Large monochromatic components in almost complete graphs and bipartite graphs

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Abstract

Gyárfás proved that every coloring of the edges of $K_n$ with $t + 1$ colors contains a monochromatic connected component of size at least $n/t$. Later, Gyárfás and Sárközy asked for which values of $\gamma = \gamma(t)$ does the following strengthening for almost complete graphs hold: if $G$ is an $n$-vertex graph with minimum degree at least $(1 - \gamma)n$, then every $(t + 1)$-edge coloring of $G$ contains a monochromatic component of size at least $n/t$. We show $\gamma = 1/(6t^3)$ suffices, improving a result of DeBiasio, Krueger, and Sárközy.

Mathematics Subject Classifications: 05C55, 05D10

1 Introduction, a stability of edge colorings

Erdős and Rado observed that every 2-edge-coloring of the complete graph $K_n$ has a monochromatic spanning tree. Generalizing this result, Gyárfás [5] proved that every $(t + 1)$-edge-coloring of the edge set $E(K_n)$ contains a monochromatic connected component of size at least $n/t$. This bound is the best possible when $n$ is divisible by $t^2$ and an affine plane of order $t$ exists.

Gyárfás and Sárközy [7] proved that Gyárfás’ theorem has a remarkable stability property, the complete graph $K_n$ can be replaced with graphs of high minimum degree.

Question 1 (Gyárfás and Sárközy [7]). Let $t \geq 2$. Which values of $\gamma = \gamma(t)$ guarantee that every $(t + 1)$-edge-coloring of any $n$-vertex graph with minimum degree at least $(1 - \gamma)n$ contains a monochromatic component of size at least $n/t$?

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Let $\gamma(t)$ denote the best value we can have. The case for $t = 1$ is trivial, $\gamma(1) = 0$. It is observed in [6] that any non-complete graph has a 2-edge-coloring without a monochromatic spanning tree: if $xy$ is a non-edge, consider any edge-coloring where every edge incident to $x$ is red and every edge incident to $y$ is blue. Then there does not exist a monochromatic component containing both $x$ and $y$.

The case for at least three colors (i.e., $t \geq 2$) is more interesting. Gyárfás and Sárközy [7] showed that $\gamma \leq 1/(1000t^3)$ suffices. This was improved to $1/(3072t^5)$ by DeBiasio, Krueger, and Sárközy [2].


**Question 2** (DeBiasio, Krueger, and Sárközy [2]). Let $t \geq 2$ and $n_1 \leq n_2$. Determine for which values of $\gamma = \gamma(t, n_1, n_2)$ the following is true: let $G$ be an $X_1, X_2$-bipartite graph such that $|X_i| = n_i$ for $i \in \{1, 2\}$, for every $x \in X_1$, $d(x) \geq (1 - \gamma)n_2$, and for every $y \in X_2$, $d(y) \geq (1 - \gamma)n_1$. Then every $t$-edge-coloring of $G$ contains a monochromatic component of order at least $n/t$.

They proved that $\gamma(t, n_1, n_2) \leq (n_1/n_2)^3/(128t^5)$ suffices. For both Questions 1 and 2 the $t = 2$ case is solved completely in [4, 8] and [2], respectively. They obtained $\gamma(2) = 1/6$, $\gamma(1, n_1, n_2) < 1/2$, and $\gamma(2, n_1, n_2) < 1/3$ (independently of $n$), and these constants are the best possible. So from now on, we only consider $t \geq 3$.

It was conjectured in [2] that for general $t$, $\gamma(t, n_1, n_2) < 1/t^3$. This would be best possible when $n_1$ and $n_2$ are divisible by $t + 1$ by the following construction. Consider $t + 1$ perfect matchings of $K_{t+1,t+1}$ with partite sets $X \cup Y$. Delete all the edges of the $(t + 1)$th matching. Now let $G$ be a graph obtained by blowing up each vertex in $X$ into $n_1/(t + 1)$ new vertices and each vertex in $Y$ into $n_2/(t + 1)$ vertices. Color an edge with color $i$ if its endpoints were obtained by blowing up two vertices which were matched in the $i$th matching. It is easy to see the degrees of vertices are either $(1 - 1/t)n_2$ or $(1 - 1/t)n_1$, and a largest monochromatic component has size $(n_1 + n_2)/(t + 1)$.

Our main result is an improvement for the bound on $\gamma(t, n_1, n_2)$ in Question 2 which in turn implies a better bound for $\gamma(t)$ in Question 1.

