A subexponential upper bound for van der Waerden numbers $W(3,k)$

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Abstract

We show an improved upper estimate for van der Waerden number $W(3,k)$: there is an absolute constant $c>0$ such that if $\{1,\dots,N\}=X\cup Y$ is a partition such that X does not contain any arithmetic progression of length 3 and Y does not contain any arithmetic progression of length k then

$$N \leqslant \exp(O(k^{1-c}))\,.$$

Mathematics Subject Classifications: 05D10, 11B25

1 Introduction

Let k and l be positive integers. The van der Waerden number $W(k,l)$ is the smallest positive integer N such that in any partition $\{1,\dots,N\}=X\cup Y$ there is an arithmetic progression of length k in X or an arithmetic progression of length l in Y. The existence of such numbers was established by van der Waerden [22], however the order of magnitude of $W(k,l)$ is unknown for $k,l\geqslant 3$. Clearly, $W(k,l)$ is related to Szemerédi's theorem on arithmetic progressions [20] and any effective estimate in this theorem leads to an upper bound on the van der Waerden numbers. Currently the best known bounds in the most important diagonal case are

$$(1-o(1))\frac{2^{k-1}}{ek}\leqslant W(k,k)\leqslant 2^{2^{2^{2^{k+9}}}}\,.$$

The upper bound follows from the famous work of Gowers [12] and the lower bound was proved by Szabó [19] using a probabilistic argument. Furthermore, Berlekamp [3] showed that if $k-1$ is a prime number then

$$W(k,k) \geqslant (k-1)2^{k-1}\,.$$

Another very intriguing instance of the problem is the estimation of the numbers $W(3,k)$, as these are related to Roth's theorem [15] concerning estimates for sets avoiding three-term arithmetic progressions. Let us denote by $r(N)$ the size of the largest progression-free subset of $\{1,\dots,N\}$. We know that

$$r(N) \ll \frac{N}{\log N^{1-o(1)}}\,, \tag{1}$$

see [4, 5, 17, 18], which implies $W(3,k) \leqslant \exp(O(k^{1+o(1)}))$.

Green [13] proposed a very clever argument based on arithmetic properties of sumsets to bound $W(3,k)$. Building on this method and applying results from [10] it was shown in [11] that

$$W(3,k) \leqslant \exp(O(k\log k))\,.$$

The best known lower bound was obtained by Li and Shu [14] (see also [8]), who showed that

$$W(3,k) \gg \Big(\frac{k}{\log k}\Big)^2.$$

The purpose of this paper is to prove a subexponential bound on $W(3,k)$.

Theorem 1. *There are absolute constants $C,c>0$ such that for every k we have*

$$W(3,k) \leqslant \exp(Ck^{1-c})\,.$$

Our argument is based on the method of [18], which explores in details the structure of a large spectrum. This method can be partly applied (see Lemma 5) in our approach and it deals only with a progression-free partition class. The second part of the proof exploits the structure of both partition classes and in this case the argument of [18] has to be significantly modified.

Let us remark that during the review process a preprint of Bloom and Sisask [6], which improves an upper bound in Roth's theorem to $N/(\log N)^{1+c}$ for $c \approx 2^{-2^{2^{1000}}}$, has appeared. That result implies directly that $W(3,k) \leqslant \exp(Ck^{1-c})$ with $c \approx 2^{-2^{2^{1000}}}$.

2 Notation

Given functions $f,g:\mathbb{Z}/N\mathbb{Z}\to\mathbb{C}$, the convolution of f and g is defined by

$$(f*g)(x) = \sum_{t\in\mathbb{Z}/N\mathbb{Z}} f(t)g(x-t).$$

The Fourier coefficients of a function $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ are defined by

$$\widehat{f}(r) = \sum_{x=0}^{N-1} f(x)e^{-2\pi ixr/N},$$

where $r \in \mathbb{Z}/N\mathbb{Z}$. The inversion formula states that

$$f(x) = \frac{1}{N}\sum_{x=0}^{N-1} \widehat{f}(r)e^{2\pi ixr/N}.$$

We denote by $1_A(x)$ the indicator function of set A. Thus using the inversion formula and the fact that $\widehat{(f*g)}(r) = \widehat{f}(r)\widehat{g}(r)$ one can express the number of three–term arithmetic progressions (including trivial ones) in a set $A \subseteq \mathbb{Z}/N\mathbb{Z}$ by

