nice set with dichromatic number at most c, then $\vec{\chi}(\mathcal{C}) \leqslant 2c$.

Proof. Let $\mathcal{C}$ be a class of digraph as in the statement. Let $D \in \mathcal{C}$ be a minimal counterexample, that is: $\vec{\chi}(D) = c_1+c_2+1$ and for every proper subdigraph H of D, $\vec{\chi}(H) \leqslant c_1+c_2$. By the hypothesis, D admits a nice set S, with in-part S_1 and out-part S_2 such that $\vec{\chi}(S_1) \leqslant c_1$ and $\vec{\chi}(S_2) \leqslant c_2$.

The key observation is that any directed cycle that intersects S and $V(D) \setminus S$ must intersect both S_1 and S_2. Hence, by the minimality of D, we can dicolor the subdigraph of D induced by $V(D) \setminus S$ with $c_1 + c_2$ colors. We can then extend this dicoloring to D by using colors $1, \dots, c_1$ for S_1 and $c_1 + 1, \dots, c_1 + c_2$ for S_2. $\square$

Let D be an oriented graph and let x, y be two vertices of D. The *distance* between x and y is the distance between x and y in the underlying graph of D. The *out-distance* from x to y is the length of a shortest directed path from x to y. The *in-distance* from x to y is the length of a shortest directed path from y to x.

As mentionned at the beginning of this subsection, $Forb_{ind}(\overleftrightarrow{K_2}, K_3, P^+(3))$ has chromatic number at most 2, here we include a short proof of the weaker fact that the dichromatic number is at most 2, which is what we need.

Theorem 18. $\vec{\chi}(Forb_{ind}(\overleftrightarrow{K_2}, K_3, P^+(3)) = 2$.

Proof. Let $D \in Forb_{ind}(\overleftrightarrow{K_2}, K_3, P^+(3))$. Assume D is strongly connected (otherwise just take the strong connected component with largest dichromatic number). Let $x \in V(D)$. For $i \geqslant 0$, set L_i to be the set of vertices at out-distance i from x. Since D is strongly connected, the collection of L_i's is a partition of $V(D)$. We are going to prove that each layer induces a stable set. Let k be the maximum integer such that each L_i is a stable set for $i = 1, \dots, k$. Since D is K_3-free, L_1 is a stable set, so $k \geqslant 1$. If L_{k+1} has at most one vertex, we are done. So assume $|L_{k+1}| \geqslant 2$, and by maximality of k, L_{k+1} contains an arc ab. There exists $a_1 \in L_k$ and $a_2 \in L_{k-1}$ such that $a_2 \to a_1 \to a$. Since $a_2 \to a_1 \to a \to b$ cannot be induced and a_1 and b are non-adjacent (because D is triangle-free), $b \to a_2$. There exists $b_1 \in L_k$ and $b_2 \in L_{k-1}$ such that $b_2 \to b_1 \to b$. Since D is K_3-free, $b_1 \neq a_1$, $b_2 \neq a_2$ and there is no arc between b_1 and a_2. Moreover, since L_{k-1} is a stable set, there is no arc between a_2 and b_2. Hence $b_2b_1ba_2$ is an induced $P^+(3)$, a contradiction.

Now, color every vertex at odd out-distance from x with color 1, and every vertex at even out-distance from x, including x itself, with color 2. It is easy to check that this gives a dicoloring of D. $\square$

Our next goal is to prove that $\{\overleftrightarrow{K_2}, K_4, P^+(3)\}$ is a heroic set. In order to do so, we first prove that two larger sets are heroic, namely $\{\overleftrightarrow{K_2}, K_4, P^+(3), \overrightarrow{C_3}\}$ (see Lemma 19) and $\{\overleftrightarrow{K_2}, K_4, P^+(3), R\}$, where R is the graph depicted in Figure 4 (Lemma 20).

Lemma 19. $\vec{\chi}(Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3), \overrightarrow{C_3})) \leqslant 8$

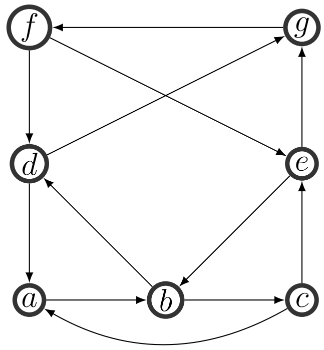

Figure 4: The oriented graph R. Observe that R is invariant under reversing all its arcs

Proof. We are going to prove that every graph in $Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3), \overrightarrow{C_3}))$ contains a nice set whose in-part and out–part are each of dichromatic number at most 4. This would imply the claim of this Lemma using Lemma 17.

