A note on transitive union-closed families.

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Abstract
We show that the Union-Closed Conjecture holds for the union-closed family generated by the cyclic translates of any fixed set.

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1 Introduction

If $X$ is a set, a family $\mathcal{F}$ of subsets of $X$ is said to be union-closed if the union of any two sets in $\mathcal{F}$ is also in $\mathcal{F}$. The celebrated Union-Closed Conjecture (a conjecture of Frankl [2]) states that if $X$ is a finite set and $\mathcal{F}$ is a union-closed family of subsets of $X$ (with $\mathcal{F} \neq \{\emptyset\}$), then there exists an element $x \in X$ such that $x$ is contained in at least half of the sets in $\mathcal{F}$. Despite the efforts of many researchers over the last forty-five years, and a recent Polymath project [5] aimed at resolving it, this conjecture remains wide open. It has only been proved under very strong constraints on the ground-set $X$ or the family $\mathcal{F}$; for example, Balla, Bollobás and Eccles [1] proved it in the case where $|\mathcal{F}| \geq \frac{2^{2|X|}}{3}$; more recently, Karpas [4] proved it in the case where $|\mathcal{F}| \geq (\frac{1}{2} - c)2^{|X|}$ for a small absolute constant $c > 0$; and it is also known to hold whenever $|X| \leq 12$ or $|\mathcal{F}| \leq 50$, from work of Vučković and Živković [8] and of Roberts and Simpson [7]. We note that Reimer [6] proved that the average size of a set in an arbitrary finite union-closed family $\mathcal{F}$ is at least $\frac{1}{2}\log_2(|\mathcal{F}|)$; this yields (by averaging) a good approximation to the Union-Closed
Conjecture in the case where \( F \) is large, e.g. it implies that there is an element contained in at least an \( \Omega(1) \)-fraction of the sets in \( F \), in the case where \( |F| = 2^\Omega(n) \).

If \( X \) is a set and \( F \) is a family of subsets of \( X \), we say \( F \) is transitive if the automorphism group of \( F \) acts transitively on \( X \). (The automorphism group of \( F \) is the set of all permutations of \( X \) that preserve \( F \).) Informally, \( F \) is transitive if all points of \( X \) ‘look the same’ with respect to \( F \). Even the special case of the Union-Closed Conjecture for transitive families is wide open.

In this note, we prove the conjecture in the special case where \( X \) is \( \mathbb{Z}_n \), the cyclic group of order \( n \), and \( F \) is the (transitive) union-closed family consisting of all unions of cyclic translates of some fixed set. This is a question asked in the Polymath project [5].

**Theorem 1.** Let \( n \in \mathbb{N}, \) and let \( R \subseteq \mathbb{Z}_n \) with \( R \neq \emptyset \). Let \( F = \{A + R : A \subseteq \mathbb{Z}_n\} \) be the set of all unions of cyclic translates of \( R \). Then the average size of a set in \( F \) is at least \( n/2 \). In particular, the Union-Closed Conjecture holds for \( F \).

Our proof is surprisingly short. In fact, we establish the following slightly more general result.

**Theorem 2.** Let \((G, +)\) be a finite Abelian group, and let \( R \subseteq G \) with \( R \neq \emptyset \). Let \( F = \{A + R : A \subseteq G\} \) be the set of all unions of translates of \( R \). Then the average size of a set in \( F \) is at least \( |G|/2 \). In particular, the Union-Closed Conjecture holds for \( F \).

We note that the family \( F \) in the statement of Theorem 2 is clearly transitive and union-closed, since \( x \mapsto x + x_0 \) is an automorphism of \( F \) for any \( x_0 \in G \), and \((A_1 + R) \cup (A_2 + R) = (A_1 \cup A_2) + R \) for any \( A_1, A_2 \subseteq G \).

We remark that it is possible to deduce a slightly weaker form of Theorem 2 from a theorem of Johnson and Vaughan (Theorem 2.10 in [3]). In fact, the result of Johnson and Vaughan, after applying a quotienting argument, yields that there is an element of \( G \) contained in at least \((|F| - 1)/2\) of the sets in \( F \). (Since \( F \) may have odd size, for example when \( G \) is \( \mathbb{Z}_3 \) and \( R = \{0, 1\} \), this is not quite enough to deduce Theorem 2.)

We are indebted to Zachary Chase for bringing this paper of Johnson and Vaughan to our attention.

