Elastic Elements in 3-Connected Matroids

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Abstract

It follows by Bixby's Lemma that if e is an element of a 3-connected matroid M, then either $\text{co}(M\backslash e)$, the cosimplification of $M\backslash e$, or $\text{si}(M/e)$, the simplification of M/e, is 3-connected. A natural question to ask is whether M has an element e such that both $\text{co}(M\backslash e)$ and $\text{si}(M/e)$ are 3-connected. Calling such an element "elastic", in this paper we show that if $|E(M)| \geqslant 4$, then M has at least four elastic elements provided M has no 4-element fans.

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1 Introduction

A result widely used in the study of 3-connected matroids is due to Bixby [1]: if e is an element of a 3-connected matroid M, then either $M\backslash e$ or M/e has no non-minimal 2-separations, in which case, $\text{co}(M\backslash e)$, the cosimplification of M, or $\text{si}(M/e)$, the simplification of M, is 3-connected. A 2-separation (X, Y) is *minimal* if $\min\{|X|, |Y|\} = 2$.

This result is commonly referred to as Bixby's Lemma. Thus, although an element e of a 3-connected matroid M may have the property that neither $M\backslash e$ nor M/e is 3-connected, Bixby's Lemma says that at least one of $M\backslash e$ and M/e is close to being 3-connected in a very natural way. In this paper, we are interested in whether or not there are elements e in M such that both $\mathrm{co}(M\backslash e)$ and $\mathrm{si}(M/e)$ are 3-connected, in which case, we say e is *elastic*. In general, a 3-connected matroid M need not have any elastic elements. For example, all wheels and whirls of rank at least four have no elastic elements. The reason for this is that every element of such a matroid is in a 4-element fan and the way, geometrically, every 4-element fan is positioned in relation to the rest of the elements of the matroid. However, as signalled by the next theorem, 4-element fans are the only possible obstacles to M having elastic elements.

A 3-separation (A, B) of a matroid is *vertical* if $\min\{r(A), r(B)\} \geqslant 3$. Now, let M be a matroid and let $(X, \{e\}, Y)$ be a partition of $E(M)$. We say that $(X, \{e\}, Y)$ is a *vertical 3-separation* of M if $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both vertical 3-separations and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Furthermore, $Y \cup \{e\}$ is *maximal* in this separation if there exists no vertical 3-separation $(X', \{e'\}, Y')$ of M such that $Y \cup \{e\}$ is a proper subset of $Y' \cup \{e'\}$. Essentially, all of the work in the paper goes into establishing the following theorem.

Theorem 1. *Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is maximal. If $X \cup \{e\}$ is not a 4-element fan, then X contains at least two elastic elements.*

Note that, in the context of Theorem 1, if $X \cup \{e\}$ is a 4-element fan, then it is possible that X contains two elastic elements. For example, consider the rank-4 matroids M_1 and M_2 for which geometric representations are shown in Fig. 1. For each $\in \{1, 2\}$, the tuple $F = (e_1, e_2, e_3, e_4)$ is a 4-element fan of M_i and $(F - \{e_1\}, \{e_1\}, E(M_i) - F)$ is a vertical 3-separation of M_i. In M_1, none of e_2, e_3, and e_4 is elastic, while in M_2, both e_2 and e_3 are elastic. However, provided $X \cup \{e\}$ is a maximal fan, the instance illustrated in Fig. 1(i) is essentially the only way in which X does not contain two elastic elements. This is made more precise in Section 3.

An almost immediate consequence of Theorem 1 is the following corollary.

Corollary 2. *Let M be a 3-connected matroid. If $|E(M)| \geqslant 7$, then M contains at least four elastic elements provided M has no 4-element fans. Moreover, if $|E(M)| \leqslant 6$, then every element of M is elastic.*

Like Bixby's Lemma, Corollary 2 is an inductive tool for handling the removal of elements of 3-connected matroids while preserving connectivity. The most well-known examples of such tools are Tutte's Wheels-and-Whirls Theorem [8] and Seymour's Splitter Theorem [7]. In both theorems, this removal preserves 3-connectivity. More recently, there have been analogues of these theorems in which the removal of elements preserves 3-connectivity up to simplification and cosimplification. These analogues have additional conditions on the elements being removed. Let B be a basis of a 3-connected matroid M, and suppose that M has no 4-element fans. Say M is representable over some field $\mathbb{F}$ and that we are given a standard representation of M over $\mathbb{F}$. To keep the information

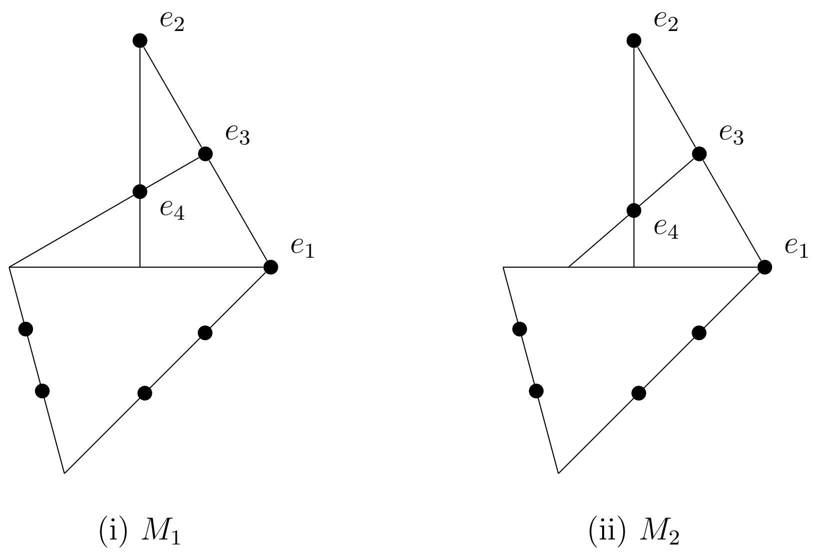

Figure 1: For each $i \in \{1,2\}$, the tuple (e_1, e_2, e_3, e_4) is a 4-element fan and the partition $(\{e_2, e_3, e_4\}, \{e_1\}, E(M_i) - \{e_1, e_2, e_3, e_4\})$ of $E(M_i)$ is a vertical 3-separation of M_i. Furthermore, in M_1, none of e_2, e_3, and e_4 are elastic, while in M_2, both e_2 and e_3 are elastic.

displayed by the representation in an $\mathbb{F}$-representation of a single-element deletion or a single element contraction of M, we need to avoid pivoting. To do this, we want to either contract an element in B or delete an element in $E(M) - B$. Whittle and Williams [10] showed that if $|E(M)| \geqslant 4$, then M has at least four elements e such that either $\mathrm{si}(M/e)$ is 3-connected if $e \in B$ or $\mathrm{co}(M \backslash e)$ is 3-connected if $e \in E(M) - B$. Brettell and Semple [2] establish a Splitter Theorem counterpart to this last result where, again, 3-connectivity is preserved up to simplification and cosimplification. These last two results are related to an earlier result of Oxley et al. [5]. Indeed, the starting point for the proof of Theorem 1 is [5].