Let $D \in Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3), \overrightarrow{C_3})$. Let $x \in V(D)$. Let X_1 (resp. X_2) be the vertices at distance 1 (resp. at distance 2) from x. Let $X_1^+ = N^+(x)$ and let $X_1^- = N^-(x)$ (so $X_1 = X_1^+ \cup X_1^-$). Let $X_2^+ = X_2 \cap N^+(X_1^+)$ and $X_2^- = X_2 \cap N^-(X_1^-)$. Observe that $X_2^+ \cup X_2^-$ does not need to be equal to X_2.

Let us prove that $S = \{x\} \cup X_1^+ \cup X_1^- \cup X_2^+ \cup X_2^-$ is a nice set with in-part $\{x\} \cup X_1^+ \cup X_2^+$ and out-part $X_1^- \cup X_2^-$. By definition of S, it is clear that x has no neighbor in $V(D) \setminus S$, that vertices in X_1^+ have no out-neighbor in $V(D) \setminus S$ and that vertices in X_1^- have no in-neighbor in $V(D) \setminus S$. Let $x_2 \in X_2^+$ and let us prove that x_2 have no out-neighbor in $V(D) \setminus S$. Assume for a contradiction that there exists $x_3 \in V(D) \setminus S$ such that $x_2 \to x_3$. By definition of X_2^+ there exists a vertex $x_1 \in X_1^+$ such that $x \to x_1 \to x_2$. If $x_3 \to x_1$, then $x_1x_2x_3$ is a $\overrightarrow{C_3}$ and if $x_1 \to x_3$, then $x_2 \in X_2^+$, a contradiction in both cases, so x_1 and x_3 are non-adjacent and thus $x \to x_1 \to x_2 \to x_3$ is induced, a contradiction. This proves that vertices in X_2^+ have no out-neighbor in $V(D) \setminus S$. Similarly (because $P^+(3)$ is invariant under reversing all edges), X_2^- have no in-neighbor in $V(D) \setminus S$. This proves that S is a nice set with in and out-part as announced.

We now prove that $\{x\} \cup X_1^+ \cup X_2^+$ and $X_1^- \cup X_2^-$ are 4-dicolorable. Since D is K_4-free, X_1 is triangle-free and is thus 2-dicolorable by Theorem 18. Assume that X_2^+ has a TT_3, say $a \to b \to c \leftarrow a$. By definition of X_2^+, there is a vertex $x_1 \in X_1^+$ such that $x_1 \to a$. Since $x \to x_1 \to a \to b$ cannot be induced, x_1 and b must be adjacent and since D has no $\overrightarrow{C_3}$, $x_1 \to b$. Since $x \to x_1 \to b \to c$ cannot be induced, there is an arc between x_1 and c, and thus $\{a, b, c, x_1\}$ induces a K_4, a contradiction. Hence X_2^+ is triangle-free and thus 2-dicolorable. Similarly, X_2^- is 2-dicolorable. We may now use two colors on X_1^+ and a distinct set of two colors on X_2^+, then use any of the four colors to color x. As x is in no directed 4-cycle induced by $\{x\} \cup X_1^+ \cup X_2^+$, this is 4-dicoloring of this induced subgraphs. That $X_1^- \cup X_2^-$ is 4-dicolorable is proved analogously. $\square$

We now need a second technical lemma. Let us first define a particular class of oriented graphs named $\mathcal{F}$. An oriented graph F belongs to $\mathcal{F}$ if there exists $D_F \in Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3))$ such that D_F is made of a copy F' of F, a stable set of vertices L (disjoint from $V(F')$) such that every vertex in F' has at least one neighbor in L, and two more vertices u and v (outside $V(L) \cup V(F')$) such that $u \to v$, and for every vertex x in L we have $v \to x \to u$, and there is no arc between $\{u, v\}$ and $V(F')$. Observe that $\mathcal{F}$ is hereditary and is a subclass of $Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3))$.

Lemma 20. *Every graph in $\mathcal{F}$ is R-free, where R is the graph depicted in Figure 4.*

Proof. Let $F \in \mathcal{F}$ and let D_F be the graph described as in the definition of $\mathcal{F}$ above. Assume for a contradiction that F contains an induced copy of R (with same name for vertices as in Figure 4). By the definition of D_F, b has a neighbor x in L. Recall that there are two vertices u and v in D_F such that $u \to v \to x \to u$ and such that there is no arc between $\{u, v\}$ and $V(R)$. By reversing all edges of D_F, if necessary, we may assume that $x \to b$ (this is legitimate since, R, $\overrightarrow{C_3}$ and $P^+(3)$ are invariant under reversing all edges). In the upcoming case analysis, we will be using the fact that there is no arc between $\{u, v\}$ and $V(R)$ without recalling this fact.

