Tomaszewski's problem on randomly signed sums, revisited

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Abstract

Let $v_1, v_2, \ldots, v_n$ be real numbers whose squares add up to 1. Consider the 2^n signed sums of the form $S = \sum \pm v_i$. Boppana and Holzman (2017) proved that at least $\frac{13}{32}$ of these sums satisfy $|S| \leqslant 1$. Here we improve their bound to 0.427685.

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1 Introduction

Let $v_1, v_2, \ldots, v_n$ be real numbers such that the sum of their squares is at most 1. Consider the 2^n signed sums of the form $S = \pm v_1 \pm v_2 \pm \cdots \pm v_n$. In 1986, B. Tomaszewski (see Guy [4]) asked the following question: is it always true that at least $\frac{1}{2}$ of these sums satisfy $|S| \leqslant 1$?

Boppana and Holzman [2] proved that at least $\frac{13}{32} = 0.40625$ of the sums satisfy $|S| \leqslant 1$. Actually, they proved a slightly better bound of 0.406259. See their paper for a discussion of earlier work on Tomaszewski's problem.

In this note, we will improve the lower bound to 0.427685. We will sharpen the Boppana-Holzman argument by using a Gaussian bound due to Bentkus and Dzindzalieta [1].

After we wrote this note, two further improvements appeared. Dvořák, van Hintum, and Tiba [3] strengthened the lower bound to 0.46. Keller and Klein [5] completely solved Tomaszewski's problem by proving a lower bound of $\frac{1}{2}$.

We will use the language of probability. Let $\Pr[A]$ be the probability of an event A. A *random sign* is a random variable whose probability distribution is the uniform distribution on the set $\{-1, +1\}$. With this language, we can state our main result.

Main Theorem. *Let v_1, v_2, ..., v_n be real numbers such that $\sum_{i=1}^n v_i^2 \leqslant 1$. Let a_1, a_2, ..., a_n be independent random signs. Let S be $\sum_{i=1}^n a_i v_i$. Then $\Pr[|S| \leqslant 1] > 0.427685$.*

2 Proof of the improved bound

In this section, we will prove the bound of 0.427685. We will follow the approach of Boppana and Holzman [2], replacing their fourth-moment method with a Gaussian bound.

Let Q be the upper tail function of the standard normal (Gaussian) distribution:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\,dt.$$

Note that Q is a decreasing, positive function.

Bentkus and Dzindzalieta [1] proved the following Gaussian bound on randomly-signed sums. See their paper for a discussion of earlier work on such bounds.

Theorem 1 (Bentkus and Dzindzalieta)**.** *Let x be a real number. Let v_1, v_2, ..., v_n be real numbers such that $\sum_{i=1}^n v_i^2 \leqslant 1$. Let a_1, a_2, ..., a_n be independent random signs. Let S be $\sum_{i=1}^n a_i v_i$. Then*

$$\Pr[S \geqslant x] \leqslant \frac{Q(x)}{4Q(\sqrt{2})}\,.$$

Given a positive number c, define $F(c)$ by

$$F(c) := \frac{1}{2} - \frac{Q(1/\sqrt{c}\,)}{4Q(\sqrt{2})}\,.$$

Note that F is a decreasing function bounded above by $\frac{1}{2}$. A calculation shows that $F(\frac{1}{4}) > 0.427685$.

We will need the following lemma, which quantitatively improves Lemma 3 of Boppana and Holzman [2]. Roughly speaking, this lemma is used to show that if a partial sum is a little less than 1 in absolute value, then the final sum has a decent chance of remaining less than 1 in absolute value.

Lemma 2. *Let c be a positive number. Let x be a real number such that $|x| \leqslant 1$. Let v_1, v_2, ..., v_n be real numbers such that*

$$\sum_{i=1}^n v_i^2 \leqslant c(1 + |x|)^2.$$

Let a_1, a_2, ..., a_n be independent random signs. Let Y be $\sum_{i=1}^n a_i v_i$. Then

$$\Pr[|x+Y| \leqslant 1] \geqslant F(c).$$

Proof. By symmetry, we may assume that $x \geqslant 0$. Let w_i be $\frac{-v_i}{\sqrt{c}\,(1+x)}$. Then $\sum_{i=1}^n w_i^2 \leqslant 1$. Let S be $\sum_{i=1}^n a_i w_i$. Then $Y = -\sqrt{c}\,(1+x)S$. Because Y has a symmetric distribution, we have

$$\Pr[Y > 1-x] \leqslant \Pr[Y > 0] \leqslant \frac{1}{2}\,.$$

By the Bentkus-Dzindzalieta inequality (Theorem 1), we have

$$\Pr[Y < -(1+x)] = \Pr\Big[S > \frac{1}{\sqrt{c}}\Big] \leqslant \frac{Q(1/\sqrt{c}\,)}{4Q(\sqrt{2})}\,.$$