$$\frac{1}{N}\sum_{r=0}^{N-1} \widehat{1_A}(r)^2\widehat{1_A}(-2r)\,.$$

Parseval's identity asserts in particular that

$$\sum_{r=0}^{N-1} |\widehat{1_A}(r)|^2 = |A|N\,.$$

Let $\theta \geqslant 0$ be a real number. The $\theta-$spectrum of A is defined by

$$\Delta_\theta(A) = \big\{r \in \mathbb{Z}/N\mathbb{Z} : |\widehat{1_A}(r)| \geqslant \theta|A|\big\}.$$

If A is specified then we write Δ_θ instead of $\Delta_\theta(A)$.

By the span of a finite set S we mean

$$\mathrm{Span}\,(S) = \Big\{\sum_{s\in S} \varepsilon_s s : \varepsilon_s \in \{-1,0,1\} \text{ for all } s \in S\Big\}$$

and the dimension of A is defined by

$$\dim(A) = \min\big\{|S| : A \subseteq \mathrm{Span}\,(S)\big\}\,.$$

Chang's Spectral Lemma provides an upper bound for the dimension of a spectrum.

Lemma 2. [9] *Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$ be a set of size $|A| = \delta N$ and let $\theta > 0$. Then*

$$\dim(\Delta_\theta(A)) \ll \theta^{-2}\log(1/\delta)\,.$$

We are going to use Bohr sets [7] to prove the main result. Let $\Gamma \subseteq \widehat{G}$ and $\gamma \in (0, \frac{1}{2}]$ then the Bohr set generated by Γ with radius γ is

$$B(\Gamma,\gamma) = \big\{x \in \mathbb{Z}/N\mathbb{Z} :\ \|tx/N\| \leqslant \gamma \text{ for all } t \in \Gamma\big\}\,,$$

where $\|x\| = \min_{y\in\mathbb{Z}}|x-y|$. The rank of B is the size of Γ and we denote it by $\mathrm{rk}(B)$. Given $\eta > 0$ and a Bohr set $B = B(\Gamma,\gamma)$, by B_η we mean the Bohr set $B(\Gamma,\eta\gamma)$. We will also use the notation $\beta = \frac{1}{|B|}1_B$. We will use two basic properties of Bohr sets concerning their size and regularity, see [21].

Lemma 3. [7] *For every $\gamma \in (0, \frac{1}{2}]$ we have*

$$\gamma^{|\Gamma|} N \leqslant |B(\Gamma, \gamma)| \leqslant 8^{|\Gamma|+1} |B(\Gamma, \gamma/2)| \,.$$

We call a Bohr set $B(\Gamma, \gamma)$ regular if for every η, where $|\eta| \leqslant 1/(100|\Gamma|)$ we have

$$(1 - 100|\Gamma||\eta|)|B| \leqslant |B_{1+\eta}| \leqslant (1 + 100|\Gamma||\eta|)|B|.$$

Bourgain [7] showed that regular Bohr sets are ubiquitous.

Lemma 4. [7] *For every Bohr set $B(\Gamma, \gamma)$, there exists γ' such that $\frac{1}{2}\gamma \leqslant \gamma' \leqslant \gamma$ and $B(\Gamma, \gamma')$ is regular.*

3 Proof of Theorem 1

Our main tool is the next lemma, which can be extracted from [18], by conjugation of results concerning the case of 'small' Fourier coefficients (see Lemmas 7 and 9 of [18]) and the case of 'middle' Fourier coefficients (Lemmas 12 and 13 of [18]). Its proof makes use of the deep result by Bateman and Katz in [1, 2] describing the structure of the large spectrum. The case of 'large' Fourier coefficients is treated similarly as in [18], however using partition properties we will be able to obtain much better estimate.