**Theorem 1.1.** Fix integers $t \geq 3, n_1, n_2$ such that $n_2 \geq n_1 \geq 1$ and let $\gamma \leq (n_1/n_2)^3/t^3$. Let $G$ be an $X_1, X_2$-bipartite graph such that $|X_i| = n_i$ for $i \in \{1, 2\}$, for every $x \in X_1$, $d(x) \geq (1 - \gamma)n_2$, and for every $y \in X_2$, $d(y) \geq (1 - \gamma)n_1$. Then every $t$-edge-coloring of $G$ contains a monochromatic component of order at least $n/t$.

**Corollary 1.2.** Fix integers $n, t \geq 3$, and let $\gamma \leq 1/(6t^3)$. Suppose $G$ is an $n$-vertex graph with minimum degree at least $(1 - \gamma)n$. Then any coloring of $E(G)$ with $t + 1$ colors contains a monochromatic connected component with at least $n/t$ vertices.
Our method is very similar to that in [7] or in [2]. The major difference is that we will first collect a series of general inequalities in the next section. While these tight inequalities are seemingly unrelated to graphs, we use them to lower bound the size of a “typical” monochromatic component. Our results will imply that in every color class there exists $t$ components that are close in size to $(n_1 + n_2)/t$, and the remaining components are very small. We prove Theorem 1.1 in Section 3 and Corollary 1.2 in Section 4.

We use standard graph theory notation. The degree of a vertex $v$ in $G$ is denoted by $d_G(v)$ or simply $d(v)$ when there is no room for ambiguity. We denote the set of integers $\{1, 2, \ldots, s\}$ by $[s]$.

## 2 Inequalities

In this section, we prove some inequalities for sequences of integers. While our results hold in general, the reader should think of the sequences of integers as the sizes of each part (determined by a bipartition) of a monochromatic component for a fixed color.

It was pointed out by the anonymous referee that the following lemma is an easy consequence of a result called Milne’s Inequality (see [9]). We include its short proof for completeness.

**Lemma 2.1.** Let $a_1, \ldots, a_s, b_1, \ldots, b_s, E, M, A, B$ be non-negative real numbers such that

- $\sum_{i=1}^{s} a_i b_i \geq E$,
- for all $i \in [s]$, $a_i + b_i \leq M$,
- $\sum_{i=1}^{s} a_i \leq A$, and $\sum_{i=1}^{s} b_i \leq B$.

Then $E(A + B) \leq MAB$.

**Proof.** The case $EAB = 0$ is easy, so we may suppose $A, B, E > 0$. Apply Jensen’s inequality for the convex function $x^2$

$$
\left( \frac{\sum_{i=1}^{s} b_i a_i}{\sum_{i=1}^{s} b_i} \right)^2 \leq \frac{\sum_{i=1}^{s} b_i a_i^2}{\sum_{i=1}^{s} b_i}.
$$

Therefore

$$
\frac{(\sum_{i=1}^{s} a_i b_i)^2}{\sum_{i=1}^{s} b_i} \leq \sum_{i=1}^{s} a_i^2 b_i,
$$

and similarly $\frac{(\sum_{i=1}^{s} a_i b_i)^2}{\sum_{i=1}^{s} a_i} \leq \sum_{i=1}^{s} a_i b_i^2$. So we have

$$
E \sum_{i=1}^{s} a_i b_i \left( \frac{1}{A} + \frac{1}{B} \right) \leq \left( \sum_{i=1}^{s} a_i b_i \right)^2 \left( \frac{1}{\sum_{i=1}^{s} a_i} + \frac{1}{\sum_{i=1}^{s} b_i} \right)
$$

$$
\leq \sum_{i=1}^{s} a_i^2 b_i + a_i b_i^2 = \sum_{i=1}^{s} (a_i b_i)(a_i + b_i) \leq M \sum_{i=1}^{s} a_i b_i.
$$

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Dividing by $(\sum_{i=1}^{s} a_i b_i)$ and simplifying, we have $E(A^{-1} + B^{-1}) = E(A + B)/(AB) \leq M$. \hfill \Box

Lemma 2.2. Fix $n_1, n_2, t, a_1, \ldots, a_s, b_1, \ldots, b_s \geq 0, \epsilon \geq 0$. Suppose $t > 1, n_1, n_2 > 0$,

• $\sum_{i=1}^{s} a_i b_i \geq (1 - \epsilon)^{n_1 n_2}$,
• $\sum_{i=1}^{s} a_i \leq n_1$, $\sum_{i=1}^{s} b_i \leq n_2$, and
• $a_i + b_i < (n_1 + n_2)/t$ for all $i \in [s]$.

