On the e-positivity of $(claw, 2K_2)$-free graphs

Grace M. X. Li   Arthur L. B. Yang*

Center for Combinatorics, LPMC
Nankai University
Tianjin 300071, P. R. China

limengxing@mail.nankai.edu.cn   yang@nankai.edu.cn

Submitted: Sep 29, 2020; Accepted: Jun 1, 2020; Published: Jun 18, 2021

Abstract

Motivated by Stanley and Stembridge's conjecture about the e-positivity of claw-free incomparability graphs, Hamel and her collaborators studied the e-positivity of $(claw, H)$-free graphs, where H is a four-vertex graph. In this paper we establish the e-positivity of generalized pyramid graphs and $2K_2$-free unit interval graphs, which are two important families of $(claw, 2K_2)$-free graphs. Hence we affirmatively solve one problem proposed by Hamel, Hoàng and Tuero, and another problem considered by Foley, Hoàng and Merkel.

Mathematics Subject Classifications: 05E05, 05C15

1 Introduction

Given a finite simple graph G with vertex set V and edge set E, a proper coloring of G is a function κ from V to $\mathbb{P} = \{1, 2, \ldots\}$ such that $\kappa(u) \neq \kappa(v)$ whenever $uv \in E$. Stanley [13] defined the chromatic symmetric function X_G as

$$X_G = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)}, \tag{1}$$

where κ ranges over all proper colorings of G. It is clear that X_G is a homogeneous symmetric function of degree n, where n is the cardinality of V. There have been many works focusing on the expansion of X_G in terms of various bases of symmetric functions. A well known basis is composed of elementary symmetric functions which are indexed by integer partitions. Recall that an integer partition of n is a weakly decreasing sequence $\lambda =$

*Supported in part by the Fundamental Research Funds for the Central Universities and the National Science Foundation of China (Nos. 11522110, 11971249).

$(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$, denoted by $\lambda \vdash n$. Sometimes we consider λ as an infinite sequence by appending infinite 0's. The elementary symmetric function e_λ is defined as

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k},$$

where

$$e_0 = 1 \text{ and } e_i = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i} \text{ for } i \geqslant 1.$$

It is well known that the set $\{e_\lambda \mid \lambda \vdash n\}$ forms a basis of homogeneous symmetric functions of degree n. A celebrated conjecture of Stanley and Stembridge states that the chromatic symmetric function X_G of a claw-free incomparability graph G is e-positive, namely, X_G can be written as a nonnegative linear combination of e_λ's, see [13] and [15]. If X_G is e-positive, we also say that G is e-positive for convenience. Stanley and Stembridge's conjecture has been extensively studied, see for instance [1, 3, 4, 9, 12]. The main objective of this paper is to prove the e-positivity of two families of $(claw, 2K_2)$-free graphs.

Let us first recall some related concepts and give an overview of some background. Let H be a set of graphs. A graph G is said to be H-free if it does not contain any graph of H as an induced subgraph. Hamel, Hoàng and Tuero [8] studied the e-positivity of H-free graphs, where H is composed of one claw and another four-vertex graph. There are eleven graphs on four vertices, see Figure 1. Concerning the e-positivity of $(claw, F)$-free

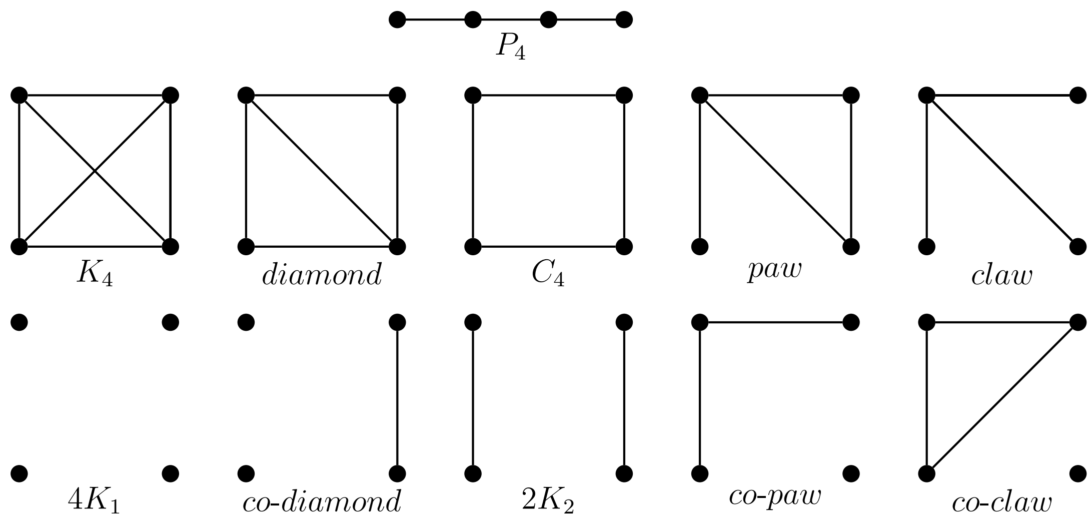

Figure 1: List of four-vertex graphs.

graphs with F being a four-vertex graph other than claw, some progress has been made. Tsujie [16] proved the e-positivity for the case $F = P_4$. Hamel, Hoàng and Tuero proved the e-positivity for $F = paw$ and $F = co\text{-}paw$. They also showed that a $(claw, F)$-free graph is not necessarily e-positive if F is a diamond, co-claw, K_4, $4K_1$, $2K_2$ or C_4. It remains to study the case that F is a co-diamond, and Hamel, Hoàng and Tuero proposed the following open problem.