Since $v \to x \to b \to d$ cannot be induced, x and d are adjacent. Since D_F is K_4-free, x and a are non-adjacent. If $x \to d$, then $v \to x \to d \to a$ is induced, a contradiction. Hence $d \to x$ holds.

Since $f \to d \to x \to u$ cannot be induced, x and f must be adjacent. Since D_F is K_4-free, x and g are non-adjacent. If $f \to x$, then $g \to f \to x \to u$ is induced, a contradiction. Hence $x \to f$ holds.

Finally, since $v \to x \to f \to e$ cannot be induced, x and e are adjacent, and since $v \to x \to b \to c$ cannot be induced, x and c are adjacent. Hence $\{x, b, c, e\}$ induces a K_4, a contradiction. $\square$

Lemma 21. $\vec{\chi}(Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3), R)) \leqslant 66$

Proof. We are going to prove that every graph in $Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3), R)$ contains a nice set of dichromatic number at most 33, which, by Lemma 17, implies our claim.

Let $D \in Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3), R)$. If D is $\overrightarrow{C_3}$-free, we are done by Theorem 19. So we may assume that D contains $C = u \to v \to w \to u$. Let X_1^{uv} (resp. X_1^{vw}, resp. X_1^{wu}) be the set of vertices $x \in N(C)$ such that $v \to x \to u$ (resp. such that $w \to x \to v$, resp. $u \to x \to w$). Let $X_1 = N(C) \setminus (X_1^{uv} \cup X_1^{vw} \cup X_1^{wu})$. Let X_2^{uv} (resp. X_2^{vw}, resp. X_2^{wv}) be the set of vertices in $V(D) \setminus (V(C) \cup N(C))$ having at least one neighbor in X_1^{uv} (resp. in X_1^{vw}, resp. in X_1^{wv}).

Let us prove that the set $S = \{u, v, w\} \cup X_1^{uv} \cup X_1^{vw} \cup X_1^{wu} \cup X_1 \cup X_2^{uv} \cup X_2^{vw} \cup X_2^{wu}$ is a nice set. We say that a vertex is *nice* if it has no out-neighbor or no in-neighbor in $V(D) \setminus S$. First observe that the neighborhood of a vertex in $\{u, v, w\} \cup X_1^{uv} \cup X_1^{vw} \cup X_1^{wu}$ is included in S, and thus every vertex in $\{u, v, w\} \cup X_1^{uv} \cup X_1^{vw} \cup X_1^{wu}$ is nice.

Let $x_1 \in X_1$ and let us prove that x_1 is nice. We are in one of the three following situations:

- x_1 has only in-neighbors in C. Assume without loss of generality that $u \to x_1$. We claim that x_1 has no out-neighbor in $V(D) \setminus S$. Assume for a contradiction that there exists $x_2 \in V(D) \setminus S$ such that $x_1 \to x_2$. By construction of S, $x_2 \notin N(C)$. Since $w \to u \to x_1 \to x_2$ cannot be induced, w and x_1 must be adjacent and thus $w \to x_1$. Since $v \to w \to x_1 \to x_2$ cannot be induced, v and x_1 must be adjacent, but then $\{u, v, w, x_1\}$ induces a K_4, a contradiction. This proves the announced claim.

- x_1 has only out-neighbors in C. In this case, x_1 has no in-neighbor in $V(D) \setminus S$, we skip the proof that is similar to the one of the previous case.

- x_1 has both an in-neighbor and an out-neighbor in C, and is not forming a $\overrightarrow{C_3}$ with arcs of C. We may assume, without loss of generality, that $u \to x_1 \to v$, and observe that in this case w and x_1 are not adjacent because the underlying graph of D is K_4-free. We claim that x_1 has no out-neighbor in $V(D) \setminus S$ (it has actually no neighbor at all in $V(D) \setminus S$). Assume for a contradiction that there exists x_2 in $V(D) \setminus S$ such that $x_1 \to x_2$. By construction of S, $x_2 \notin V(C) \cup N(C)$, and thus $w \to u \to x_1 \to x_2$ is induced, a contradiction.

This proves that every vertex in X_1 is nice.

Let us now prove that every vertex in X_2^{uv} is nice. Let $x_2 \in X_2^{uv}$. By definition of X_2^{uv}, there is a vertex $x_1 \in X_1^{uv}$ such that x_1 and x_2 are adjacent. Observe that x_1 and w are non-adjacent (because D is K_4-free). Let x_3 be a neighbor of x_2 in $V(D) \setminus S$. Observe that, by the definition of S, x_1 and x_3 are non-adjacent. Hence, if $x_1 \to x_2$, then $x_2 \to x_3$ cannot hold, as otherwise $v \to x_1 \to x_2 \to x_3$ is induced. Similarly, if $x_2 \to x_1$, then $x_3 \to x_2$ cannot hold, as otherwise $x_3 \to x_2 \to x_1 \to u$ is induced. This proves that x_2 is nice. The situation in X_2^{vw} and X_2^{wu} being exactly the same, every vertex in $X_2^{vw} \cup X_2^{wu}$ is also nice, and thus S is a nice set.

