A generalization of Stiebitz-type results on graph decomposition

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Abstract

In this paper, we consider the decomposition of multigraphs under minimum degree constraints and give a unified generalization of several results by various researchers. Let G be a multigraph in which no quadrilaterals share edges with triangles and other quadrilaterals and let $\mu_G(v) = \max\{\mu_G(u, v) : u \in V(G) \setminus \{v\}\}$, where $\mu_G(u, v)$ is the number of edges joining u and v in G. We show that for any two functions $a, b : V(G) \to \mathbb{N} \setminus \{0, 1\}$, if $d_G(v) \geqslant a(v) + b(v) + 2\mu_G(v) - 3$ for each $v \in V(G)$, then there is a partition (X, Y) of $V(G)$ such that $d_X(x) \geqslant a(x)$ for each $x \in X$ and $d_Y(y) \geqslant b(y)$ for each $y \in Y$. This extends the related results due to Diwan, Liu–Xu and Ma–Yang on simple graphs to the multigraph setting.

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1 Introduction

All graphs considered in this paper are finite, undirected and may have multiple edges but no loops. Let G be a graph. For a subset $X \subset V(G)$, let $G[X]$ be the subgraph of G induced by X. For each $v \in V(G)$, denote $N_X(v)$ the set of neighbors of v contained in X and $d_X(v)$ the number of edges between v and $X \setminus \{v\}$. When $X = V(G)$, we simplify $N_{V(G)}(v)$ and $d_{V(G)}(v)$ to $N_G(v)$ and $d_G(v)$, respectively. The *multiplicity* $\mu_G(u, v)$ of two different vertices u and v in G is the number of edges joining u and v, and the weight $\mu_G(v)$ of a vertex v is defined as $\mu_G(v) = \max\{\mu_G(u, v) : u \in V(G) \setminus \{v\}\}$. Call a graph G *simple* if $\mu_G(v) \leqslant 1$ for each $v \in V(G)$. By a *partition* (X, Y) of $V(G)$, we mean that X, Y are two disjoint nonempty sets with $X \cup Y = V(G)$. For a set $\mathscr{H}$ of graphs, we say that a graph is $\mathscr{H}$-free if it contains no member of $\mathscr{H}$ as a subgraph. We also denote by $\mathbb{N}$ the set of nonnegative integers.

Many problems raised in graph theory concern graph partitioning and one popular direction of them is to partition graphs under minimum degree constraints. For a graph G and two functions $a, b : V(G) \to \mathbb{N}$, a partition (X, Y) of $V(G)$ is called an (a, b)-*feasible* partition if $d_X(x) \geqslant a(x)$ for each $x \in X$ and $d_Y(y) \geqslant b(y)$ for each $y \in Y$. In 1996, Stiebitz [15] proved the following celebrated result for simple graphs, solving a conjecture due to Thomassen [16].

Theorem 1 (Stiebitz [15]). *Let G be a simple graph and $a, b : V(G) \to \mathbb{N}$ be two functions. If $d_G(x) \geqslant a(x) + b(x) + 1$ for each $x \in V(G)$, then there is an (a, b)-feasible partition of G.*

For special families of simple graphs, the minimum degree condition can be further sharpen (see [4, 6, 7, 8, 11]). In particular, for $s, t \geqslant 2$, Diwan [4] showed that every simple graph with neither triangles nor quadrilaterals and minimum degree at least $s + t - 1$ can already force a partition (X, Y) as above. Later, Liu and Xu [8] generalized this result by considering triangle-free simple graphs in which no two quadrilaterals share edges.

Theorem 2 (Liu and Xu [8]). *Let G be a triangle-free simple graph in which no two quadrilaterals share edges, and $a, b : V(G) \to \mathbb{N} \setminus \{0, 1\}$ be two functions. If $d_G(x) \geqslant a(x) + b(x) - 1$ for each $x \in V(G)$, then G admits an (a, b)-feasible partition.*

Recently, Ma and Yang [11] obtained the following strengthening of Diwan's result.