Problem 1 ([8, Open problem 6.2])**.** Are $(claw, \textit{co-diamond})$-free graphs e-positive?

By considering the structure of $(claw, \textit{co-diamond})$-free graphs, they reduced the above problem to determine the e-positivity of certain peculiar graphs, as illustrated in [8, Figure 3].

They further explored the e-positivity of $(claw, \textit{co-diamond}, F)$-free graphs where F is a four-vertex graph. The e-positivity of $(claw, \textit{co-diamond}, F)$-free graphs is unknown for the cases $F = C_4$, $F = 2K_2$ and $F = K_4$. Hamel, Hoàng and Tuero showed that if a peculiar graph is $(claw, \textit{co-diamond}, 2K_2)$-free, then it can be characterized as a generalized pyramid $\mathrm{GP}(r, s, t)$, as illustrated in Figure 2, where a, b, c are three pairwise nonadjacent vertices, the vertices of $S_{a,b}$ ($S_{a,c}$ or $S_{b,c}$) form a clique of size r (resp. s or t), and each vertex of $S_{a,b}$ ($S_{a,c}$ or $S_{b,c}$) is adjacent to every vertex of $\mathrm{GP}(r, s, t)$ other than c (resp. b or a). In particular, they came up with the following problem.

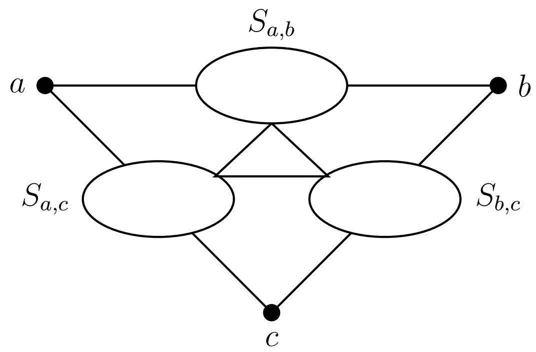

Figure 2: The generalized pyramid graph $\mathrm{GP}(r, s, t)$.

Problem 2 ([8, Open problem 6.1])**.** Are generalized pyramids e-positive?

In this paper we give an affirmative answer to this problem.

The second part of this paper is devoted to the study of the e-positivity of $2K_2$-free unit interval graphs. Guay-Paquet [7] proved that if unit interval graphs are e-positive, then any claw-free incomparability graph G is e-positive, as conjectured by Stanley and Stembridge. Based on Guay-Paquet's work, Foley, Hoàng and Merkel [5] considered the e-positivity of F-free unit interval graphs, where F is a four-vertex graph. It was shown that for any four-vertex graph F other than *co-diamond*, K_4, $4K_1$ and $2K_2$, each F-free unit interval graph is e-positive. Foley, Hoàng and Merkel proved some special cases of $2K_2$-free unit interval graphs are e-positive. In this paper we show that any $2K_2$-free unit interval graph is e-positive, which provides further evidence in favor of Stanley and Stembridge's conjecture.

The paper is organized as follows. In Section 2 we prove the e-positivity of generalized pyramid graphs based on the monomial expansion of the corresponding chromatic symmetric functions. In Section 3 we prove the e-positivity of $2K_2$-free unit interval graphs by showing that such graphs must be co-triangle free graphs or generalized bull graphs.

2 Generalized pyramid graphs

This section is devoted to proving the e-positivity of the generalized pyramid graphs $\mathrm{GP}(r,s,t)$. By using Stanley's result on the monomial expansion of the chromatic symmetric function of a graph, we first obtain the monomial expression of $X_{\mathrm{GP}(r,s,t)}$. Then based on the transition matrix between the monomial basis and the elementary basis, we explicitly determine the coefficients in the expansion of $X_{\mathrm{GP}(r,s,t)}$ in terms of elementary symmetric functions. Finally, we prove that all these coefficients are nonnegative.

Now let us recall some related definitions and results. Given an integer partition λ, the monomial symmetric function m_λ is defined as

$$m_\lambda = \sum_{\alpha} x^\alpha,$$

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ arranges over all distinct permutations of $\lambda = (\lambda_1, \lambda_2, \ldots)$. If λ has r_i parts equal to i, we also use $\langle 1^{r_1} 2^{r_2} \ldots \rangle$ to represent λ. The augmented monomial symmetric function $\tilde{m}_\lambda$ is defined as

$$\tilde{m}_\lambda = r_1! r_2! \cdots m_\lambda.$$

It is clear that $\{m_\lambda \mid \lambda \vdash n\}$ forms a basis of homogeneous symmetric functions of degree n, and hence so does $\{\tilde{m}_\lambda \mid \lambda \vdash n\}$. Let G be a graph with vertex set V and edge set E. By using the notion of stable partitions of G, Stanley [13] gave a combinatorial interpretation of the coefficients in the expansion of X_G in terms of $\{\tilde{m}_\lambda\}$. Recall that a stable set of G is a subset S of V such that no two vertices of S are adjacent, and a stable partition π of G is a set partition of V such that each block of π is a stable set. The type of π is defined to be the integer partition obtained by rearranging the block sizes of π in decreasing order. Stanley's result can be stated as follows.

Lemma 3. *[13, Proposition 2.4] Let G be a graph with n vertices and a_λ be the number of stable partitions of G of type λ. Then*

$$X_G = \sum_{\lambda \vdash n} a_\lambda \tilde{m}_\lambda.$$

We now consider the monomial expansion of the chromatic symmetric function of a generalized pyramid graph $\mathrm{GP}(r,s,t)$ in Figure 2.