The paper is organised as follows. The next section contains some necessary preliminaries on connectivity, while Section 3 considers fans and determines exactly which elements of a fan are elastic. Section 4 establishes several results concerning when an element in a rank-2 restriction of a 3-connected matroid is deletable or contractible. Section 5 consists of the proofs of Theorem 1 and Corollary 2. Throughout the paper, the notation and terminology follows [3].

2 Preliminaries

Connectivity

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

A subset X of E or a partition $(X, E-X)$ is *k-separating* if $\lambda_M(X) \leqslant k-1$ and *exactly k-separating* if $\lambda_M(X) = k-1$. A k-separating partition $(X, E-X)$ is a *k-separation* if $\min\{|X|, |E-X|\} \geqslant k$. A matroid is *$n$-connected* if it has no k-separations for all $k < n$.

Let e be an element of a 3-connected matroid M. We say e is *deletable* if $\mathrm{co}(M\backslash e)$ is 3-connected, and e is *contractible* if $\mathrm{si}(M/e)$ is 3-connected. Thus, e is elastic if it is both deletable and contractible.

Two k-separations (X_1, Y_1) and (X_2, Y_2) *cross* if each of the intersections $X_1 \cap Y_1$, $X_1 \cap Y_2$, $X_2 \cap Y_1$, $X_2 \cap Y_2$ are non-empty. The next lemma is a standard tool for dealing with crossing separations. It is a straightforward consequence of the fact that the connectivity function λ of a matroid M is submodular, that is,

$$\lambda(X) + \lambda(Y) \geqslant \lambda(X \cap Y) + \lambda(X \cup Y)$$

for all $X, Y \subseteq E(M)$. An application of this lemma will be referred to as *by uncrossing.*

Lemma 3. *Let M be a k-connected matroid, and let X and Y be k-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geqslant k-1$, then $X \cup Y$ is k-separating.*

(ii) *If $|E(M) - (X \cup Y)| \geqslant k-1$, then $X \cap Y$ is k-separating.*

The next five lemmas are used frequently throughout the paper. The first follows from orthogonality, while the second follows from the first. The third follows from the first and second. A proof of the fourth and fifth can be found in [9] and [2], respectively.

Lemma 4. *Let e be an element of a matroid M, and let X and Y be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Lemma 5. *Let X be an exactly 3-separating set in a 3-connected matroid M, and suppose that $e \in E(M) - X$. Then $X \cup \{e\}$ is 3-separating if and only if $e \in \mathrm{cl}(X) \cup \mathrm{cl}^*(X)$.*

Lemma 6. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M, and suppose that $|X| \geqslant 3$ and $e \in X$. Then $(X - \{e\}, Y \cup \{e\})$ is exactly 3-separating if and only if e is in exactly one of $\mathrm{cl}(X - \{e\}) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X - \{e\}) \cap \mathrm{cl}^*(Y)$.*

Lemma 7. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M. If $e \in C^*$ has the property that $\mathrm{cl}(C^*) - \{e\}$ contains a triangle of M/e, then $\mathrm{si}(M/e)$ is 3-connected.*

Lemma 8. *Let (X, Y) be a 3-separation of a 3-connected matroid M. If $X \cap \mathrm{cl}(Y) \neq \emptyset$ and $X \cap \mathrm{cl}^*(Y) \neq \emptyset$, then $|X \cap \mathrm{cl}(Y)| = |X \cap \mathrm{cl}^*(Y)| = 1$.*

Vertical connectivity

A k-separation (X, Y) of a matroid M is *vertical* if $\min\{r(X), r(Y)\} \geqslant k$. As noted in the introduction, we say a partition $(X, \{e\}, Y)$ of $E(M)$ is a *vertical 3-separation* of M if $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both vertical 3-separations of M and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$.

Furthermore, $Y \cup \{e\}$ is *maximal* if there is no vertical 3-separation $(X', \{e'\}, Y')$ of M such that $Y \cup \{e\}$ is a proper subset of $Y' \cup \{e'\}$. A k-separation (X, Y) of M is *cyclic* if both X and Y contain circuits. The next lemma gives a duality link between the cyclic k-separations and vertical k-separations of a k-connected matroid.

Lemma 9. *Let (X, Y) be a partition of the ground set of a k-connected matroid M. Then (X, Y) is a cyclic k-separation of M if and only if (X, Y) is a vertical k-separation of M^*.*

Proof. Suppose that (X, Y) is a cyclic k-separation of M. Then (X, Y) is a k-separation of M^*. Since (X, Y) is a k-separation of a k-connected matroid, (X, Y) is exactly k-separating, and so $r(X)+r(Y)-r(M) = k-1$. Therefore, as $r^*(X) = r(Y)+|X|-r(M)$, it follows that

$$r^*(X) = ((k-1) - r(X) + r(M)) + |X| - r(M) = (k-1) + |X| - r(X).$$

As X contains a circuit, X is dependent, so $|X| - r(M) \geqslant 1$. Hence $r^*(X) \geqslant k$. By symmetry, $r^*(Y) \geqslant k$, and so (X, Y) is a vertical k-separation of M^*. A similar argument establishes the converse. □

Following Lemma 9, we say a partition $(X, \{e\}, Y)$ of the ground set of a 3-connected matroid M is a *cyclic* 3*-separation* if $(X, \{e\}, Y)$ is a vertical 3-separation of M^*.

Of the next two results, the first combines Lemma 9 with a straightforward strengthening of [5, Lemma 3.1] and, in combination with Lemma 9, the second follows easily from Lemma 6.

Lemma 10. *Let M be a* 3*-connected matroid, and suppose that $e \in E(M)$. Then* $\mathrm{si}(M/e)$ *is not* 3*-connected if and only if M has a vertical* 3*-separation $(X, \{e\}, Y)$. Dually,* $\mathrm{co}(M\backslash e)$ *is not* 3*-connected if and only if M has a cyclic* 3*-separation $(X, \{e\}, Y)$.*

Lemma 11. *Let M be a* 3*-connected matroid. If $(X, \{e\}, Y)$ is a vertical* 3*-separation of M, then $(X - \mathrm{cl}(Y), \{e\}, \mathrm{cl}(Y) - e)$ is also a vertical* 3*-separation of M. Dually, if $(X, \{e\}, Y)$ is a cyclic* 3*-separation of M, then $(X - \mathrm{cl}^*(Y), \{e\}, \mathrm{cl}^*(Y) - \{e\})$ is also a cyclic* 3*-separation of M.*

Note that an immediate consequence of Lemma 11 is that if $(X, \{e\}, Y)$ is a vertical 3-separation such that $Y \cup \{e\}$ is maximal, then $Y \cup \{e\}$ must be closed. We will make repeated use of this fact.

3 Fans

Let M be a 3-connected matroid. A subset F of $E(M)$ with at least three elements is a *fan* if there is an ordering $(f_1, f_2, \ldots, f_k)$ of F such that

(i) for all $i \in \{1, 2, \ldots, k-2\}$, the triple $\{f_i, f_{i+1}, f_{i+2}\}$ is either a triangle or a triad, and

(ii) for all $i \in \{1, 2, \ldots, k-3\}$, if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad, while if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle.