Theorem 3 (Ma and Yang [11]). *Let G be a quadrilateral-free simple graph and $a, b : V(G) \to \mathbb{N} \setminus \{0, 1\}$ be two functions. If $d_G(x) \geqslant a(x) + b(x) - 1$ for each $x \in V(G)$, then G admits an (a, b)-feasible partition.*

In 2017, Ban [1] proved a conclusion related to Theorem 1 on weighted simple graphs. Later, Schweser and Stiebitz [12] further studied this problem on graphs, and generalized the results of Stiebitz [15] and Liu and Xu [8] from simple graphs to graphs. Very recently, confirming two conjectures of Schweser and Stiebitz, Liu and Xu [9] obtained a graph version of Theorem 2.

Theorem 4 (Liu and Xu [9])**.** *Let G be a triangle-free graph in which no two quadrilaterals share edges, and $a, b : V(G) \to \mathbb{N} \setminus \{0, 1\}$ be two functions. If $d_G(x) \geqslant a(x) + b(x) + 2\mu_G(x) - 3$ for each $x \in V(G)$, then G admits an (a, b)-feasible partition.*

For related problems on graph partitioning under degree constraints or other variances, we refer readers to [2, 3, 5, 10, 13, 14]. In this paper, we consider partitions of graphs and give a unified generalization of Theorems 2, 3 and 4 as well as the result of Diwan [4]. Precisely, we establish the following theorem.

Theorem 5. *Let G be a graph in which no quadrilaterals share edges with triangles and other quadrilaterals, and let $a, b : V(G) \to \mathbb{N} \setminus \{0, 1\}$ be two functions. If $d_G(x) \geqslant a(x) + b(x) + 2\mu_G(x) - 3$ for each $x \in V(G)$, then G admits an (a, b)-feasible partition.*

Note that this is tight for cycles in the following two perspectives. Firstly, the ranges of the functions a, b cannot be relaxed to the set of integers at least one by choosing the constant functions $a = b - 1 = 1$. Secondly, one also cannot lower the degree condition further by choosing the constant functions $a = b = 2$. We also mention that G is actually $\{K_4^-, C_5^+, K_{2,3}, L_3\}$-free in Theorem 5, where K_4^- is the graph obtained from K_4 by removing one edge, C_5^+ is the graph obtained from C_5 by adding one edge between two nonadjacent vertices, and L_3 is the graph consisting of two quadrilaterals sharing exactly one common edge. Additionally, we use the condition that G is L_3-free exactly once (see Claim 14) in our proof; however, this condition is necessary as shown by the graph constructed in [17].

2 Notations and Propositions

Let G be a graph and $f : V(G) \to \mathbb{N}$ be a function. For a subset $X \subseteq V(G)$, we say that (i) X is *f-nice* if $d_X(x) \geqslant f(x) + \mu_G(x) - 1$ for each $x \in X$, (ii) X is *f-feasible* if $d_X(x) \geqslant f(x)$ for each $x \in X$, (iii) X is *f-meager* if for each nonempty subset $X' \subseteq X$ there exists a vertex $x \in X'$ such that $d_{X'}(x) \leqslant f(x) + \mu_G(x) - 1$, and (iv) X is *f-degenerate* if for each nonempty subset $X' \subseteq X$ there exists a vertex $x \in X'$ such that $d_{X'}(x) \leqslant f(x)$. We have the following propositions immediately from the definitions.

Proposition 6. *If $\mu_G(x) \geqslant 1$ for each $x \in V(G)$, then each f-nice subset is also f-feasible and each f-degenerate subset is also f-meager.*

Proposition 7. *A subset of $V(G)$ does not contain any f-feasible subset if and only if it is $(f - 1)$-degenerate.*

For a graph G and two functions $a, b : V(G) \to \mathbb{N}$, a pair (X, Y) of disjoint subsets of $V(G)$ is called an *(a, b)-feasible* pair if X is a-feasible and Y is b-feasible; if in addition (X, Y) is a partition of $V(G)$, then we call it an (a, b)-feasible partition. Similarly, a partition (X, Y) of $V(G)$ is called an *(a, b)-meager* partition if X is a-meager and Y is b-meager. The following proposition due to Schweser and Stiebitz [12] plays a vital role in our proof of Theorem 5.