Lemma 5. [18] *There exists an absolute constant $c > 0$ such that the following holds. Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$, $|A| = \delta N$ be a set such that*

$$\sum_{r:\, \delta^{1+c}|A| \leqslant |\widehat{1_A}(r)| \leqslant \delta^{1/10}|A|} |\widehat{1_A}(r)|^3 \geqslant \frac{1}{10} \delta^{c/5} |A|^3 \,. \tag{2}$$

Then there is a regular Bohr set B with $\mathrm{rk}(B) \ll \delta^{-1+c}$ and radius $\Omega(\delta^{1-c})$ such that for some t

$$|(A + t) \cap B| \gg \delta^{1-c}|B|.$$

Furthermore, we apply Bloom's iterative lemma, that provides a density increment by a constant factor greater than 1 for progression-free sets and Sanders' lemma on a containment of long arithmetic progressions in dense subsets of regular Bohr sets.

Lemma 6. [4] *There exists an absolute constant $c_1 > 0$ such that the following holds. Let $B \subseteq \mathbb{Z}/N\mathbb{Z}$ be a regular Bohr set of rank d. Let $A_1 \subseteq B$ and $A_2 \subseteq B_\varepsilon$, each with relative densities α_i. Let $\alpha = \min(c_1, \alpha_1, \alpha_2)$ and assume that $d \leqslant \exp(c_1(\log^2(1/\alpha))$. Suppose that B_ε is also regular and $c_1\alpha/(4d) \leqslant \varepsilon \leqslant c_1\alpha/d$. Then either*

(i) there is a regular Bohr set B' of rank $\mathrm{rk}(B') \leqslant d + O(\alpha^{-1}\log(1/\alpha))$ and size

$$|B'| \geqslant \exp\big(-O(\log^2(1/\alpha)(d + \alpha^{-1}\log(1/\alpha)))\big)|B|$$

such that

$$|(A_1 + t) \cap B'| \gg (1 + c_1)\alpha_1|B'|$$

for some $t \in \mathbb{Z}/N\mathbb{Z}$;

(ii) or there are $\Omega(\alpha_1^2\alpha_2|B||B_\varepsilon|)$ *three-term arithmetic progressions* $x+y=2z$ *with* $x,y\in A_1, z\in A_2$*;*

Lemma 7. [16] *Let* $B(\Gamma,\gamma)\subseteq \mathbb{Z}/N\mathbb{Z}$ *be a regular Bohr set of rank* d *and let* ε *be a positive number satisfying* $\varepsilon^{-1}\ll \gamma d^{-1}N^{1/d}$*. Suppose that* $A\subseteq B$ *contains at least a proportion* $1-\varepsilon$ *of* $B(\Gamma,\gamma)$*. Then* A *contains an arithmetic progression of length at least* $1/(4\varepsilon)$*.*

Furthermore, to apply Bloom's result we will need to prove a standard fact on Bohr sets. Let us also remark that it follows from the proof of Bloom's lemma that we can take $c_1<1/1000$. We will use the following basic property of Bohr sets.

Lemma 8. [11] *Let* B *be a regular Bohr set of rank* d *and radius* γ *and let* A *be a set with* $|A|=\alpha|B|$*. Suppose that* $\varepsilon<\kappa\alpha/(100d)$ *for some* $\kappa\in(0,1)$*. Then*

$$\sum_{x\in B}(1_A * 1_{B_\varepsilon})(x)\geqslant (1-\kappa)\alpha|B||B_\varepsilon|\,.$$

Lemma 9. *Let* B *be a regular Bohr set of rank* d *and suppose that* $A\subseteq B$ *and* $|A|=\alpha|B|$*. Then there is* ε *with* $c_1\alpha/(4d)\leqslant\varepsilon\leqslant c_1\alpha/d$ *(*c_1 *is a constant given by Lemma 6), and regular Bohr sets* B' *and* B'_ε *of rank* d *and size*

$$|B'|\geqslant \exp(-O(d\log(1/\alpha)\log(d/\alpha)))|B|$$

such that

$$|(A+t)\cap B'|\geqslant \frac{1}{2}\alpha|B'| \quad \textit{and} \quad |(A+t)\cap B'_\varepsilon|\geqslant\frac{1}{2}\alpha|B'_\varepsilon|$$