If $k \geqslant 4$, then the elements f_1 and f_k are the *ends* of F. Furthermore, if $\{f_1, f_2, f_3\}$ is a triangle, then f_1 is a *spoke-end*; otherwise, f_1 is a *rim-end*. Observe that if F is a 4-element fan (f_1, f_2, f_3, f_4), then either f_1 or f_4 is the unique spoke-end of F depending on whether $\{f_1, f_2, f_3\}$ or $\{f_2, f_3, f_4\}$ is a triangle, respectively. The proof of the next lemma is straightforward and omitted.

Lemma 12. *Let M be a 3-connected matroid, and suppose that $F = (f_1, f_2, f_3, f_4)$ is a 4-element fan of M with spoke-end f_1. Then $(\{f_2, f_3, f_4\}, \{f_1\}, E(M) - F)$ is a vertical 3-separation of M provided $r(M) \geqslant 4$, in which case, $E(M) - \{f_2, f_3, f_4\}$ is maximal.*

We end this section by determining when an element in a fan of size at least four is elastic. For subsets X and Y of a matroid, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined by

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

Let M be a 3-connected matroid and let k be a positive integer. A *flower* Φ of M is an (ordered) partition $(P_1, P_2, \ldots, P_k)$ of $E(M)$ such that each P_i has at least two elements and is 3-separating, and each $P_i \cup P_{i+1}$ is 3-separating, where all subscripts are interpreted modulo k. If $k \geqslant 4$, we say Φ is *swirl-like* if $\bigcup_{i \in I} P_i$ is exactly 3-separating for all proper subsets I of $\{1, 2, \ldots, k\}$ whose members form a consecutive set in the cyclic order $(1, 2, \ldots, k)$, and

$$\sqcap(P_i, P_j) = \begin{cases} 1, & \text{if } P_i \text{ and } P_j \text{ are consecutive;} \\ 0, & \text{if } P_i \text{ and } P_j \text{ are not consecutive} \end{cases}$$

for all distinct $i, j \in \{1, 2, \ldots, k\}$. For further details of swirl-like flowers and, more generally flowers, we refer the reader to [4].

Lemma 13. *Let M be a 3-connected matroid such that $r(M), r^*(M) \geqslant 4$, and let $F = (f_1, f_2, \ldots, f_n)$ be a maximal fan of M.*

(i) *If $n \geqslant 6$, then F contains no elastic elements of M.*

(ii) *If $n = 5$, then F contains either exactly one elastic element, namely f_3, or no elastic elements of M.*

(iii) *If $n = 4$, then F contains either exactly two elastic elements, namely f_2 and f_3, or no elastic elements of M.*

Moreover, if $n \in \{4, 5\}$ and F contains no elastic elements, then M has a swirl-like flower $(A, \{f_1, f_2\}, F - \{f_1, f_2\}, B)$ as shown geometrically in Fig. 2.

Proof. It follows by Lemma 12 that the ends of a 4-element fan in M are not elastic. Thus, if $n \geqslant 6$, then, as every element of F is the end of a 4-element fan, F contains no elastic elements, and if $n = 5$, then, as every element of F, except f_3, is the end of a 4-element fan, F contains no elastic elements except possibly f_3. Thus (i) and (ii) hold. We next prove the lemma for when $n = 4$. The remaining part of the lemma for when $n = 5$ is proved similarly.

Suppose that $n = 4$ and either f_2 or f_3 is not elastic. Since (f_1, f_3, f_2, f_4) is also a fan ordering for F, we may assume that f_3 is not elastic. Up to duality, we may also assume that $\text{si}(M/f_3)$ is not 3-connected. Then, by Lemma 10,

$$(A \cup \{f_1, f_2\}, \{f_3\}, B \cup \{f_4\})$$

is a vertical 3-separation of M, where $|A| \geqslant 1$ and $|B| \geqslant 2$. Say $|A| = 1$, where $A = \{f_0\}$. Then $A \cup \{f_1, f_2\}$ is a triad, and so $(f_0, f_1, f_2, f_3, f_4)$ is a 5-element fan, contradicting the maximality of F. Thus $|A| \geqslant 2$. Since $A \cup B$ and $B \cup \{f_4\}$ are 3-separating in M, it follows by uncrossing that B is 3-separating in M. Similarly, A is 3-separating in M. Hence,

$$(A, \{f_1, f_2\}, \{f_3, f_4\}, B)$$

is a flower Φ. Since $\sqcap(\{f_1, f_2\}, \{f_3, f_4\}) = 1$, it follows by [4, Theorem 4.1] that

$$\sqcap(A, \{f_1, f_2\}) = \sqcap(\{f_3, f_4\}, B) = \sqcap(A, B) = 1.$$

To show that Φ is a swirl-like flower, it remains to show that

$$\sqcap(\{A, \{f_3, f_4\}) = \sqcap(B, \{f_1, f_2\}) = 0.$$

If $f_1 \notin \text{cl}(A)$, then, as $f_2 \notin \text{cl}(A \cup \{f_1\})$, it follows that $r(A \cup \{f_1, f_2\}) = r(A) + 2$. But then $\sqcap(A, \{f_1, f_2\}) = 0$, a contradiction. Thus $f_1 \in \text{cl}(A)$. Furthermore, $f_3 \notin \text{cl}(A)$. Assume that $f_4 \in \text{cl}(A \cup \{f_3\})$. Then, as $\sqcap(\{f_3, f_4\}, B) = 1$,

$$\begin{aligned}
1 &= r_{M/f_3}(A \cup \{f_1, f_2\}) + r_{M/f_3}(B \cup \{f_4\}) - r(M/f_3) \\
&= r_{M/f_3}(A \cup \{f_1, f_2, f_4\}) + r_{M/f_3}(B) - r(M/f_3) \\
&= r(A \cup F) - 1 + r(B) - (r(M) - 1) \\
&= r(A \cup F) + r(B) - r(M),
\end{aligned}$$

and so B is 2-separating in M, a contradiction. Thus $f_4 \notin \text{cl}(A \cup \{f_3\})$, and so $\sqcap(A, \{f_3, f_4\}) = 0$. To see that $\sqcap(B, \{f_1, f_2\}) = 0$, first assume that $f_1 \in \text{cl}(B)$. Then, as $f_1 \in \text{cl}(A)$,

$$\begin{aligned}
1 &= r_{M/f_3}(A \cup \{f_1, f_2\}) + r_{M/f_3}(B \cup \{f_4\}) - r(M/f_3) \\
&= r_{M/f_3}(A) + r_{M/f_3}(B \cup \{f_1, f_2, f_4\}) - r(M/f_3) \\
&= r(A) + r(B \cup F) - 1 - (r(M) - 1) \\
&= r(A) + r(B \cup F) - r(M),
\end{aligned}$$

and so A is 2-separating in M. This contradiction implies that $f_1 \notin \mathrm{cl}(B)$. It follows that $r(B \cup \{f_1, f_2\}) = r(B) + 2$, that is $\sqcap(B, \{f_1, f_2\}) = 0$. We deduce that $(A, \{f_1, f_2\}, \{f_3, f_4\}, B)$ is a swirl-like flower.