Theorem 4. *For any nonnegative integers r, s, t, we have*

$$\begin{aligned} X_{\mathrm{GP}(r,s,t)} =& \tilde{m}_{(3,1^{r+s+t})} + (rst)\tilde{m}_{(2,2,2,1^{r+s+t-3})} + (rt+rs+st+r+s+t)\cdot \\ & \tilde{m}_{(2,2,1^{r+s+t-1})} + (r+s+t+3)\tilde{m}_{(2,1^{r+s+t+1})} + \tilde{m}_{(1^{r+s+t+3})}. \end{aligned} \tag{2}$$

Proof. From Figure 2 we see that there exists no stable set of size greater than or equal to 4. Moreover, there exists a unique stable set of size 3, namely $\{a, b, c\}$. A stable set of size 2 can only be of the form $\{a, u\}$ with $u \in S_{b,c} \cup \{b, c\}$, or $\{b, v\}$ with $v \in S_{a,c} \cup \{a, c\}$, or $\{c, w\}$ with $w \in S_{a,b} \cup \{a, b\}$. Thus, any admissible stable partition of $\mathrm{GP}(r, s, t)$ is of type $(3, 1^{r+s+t})$, $(2, 1^{r+s+t+1})$, $(2, 2, 1^{r+s+t-1})$, $(2, 2, 2, 1^{r+s+t-3})$ or $(1^{r+s+t+3})$. Moreover, we will show that

$$
\begin{aligned}
&a_{(3,1^{r+s+t})} = 1,\\
&a_{(2,1^{r+s+t+1})} = r + s + t + 3,\\
&a_{(2,2,1^{r+s+t-1})} = rt + rs + st + r + s + t,\\
&a_{(2,2,2,1^{r+s+t-3})} = rst,\\
&a_{(1^{r+s+t+3})} = 1.
\end{aligned}
$$

The above formulas can be proven in the same manner. As an example we prove the fourth formula. Note that a stable partition of type $(2, 2, 2, 1^{r+s+t-3})$ is uniquely determined by the set of three stable sets of size 2, which can only be of the form $\{\{a, u\}, \{b, v\}, \{c, w\}\}$ with $u \in S_{b,c}$, $v \in S_{a,c}$, $w \in S_{a,b}$. It is clear that u has t choices, v has s choices and w has r choices. Hence the fourth formula holds. This completes the proof. $\square$

Next we shall give the expansion of $X_{\mathrm{GP}(r,s,t)}$ in terms of elementary symmetric functions. To this end, we need to use some results concerning the transition matrix between the bases $\{m_\lambda : \lambda \vdash n\}$ and $\{e_\lambda : \lambda \vdash n\}$. Let $\mathrm{Par}(n)$ denote the set of all partitions of n. Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of $\mathrm{Par}(n)$, we say that $\mu \leqslant \lambda$ if

$$\mu_1 + \mu_2 + \cdots + \mu_i \leqslant \lambda_1 + \lambda_2 + \cdots + \lambda_i \quad \text{for all } i \geqslant 1.$$

The conjugate of $\lambda = (\lambda_1, \lambda_2, \ldots)$ is defined as the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ where $\lambda'_i = |\{j : \lambda_j \geqslant i\}|$. We have the following result.

Lemma 5. *[14, Chapter 7] Let $\lambda \vdash n$. If*

$$e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu,$$

then $M_{\lambda\mu}$ is equal to the number of $(0, 1)$-matrices $A = (a_{ij})_{i,j\geqslant 1}$ satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$, where $\mathrm{row}(A)$ (resp., $\mathrm{col}(A)$) is the vector of row sums (resp., column sums) of A. Moreover, $M_{\lambda\mu} = 0$ unless $\lambda \leqslant \mu'$, and $M_{\lambda\lambda'} = 1$.

Combining Theorem 4 and Lemma 5, we obtain the following result.

Theorem 6. *For any nonnegative integers r, s, t, we have*

$$
\begin{aligned}
X_{\mathrm{GP}(r,s,t)} =& A \cdot e_{(r+s+t+1,1,1)} + B \cdot e_{(r+s+t,3)} + C \cdot e_{(r+s+t+1,2)}\\
&+ D \cdot e_{(r+s+t+2,1)} + E \cdot e_{(r+s+t+3)},
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
A =&(r+s+t)!,\\
B =&(r+s+t-3)!\cdot 6rst,\\
C =&(r+s+t-3)!\cdot 2(r+s+t-1)\\
&\cdot[(r^2s+rs^2-2rs)+(rt^2+r^2t-2rt)+(s^2t+st^2-2st)],\\
D =&(r+s+t-2)!\cdot[(r^4+r^3-2r^2)+(3r^2s-2rs)+(3rs^2-2s^2)\\
&+(3r^2t-2rt)+(9rst-2st)+(3rt^2-2t^2)+3s^2t+5rs^2t\\
&+2s^3t+5r^2st+2r^3t+2r^2t^2+3st^2+5rst^2+2s^2t^2\\
&+t^3+2rt^3+2st^3+t^4+2r^3s+2r^2s^2+s^3+2rs^3+s^4],\\
E =&(r+s+t-1)!\cdot(3+r+s+t)(r+s)(r+t)(s+t).
\end{aligned}
$$