Therefore

$$\Pr[|x+Y| > 1] = \Pr[Y > 1-x] + \Pr[Y < -(1+x)] \leqslant \frac{1}{2} + \frac{Q(1/\sqrt{c}\,)}{4Q(\sqrt{2})}\,.$$

Taking the complement, we obtain

$$\Pr[|x+Y| \leqslant 1] = 1 - \Pr[|x+Y| > 1] \geqslant \frac{1}{2} - \frac{Q(1/\sqrt{c}\,)}{4Q(\sqrt{2})} = F(c). \qquad \square$$

We will also need the following lemma, which says that F satisfies a certain weighted-average inequality. This lemma is used to show that a weighted average of lower bounds from Lemma 2 is still a good lower bound.

Lemma 3. *Let K be an integer such that $K \geqslant 2$. Then*

$$\frac{1}{2^{K-1}} F\left(\frac{(K+1)^2 - K}{(2K+1)^2}\right) + \Big(1 - \frac{1}{2^{K-1}}\Big) F\left(\frac{(K+1)^2 - (K+2)}{(2K+1)^2}\right) \geqslant F\Big(\frac{1}{4}\Big).$$

Proof. Let

$$c_1 = \frac{(K+1)^2 - K}{(2K+1)^2} = \frac{1}{4} + \frac{3}{4}\,\frac{1}{(2K+1)^2};\; c_2 = \frac{(K+1)^2 - (K+2)}{(2K+1)^2} = \frac{1}{4} - \frac{5}{4}\,\frac{1}{(2K+1)^2}.$$

Since $c_1 \geqslant c_2$ and F is a decreasing function, we see that for $K \geqslant 2$ we have

$$\frac{1}{2^{K-1}} F(c_1) + \left(1 - \frac{1}{2^{K-1}}\right) F(c_2) \geqslant \frac{1}{2} F(c_1) + \frac{1}{2} F(c_2).$$

Therefore it is sufficient to show that the following inequality holds for $0 \leqslant \xi \leqslant 1/25$:

$$\frac{1}{2} F\left(\frac{1}{4} + \frac{3}{4}\xi\right) + \frac{1}{2} F\left(\frac{1}{4} - \frac{5}{4}\xi\right) \geqslant F\left(\frac{1}{4}\right). \tag{1}$$

Once we show that $F(x)$ is a concave function in the region $0 < x \leqslant 1/4 + 3/100$, we conclude that the left hand side of the inequality is also concave in ξ in the region $0 \leqslant \xi \leqslant 1/25$ and we need only check the inequality for $\xi = 0$ and for $\xi = 1/25$. We will show that $Q(1/\sqrt{x})$ is convex in x in the region $0 < x \leqslant 1/3$. Recall that Q satisfies the ordinary differential equation $Q''(x) = -xQ'(x)$ and that $Q'(x) < 0$ for all x. Thus, for $x > 0$

$$\begin{aligned}\frac{d^2}{dx^2}Q(x^{-1/2}) &= Q''(x^{-1/2})\Big(-\frac{1}{2}x^{-3/2}\Big)^2 + Q'(x^{-1/2})\Big(\frac{3}{4}x^{-5/2}\Big)\\ &= -\frac{1}{4}Q'(x^{-1/2})x^{-7/2}(1-3x),\end{aligned}$$

which is positive if $1-3x > 0$. It follows that $Q(x^{-1/2})$ is convex in the region $0 < x \leqslant 1/3$. Therefore $F(x)$ is concave in the region $0 < x \leqslant 1/3$. Inequality (1) holds trivially for $\xi = 0$, and one can check by calculation that it also holds for $\xi = 1/25$ (and even for $\xi = 1/9$). □

Finally, we will use these two lemmas to prove our main theorem.

Proof of Main Theorem. We will follow the proof of Theorem 4 of Boppana and Holzman [2] nearly line for line. Their proof uses a different function F. Closely examining their proof, we see that they use four properties of F: it is bounded above by $\frac{1}{2}$, satisfies their Lemma 3 (our Lemma 2), is a nonincreasing function (on the set of positive numbers), and satisfies the weighted-average inequality of Lemma 3. Our function F has those same four properties. Hence we reach the same conclusion: $\Pr[|S| \leqslant 1] \geqslant F(\frac{1}{4})$. A calculation shows that $F(\frac{1}{4}) > 0.427685$. □

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