To complete the proof for when $n = 4$, since f_2 is contained in the triangle $\{f_1, f_2, f_3\}$ and F is maximal, it is easily seen that f_2 is contained in exactly one triad. Therefore $\mathrm{co}(M\backslash f_2) \cong M/f_3\backslash f_2$, and so, as $\mathrm{si}(M/f_3)$ is not 3-connected, $\mathrm{co}(M\backslash f_2)$ is not 3-connected. Hence, if f_3 is not elastic, f_2 is not elastic. In particular, if F contains an elastic element, then it contains exactly two elastic elements. $\square$

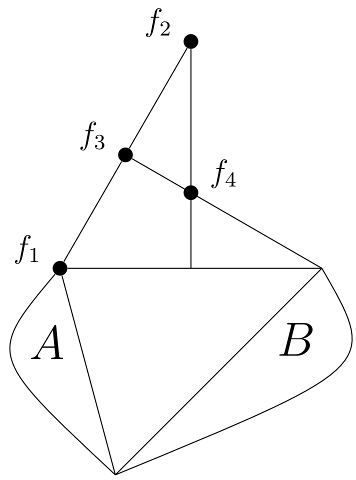

Figure 2: The swirl-like flower $(A, \{f_1, f_2\}, F - \{f_1, f_2\}, B)$ of Lemma 13.

4 Elastic Elements in Segments

Let M be a matroid. A subset L of $E(M)$ of size at least two is a *segment* if $M|L$ is isomorphic to a rank-2 uniform matroid. In this section we consider when an element in a segment is deletable or contractible. We begin with the following elementary lemma.

Lemma 14. *Let L be a segment of a 3-connected matroid M. If L has at least four elements, then $M\backslash \ell$ is 3-connected for all $\ell \in L$.*

In particular, Lemma 14 implies that, in a 3-connected matroid, every element of a segment with at least four elements is deletable. We next establish a sufficient condition for when almost every element of a segment in a 3-connected matroid is contractible.

Lemma 15. *Let M be a 3-connected matroid, and suppose that $L \cup \{w\}$ is a rank-3 cocircuit of M, where L is a segment. Then at least $|L| - 1$ elements of L are contractible.*

Proof. Suppose that the lemma does not hold, and let y_1 and y_2 be distinct elements of L that are not contractible. For each $i \in \{1, 2\}$, it follows by Lemma 10 that there exists a vertical 3-separation $(X_i, \{y_i\}, Y_i)$ of M such that $y_j \in Y_i$, where $\{i, j\} = \{1, 2\}$. By

Lemma 11, we may assume $Y_i \cup \{y_i\}$ is closed, in which case, $L - y_i \subseteq Y_i$. If $w \in Y_i$, then, as $L \cup \{w\}$ is a cocircuit, X_i is contained in the hyperplane $E(M) - (L \cup \{w\})$, and so $y_i \notin \mathrm{cl}(X_i)$. This contradiction implies that $w \in X_i$. Thus, for each $i \in \{1, 2\}$, we deduce that M has a vertical 3-separation

$$(U_i \cup \{w\}, \{y_i\}, V_i \cup (L - y_i)),$$

where $U_i, V_i \subseteq E(M) - (L \cup \{w\})$. Next we show the following.

15.1. For each $i \in \{1, 2\}$, we have $w \in \mathrm{cl}_M(U_i \cup \{y_i\}) - \mathrm{cl}_M(U_i)$.

Since $L \cup \{w\}$ is a cocircuit, the elements $y_i, w \notin \mathrm{cl}_M(U_i)$. But $y_i \in \mathrm{cl}_M(U_i \cup \{w\})$, and so $y_i \in \mathrm{cl}_M(U_i \cup \{w\}) - \mathrm{cl}_M(U_i)$. Thus, by the MacLane-Steinitz exchange property, $w \in \mathrm{cl}_M(U_i \cup \{y_i\}) - \mathrm{cl}_M(U_i)$.

15.2. For each $i \in \{1, 2\}$, we have $y_i \notin \mathrm{cl}_M(U_j \cup \{w\})$, where $\{i, j\} = \{1, 2\}$.

By Lemma 11,

$$(\mathrm{cl}(U_j \cup \{w\}) - \{y_j\}, \{y_j\}, (V_j \cup (L - y_j)) - \mathrm{cl}(U_j \cup \{w\}))$$

is a vertical 3-separation of M. If $y_i \in \mathrm{cl}(U_j \cup \{w\})$, then, as $y_j \in \mathrm{cl}(U_j \cup \{w\})$, the segment L is contained in $\mathrm{cl}(U_j \cup \{w\})$. Therefore $L \cup \{w\} \subseteq \mathrm{cl}(U_j \cup \{w\})$, and so $(V_j \cup (L - \{y_i\})) - \mathrm{cl}(U_j \cup \{w\}) = V_j - \mathrm{cl}(U_j \cup \{w\})$. Since $V_j - \mathrm{cl}(U_j \cup \{w\})$ is contained in the hyperplane $E(M) - (L \cup \{w\})$, it follows that $y_j \notin V_j - \mathrm{cl}(U_j \cup \{w\})$, a contradiction. Thus (15.2) holds.

Since M is 3-connected and $(U_i \cup \{w\}, \{y_i\}, V_i \cup (L - y_i))$ is a vertical 3-separation, it follows by (15.1) that

$$r(U_i) + r(V_i \cup L) - r(M \backslash w) = r(U_i \cup \{w\}) - 1 + r(V_i \cup L) - r(M) = 1.$$

Thus $(U_i, V_i \cup L)$ is a 2-separation of $M \backslash w$ for each $i \in \{1, 2\}$.

15.3. Either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

Consider the 2-connected matroid $M \backslash w$, and suppose that $U_1 \nsubseteq U_2$. Then $|(U_1 \cap (V_2 \cup L)| \geqslant 1$, and so, by uncrossing the two 2-separating sets U_1 and U_2, we deduce that $U_1 \cup V_2 \cup L$ is 2-separating in $M \backslash w$. But, by (15.1), $w \in \mathrm{cl}_M(U_1 \cup L)$ and so $U_1 \cup V_2 \cup (L \cup \{w\})$ is 2-separating in M. Since M is 3-connected, it follows that $|U_2 \cap (V_1 \cup L)| = 0$, that is, $U_2 \subseteq U_1$.