Proof. Let $i=r+s+t$ and $P=\{(2^3,1^{i-3}),\,(3,1^i),\,(2^2,1^{i-1}),\,(2,1^{i+1}),\,(1^{i+3})\}$. In order to give the elementary expansion of $X_{\mathrm{GP}(r,s,t)}$, by Theorem 4 and Lemma 5 it suffices to consider the monomial expansion of those e_λ's such that $\lambda' \leqslant \mu$ for some $\mu \in P$. It is straightforward to verify that the set of such partitions λ is composed of $\{(i,3),\,(i+1,1,1),\,(i+1,2),\,(i+2,1),\,(i+3)\}$. Using Lemma 5, we will show

$$e_{(i,3)} = m_{(2,2,2,1^{i-3})} + (i-1)m_{(2,2,1^{i-1})} + \binom{i+1}{2}m_{(2,1^{i+1})} + \binom{i+3}{3}m_{(1^{i+3})}, \tag{4}$$

$$e_{(i+1,1,1)} = m_{(3,1^i)} + (2i+3)m_{(2,1^{i+1})} + 2m_{(2,2,1^{i-1})} + 2\binom{i+3}{2}m_{(1^{i+3})}, \tag{5}$$

$$e_{(i+1,2)} = m_{(2,2,1^{i-1})} + (i+1)m_{(2,1^{i+1})} + \binom{i+3}{2}m_{(1^{i+3})}, \tag{6}$$

$$e_{(i+2,1)} = m_{(2,1^{i+1})} + (i+3)m_{(1^{i+3})}, \tag{7}$$

$$e_{i+3} = m_{(1^{i+3})}. \tag{8}$$

The above formulas are easy to prove. As an example we prove that the coefficient of $m_{(2,1^{i+1})}$ in $e_{(i+1,2)}$ is $i+1$. By Lemma 5, we only need to count the number of $(0,1)$-matrices $A=(a_{pq})_{p,q\geqslant 1}$ with $\mathrm{row}(A)=(i+1,2)$ and $\mathrm{col}(A)=(2,1^{i+1})$. Since $\mathrm{row}(A)=(i+1,2)$, there are $i+1$ entries equal to 1 in the first row of matrix A and two entries equal to 1 in the second row. Since $\mathrm{col}(A)=(2,1^{i+1})$, we must have $a_{11}=a_{21}=1$ and $a_{pq}=0$ for $p\geqslant 3$ or $q\geqslant i+3$. Moreover, the submatrix

$$\begin{pmatrix} a_{12} & a_{13} & \cdots & a_{1,i+2}\\ a_{22} & a_{23} & \cdots & a_{2,i+2}\end{pmatrix}$$

can be any $2\times(i+1)$ matrix composed of i column vectors $\binom{1}{0}$'s and one column vector $\binom{0}{1}$. Hence we have $M_{(i+1,2),(2,1^{i+1})}=i+1$.

By using the above m-expansion formulas we can get the e-expansion of those monomial symmetric functions appearing in (2). Substituting the resulted e-expansion formulas into (2), we complete the proof. □

We proceed to prove the main result of this section.

Theorem 7. *For any nonnegative integers $r, s, t \geqslant 0$ the generalized pyramid graph $\mathrm{GP}(r, s, t)$ is e-positive.*

Proof. Note that if $r = s = t = 0$, then $X_{\mathrm{GP}(r,s,t)} = e_1^3$, which is obviously e-positive. If only two of r, s, t are zero, then $\mathrm{GP}(r, s, t)$ belongs to one class of e-positive graphs studied by Hamel, Hoàng and Tuero, see [8, Lemma 9]. If exactly one of r, s, t is zero, then $\mathrm{GP}(r, s, t)$ is a generalized bull graph in Figure 3, whose positivity is already known, see Foley, Hoàng and Merkel [5, Theorem 11] and Cho, Huh [2, Theorem 3.7].

From now on we assume that r, s, t are positive integers. In order to show the e-positivity of $X_{\mathrm{GP}(r,s,t)}$, it suffices to show that the coefficients A, B, C, D, E in (3) are nonnegative. Clearly, A, B and E are always nonnegative.

We continue to prove $C \geqslant 0$. Since $r, s \geqslant 1$, we have

$$r^2s + rs^2 - 2rs \geqslant r^2 + s^2 - 2rs \geqslant 0,$$

Similarly, we have

$$r^2t + rt^2 - 2rt \geqslant 0,$$

and

$$st^2 + s^2t - 2st \geqslant 0.$$

Therefore, $C \geqslant 0$.

Finally, we prove that $D \geqslant 0$. Since $r, s, t \geqslant 1$, it is straightforward to verify that $r^4 + r^3 - 2r^2, 3r^2s - 2rs, 3rs^2 - 2s^2, 3r^2t - 2rt, 9rst - 2st, 3rt^2 - 2t^2$ are all nonnegative. Thus, $D \geqslant 0$. This completes the proof. □

3 $2K_2$-free unit interval graphs

The aim of this section is to prove that $2K_2$-free unit interval graphs are e-positive. Our proof is based on the characterization of $2K_2$-free unit interval graphs due to Hempel and Kratsch [10], who actually gave a characterization of a larger family of graphs. Using their result, we show that $2K_2$-free unit interval graphs can only be either co-triangle-free graphs or generalized bull graphs, which are already known to be e-positive.

Let us first recall some related definitions and results. A co-triangle means a stable set of size 3. Stanley and Stembridge [15] proved the e-positivity of the complement graphs of bipartite graphs, which are a special class of co-triangle-free graphs. Stanley [13] gave a different proof of their result, and his arguments can also be applied to the following general case.

Lemma 8. *[14, Exercise 7.47] If G is a co-triangle-free graph, then X_G is e-positive.*

The generalized bull graphs were introduced by Foley, Hoàng and Merkel [5], but their e-positivity was first proved by Cho and Huh [2]. A generalized bull graph can be characterized as Figure 3, where K_r, K_s, K_t form a clique of size $r+s+t$, a is adjacent to each vertex of K_r, and b is adjacent to each vertex of K_s. We denote such a graph by $\mathrm{GB}(r,s,t)$.