Then for all $i \in [s]$,

$$a_i < \frac{n_1}{t} + \frac{\sqrt{\epsilon(t - 1)n_1 n_2}}{t} \quad \text{and} \quad b_i < \frac{n_2}{t} + \frac{\sqrt{\epsilon(t - 1)n_1 n_2}}{t}. \quad (1)$$

**Proof.** We prove the statement only for $a_1$, as the proofs for other $a_i$’s and $b_i$’s are symmetric.

First, we handle the case $a_1 = n_1$. Then $a_2 = \cdots = a_s = 0$ so the first constraint gives $a_1 b_1 = n_1 b_1 \geq (1 - \epsilon)^{n_1 n_2}$. Hence $(1 - \epsilon)n_2/t \leq b_1$. Combining this with the last constraint we get

$$n_1 + (n_2/t) - (\epsilon n_2)/t \leq a_1 + b_1 < (n_1/t) + (n_2/t).$$

Rearranging we have $(t - 1)n_1 < \epsilon n_2$. Multiplying each side by $(t - 1)n_1$ and taking square roots, we get $(t - 1)n_1 < \sqrt{\epsilon(t - 1)n_1 n_2}$ and therefore

$$a_1 = n_1 < \frac{n_1}{t} + \frac{\sqrt{\epsilon(t - 1)n_1 n_2}}{t},$$

as desired.

Second, consider the case $b_1 = n_2$. Then the last constraint implies $a_1 < (n_1 + n_2)/t - b_1 = (n_1 + n_2)/t - n_2 < n_1/t$, so (1) holds. From now on, we may suppose that $n_1 - a_1$ and $n_2 - b_1$ are both positive.

Third, suppose that $\sum_{i=2}^{s} a_i b_i \geq (n_1 - a_1)(n_2 - b_1)$. Let $M := \max_{2 \leq i \leq s} \{a_i + b_i\}$, $A = n_1 - a_1$, $B = n_2 - b_1$. Then by Lemma 2.1, we obtain

$$\frac{(n_1 - a_1)(n_2 - b_1)}{t - 1}(n_1 - a_1 + n_2 - b_1) \leq M(n_1 - a_1)(n_2 - b_1).$$

Simplify by the positive term $(n_1 - a_1)(n_2 - b_1)$

$$M \geq \frac{n_1 - a_1 + n_2 - b_1}{t - 1} \geq \frac{n_1 + n_2 - (n_1 + n_2)/t}{t - 1} = \frac{n_1 + n_2}{t},$$

a contradiction.
Therefore, in the last case we consider, we may assume
\[
\frac{(n_1 - a_1)(n_2 - b_1)}{t - 1} + a_1 b_1 > \sum_{i=1}^{s} a_i b_i \geq (1 - \varepsilon) \frac{n_1 n_2}{t}.
\]
Rearranging, we get
\[
(n_1 - a_1)(n_2 - b_1) + (t - 1)(a_1 b_1) > (1 - \varepsilon) \frac{(t - 1)(n_1 n_2)}{t}
\Rightarrow n_1 n_2 - n_1 b_1 - n_2 a_1 + t a_1 b_1 > n_1 n_2 - \frac{n_1 n_2}{t} - \varepsilon \frac{(t - 1) n_1 n_2}{t}
\Rightarrow \frac{n_1 n_2}{t} + \varepsilon \frac{(t - 1) n_1 n_2}{t} > n_2 a_1 - b_1 (t a_1 - n_1).
\]
If \( a_1 < n_1 / t \), then we are done. So assume \( a_1 \geq n_1 / t \) (so \( t a_1 - n_1 \geq 0 \)). We add the non-positive term \((a_1 + b_1 - (n_1 + n_2) / t)(t a_1 - n_1)\) to the right hand side to obtain
\[
\frac{n_1 n_2}{t} + \varepsilon \frac{(t - 1) n_1 n_2}{t} > n_2 a_1 - b_1 (t a_1 - n_1) + (a_1 + b_1 - \frac{n_1 + n_2}{t})(t a_1 - n_1)
= n_2 a_1 + t a_1^2 - a_1 n_1 - n_1 a_1 + \frac{n_1^2}{t} - n_2 a_1 + \frac{n_1 n_2}{t}
\Rightarrow 0 > t a_1^2 - 2 n_1 a_1 + \left( \frac{n_1^2}{t} - \varepsilon \frac{(t - 1) n_1 n_2}{t} \right)
\]
Solving for \( a_1 \), we obtain
\[
a_1 < \frac{2 n_1 + \sqrt{4 n_1^2 - 4(n_1^2 - \varepsilon (t - 1) n_1 n_2)}}{2t} = \frac{n_1 + \sqrt{\varepsilon (t - 1) n_1 n_2}}{t}. \]