By (15.3), we may assume without loss of generality that $U_1 \subseteq U_2$. Thus

$$y_1 \in \mathrm{cl}(U_1 \cup \{w\}) \subseteq \mathrm{cl}(U_2 \cup \{w\}),$$

contradicting (15.2). This completes the proof of the lemma. □

Corollary 16. *Let L be a segment in a 3-connected matroid M, and suppose that L is not coclosed. Then at least $|L| - 2$ elements of L are contractible.*

Proof. Let E denote the ground set of M. The lemma certainly holds if $r(M) = 2$, so we may assume $r(M) \geqslant 3$. Since L is not coclosed, there exists an element $w \in \mathrm{cl}^*(L) - L$. Since $r(M) \geqslant 3$ and M is 3-connected, $w \notin \mathrm{cl}(L) - L$. Observing that $(L, E - (L \cup \{w\}))$ is a partition of $E - \{w\}$, it follows by Lemma 4 that $w \notin \mathrm{cl}(E - (L \cup \{w\}))$. If $|\mathrm{cl}(L) - L| \geqslant 2$, then L is contained in $\mathrm{cl}(E - (L \cup \{w\}))$ and so, as M is 3-connected, $w \in \mathrm{cl}(E - (L \cup \{w\}))$, a contradiction. Thus $|\mathrm{cl}(L) - L| \leqslant 1$. Furthermore, if $\ell \in \mathrm{cl}(L) - L$, then $\ell \in \mathrm{cl}(E - (\mathrm{cl}(L) \cup \{w\}))$. To see this, observe that $r(\mathrm{cl}(L) \cup w) = 3$ and so, as M is 3-connected, $r(E - (\mathrm{cl}(L) \cup w)) = r(M) - 1$. If $\ell \notin \mathrm{cl}(E - (\mathrm{cl}(L) \cup \{w\}))$, then $\mathrm{cl}((E - (\mathrm{cl}(L) \cup \{w\})) \cup \{\ell\}) = r(M)$, and so $w \in \mathrm{cl}(E - (L \cup \{w\}))$, a contradiction. It now follows that either

$$|L \cap \mathrm{cl}(E(M) - (L \cup \{w\}))| = 0,$$

in which case $L \cup \{w\}$ is a cocircuit, or

$$|L \cap \mathrm{cl}(E(M) - (L \cup \{w\}))| = 1,$$

in which case $(L - \{\ell\}) \cup \{w\}$ is a cocircuit for some $\ell \in L$. Note that, in the latter case, $|L| \geqslant 3$; otherwise, M has a series pair consisting of the unique element in $L - \{\ell\}$ and w. The corollary now follows from Lemma 15. □

Combining Corollary 16 with Lemma 14 gives the following result.

Corollary 17. *Let L be a segment with at least four elements in a 3-connected matroid M. If L is not coclosed, then at least $|L| - 2$ elements of L are elastic.*

5 Proofs of Theorem 1 and Corollary 2

In this section, we prove Theorem 1 and Corollary 2. However, almost all of the section consists of the proof of Theorem 1. The proof of this theorem is essentially partitioned into two lemmas, Lemmas 19 and 20. Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is maximal, and suppose that $X \cup \{e\}$ is not a 4-element fan. Lemma 19 establishes Theorem 1 for when X contains at least one non-contractible element, while Lemma 20 establishes the theorem for when every element in X is contractible.

To prove Lemma 19, we will make use of the following technical result which is extracted from the proof of Lemma 3.2 in [5].

Lemma 18. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{e_1\}, Y_1)$ such that $Y_1 \cup \{e_1\}$ is maximal. Suppose that $(X_2, \{e_2\}, Y_2)$ is a vertical 3-separation of M such that $e_2 \in X_1$, $e_1 \in Y_2$, and $Y_2 \cup \{e_2\}$ is closed. Then each of the following holds:*

(i) *None of $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ are empty.*

(ii) $r((X_1 \cap X_2) \cup \{e_2\}) = 2$.

(iii) *If $|Y_1 \cap X_2| = 1$, then X_2 is a rank-3 cocircuit.*

(iv) *If* $|Y_1 \cap X_2| \geqslant 2$*, then* $r((X_1 \cap Y_2) \cup \{e_1, e_2\}) = 2$.

Lemma 19. *Let* M *be a* 3*-connected matroid with a vertical* 3*-separation* $(X_1, \{e_1\}, Y_1)$ *such that* $Y_1 \cup \{e_1\}$ *is maximal, and suppose that* $X_1 \cup \{e_1\}$ *is not a* 4*-element fan. If at least one element of* X_1 *is not contractible, then* X_1 *contains at least two elastic elements.*

Proof. Let e_2 be an element of X_1 that is not contractible. Then, by Lemma 10, there exists a vertical 3-separation $(X_2, \{e_2\}, Y_2)$ of M. Without loss of generality, we may assume $e_1 \in Y_2$. Furthermore, by Lemma 11, we may also assume that $Y_2 \cup \{e_2\}$ is closed. By Lemma 18, each of $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ is non-empty. The proof is partitioned into two cases depending on the size of $Y_1 \cap X_2$. Both cases use the following:

19.1. If $X_1 \cap X_2$ contains two contractible elements, then either X_1 has at least two elastic elements, or $|X_1 \cap X_2| = 2$ and there exists a triangle $\{x, y_1, y_2\}$, where $x \in X_1 \cap X_2$, $y_1 \in Y_1 \cap X_2$, and $y_2 \in X_1 \cap Y_2$.

By Lemma 18(ii), $r((X_1 \cap X_2) \cup \{e_2\}) = 2$. Let x_1 and x_2 be distinct contractible elements of $X_1 \cap X_2$. If $|X_1 \cap X_2| \geqslant 3$, then, by Lemma 14 each of x_1 and x_2 is elastic. Thus we may assume that $|X_1 \cap X_2| = 2$ and that either x_1 or x_2, say x_1, is not deletable. Let (W, Z) be a 2-separation of $M \backslash x_1$ such that neither $r^*(W) = 1$ nor $r^*(Z) = 1$. Since x_1 is not deletable, such a separation exists. Observe that $|W|, |Z| \geqslant 3$; otherwise, either W or Z is a series pair. If $x_1 \in \mathrm{cl}(W)$ or $x_1 \in \mathrm{cl}(Z)$, then either $(W \cup \{x_1\}, Z)$ or $(W, Z \cup \{x_1\})$, respectively, is a 2-separation of M, a contradiction. So $\{x_2, e_2\} \not\subseteq W$ and $\{x_2, e_2\} \not\subseteq Z$. Therefore, without loss of generality, we may assume $x_2 \in W - \mathrm{cl}(Z)$ and $e_2 \in Z - \mathrm{cl}(W)$. Since (W, Z) is a 2-separation of $M \backslash x_1$ and $x_2 \notin \mathrm{cl}(Z)$, we deduce that $(W - \{x_2\}, Z \cup \{x_1\})$ is a 2-separation of M / x_2. Thus, as x_2 is contractible, $\mathrm{si}(M/x_2)$ is 3-connected, and so $r(W) = 2$. In turn, as $Y_1 \cup \{e_1\}$ and $Y_2 \cup \{e_2\}$ are both closed, this implies that $|W \cap (Y_1 \cup \{e_1\})| \leqslant 1$ and $|W \cap (Y_2 \cup \{e_2\})| \leqslant 1$; otherwise, $W \subseteq Y_1 \cup \{e_1\}$ or $W \subseteq Y_2 \cup \{e_2\}$. Thus $|W| = 3$ and, in particular, W is the desired triangle. Hence 19.1 holds.