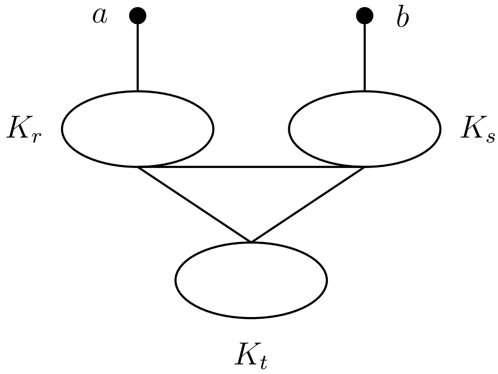

Figure 3: The generalized bull graph $\mathrm{GB}(r,s,t)$.

Cho and Huh [2] obtained the following result.

Lemma 9. *[2, Theorem 3.7] For any positive integers r,s,t, the generalized bull graph $\mathrm{GB}(r,s,t)$ is e-positive.*

Note that Cho and Huh proved the above result based on the Schur expansion of $X_{\mathrm{GB}(r,s,t)}$. To be self-contained, we would like to give a new proof, which parallels that of Theorem 7.

Proof of Lemma 9. We first give the monomial expansion of $X_{\mathrm{GB}(r,s,t)}$. Using the same method as in the proof of Theorem 4, we get that

$$\begin{aligned} X_{\mathrm{GB}(r,s,t)} =& t\cdot \tilde{m}_{(3,1^{r+s+t-1})} + (t(t-1)+tr+sr+st)\cdot \tilde{m}_{(2,2,1^{r+s+t-2})} \\ &+ (1+2t+s+r)\cdot \tilde{m}_{(2,1^{r+s+t})} + \tilde{m}_{(1^{r+s+t+2})}. \end{aligned} \tag{9}$$

Setting $k=r+s+t$ and $i=k-1$ in (5), (6), (7) and (8), and then substituting these four equations into (9), we obtain

$$\begin{aligned} X_{\mathrm{GB}(r,s,t)} =&(r+s+t-2)!\cdot [(r+s+t-1)t\cdot e_{(r+s+t,1,1)} + 2rs\cdot e_{(r+s+t,2)} \\ &+ (r^3+r^2s+rs^2+s^3+2r^2t+2rst+2s^2t+rt^2+st^2-r-s)\cdot \\ & e_{(r+s+t+1,1)} + (r+s+t+2)(r+s+t-1)rs\cdot e_{(r+s+t+2)}]. \end{aligned}$$

Since $r,s,t\geqslant 1$, the e-positivity of $X_{\mathrm{GB}(r,s,t)}$ is obvious. □

We proceed to recall Hempel and Kratsch's characterization of $2K_2$-free unit interval graphs. As will be shown below, $2K_2$-free unit interval graphs are a special class of (claw, AT)-free graphs. Recall that an interval graph is formed from a set of intervals on the real line, with a vertex for each interval and an edge between vertices whose intervals intersect. A unit interval graph is an interval graph for which each of its intervals has unit length. It is well known that unit interval graphs must be claw-free and C_4-free. The notion of AT-free graphs was introduced by Lekkerkerker and Boland [11]. A co-triangle in a graph G is called an asteroidal triple, denoted by AT for short, if for any pair of its vertices there exists a path between them which does not intersect with the neighborhood of the third vertex. It has been shown in [11] that interval graphs are exactly the class of chordal AT-free graphs, where a chordal graph is a graph such that every induced cycle in the graph has exactly three vertices. Meanwhile, unit interval graphs have been shown to be exactly the class of claw-free interval graphs [6]. Hence, $2K_2$-free unit interval graphs are equivalent to ($2K_2$, claw, AT)-free chordal graphs. Given a graph G with vertex set V and edge set E and a pair of vertices u and v, let $\alpha(G)$ denote the maximum size of stable sets and let $d(u, v)$ denote the number of edges of the shortest path between u and v. For any vertex $w \in V$, let $N_i(w) = \{x \in V \mid d(x, w) = i\}$ and $[N_i(w)]$ denote the induced subgraph on $N_i(w)$. In particular, $N_1(w)$ is the neighborhood of w, also denoted by $N(w)$. With these notations, Hempel and Kratsch's characterization of (claw, AT)-free graphs can be stated as follows.

Lemma 10. *[10, Lemma 6] For any connected (claw, AT)-free graph G, there exists a vertex w such that $\alpha([N(w)]) \leqslant 2$ and for any $i \geqslant 2$ each $[N_i(w)]$ is a clique (which might be empty).*

It is well known that $X_{G \uplus H} = X_G X_H$, where $G \uplus H$ is a disjoint union of graphs G and H. Given a $2K_2$-free unit interval graph G, it is clear that every connected component of G is also a $2K_2$-free unit interval graph. Thus when studying the e-positivity of X_G, we may assume that G is connected. Based on the above result, we could give a characterization of connected $2K_2$-free unit interval graphs.

Corollary 11. *If G is a connected $2K_2$-free unit interval graph, then there exists a vertex w such that $\alpha([N(w)]) \leqslant 2$, $[N_2(w)]$ is a clique, $|N_3(w)| \leqslant 1$, and $N_i(w) = \emptyset$ for any $i \geqslant 4$. Moreover, if $[N(w)]$ is connected and $\alpha([N(w)]) = 2$, then $|N_2(w)| \leqslant 2$ and $[N(p) \cap N(w)]$ is a clique for any $p \in N_2(w)$.*

Proof of Corollary 11. Since G is a $2K_2$-free unit interval graph, thus it must be (claw, AT)-free, as mentioned before Lemma 10. Thus, there exists w such that $\alpha([N(w)]) \leqslant 2$ and for any $i \geqslant 2$ each $[N_i(w)]$ is a clique.