for some t*.*

Proof. Let ε_1 and ε_2 be any numbers satisfying $c_1\alpha/(4d)\leqslant\varepsilon_1,\varepsilon_2\leqslant c_1\alpha/d$ and such that the Bohr sets $B^1=B_{\varepsilon_1}$ and $B^2=B_{\varepsilon_1\varepsilon_2}$ are regular. Put $\beta_1=\frac{1}{|B_1|}1_{B_1}$ and $\beta_2=\frac{1}{|B_2|}1_{B_2}$ then by Lemma 8 we have

$$\sum_x\big((\beta_1*1_A)(x)+(\beta_2*1_A)(x)\big)\geqslant\frac{9}{5}\alpha|B|\,,$$

hence for some x we have

$$(\beta_1*1_A)(x)+(\beta_2*1_A)(x)\geqslant\frac{9}{5}\alpha.$$

If

$$(\beta_1*1_A)(x)\geqslant\frac{1}{2}\alpha \quad\text{and}\quad (\beta_2*1_A)(x)\geqslant\frac{1}{2}\alpha \tag{3}$$

then we can put $B'=B^1$. Otherwise, for some $i\in\{1,2\}$ we have $(\beta_i*1_A)(x)\geqslant(6/5)\alpha$ and we put $B=B^i$ and apply the same procedure again. Clearly, we can iterate this procedure $O(\log(1/\alpha))$ times, as the density of a set can not exceed 1. Thus after $O(\log(1/\alpha))$ iterative steps we obtain a pair of Bohr sets B',B'_ε satisfying (3). From Lemma 3 it follows that

$$|B'|\geqslant\exp(-O(d\log(1/\alpha)\log(d/\alpha)))|B|\,,$$

which concludes the proof. □

Proof of Theorem 1. Put $M = W(3,k)-1$ and let $\{1,\dots,M\} = X \cup Y$ be a partition such that X and Y avoid 3 and k-term arithmetic progressions respectively. Clearly, we may assume that $M \geqslant 100k$ hence

$$|Y| \leqslant M - \lfloor M/k \rfloor \leqslant M - M/(2k),$$

as no block of k consecutive numbers is contained in Y and therefore $|X| \geqslant M/(2k)$. Let N be any prime number satisfying $2M < N \leqslant 4M$. We embed $\{1,\dots,M\} = X \cup Y$ in $\mathbb{Z}/N\mathbb{Z}$ in a natural way and observe that $\{1,\dots,M\}$ considered as a subset of $\mathbb{Z}/N\mathbb{Z}$ is 2-Freiman isomorphic to $\{1,\dots,M\}$ considered as a subset of $\mathbb{Z}$ and therefore any arithmetic progression in $\{1,\dots,M\} \subseteq \mathbb{Z}/N\mathbb{Z}$ is a genuine progression in $\mathbb{Z}$. Put $|X| = \delta N$ and note that we can assume that $\delta \gg (\log N)^{-1.1}$. First let us assume that

$$\sum_{r:\, \delta^{1+c}|X| \leqslant |\widehat{1_X}(r)| \leqslant \delta^{1/10}|X|} |\widehat{1_X}(r)|^3 \geqslant \delta^{c/5}|X|^3\,, \tag{4}$$

where $c > 0$ is the fixed absolute constant from Lemma 5. Then by Lemma 5 there is $t \in \mathbb{Z}/N\mathbb{Z}$ and a regular Bohr set B^0 with $\mathrm{rk}(B^0) = d \ll \delta^{-1+c}$ and radius $\Omega(\delta^{1-c})$ such that

$$|(X+t) \cap B^0| \gg \delta^{1-c}|B|$$

for some absolute constant $c > 0$. Writing $X_0 = (X+t) \cap B_0$ we have

$$|X_0 \cap B^0| \gg \alpha |B^0|\,,$$

where

$$\alpha \gg \delta^{1-c}\,,$$

and by Lemma 3

$$|B^0| \geqslant \exp\big(-O(\delta^{-1+c}\log(1/\delta))\big)N\,.$$

By Lemma 9 there is $\varepsilon \gg \alpha^2$ and regular Bohr sets B' and B'_ε of rank d such that

$$|(A+t) \cap B'| \geqslant \frac{1}{2}\alpha|B'| \quad \text{and} \quad |(A+t) \cap B'_\varepsilon| \geqslant \frac{1}{2}\alpha|B'_\varepsilon|\,,$$

and

$$|B'| \geqslant \exp(-O(\alpha^{-1}\log^2(1/\alpha)))|B^0| \geqslant \exp\big(-O(\delta^{-1+c}\log^2(1/\delta))\big)N\,.$$