We now distinguish two cases depending on the size of $Y_1 \cap X_2$:

(I) $|Y_1 \cap X_2| = 1$; and

(II) $|Y_1 \cap X_2| \geqslant 2$.

Consider (I). Let w be the unique element in $Y_1 \cap X_2$. By Lemma 18, $(X_1 \cap X_2) \cup \{e_2\}$ is a segment and $(X_1 \cap X_2) \cup \{w\}$ is a rank-3 cocircuit. Let $L_1 = (X_1 \cap X_2) \cup \{e_2\}$. If $|L_1| \geqslant 4$, then, as $w \in \mathrm{cl}^*(L_1)$ and e_2 is not elastic, it follows by Corollary 17 that X_1 contains at least two elastic elements. Thus, as $|Y_1 \cap X_2| = 1$, we may assume L_1 is closed and $|L_1| = 3$, and so $(L_1 - \{e_2\}) \cup \{w\}$ is a triad. Let $L_1 = \{x_1, x_2, e_2\}$ and let $\{i, j\} = \{1, 2\}$.

19.2. For each $i \in \{1, 2\}$, the element x_i is contractible.

If x_i is not contractible, then, by Lemma 10, M has a vertical 3-separation $(U_i, \{x_i\}, V_i)$, where $e_1 \in V_i$. By Lemma 11, we may assume that $V_i \cup x_i$ is closed. By Lemma 18, $Y_1 \cap U_i$ is non-empty and $r((X_1 \cap U_i) \cup \{x_i\}) = 2$. First assume that $|Y_1 \cap U_i| = 1$.

Then $|(X_1 \cap U_i) \cup \{x_i\}| \geqslant 3$, and so x_i is contained in a triangle $T \subseteq (X_1 \cap U_i) \cup \{x_i\}$. If $x_j \in V_i$, then, as $V_i \cup \{x_i\}$ is closed, $e_2 \in V_i$. Thus $x_j, e_2 \notin T$ and so, by orthogonality, as $\{x_i, x_j, w\}$ is a triad, $w \in T$. This contradicts $w \in Y_1$. It now follows that $x_j \in X_1 \cap U_i$ and so $e_2 \in X_1 \cap U_i$. Thus, as L_1 is closed and $L_1 \subseteq (X_1 \cap U_i) \cup \{x_i\}$, we have $|(X_1 \cap U_i) \cup \{x_i\}| = 3$, and therefore $T = \{x_1, x_2, e_2\}$. Let z be the unique element in $Y_1 \cap U_i$. Then, by Lemma 18 again, $\{x_j, e_2, z\}$ is a triad, and so $z \in \text{cl}^*(X_1)$. Furthermore, $w \in \text{cl}^*(X_1)$ and $e_1 \in \text{cl}(X_1)$, and so, by Lemma 8, we deduce that $z = w$. This implies that $Y_2 = V_i$. But then $\text{cl}(Y_2 \cup \{e_2\})$ contains x_i, contradicting that $Y_2 \cup \{e_2\}$ is closed.

Now assume that $|Y_1 \cap U_i| \geqslant 2$. By Lemma 18, $r((X_1 \cap V_i) \cup \{x_i, e_1\}) = 2$. If $x_j \in V_i$, then, as $V_i \cup \{x_i\}$ is closed, $e_2 \in X_1 \cap V_i$, and so $\{x_j, e_1, e_2\}$ is a triangle. Since $\{x_1, x_2, w\}$ is a triad, this contradicts orthogonality. Thus $x_j \in U_i$. Also, $e_2 \in U_i$; otherwise, as $V_i \cup \{x_i\}$ is closed, $x_j \in V_i$, a contradiction. By Lemma 18, $X_1 \cap V_i$ is non-empty, and so M has a triangle $T' = \{x_i, e_1, y\}$, where $y \in X_1 \cap V_i$. As $\{x_i, x_j, w\}$ is a triad, T' contradicts orthogonality unless $y = w$. But $w \in Y_1$ and therefore cannot be in $X_1 \cap V_i$. Hence x_i is contractible, and so (19.2) holds.

Since x_1 and x_2 are both contractible, it follows by (19.1) that either X_1 contains two elastic elements or w is in a triangle with two elements of X_1. If the latter holds, then $w \in \text{cl}(X_1)$. As $\{x_1, x_2, w\}$ is a triad and $(Y_1 \cup \{e_1\}) - \{w\}$ is contained in $Y_2 \cup e_2$, it follows that $w \notin \text{cl}((Y_1 \cup \{e_1\}) - \{w\})$. Therefore

$$(X_1 \cup \{w\}, (Y_1 \cup \{e_1\}) - \{w\})$$

is a 2-separation of M, a contradiction. Thus X_1 contains two elastic elements. This concludes (I).

Now consider (II). Let $L_1 = (X_1 \cap X_2) \cup \{e_2\}$ and $L_2 = (X_1 \cap Y_2) \cup \{e_1, e_2\}$. By parts (ii) and (iv) of Lemma 18, L_1 and L_2 are both segments. Since M is 3-connected, X_1 is 3-separating, and $Y_1 \cup \{e_1\}$ is closed, it follows that X_1 is a rank-3 cocircuit of M. Say $|L_2| \geqslant 4$. If $|L_1| \geqslant 3$, then, by Lemma 7, each element of $L_2 - \{e_1, e_2\}$ is contractible. Moreover, as $|L_2| \geqslant 4$, Lemma 14 implies that each element of $L_2 - \{e_1, e_2\}$ is deletable, and so each element of $L_2 - \{e_1, e_2\}$ is elastic. Since $|L_2| \geqslant 4$, it follows that X_1 has at least two elastic elements. Thus we may assume that $|L_1| = 2$, that is $|X_1 \cap X_2| = 1$. Then, as $\text{cl}(Y_1 \cup \{e_1\}) \cap (X_1 \cap X_2)$ is empty, it follows by Lemma 4 that the unique element in $L_1 - \{e_2\}$ is contained in $\text{cl}^*(L_2)$, and so L_2 is not coclosed. Thus, as $|L_2| \geqslant 4$ and e_1 is not elastic, we deduce by Corollary 17 that X_1 has at least two elastic elements. Hence, as $X_1 \cap Y_2$ is non-empty, we may now assume that $|L_2| = 3$.