We proceed to show that $|N_3(w)| \leqslant 1$ and $N_i(w) = \emptyset$ for any $i \geqslant 4$. We first show that $N_i(w) = \emptyset$ for any $i \geqslant 4$. Otherwise, if $N_i(w) \neq \emptyset$ for some $i \geqslant 4$, then $N_j(w) \neq \emptyset$ for any $1 \leqslant j \leqslant i-1$. Thus there exist $x \in N(w)$, $y \in N_{i-1}(w)$ and $z \in N_i(w)$ such that the set $\{w, x, y, z\}$ induces a $2K_2$, a contradiction. We next show that $|N_3(w)| \leqslant 1$. Otherwise if $|N_3(w)| > 1$, then there exist $u, v \in N_3(w)$ such that $uv \in E$, since $[N_3(w)]$

is a clique. Then for any x in $N(w)$, the set $\{w, x, u, v\}$ induces a $2K_2$, a contradiction. Hence $|N_3(w)| \leqslant 1$.

It remains to show that if $[N(w)]$ is connected, $|N_3(w)| = 0$ and $\alpha([N(w)]) = 2$, then $|N_2(w)| \leqslant 2$ and $[N(p) \cap N(w)]$ is a clique for any $p \in N_2(w)$. Note that by definition a unit interval graph must be C_4-free. We first show that $[N(p) \cap N(w)]$ is a clique for any $p \in N_2(w)$. Suppose to the contrary there exist $p \in N_2(w)$ and non-adjacent $a, b \in N(p) \cap N(w)$. Then $\{p, a, b, w\}$ induces a C_4, a contradiction. We next show that $|N_2(w)| \leqslant 2$. Suppose $|N_2(w)| = s$. We claim that for any $a \in N(w)$ there are at least $s - 1$ vertices in $N_2(w)$ which are adjacent to a, namely, $|N(a) \cap N_2(w)| \geqslant s - 1$. Suppose to the contrary there exist $a \in N(w)$ and $x, y \in N_2(w)$ such that neither x nor y is adjacent to a, and thus $\{x, y, a, w\}$ induces a $2K_2$ in G since $[N_2(w)]$ is a clique, a contradiction. Since $\alpha([N(w)]) = 2$, there exist $a, b \in N(w)$ which are not adjacent. Moreover, a, b can not be adjacent to the same vertex x in $N_2(w)$ for otherwise the set $\{x, a, b, w\}$ induces a C_4, a contradiction. This means that

$$(N(a) \cap N_2(w)) \cap (N(b) \cap N_2(w)) = \emptyset.$$

Hence

$$s = |N_2(w)| \geqslant |N(a) \cap N_2(w)| + |N(b) \cap N_2(w)| \geqslant (s - 1) + (s - 1),$$

yielding $s \leqslant 2$. Hence $|N_2(w)| \leqslant 2$. This completes the proof. □

We would like to point out that the first part of Corollary 11 is already known to Foley, Hoàng and Merkel [5], and the second part tells more information of a $2K_2$-free unit interval graph G. In fact, if more constraints are added, we could get a clearer characterization of G. The following result will be used to check the e-positivity of some special $2K_2$-free unit interval graphs.

Corollary 12. *Given a connected $2K_2$-free unit interval graph G, let w be as in Corollary 11. Suppose that $[N(w)]$ is connected, $|N_2(w)| = 1$, $|N_3(w)| = 0$ and $\alpha([N(w)]) = 2$. Let $N_2(w) = \{p\}$, $A = N(p) \cap N(w)$ and $B = N(w) \setminus A$, then $|N(a) \cap B| \geqslant |B| - 1$ and $[N(a) \cap B]$ is a clique for any $a \in A$.*

Proof. Let us first prove that $|N(a) \cap B| \geqslant |B| - 1$ for any $a \in A$. Suppose the contrary. Then there exist $a \in A$ and $b_1, b_2 \in B$ such that b_1 and b_2 are not adjacent to a. If b_1 and b_2 are not adjacent in G, then $\{a, b_1, b_2\}$ is a stable set, contradicting $\alpha([N(w)]) = 2$. If b_1 and b_2 are adjacent, then $\{a, p, b_1, b_2\}$ induces a $2K_2$, a contradiction. Thus a is adjacent to at least $|B| - 1$ vertices in B. Next we show that $[N(a) \cap B]$ is a clique for any $a \in A$. Suppose to the contrary there exist some $a \in A$ and non-adjacent $b, b' \in N(a) \cap B$. Note that the set $\{a, p, b, b'\}$ induces a claw, which leads to a contradiction. This completes the proof. □

Finally we come to the main result of this section.

Theorem 13. *If G is a $2K_2$-free unit interval graph, then X_G is e-positive.*

Proof. Without loss of generality, we may assume that G is connected. By Corollary 11, there are six cases to check:

(1) $[N(w)]$ is not connected;

(2) $[N(w)]$ is connected and $|N_3(w)| = 1$;

(3) $[N(w)]$ is connected, $|N_3(w)| = 0$ and $\alpha([N(w)]) = 1$;

(4) $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 2$;

(5) $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 1$;

(6) $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 0$;

where w is given as in Corollary 11.

Foley, Hoàng and Merkel [5] showed that the theorem is true for the first three cases. Indeed, they showed that G must be a co-triangle free graph or a generalized bull graph. Hence we only need to consider the remaining three cases.