**Lemma 2.3.** Fix \( \varepsilon \geq 0 \), integers \( 1 \leq t \leq s \), and reals \( a_1, \ldots, a_s, b_1, \ldots, b_s \geq 0 \) such that
\begin{itemize}
  \item \( a_1 \geq \ldots \geq a_s \geq 0 \),
  \item \( \sum_{i=1}^{s} a_i = n_1 \), \( \sum_{i=1}^{s} b_i = n_2 > 0 \),
  \item for all \( i \in [s] \), \( a_i + b_i \leq (n_1 + n_2) / t \),
  \item \( \sum_{i=1}^{s} a_i b_i \geq (1 - \varepsilon) n_1 n_2 / t \).
\end{itemize}
Let \( a := a_{t+1} + \ldots + a_s \). Then
\[
a \leq \varepsilon n_1 \frac{n_1 + n_2}{n_2}.
\]
In particular, if \( n_1 \leq n_2 \), then \( a \leq 2 \varepsilon n_1 \).

**Proof.** We construct a new sequence \( b'_1, \ldots, b'_s \) with \( b'_i \geq b_i \) for \( i \in [t] \), \( b'_j = 0 \) for \( t < j \leq s \), such that \( \sum_{i=1}^{t} b'_i = \sum_{i=1}^{s} b_i = n_2 \), and \( a_i + b'_i \leq (n_1 + n_2) / t =: M \) for all \( i \in [t] \). Note that these conditions together with the fact that the \( a_i \)'s are non-increasing imply that
We build our sequence greedily starting with \( b_1, \ldots, b_s \). Define a set \( I \subseteq [s] \) as follows

\[
I(b_1, \ldots, b_s) := \{i \in [t], a_i + b_i < M\} \cup \{j : j > t, b_j > 0\}.
\]

If for all \( j \geq t + 1, b_j = 0 \), then we let \( b'_1, \ldots, b'_s = b_1, \ldots, b_s \) and we are done. So suppose some \( j \geq t + 1 \) satisfies \( b_j \neq 0 \), and hence \( j \in I(b_1, \ldots, b_s) \). Then there exists \( i \in [t] \) with \( b_i + a_i < M \) (i.e., \( i \in I(b_1, \ldots, b_s) \)) because \( \sum_{i=1}^s (a_i + b_i) \leq n_1 + n_2 - b_j = tM - b_j \). If \( a_i + b_i + b_j \leq M \) then we update \( b'_i = b_i + b_j \), \( b'_j = 0 \) and \( b'_k = b_k \) for all \( k \in [s] \setminus \{i, j\} \). Note that \( j \notin I(b'_1, \ldots, b'_s) \).

If \( a_i + b_i + b_j > M \) then we update \( b'_i = b_i + M - (a_i + b_i) = M - a_i, b'_j = b_j - (M - (a_i + b_i)) \) and \( b'_k = b_k \) for \( k \in [s] \setminus \{i, j\} \). In this case, we get \( i \notin I(b'_1, \ldots, b'_s) \). Therefore in both cases we get \( |I(b'_1, \ldots, b'_s)| \leq |I(b_1, \ldots, b_s)| - 1 \), so one can continue this process at most \( s \) steps until we get \( I(b'_1, \ldots, b'_s) \subseteq [t] \).

So suppose we have found a sequence \( b'_1, \ldots, b'_s \) as desired. Apply Lemma 2.1 on the sequences \( a_1, \ldots, a_t \) and \( b'_1, \ldots, b'_t \). We have \( \sum_{i=1}^t a_i = n_1 - a =: A, \sum_{i=1}^t b'_i = n_2 =: B, \sum_{i=1}^s a_i b'_i \geq \sum_{i=1}^s a_i b_i = (1 - \varepsilon)n_1n_2/t =: E, \) and \( a_i + b'_i \leq M \) for all \( i \in [t] \). Therefore,

\[
\frac{(1 - \varepsilon)n_1n_2}{t}(n_1 + n_2 - a) \leq \frac{n_1 + n_2}{t}(n_1 - a)n_2
\]