Let $L_2 = \{e_2, a, e_1\}$. If $|X_1 \cap X_2| = 1$, then $|X_1| = 3$, and so X_1 is a triad. In turn, this implies that $X_1 \cup \{e_1\}$ is a 4-element fan, a contradiction. Thus $|X_1 \cap X_2| \geqslant 2$. Let x_1 and x_2 be distinct elements in $X_1 \cap X_2$. Since $\{e_1, a, e_2\}$ is a triangle in M/x_i for each $i \in \{1, 2\}$, it follows by Lemma 7 that x_i is contractible for each $i \in \{1, 2\}$. Thus, by (19.1), either X_1 contains two elastic elements, or $X_1 \cap X_2 = \{x_1, x_2\}$ and a is in a triangle with two elements of X_2. The latter implies that $a \in \text{cl}(X_2 \cup \{e_2\})$. As $a \notin \text{cl}(Y_1 \cup \{e_1\})$ and $Y_2 - \{a\}$ is contained in $Y_1 \cup \{e_1\}$, it follows that $a \notin \text{cl}(Y_2 - \{a\})$. Hence

$$(X_2 \cup \{a, e_2\}, Y_2 - \{a\})$$

is a 2-separation of M, a contradiction. Thus X_1 contains two elastic elements. This concludes (II) and the proof of the lemma. □

Lemma 20. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{e_1\}, Y_1)$ such that $Y_1 \cup \{e_1\}$ is maximal, and suppose that $X_1 \cup \{e_1\}$ is not a 4-element fan. If every element of X_1 is contractible, then X_1 contains at least two elastic elements.*

Proof. First suppose that X_1 is independent. Then, as $r(X_1) = |X_1|$ and $\lambda(X_1) = r(X_1) + r^*(X_1) - |X_1|$, we have $r^*(X_1) = 2$. That is, X_1 is a segment in M^*. Therefore, as $e_1 \in \mathrm{cl}(X_1)$, it follows by the dual of Corollary 17 that, if $|X_1| \geqslant 4$, then X_1 has at least two elastic elements. Furthermore, if $|X_1| = 3$, then, as $X_1 \cup \{e_1\}$ is not a 4-element fan, $X_1 \cup \{e_1\}$ is a circuit. Thus, $X_1 \cup \{e_1\}$ is a rank-3 cocircuit of M_1^*, where X_1 is a segment. Therefore, by Lemma 15, at least two elements of X_1 are contractible in M^*. In turn, this implies that at least two elements of X_1 are deletable in M. Hence, again, X_1 has at least two elastic elements.

Now suppose that X_1 is dependent, and let C be a circuit in X_1. As M is 3-connected, $|C| \geqslant 3$. If every element in C is deletable, then X_1 contains at least two elastic elements. Thus we may assume that there is an element, say g, in C that is not deletable. By Lemma 10, there exists a cyclic 3-separation $(U, \{g\}, V)$ in M, where $e_1 \in V$. By Lemma 11, we may also assume that $V \cup \{g\}$ is coclosed. Note that, as $(U, \{g\}, V)$ is a cyclic 3-separation, $r^*(U) \geqslant 3$, and so $|U| \geqslant 3$.

We next show that

20.1. $|X_1 \cap U|, |X_1 \cap V| \geqslant 2$.

If either $C - \{g\} \subseteq U$ or $C - \{g\} \subseteq V$, then $g \in \mathrm{cl}(U)$ or $g \in \mathrm{cl}(V)$, respectively, in which case either $(U \cup \{g\}, V)$ or $(U, V \cup \{g\})$ is a 2-separation of M, a contradiction. Thus $C \cap (X_1 \cap U)$ and $C \cap (X_1 \cap V)$ are both non-empty, and so $|X_1 \cap U|, |X_1 \cap V| \geqslant 1$. Say $X_1 \cap U = \{g'\}$, where $g' \in C$. Since C is a circuit, $g \in \mathrm{cl}_{M/g'}(V)$. Therefore, as $Y_1 \cup \{e_1\}$ is closed and so $g' \notin \mathrm{cl}(Y_1)$, and (U, V) is a 2-separation of $M\backslash g$, we have

$$\begin{aligned}
\lambda_{M/g'}(U \cap Y_1) &= r_{M/g'}(U \cap Y_1) + r_{M/g'}(V \cup \{g\}) - r(M/g') \\
&= r_M(U \cap Y_1) + r_M(V) - (r(M) - 1) \\
&= r_M(U \cap Y_1) + r_M(V) - r(M\backslash g) + 1 \\
&= r_M(U) - 1 + r_M(V) - r(M\backslash g) + 1 \\
&= r_M(U) + r_M(V) - r(M\backslash g) \\
&= 1.
\end{aligned}$$

Thus $(U \cap Y_1, V \cup \{g\})$ is a 2-separation of M/g'. Since every element in X_1 is contractible, g' is contractible, and so $r(U) = 2$. Since $|U| \geqslant 3$, it follows that $|U \cap Y_1| \geqslant 2$, and so $g' \in \mathrm{cl}(Y_1 \cup \{e_1\})$, a contradiction as $Y_1 \cup \{e_1\}$ is closed. Hence $|X_1 \cap U| \geqslant 2$. An identical argument interchanging the roles of U and V establishes that $|X_1 \cap V| \geqslant 2$, thereby establishing (20.1).

20.2. If $|Y_1 \cap U| \geqslant 2$, then X_1 has at least two elastic elements.

Say $|Y_1 \cap U| \geqslant 2$. It follows by two applications of uncrossing that each of $X_1 \cap V$ and $(X_1 \cap V) \cup \{e_1\}$ is 3-separating. Since $|X_1 \cap V| \geqslant 2$ and M is 3-connected, $X_1 \cap V$ and $(X_1 \cap V) \cup \{e_1\}$ are exactly 3-separating. Therefore, by Lemma 5, $e_1 \in \mathrm{cl}(X_1 \cap V)$ or $e_1 \in \mathrm{cl}^*(X_1 \cap V)$. Since $e_1 \in \mathrm{cl}(Y_1)$, it follows that $e_1 \in \mathrm{cl}(E - ((X_1 \cap V) \cup \{e_1\}))$ and so, by Lemma 4, $e_1 \notin \mathrm{cl}^*(X_1 \cap V)$. So $e_1 \in \mathrm{cl}(X_1 \cap V)$. Thus, if $r(X_1 \cap V) \geqslant 3$, then $(X_1 \cap V, \{e_1\}, Y_1 \cup U)$ is a vertical 3-separation, contradicting the maximality of $Y_1 \cup \{e_1\}$. Therefore $r(X_1 \cap V) = r((X_1 \cap V) \cup \{e_1\}) = 2$. If $|(X_1 \cap V) \cup \{e_1\}| \geqslant 4$, then, as e_1 is not contractible, it follows by Corollary 17 that $X_1 \cap V$, and therefore X_1, contains at least two elastic elements. Thus we may assume that $|(X_1 \cap V) \cup \{e_1\}| = 3$. Again, as $|Y_1 \cap U| \geqslant 2$, an application of uncrossing implies $(X_1 \cap V) \cup \{g\}$ is 3-separating. Since $X_1 \cap V$ is exactly 3-separating and $g \notin \mathrm{cl}(X_1 \cap V)$, it follows by Lemma 5 that $g \in \mathrm{cl}^*(X_1 \cap V)$. Therefore $(X_1 \cap V) \cup \{g\}$ is a triad, and so $(X_1 \cap V) \cup \{e_1, g\}$ is a 4-element fan with spoke-end e_1. But then, by Lemma 12, $((X_1 \cap V) \cup \{g\}, \{e_1\}, E - ((X_1 \cap V) \cup \{e_1, g\}))$ is a vertical 3-separation that contradicts the maximality of $Y_1 \cup \{e_1\}$. Hence (20.2) holds.