Let us first deal with Case (6). In this case, it is clear that G is co-triangle-free. Thus X_G is e-positive by Lemma 8.

Next we consider Case (4). Set $N_2(w) = \{p, q\}$, $A = N(p) \cap N(w)$ and $B = N(w) \setminus A$. By Corollary 11, both $[A]$ and $[N_2(w)]$ are cliques. We claim that any vertex $b \in B$ is adjacent to q. Otherwise if there exists some $b \in B$ such that q and b are not adjacent, then $\{p, q, b, w\}$ induces a $2K_2$, a contradiction. Hence all vertices of B are adjacent to q. By Corollary 11 the induced subgraph $[N(q) \cap N(w)]$ is a clique and hence $[B]$ is a clique. Thus G can be characterized as a co-triangle-free graph, as depicted in Figure 4, where the dashed lines represent that there may exist some edges between A and B, as well as between q and A. Again by Lemma 8, we obtain the e-positivity of X_G.

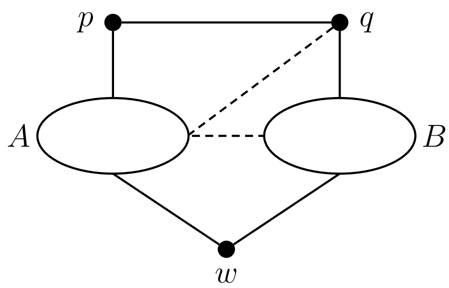

Figure 4: The structure of G in Case (4).

Finally, we prove that the theorem holds for Case (5). Now set $N_2(w) = \{p\}$, $A = N(p) \cap N(w)$ and $B = N(w) \setminus A$. By Corollary 11, $[A]$ is a clique. If $[B]$ is a clique, then it is easy to see that G is co-triangle-free, see Figure 5, where the dashed line represents that there may exist some edges between A and B. Hence X_G is e-positive by Lemma 8.

From now on we assume that $[B]$ is not a clique. Then there exist non-adjacent vertices $x, y \in B$. Now set $A_1 = N(x) \cap A$, $A_2 = N(y) \cap A$ and $A_3 = B \setminus \{x, y\}$. We claim that

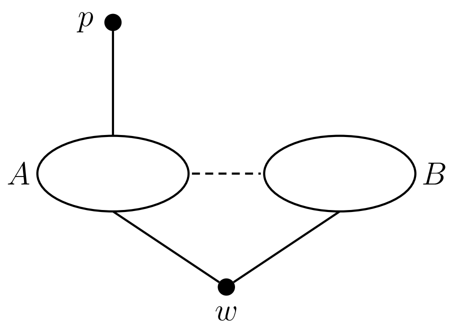

Figure 5: The structure of G in Case (5) when B is a clique.

either $A_1 = \emptyset$ or $A_2 = \emptyset$. Suppose to the contrary that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Now we have $A_1 \cap A_2 = \emptyset$, otherwise there exists $a \in A_1 \cap A_2$ and then $\{a, x, y, p\}$ induces a claw, a contradiction. Moreover, we have $A_1 \cup A_2 = A$, otherwise there exists $b \in A \setminus (A_1 \cup A_2)$ such that $\{b, x, y\}$ is a stable set, contradicting $\alpha([N(w)]) \leqslant 2$. By Corollary 12, both $[\{x\} \cup A_1 \cup A_3]$ and $[\{y\} \cup A_2 \cup A_3]$ are cliques. A little thought shows that $\{x, y, p\}$ is an asteroidal triple, as shown in Figure 6. This contradicts the fact that G is AT-free. Thus at least one of A_1 and A_2 is empty.

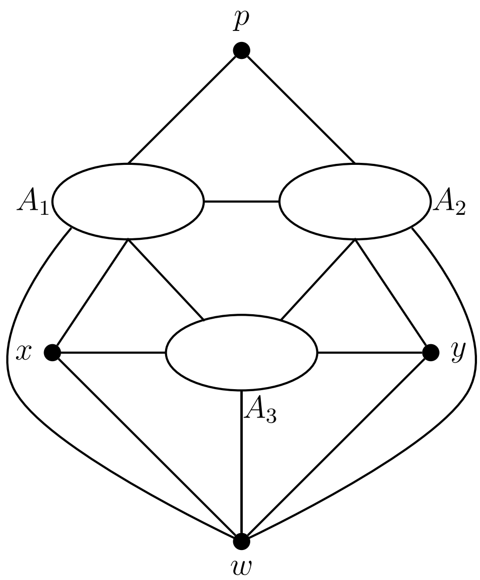

Figure 6: $\{x, y, p\}$ induces an asteroidal triple.

Without loss of generality, we may assume that $A_1 = \emptyset$. We proceed to show that $N(w) \setminus \{x\}$ induces a clique. By observing that $N(w) = A \cup B$ and A is a clique, it suffices to show that each $a \in A$ and each $z \in B \setminus \{x\}$ are adjacent and $[B \setminus \{x\}]$ is a clique. For the former assertion, assume to the contrary that there exist non-adjacent $a \in A$ and $z \in B \setminus \{x\}$. If x, z are not adjacent, then $\{w, x, a, z\}$ induces a claw, a contradiction. If x, z are adjacent, then $\{a, p, x, z\}$ induces a $2K_2$, again a contradiction.

Hence a and z are adjacent. For the latter assertion, assume to the contrary that there exist non-adjacent vertices $b_1, b_2 \in B \setminus \{x\}$, but then for any $a \in A$ the set $\{a, p, b_1, b_2\}$ induces a claw, a contradiction. Thus $[N(w) \setminus \{x\}]$ is a clique.