Next, we iteratively apply Lemma 6 and Lemma 9. Since after each step the density increases by factor $1+c_1$ it follows that after $l \ll \log(1/\alpha)$ steps case (ii) of Lemma 6 holds. Let B^i be Bohr sets obtained in the iterative procedure. Note that B^i has rank $\mathrm{rk}(B^i) = d_i \ll \alpha^{-1}\log^2(1/\alpha)$ for every $i \leqslant l$ and

$$|B^{i+1}| \geqslant \exp\big(-O(\log^2(1/\alpha)(d_i + \alpha^{-1}\log(1/\alpha)))\big)|B_i|$$

for every $i < l$. Therefore, there are

$$\Omega(\alpha^3|B^l||B^l_\varepsilon|)$$

three-term arithmetic progressions in X, where $\varepsilon \geqslant c_1\alpha/(4\mathrm{rk}(B^l)) \gg \alpha^2\log^2(1/\alpha)$. By Lemma 6 and Lemma 3 we have

$$|B^l| \geqslant \exp\big(-O(\alpha^{-1}\log^4(1/\alpha))\big)N \geqslant \exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N\,,$$

and

$$\begin{aligned}|B^l_\varepsilon| &\geqslant \exp\big(-O(\alpha^{-1}\log^3(1/\alpha))\big)\exp\big(-O(\alpha^{-1}\log^4(1/\alpha))\big)N\\ &\geqslant \exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N\,.\end{aligned}$$

Thus, X contains

$$\Omega(\delta^{3-3c}\exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N^2)$$

arithmetic progressions of length three. Since there are only $|X|$ trivial progressions in X it follows that

$$|X| \gg \delta^{3-3c}\exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N^2\,,$$

so

$$W(3,k) \ll N \ll \exp\big(O(\delta^{-1+c}\log^4(1/\delta)\big) \leqslant \exp\big(O(k^{1-c}\log^4 k)\big)\,.$$

Next let us assume that (4) does not hold. Let us define $\Delta' = \Delta_{\delta^{1/10}} \cup 2^{-1}\cdot\Delta_{\delta^{1/10}}$ and observe that $r \notin \Delta'$ is equivalent to $r \notin \Delta_{\delta^{1/10}}$ and $2r \notin \Delta_{\delta^{1/10}}$. By Chang's lemma

$$\dim(\Delta') \leqslant 2\dim(\Delta_{\delta^{1/10}}) \ll \delta^{-1/5}\log(1/\delta)\,,$$

and let Λ be any set such that $1 \in \Lambda$, $|\Lambda| \ll \delta^{-1/5}\log(1/\delta)$ and $\Delta' \subseteq \mathrm{Span}\,(\Lambda)$. Let $B = B(\Lambda,\gamma)$ be a regular Bohr set with radius $\delta^3 \ll \gamma \leqslant \delta^3$. Since $1 \in \Lambda$ it follows that for every $b \in B$ we

$$\|b/N\| \leqslant \gamma N \leqslant 4\gamma M\,.$$

Recall that $\beta = \frac{1}{|B|}1_B$ then for every $r \in \Delta'$ we have

$$\big|\widehat{\beta}(r) - 1\big| \leqslant \frac{1}{|B|}\sum_{b\in B}|e^{-2\pi i r b/N} - 1| \leqslant \frac{2\pi}{|B|}\sum_{b\in B}\sum_{\lambda\in\Lambda}\|rb/N\| \leqslant 2\pi\delta^2\,, \qquad (5)$$

and similarly $|\widehat{\beta}(2r) - 1| \leqslant 4\pi\delta^2$. For $t \in \mathbb{Z}/N\mathbb{Z}$ put

$$f(t) = \beta * 1_X(t)$$

and note that if for some $t \in [4\gamma M, (1-4\gamma)M]$ we have $f(t) = \frac{1}{|B|}|X \cap (B+t)| \leqslant \delta^{1+c'}$, where $c' = c/20$, then since $B + t \subseteq [1, M]$ it follows that

