A note on transitive union-closed families.

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Abstract

We show that the Union-Closed Conjecture holds for the union-closed family generated by the cyclic translates of any fixed set.

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1 Introduction

If X is a set, a family $\mathcal{F}$ of subsets of X is said to be *union-closed* if the union of any two sets in $\mathcal{F}$ is also in $\mathcal{F}$. The celebrated Union-Closed Conjecture (a conjecture of Frankl [2]) states that if X is a finite set and $\mathcal{F}$ is a union-closed family of subsets of X (with $\mathcal{F} \neq \{\varnothing\}$), then there exists an element $x \in X$ such that x is contained in at least half of the sets in $\mathcal{F}$. Despite the efforts of many researchers over the last forty-five years, and a recent Polymath project [5] aimed at resolving it, this conjecture remains wide open. It has only been proved under very strong constraints on the ground-set X or the family $\mathcal{F}$; for example, Balla, Bollobás and Eccles [1] proved it in the case where $|\mathcal{F}| \geqslant \frac{2}{3}2^{|X|}$; more recently, Karpas [4] proved it in the case where $|\mathcal{F}| \geqslant (\frac{1}{2} - c)2^{|X|}$ for a small absolute constant $c > 0$; and it is also known to hold whenever $|X| \leqslant 12$ or $|\mathcal{F}| \leqslant 50$, from work of Vučković and Živković [8] and of Roberts and Simpson [7]. We note that Reimer [6] proved that the average size of a set in an arbitrary finite union-closed family $\mathcal{F}$ is at least $\frac{1}{2}\log_2(|\mathcal{F}|)$; this yields (by averaging) a good approximation to the Union-Closed

Conjecture in the case where $\mathcal{F}$ is large, e.g. it implies that there is an element contained in at least an $\Omega(1)$-fraction of the sets in $\mathcal{F}$, in the case where $|\mathcal{F}| = 2^{\Omega(n)}$.

If X is a set and $\mathcal{F}$ is a family of subsets of X, we say $\mathcal{F}$ is *transitive* if the automorphism group of $\mathcal{F}$ acts transitively on X. (The automorphism group of $\mathcal{F}$ is the set of all permutations of X that preserve $\mathcal{F}$.) Informally, $\mathcal{F}$ is transitive if all points of X 'look the same' with respect to $\mathcal{F}$. Even the special case of the Union-Closed Conjecture for transitive families is wide open.

In this note, we prove the conjecture in the special case where X is $\mathbb{Z}_n$, the cyclic group of order n, and $\mathcal{F}$ is the (transitive) union-closed family consisting of all unions of cyclic translates of some fixed set. This is a question asked in the Polymath project [5].

Theorem 1. *Let $n \in \mathbb{N}$, and let $R \subseteq \mathbb{Z}_n$ with $R \neq \varnothing$. Let $\mathcal{F} = \{A + R : A \subseteq \mathbb{Z}_n\}$ be the set of all unions of cyclic translates of R. Then the average size of a set in $\mathcal{F}$ is at least $n/2$. In particular, the Union-Closed Conjecture holds for $\mathcal{F}$.*

Our proof is surprisingly short. In fact, we establish the following slightly more general result.

Theorem 2. *Let $(G, +)$ be a finite Abelian group, and let $R \subseteq G$ with $R \neq \varnothing$. Let $\mathcal{F} = \{A + R : A \subseteq G\}$ be the set of all unions of translates of R. Then the average size of a set in $\mathcal{F}$ is at least $|G|/2$. In particular, the Union-Closed Conjecture holds for $\mathcal{F}$.*

We note that the family $\mathcal{F}$ in the statement of Theorem 2 is clearly transitive and union-closed, since $x \mapsto x + x_0$ is an automorphism of $\mathcal{F}$ for any $x_0 \in G$, and $(A_1 + R) \cup (A_2 + R) = (A_1 \cup A_2) + R$ for any $A_1, A_2 \subseteq G$.

We remark that it is possible to deduce a slightly weaker form of Theorem 2 from a theorem of Johnson and Vaughan (Theorem 2.10 in [3]). In fact, the result of Johnson and Vaughan, after applying a quotienting argument, yields that there is an element of G contained in at least $(|\mathcal{F}| - 1)/2$ of the sets in $\mathcal{F}$. (Since $\mathcal{F}$ may have odd size, for example when G is $\mathbb{Z}_3$ and $R = \{0, 1\}$, this is not quite enough to deduce Theorem 2.) We are indebted to Zachary Chase for bringing this paper of Johnson and Vaughan to our attention.