Rearranging and solving for \( a \), we get

\[
a(n_2 + \varepsilon n_1) \leq \varepsilon n_1^2 + \varepsilon n_1 n_2
\]

\[
\Rightarrow a \leq \varepsilon n_1 \frac{n_1 + n_2}{n_2 + \varepsilon n_1} \leq \varepsilon n_1 \frac{n_1 + n_2}{n_2}.
\]

3 Proof of Theorem 1.1 for almost complete bipartite graphs

Proof. Let \( G \) be an \( X_1, X_2 \)-bipartite graph with \( |X_1| = n_1, |X_2| = n_2 \), and \( n_2 \geq n_1 \geq 1 \). Consider any coloring of the edges of \( G \) with colors \( 1, \ldots, t \). For a color \( i \in [t] \), we denote by \( G^i \) the spanning subgraph of edges colored with \( i \). Suppose that every monochromatic component has less than \( (n_1 + n_2)/t \) vertices. We claim that \( |E(G^i)| < n_1n_2/t \). Indeed, let \( D_1, \ldots, D_s \) be the connected components of \( G^i \). For \( j \in [s] \), let \( a_j = |D_j \cap X_1| \), \( b_j = |D_j \cap X_2| \). Then \( E := |E(G^i)| \leq \sum_{j=1}^s a_jb_j \). Apply Lemma 2.1 with \( A = n_1, B = n_2, M = (n_1 + n_2 - 1)/t \). We get

\[
E \leq \frac{(n_1 + n_2 - 1)/t \cdot (n_1 + n_2)^{-1}}{(n_1n_2)^2} < n_1n_2/t,
\]

as desired.

Let \( \varepsilon_i \) be such that \( |E(G^i)| = (1 - \varepsilon_i)n_1n_2/t \). By Lemma 2.2, a connected component of color \( i \) contains at most \( \frac{t}{\varepsilon_i} + \frac{\sqrt{t-1}n_1n_2}{t} \) vertices from \( X_\alpha, \alpha \in \{1, 2\} \). Therefore, for any \( i \in [t], x \in X_1 \) and \( y \in X_2 \),

\[
d_{G^i}(x) < \frac{n_2}{t} + \frac{\sqrt{\varepsilon_i(t-1)n_1n_2}}{t}, \quad d_{G^i}(y) < \frac{n_1}{t} + \frac{\sqrt{\varepsilon_i(t-1)n_1n_2}}{t}.
\]

(2)

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Since \(|E(G)| \geq (1 - \gamma)n_1n_2\), we have \(\sum_{i=1}^{t} \varepsilon_i \leq t\gamma\). Without loss of generality, suppose color 1 satisfies \(\varepsilon_1 \leq \gamma\). Let \(C_1, \ldots, C_t\) be the vertex sets of the connected components of color 1, ordered so that \(|X_1 \cap C_1| \geq \ldots \geq |X_1 \cap C_t|\). Define \(a_j, b_j\) as before. Note that \(s \geq t + 1\), since the \(C_j\)'s cover \(V(G)\) and \(|C_j| < (n_1 + n_2)/t\) for all \(j\). By Lemma 2.3, \(a := a_{t+1} + \ldots + a_s \leq 2\varepsilon_1n_1\).

**Case 1:** \(X_2 \cap (C_{t+1} \cup \ldots \cup C_r) \neq \emptyset\). Fix a vertex \(y\) in this set. Then \(d_{G^1}(y) \leq 2\varepsilon_1n_1\). We get

\[
(1 - \gamma)n_1 \leq d_G(y) \leq 2\varepsilon_1n_1 + \frac{n_1(t-1)}{t} + \sum_{i=2}^{t} \frac{\sqrt{\varepsilon_i(t-1)n_1n_2}}{t}
\leq 2\gamma n_1 + n_1 - \frac{n_1}{t} + \sqrt{(t-1)^2(\sum_{i=2}^{t} \varepsilon_i)n_1n_2}
\leq 2\gamma n_1 + n_1 - \frac{n_1}{t} + \sqrt{\gamma tn_1n_2} \cdot \frac{t-1}{t}.
\]

Here we used the fact that \(\sum_{i=2}^{t} \frac{\sqrt{i}}{i-1} \leq \sqrt{\sum_{i=1}^{t} \frac{\sqrt{i}}{i}}\) because \(\sqrt{x}\) is a concave function. Therefore

\[
\frac{n_1}{t} < n_13\gamma + \sqrt{\gamma tn_1n_2} \cdot \frac{t-1}{t} \leq n_13\left(\frac{n_1}{t}\right) + \sqrt{t\left(\frac{n_1}{t}\right)^3} \cdot \frac{t-1}{t} \leq \frac{n_1}{t} \left(3 + \frac{t-1}{t}\right),
\]

a contradiction when \(t \geq 3\).