It now remains to prove that $\vec{\chi}(S) \leqslant 33$. Observe that because D is K_4-free, the neighborhood of a vertex of D is K_3-free and thus 2-dicolorable by Theorem 18. Hence $C \cup N(C) = \{u, v, w\} \cup X_1^{uv} \cup X_1^{vw} \cup X_1^{wu} \cup X_1$ is 9-dicolorable. It is thus enough to show that $X_2^{uv} \cup X_2^{vw} \cup X_2^{wu}$ is 24-dicolorable. To this end, and by symmetry between X_2^{uv}, X_2^{vw} and X_2^{wu}, it is enough to show that X_2^{uv} is 8-dicolorable. Finally, by Lemma 19, it is enough to show that X_2^{uv} is $\overrightarrow{C_3}$-free. Assume for a contradiction that X_2^{uv} contains a $\overrightarrow{C_3}$, say $a \to b \to c \to a$. Let us prove a technical claim:

Claim: *Let $x_1 \in X_1^{uv}$ such that x_1 has a neighbor in $\{a, b, c\}$. Then x_1 has exactly two neighbors in $\{a, b, c\}$ and forms a $\overrightarrow{C_3}$ with these two neighbors.*

Proof of Claim: Assume without loss of generality that x_1 and a are adjacent. Since all the reasoning will be on $D[\{u, v, x_1, a, b, c\}]$ which is invariant under reversing all edges, and what we want to prove is also invariant under reversing all edges, we can assume, without loss of generality, that $x_1 \to a$. Since $v \to x_1 \to a \to b$ cannot be induced, x_1 and b are adjacent. Since D is K_4-free, x_1 and c are non-adjacent. If $x_1 \to b$, then

$v \to x_1 \to b \to c$ is induced, a contradiction. Hence $b \to x_1$. This completes the proof of the claim.

By definition of X_2^{uv}, there exists $x_1 \in X_1^{uv}$ such that x_1 and a are adjacent. By the claim, we can assume, without loss of generality, that $b \to x_1 \to a$ and that x_1 and c are non-adjacent. Hence, there exists a vertex x'_1 in $X_1^{uv} \setminus \{x_1\}$ such that x'_1 and c are adjacent. Observe that x_1 and x'_1 are non-adjacent, otherwise $D[\{u, v, x_1, x'_1\}]$ is a K_4, a contradiction. By the claim, we either have $c \to x'_1 \to b$ and x'_1 and a are non-adjacent, or $a \to x'_1 \to c$ and x'_1 and b are non-adjacent. In both cases $D[\{u, v, x_1, x'_1, a, b, c\}]$ induces R, a contradiction. Hence, X_2^{uv} is $\overrightarrow{C_3}$-free. $\square$

Theorem 22. $\vec{\chi}(Forb_{ind}(\overleftrightarrow{K_2}, K_4, P^+(3))) \leqslant 414$.

Proof. By Lemma 17 it would be enough to prove that $Forb_{ind}(K_4, P^+(3))$ contains a nice set of dichromatic number at most 207.

Let $D \in Forb_{ind}(K_4, P^+(3)))$. If D is $\overrightarrow{C_3}$-free, we are done by Lemma 19. So we may assume that D contains $C = u \to v \to w \to u$.

Define $S = \{u, v, w\} \cup X_1^{uv} \cup X_1^{vw} \cup X_1^{wu} \cup X_1 \cup X_2^{uv} \cup X_2^{vw} \cup X_2^{wu}$ exactly as in the proof of Lemma 19. As in the proof of Lemma 21, S is a nice set and $C \cup N(C) = \{u, v, w\} \cup X_1^{uv} \cup X_1^{vw} \cup X_1^{wu} \cup X_1$ is 9-dicolorable. It is thus enough to show that $X_2^{uv} \cup X_2^{vw} \cup X_2^{wu}$ is 198-dicolorable, and by symmetry between X_2^{uv}, X_2^{vw} and X_2^{wu}, it is enough to show that X_2^{uv} is 66-dicolorable. By the construction of X_2^{uv}, $D[X_2^{uv}]$ is in $\mathcal{F}$, hence by Lemma 20, it is R-free and thus 66-colorable by Lemma 21. $\square$

7 Remark

While this paper was under review, Conjecture 15 was verified independently in [1] and [31].

Acknowledgment

We would also like to thank Maria Abi Aad and Mekkia Kouider for participation in discussions at the early stages of the project.

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