$$|Y \cap (B+t)| \geqslant (1-\delta^{1+c'})|B|\,.$$

Therefore, by Lemma 7 either $\delta^{-1-c'} \gg \gamma d^{-1}N^{1/d}$ or Y contains an arithmetic progression of length $\frac{1}{4}\delta^{-1-c'}$. The former inequality implies that

$$k^{1+c'} \gg \gamma d^{-1}N^{1/d} \gg \delta^4 N^{O(\delta^{1/5}\log^{-1}(1/\delta))} \gg k^{-4}N^{O(k^{-1/5}\log^{-1}k)}\,,$$

so

$$W(3,k) \ll N \ll \exp\left(O(k^{1/5}\log^2 k)\right).$$

If the second alternative holds then

$$\frac{1}{4}\delta^{-1-c'} < k$$

hence by (1)

$$k^{-1/(1+c')} \ll \delta \ll (\log N)^{-1+o(1)}$$

so

$$W(3,k) \leqslant N \ll \exp\left(O(k^{\frac{1}{1+c'}+o(1)})\right).$$

Finally we can assume that for every $t \in [4\gamma M, (1-4\gamma)M]$ we have $f(t) \geqslant \delta^{1+c'}$. Let $T(X)$ denote the number of three-term arithmetic progressions in X and let

$$T(f) = \sum_{x+y=2z} f(x)f(y)f(z).$$

Then clearly

$$T(f) \gg \delta^{3+3c'}M^2 \gg \delta^{3+c/6}N^2 \tag{6}$$

and we will show that $T(X)$ does not differ much from $T(f)$

$$\begin{aligned} \left|T(X)-T(f)\right| &= \frac{1}{N}\Big|\sum_{r=0}^{N-1}\widehat{1_X}(r)^2\widehat{1_X}(-2r) - \sum_{r=0}^{N-1}\widehat{f}(r)^2\widehat{f}(-2r)\Big| \\ &\leqslant \frac{1}{N}\sum_{r=0}^{N-1}|\widehat{1_X}(r)^2\widehat{1_X}(-2r)(1-\widehat{\beta}(r)^2\widehat{\beta}(-2r))| \qquad (7) \\ &= S_1+S_2, \end{aligned}$$

where S_1 and S_2 are summations of (7) respectively over Δ' and $\mathbb{Z}/N\mathbb{Z}\setminus\Delta'$. By (5), the negation of (4), Parseval's formula and Hölder's inequality we have

$$S_1 \ll \delta^2\frac{1}{N}\sum_{r\in\Delta'}|\widehat{1_X}(r)|^3 \leqslant \delta^3\sum_{r=0}^{N-1}|\widehat{1_X}(r)|^2 = \delta^2|X|^2,$$

$$\begin{aligned} S_2 &\leqslant \frac{2}{N}\sum_{r\notin\Delta'}|\widehat{1_X}(r)^2\widehat{1_X}(-2r)| \\ &\leqslant \frac{2}{N}\Big(\sum_{r\notin\Delta'}|\widehat{1}_X(r)|^3\Big)^{2/3}\Big(\sum_{r\notin\Delta'}|\widehat{1}_X(2r)|^3\Big)^{1/3} \\ &\leqslant \frac{2}{N}\sum_{r\notin\Delta_{\delta^{1/10}}}|\widehat{1_X}(r)|^3 \\ &\leqslant \frac{2}{N}\sum_{r\in\Delta_{\delta^{1+c}}\setminus\Delta_{\delta^{1/10}}}|\widehat{1_X}(r)|^3 + \frac{2}{N}\sum_{r\notin\Delta_{\delta^{1+c}}}|\widehat{1_X}(r)|^3 \\ &\ll \delta^2|X|^2 + \delta^{1+c/5}|X|^2 + \delta^{1+c}|X|^2 \ll \delta^{1+c/5}|X|^2 \end{aligned}$$

Thus,

$$\left|T(X)-T(f)\right| \ll \delta^{3+c/5}N^2,$$

so by (6) and the fact that X avoids non-trivial three-term arithmetic progression we have

$$|X| = T(X) \gg \delta^{3+c/6}N^2\,,$$

hence

$$W(3,k) \leqslant N \ll \delta^{-2-c/6} \ll k^3$$

which concludes the proof. □

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