**Case 2:** \(X_2 \cap (C_{t+1} \cup \ldots \cup C_r) = \emptyset\). Let \(x \in X_1 \cap (C_{t+1} \cup \ldots \cup C_r)\). By the case, \(x\) is not incident to an edge of color 1. So we instead obtain

\[
(1 - \gamma)n_2 \leq d_G(x) \leq \frac{n_2(t-1)}{t} + \sum_{i=2}^{t} \frac{\sqrt{\varepsilon_i(t-1)n_1n_2}}{t}
\leq n_2 - \frac{n_2}{t} + \sqrt{\gamma tn_1n_2} \cdot \frac{t-1}{t}.
\]

This implies that

\[
\frac{n_2}{t} < n_2\gamma + \sqrt{\gamma tn_1n_2} \cdot \frac{t-1}{t} \leq n_2\left(\frac{n_1}{t}\right) + \sqrt{t\left(\frac{n_1}{t}\right)^3} \cdot \frac{t-1}{t} = \frac{n_1}{t} \left(1 + \frac{t-1}{t}\right),
\]

a contradiction since \(n_1 \leq n_2\) and \(t \geq 3\).

4 Proof of Corollary 1.2 for almost complete graphs

**Proof.** Let \(G\) be an \(n\)-vertex graph with minimum degree at least \((1 - \gamma)n\), and suppose the edges of \(G\) are colored with colors 0, 1, \ldots, \(t\) such that each monochromatic connected
component has size less than \( n/t \). Again, we use \( G^i \) to refer to the spanning subgraph of the edges of color \( i \).

Let \( V_1, \ldots, V_r \) be the vertex sets of the connected components of \( G^0 \). We will split the vertex set into two almost equal parts \( X_1 \) and \( X_2 \) such that the size of each part is in the range \([n(\frac{1}{2} - \frac{1}{2t}), n(\frac{1}{2} + \frac{1}{2t})]\), and each set \( V_i \) is contained either entirely in \( X_1 \) or entirely in \( X_2 \). To see that this is possible, arbitrarily add entire sets \( V_i \) to \( X_1 \) until \(|X_1| < n(\frac{1}{2} + \frac{1}{2t}) \) but adding any additional set to \( X_1 \) causes the size of \( X_1 \) to be at least \( n(\frac{1}{2} + \frac{1}{2t}) \). Then let \( X_2 = V(G) - X_1 \). At this point, \(|X_1| > n(\frac{1}{2} - \frac{1}{2t}) \), otherwise all sets \( V_j \) not contained in \( X_1 \) have size at least \( n/t \), a contradiction.

Now let \(|X_1| = n_1, |X_2| = n_2 \), where without loss of generality, \(|X_1| \leq |X_2| < 2|X_1| \) (and \( n = n_1 + n_2 \)). By construction, there are no edges of color 0 between \( X_1 \) and \( X_2 \). Hence, the edges of the bipartite subgraph \( G[X_1, X_2] \) are colored with \( t \) colors. (Here \( G[X, Y] \) denotes the spanning bipartite subgraph of \( G \) in which we include only edges with endpoints in both \( X \) and \( Y \).)

For simplicity, set \( G' = G[X_1, X_2] \). Let \( x \in X_1 \) and \( y \in X_2 \). Then
\[
d_{G'}(x) \geq n_2 - \gamma n = n_2 - \gamma(n_1 + n_2) \geq (1 - 2\gamma)n_2,
\]
and
\[
d_{G'}(y) \geq n_1 - \gamma n = n_1 - \gamma(n_1 + n_2) \geq n_1 - \gamma(n_1 + 2n_1) = (1 - 3\gamma)n_1.
\]

Since \( G' \) does not have a monochromatic component of size at least \( n/t = (n_1 + n_2)/t \), Theorem 1.1 implies that
\[
3\gamma \geq \frac{(n_1/n_2)}{t^3} > \frac{1/2}{t^3} = \frac{1}{2t^3}.
\]

We get a contradiction when \( \gamma \leq 1/(6t^3) \).

\[ \square \]

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**References**