By (20.2), we may assume that $|Y_1 \cap U| \leqslant 1$. Say $Y_1 \cap U$ is empty. Then $U \subseteq X_1$. Let $(U', \{h\}, V')$ be a cyclic 3-separation of M such that $V \cup \{g\} \subseteq V' \cup \{h\}$ with the property that there is no other cyclic 3-separation $(U'', \{h'\}, V'')$ in which $V' \cup \{h\}$ is a proper subset of $V'' \cup \{h'\}$. Observe that such a cyclic 3-separation exists as we can choose $(U, \{g\}, V)$ if necessary. If every element in U' is deletable, then, as $U' \subseteq X_1$ and $|U'| \geqslant 3$, it follows that X_1 has at least two elastic elements. Thus we may assume that there is an element in U' that is not deletable. By the dual of Lemma 19, either U', and thus X_1, contains at least two elastic elements or $U' \cup \{h\}$ is a 4-element fan. If the latter holds, then, by Lemma 12,

$$((U' \cup \{h\}) - \{f\}, \{f\}, E - (U' \cup \{h\}))$$

is a vertical 3-separation, where f is the spoke-end of the 4-element fan $U' \cup \{h\}$. But then, as $X_1 \cap V$ is non-empty, $Y_1 \cup \{e_1\}$ is properly contained in $E - (U' \cup \{h\})$, contradicting maximality. Hence we may assume that $|Y_1 \cap U| = 1$.

Let $Y_1 \cap U = \{y\}$. Since $|Y_1 \cap U| = 1$, we have $|Y_1 \cap V| \geqslant 2$ and so, by two applications of uncrossing, $X_1 \cap U$ and $(X_1 \cap U) \cup \{g\}$ are both 3-separating. Since M is 3-connected and $|X_1 \cap U| \geqslant 2$, these sets are exactly 3-separating. If $y \notin \mathrm{cl}(X_1 \cap U)$, then, by Lemma 4, $y \in \mathrm{cl}^*(V \cup \{g\})$. But then $V \cup \{g\}$ is not coclosed, a contradiction. Thus $y \in \mathrm{cl}(X_1 \cap U)$, and so $y \in \mathrm{cl}((X_1 \cap U) \cup \{g\})$. Now $y \notin \mathrm{cl}^*(V \cup \{g\})$, and so $y \notin \mathrm{cl}^*(V)$. Hence as $(X_1 \cap U) \cup \{g\}$ and, therefore, the complement $V \cup \{y\}$ is 3-separating, Lemma 5 implies that $y \in \mathrm{cl}(V)$. Therefore, as $(X_1 \cap U) \cup \{g\}$ and V each have rank at least three, it follows that $((X_1 \cap U) \cup \{g\}, \{y\}, V)$ is a vertical 3-separation of M. Note that $r(V) \geqslant 3$; otherwise, $(X_1 \cap V) \subseteq \mathrm{cl}(\{y, e_1\})$, in which case, $Y_1 \cup \{e_1\}$ is not closed. But $(X_1 \cap U) \cup \{g\}$ is a proper subset of X_1, a contradiction to the maximality of $Y_1 \cup \{e_1\}$. This last contradiction completes the proof of the lemma. □

We now combine Lemmas 19 and 20 to prove Theorem 1.

Proof of Theorem 1. Let $(X, \{e\}, Y)$ be a vertical 3-separation of M, where $Y \cup \{e\}$ is maximal, and suppose that $X \cup \{e\}$ is not a 4-element fan. If at least one element in X

is not contractible, then, by Lemma 19, X contains at least two elastic elements. On the other hand, if every element in X is contractible, then, by Lemma 20 X again contains at least two elastic elements, thereby completing the proof of the theorem. □

We end the paper by establishing Corollary 2.

Proof of Corollary 2. Let M be a 3-connected matroid. If every element of M is elastic, then the corollary holds. Therefore suppose that M has at least one non-elastic element, e say. Up to duality, we may assume that $\mathrm{si}(M/e)$ is not 3-connected. Then, by Lemma 10, M has a vertical 3-separation $(X, \{e\}, Y)$. As $r(X), r(Y) \geqslant 3$, this implies that $|E(M)| \geqslant 7$, and so we deduce that every element in a 3-connected matroid with at most six elements is elastic. Now let $(X', \{e'\}, Y')$ be a vertical 3-separation such that $Y' \cup \{e'\}$ is maximal and contains $Y \cup \{e\}$. Then it follows by Theorem 1 that X', and hence X, contains at least two elastic elements. But an identical argument, interchanging the roles of X and Y, gives us that Y also contains at least two elastic elements. Thus, M contains at least four elastic elements. □

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Corrigendum – Added December 3, 2021

Theorem 1 and Corollary 2 are incorrect. In particular, as well as 4-element fans, there is a family of matroids which provide obstacles to having elastic elements. The error lies in the proof of Lemma 15; this lemma is not true. A corrected version of the paper is available on the arXiv:

arXiv:2010.01797

For the purposes of clarification, Theorem 1 and Corollary 2 should be replaced with their namesakes below.

Let $n \geqslant 3$, and let $Z = \{z_1, z_2, \ldots, z_n\}$ be a basis of $PG(n-1, \mathbb{R})$. Suppose that L is a line that is freely placed relative to Z. For each $i \in \{1, 2, \ldots, n\}$, let w_i be the unique point of L contained in the hyperplane spanned by $Z - \{z_i\}$. Let $W = \{w_1, w_2, \ldots, w_n\}$, and let Θ_n denote the restriction of $PG(n-1, \mathbb{R})$ to $W \cup Z$. Note that Θ_n is 3-connected and Z is a corank-2 subset of Θ_n. For all $i \in \{1, 2, \ldots, n\}$, we denote the matroid $\Theta_n \backslash w_i$ by Θ_n^-. The matroid Θ_n^- is well defined as, up to isomorphism, $\Theta_n \backslash w_i \cong \Theta_n \backslash w_j$ for all $i, j \in \{1, 2, \ldots, n\}$. A more formal definition of Θ_n is given in the corrected version of the paper. If $n = 3$, then Θ_3 is isomorphic to $M(K_4)$. However, for all $n \geqslant 4$, the matroid Θ_n has no 4-element fans and no elastic elements.

Let M be a 3-connected matroid, and let A and B be rank-2 and corank-2 subsets of $E(M)$. We say that $A \cup B$ is a Θ-*separator* of M if $r(M) \geqslant 4$ and $r^*(M) \geqslant 4$, and either $M|(A \cup B)$ or $M^*|(A \cup B)$ is isomorphic to one of the matroids Θ_n and Θ_n^- for some $n \geqslant 3$.

Theorem 1. *Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is maximal. Then at least one of the following holds:*

(i) *X contains at least two elastic elements;*

(ii) *$X \cup \{e\}$ is a 4-element fan; or*

(iii) *X is contained in a Θ-separator.*

Corollary 2. *Let M be a 3-connected matroid. If $|E(M)| \geqslant 7$, then M contains at least four elastic elements provided M has no 4-element fans and no Θ-separators. Moreover, if $|E(M)| \leqslant 6$, then every element of M is elastic.*