A short explanation of our notation and terminology is in order. As usual, if G is an Abelian group, and $A, B \subseteq G$, we write $A + B = \{a + b : a \in A,\ b \in B\}$ for the *sumset* of A and B. Similarly, if $a \in G$ and $B \subseteq G$, we define $a + B = \{a + b : b \in B\}$. For any $x \in G$, we let $-x$ denote the inverse of x in G, and for any set $A \subseteq G$, we let $-A = \{-a : a \in A\}$. We say a subset $A \subseteq G$ is *symmetric* if $A = -A$. If X is a finite set, we write $\mathcal{P}(X)$ for the power-set of X.

2 Proof of Theorem 2.

Before proving Theorem 2, we introduce some useful concepts and notation. Let G be a fixed, finite Abelian group, and let $R \subseteq G$ be fixed. For any set $A \subseteq G$, we define its *R-neighbourhood* to be

$$N_R(A) := A + R,$$

and its *R-interior* to be

$$\mathrm{Int}_R(A) := \{x \in G :\ x + R \subseteq A\}.$$

We note that, if R is symmetric and contains the identity element 0 of G, then the R-neighbourhood of any set A is precisely the graph-neighbourhood of A in the Cayley graph of G with generating-set $R \setminus \{0\}$, and similarly, the R-interior of A is precisely the graph-interior of A with respect to this Cayley graph.

Proof of Theorem 2. Let G be a fixed, finite Abelian group and let $R \subseteq G$ be a fixed, nonempty subset of G. Let

$$\mathcal{F} = \{A + R :\ A \subseteq G\}$$

be the union-closed family consisting of all unions of translates of R.

We define a function $f : \mathcal{P}(G) \to \mathcal{P}(G)$ by

$$f(S) = -(G \setminus \mathrm{Int}_R(S)) \quad \text{for all } S \subseteq G.$$

It is clear that for any set $S \subseteq G$, $|\mathrm{Int}_R(S)| \leqslant |S|$, since for any element $r \in R$, the function $x \mapsto x + r$ is an injection from $\mathrm{Int}_R(S)$ into S. Hence,

$$|S| + |f(S)| \geqslant |G| \quad \text{for all } S \subseteq G. \tag{1}$$

Next, we observe that

$$f(S) = (-(G \setminus S)) + R \quad \text{for all } S \subseteq G. \tag{2}$$

Indeed, for any $x \in G$, it holds that $x \in f(S)$ iff $-x \notin \mathrm{Int}_R(S)$ iff $(-x + R) \cap (G \setminus S) \neq \varnothing$ iff $x \in (-(G \setminus S)) + R$. It follows that $f(\mathcal{P}(G)) \subseteq \mathcal{F}$.

Finally, we observe that the restriction $f|_{\mathcal{F}}$ is an injection. This might seem surprising at first glance, but it follows immediately from the fact that

$$N_R(\mathrm{Int}_R(A + R)) = A + R \quad \text{for all } A \subseteq G. \tag{3}$$

To see (3), let $S = A + R$ and observe that $N_R(\mathrm{Int}_R(S)) \subseteq S$ holds by definition (in fact for any set S). On the other hand, if $S = A + R$, then we have $A \subseteq \mathrm{Int}_R(S)$ and therefore $S = A + R \subseteq N_R(\mathrm{Int}_R(S))$. Hence, $S = N_R(\mathrm{Int}_R(S))$, as required.

Putting everything together, we see that $f|_{\mathcal{F}}$ is a bijection from $\mathcal{F}$ to itself and satisfies

$$|S| + |f(S)| \geqslant |G| \quad \text{for all } S \in \mathcal{F}.$$

Therefore,

$$\frac{1}{|\mathcal{F}|}\sum_{S \in \mathcal{F}} |S| = \frac{1}{2|\mathcal{F}|}\sum_{S \in \mathcal{F}} (|S| + |f(S)|) \geqslant \frac{1}{2|\mathcal{F}|}\sum_{S \in \mathcal{F}} |G| = |G|/2,$$

proving the first part of the theorem. It follows that

$$\frac{1}{|G|}\sum_{x \in G} \frac{|\{S \in \mathcal{F} :\ x \in \mathcal{F}\}|}{|\mathcal{F}|} = \frac{1}{|G|}\frac{1}{|\mathcal{F}|}\sum_{S \in \mathcal{F}} |S| \geqslant 1/2,$$

so by averaging, there exists $x \in G$ such that at least half the sets in $\mathcal{F}$ contain x, and so the Union-Closed Conjecture holds for $\mathcal{F}$. □

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