If we set $B_1 = N(x) \cap (B \setminus \{x\})$ and $B_2 = B \setminus \{\{x\} \cup B_1\}$, then G can be considered as a generalized bull graph, see Figure 7. Thus in case (5) if $[B]$ is not a clique, the graph G is also e-positive by Lemma 9.

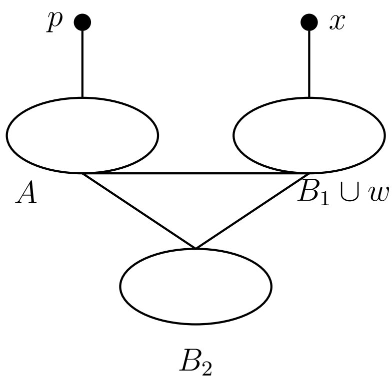

Figure 7: The structure of G in Case (5) when B is not a clique.

Combining all the above cases, we complete the proof. □

4 Future work

So far we have established the e-positivity of certain $(claw, 2K_2)$-free graphs. It is a natural problem to consider how to construct new e-positive graphs from old ones. This kind of problems have been considered by Foley, Hoàng and Merkel [5]. Given a graph G and a vertex a, let $G^{\langle a \rangle}$ be the graph obtained from G by replacing a by two adjacent vertices x, y, and then placing edges connecting every vertex b of G to x and y if ab is an edge of G. Foley, Hoàng and Merkel proposed the following conjecture.

Conjecture 14 ([5, Conjecture 23])**.** If G is e-positive, so is $G^{\langle a \rangle}$ for any vertex a.

We have proved the e-positivity of the generalized pyramid graphs $\mathrm{GP}(r, s, t)$ and the generalized bull graphs $\mathrm{GB}(r, s, t)$. Motivated by the above conjecture, we wish to consider the following problem. Given positive integers i, j, k, r, s, t, let $\mathrm{GP}(i, j, k; r, s, t)$ denote the graph obtained from the generalized pyramid $\mathrm{GP}(r, s, t)$ by replacing a (b or c) in Figure 2 by a clique K_i (resp. K_j or K_k), and placing edges connecting every vertex of K_i (resp. K_j, or K_k) to $S_{a,b}$ and $S_{a,c}$ ($S_{a,b}$ and $S_{b,c}$, or $S_{a,c}$ and $S_{b,c}$). Similarly, let $\mathrm{GB}(i, j; r, s, t)$ denote the graph obtained from the generalized bull graph $\mathrm{GB}(r, s, t)$ by replacing a (resp. b) in Figure 3 by K_i (resp. K_j), and placing edges connecting every vertex of K_i (resp. K_j) to K_r (resp. K_s). Following our approach to Theorem 6 and Lemma 9, for small values of i, j, k it is possible to get the monomial expansion of $X_{\mathrm{GP}(i,j,k;r,s,t)}$, as well as that of $X_{\mathrm{GB}(i,j;r,s,t)}$. However, the enumeration of stable partitions becomes complicated for general i, j, k. Thus it would be interesting to explore the e-positivity of $X_{\mathrm{GP}(i,j,k;r,s,t)}$ and $X_{\mathrm{GB}(i,j;r,s,t)}$.

Acknowledgements

We thank Ethan Y.H. Li for stimulating discussions.

References

[1] S. Cho and J. Hong. Positivity of chromatic symmetric functions associated with Hessenberg functions of bounce number 3. arXiv:1910.07308, 2019.

[2] S. Cho and J. Huh. On e-positivity and e-unimodality of chromatic quasisymmetric functions. *SIAM J. Discrete Math.*, 33: 2286–2315, 2019.

[3] S. Dahlberg. Triangular ladders $P_{d,2}$ are e-positive. arXiv:1811.04885, 2018.

[4] S. Dahlberg, A. She and S. van Willigenburg. Schur and e-positivity of trees and cut vertices. *Electron. J. Combin.*, 27(1): #P1.2, 2020.

[5] A. M. Foley, C. T. Hoàng and O. D. Merkel. Classes of graphs with e-positive chromatic symmetric function. *Electron. J. Combin.*, 26(3):#P3.51, 2019.

[6] F.Gardi. The Roberts characterization of proper and unit interval graphs. *Discrete Math.*, 307: 2906–2908, 2007.

[7] M. Guay-Paquet. A modular law for the chromatic symmetric functions of (3+1)-free posets. arXiv:1306.2400, 2013.

[8] A. M. Hamel, C. T. Hoàng and J. E. Tuero. Chromatic symmetric functions and H-free graphs. *Graphs Combin.*, 35: 815–825, 2019.

[9] M. Harada and M. Precup. The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture. *Algebr. Combin.*, 2:1059–1108, 2019.

[10] H. Hempel and D. Kratsch. On claw-free asteroidal triple-free graphs. *Discrete Appl. Math.*, 121: 155–180, 2002.

[11] C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51: 45–64, 1962.

[12] J. Shareshian and M. Wachs. Chromatic quasisymmetric functions. *Adv. Math.*, 295: 497–551, 2016.

[13] R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111: 166–194, 1995.

[14] R. P. Stanley. Enumerative Combinatorics, Vol. 2. Cambridge: Cambrige University Press, 1999.

[15] R. P. Stanley, J. R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted position. *J. Combin. Theory Ser. A*, 62: 261–279, 1993.

[16] S. Tsujie. The chromatic symmetric functions of trivially perfect graphs and cographs. *Graphs Combin.*, 34: 1037–1048, 2018.