A short explanation of our notation and terminology is in order. As usual, if \( G \) is an Abelian group, and \( A, B \subseteq G \), we write \( A + B = \{a + b : a \in A, b \in B\} \) for the sumset of \( A \) and \( B \). Similarly, if \( a \in G \) and \( B \subseteq G \), we define \( a + B = \{a + b : b \in B\} \). For any \( x \in G \), we let \(-x\) denote the inverse of \( x \) in \( G \), and for any set \( A \subseteq G \), we let \(-A = \{-a : a \in A\} \). We say a subset \( A \subseteq G \) is symmetric if \( A = -A \). If \( X \) is a finite set, we write \( P(X) \) for the power-set of \( X \).

### 2 Proof of Theorem 2.

Before proving Theorem 2, we introduce some useful concepts and notation. Let \( G \) be a fixed, finite Abelian group, and let \( R \subseteq G \) be fixed. For any set \( A \subseteq G \), we define its \( R \)-neighbourhood to be

\[ N_R(A) := A + R, \]

where \( R \) is a fixed subset of \( G \).
and its $R$-interior to be
\[ \text{Int}_R(A) := \{ x \in G : x + R \subseteq A \}. \]

We note that, if $R$ is symmetric and contains the identity element 0 of $G$, then the $R$-neighbourhood of any set $A$ is precisely the graph-neighbourhood of $A$ in the Cayley graph of $G$ with generating-set $R \setminus \{0\}$, and similarly, the $R$-interior of $A$ is precisely the graph-interior of $A$ with respect to this Cayley graph.

**Proof of Theorem 2.** Let $G$ be a fixed, finite Abelian group and let $R \subseteq G$ be a fixed, nonempty subset of $G$. Let
\[ \mathcal{F} = \{ A + R : A \subseteq G \} \]
be the union-closed family consisting of all unions of translates of $R$.

We define a function $f : \mathcal{P}(G) \to \mathcal{P}(G)$ by
\[ f(S) = -(G \setminus \text{Int}_R(S)) \quad \text{for all } S \subseteq G. \]

It is clear that for any set $S \subseteq G$, $|\text{Int}_R(S)| \leq |S|$, since for any element $r \in R$, the function $x \mapsto x + r$ is an injection from $\text{Int}_R(S)$ into $S$. Hence,
\[ |S| + |f(S)| \geq |G| \quad \text{for all } S \subseteq G. \] (1)

Next, we observe that
\[ f(S) = -(G \setminus S) + R \quad \text{for all } S \subseteq G. \] (2)

Indeed, for any $x \in G$, it holds that $x \in f(S)$ iff $-x \notin \text{Int}_R(S)$ iff $(-x + R) \cap (G \setminus S) \neq \emptyset$ iff $x \in -(G \setminus S) + R$. It follows that $f(\mathcal{P}(G)) \subseteq \mathcal{F}$.

Finally, we observe that the restriction $f|_\mathcal{F}$ is an injection. This might seem surprising at first glance, but it follows immediately from the fact that
\[ N_R(\text{Int}_R(A + R)) = A + R \quad \text{for all } A \subseteq G. \] (3)

To see (3), let $S = A + R$ and observe that $N_R(\text{Int}_R(S)) \subseteq S$ holds by definition (in fact for any set $S$). On the other hand, if $S = A + R$, then we have $A \subseteq \text{Int}_R(S)$ and therefore $S = A + R \subseteq N_R(\text{Int}_R(S))$. Hence, $S = N_R(\text{Int}_R(S))$, as required.

Putting everything together, we see that $f|_\mathcal{F}$ is a bijection from $\mathcal{F}$ to itself and satisfies
\[ |S| + |f(S)| \geq |G| \quad \text{for all } S \in \mathcal{F}. \]

Therefore,
\[ \frac{1}{|\mathcal{F}|} \sum_{S \in \mathcal{F}} |S| = \frac{1}{2|\mathcal{F}|} \sum_{S \in \mathcal{F}} (|S| + |f(S)|) \geq \frac{1}{2|\mathcal{F}|} \sum_{S \in \mathcal{F}} |G| = |G|/2, \]
proving the first part of the theorem. It follows that
\[ \frac{1}{|G|} \sum_{x \in G} \left| \frac{\{ S \in \mathcal{F} : x \in S \}}{|\mathcal{F}|} \right| = \frac{1}{|G|} \frac{1}{|\mathcal{F}|} \sum_{S \in \mathcal{F}} |S| \geq 1/2, \]
so by averaging, there exists $x \in G$ such that at least half the sets in $\mathcal{F}$ contain $x$, and so the Union-Closed Conjecture holds for $\mathcal{F}$. $\square$

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References