Proposition 8 (Schweser and Stiebitz [12]). *Let G be a graph without isolated vertices, and let $a, b : V(G) \to \mathbb{N}$ be two functions such that $d_G(x) \geqslant a(x) + b(x) + 2\mu_G(x) - 3$ for each $x \in V(G)$. If G has an (a, b)-feasible pair, then it admits an (a, b)-feasible partition.*

Let G be a graph and let $a, b : V(G) \to \mathbb{N}$ be two functions. For each partition (A, B) of $V(G)$, we define the weight $\omega(A, B)$ of (A, B) as

$$\omega(A, B) = |E(G[A])| + |E(G[B])| + \sum_{u \in A} b(u) + \sum_{v \in B} a(v).$$

Then, for each $u \in A$ and $v \in B$, simple calculations show that

$$\omega(A \setminus \{u\}, B \cup \{u\}) - \omega(A, B) = d_B(u) - d_A(u) + a(u) - b(u), \tag{1}$$

$$\omega(A \cup \{v\}, B \setminus \{v\}) - \omega(A, B) = d_A(v) - d_B(v) + b(v) - a(v) \tag{2}$$

and

$$\begin{aligned} &\omega(A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) - \omega(A, B) \\ &= d_B(u) - d_A(u) + a(u) - b(u) + d_A(v) - d_B(v) + b(v) - a(v) - 2\mu_G(u, v). \end{aligned} \tag{3}$$

3 Proof of Theorem 5

Throughout this section, let G be a $\{K_4^-, C_5^+, K_{2,3}, L_3\}$-free graph and $a, b : V(G) \to \mathbb{N} \setminus \{0, 1\}$ be two functions such that $d_G(x) \geqslant a(x) + b(x) + 2\mu_G(x) - 3$ for each $x \in V(G)$. Clearly, $d_G(x) \geqslant 1$ for each $x \in V(G)$. Thus, $\mu_G(x) \geqslant 1$ for each $x \in V(G)$. Since there is no danger of confusion, the reference to G in the subscript of μ_G will be dropped in the following proof.

Suppose for a contradiction that G contains no (a, b)-feasible partitions. It follows from Proposition 8 that there is no (a, b)-feasible pair in G. We may assume that

$$d_G(x) = a(x) + b(x) + 2\mu(x) - 3 \tag{4}$$

for each $x \in V(G)$. Otherwise, we can increase a, b to get functions a', b' such that $a' \geqslant a$, $b' \geqslant b$ and $d_G(x) = a'(x) + b'(x) + 2\mu(x) - 3$ for each $x \in V(G)$. Clearly, the existence of an (a', b')-feasible partition would guarantee that of an (a, b)-feasible partition in G.

Claim 9. *There exists an $(a - 1, b - 1)$-meager partition in G.*

Proof. Observe that there is an a-nice proper subset of $V(G)$. Indeed, for a fixed $u \in V(G)$ and each $x \in V(G) \setminus \{u\}$, it follows from (4) that

$$d_{V(G)\setminus\{u\}}(x) = d_G(x) - \mu(u, x) \geqslant a(x) + b(x) + \mu(x) - 3 \geqslant a(x) + \mu(x) - 1,$$

meaning that $V(G) \setminus \{u\}$ is a-nice. Let S be a minimum a-nice subset of $V(G)$ and $T = V(G) \setminus S$. Clearly, $|S| \geqslant 2$ and $T \neq \emptyset$. Note that S is a-feasible by Proposition 6.

Since G has no (a,b)-feasible pair, T contains no b-feasible subset. By Proposition 7, T is $(b-1)$-degenerate, and thus is $(b-1)$-meager. Take $v \in S$ and it follows that $S \setminus \{v\}$ is $(a-1)$-meager by the minimality of S. Note that $d_S(v) \geqslant a(v) + \mu(v) - 1$. This together with (4) yields that $d_{T\cup\{v\}}(v) = d_T(v) \leqslant b(v) + \mu(v) - 2$. Thus, $T \cup \{v\}$ is $(b-1)$-meager. If not, then there is a b-nice subset $T' \subseteq T \cup \{v\}$. Since T is $(b-1)$-meager, we have $v \in T'$ and $d_{T\cup\{v\}}(v) \geqslant d_{T'}(v) \geqslant b(v)+\mu(v)-1$, a contradiction. Consequently, $(S\setminus\{v\}, T\cup\{v\})$ is an $(a-1, b-1)$-meager partition in G, as desired. □

Let $\mathscr{P}$ be the family of all $(a-1, b-1)$-meager partitions (A, B) satisfying that $\omega(A, B)$ is maximum. For any $(A, B) \in \mathscr{P}$, let $A^- = \{u \in A \mid d_A(u) \leqslant a(u) + \mu(u) - 2\}$ and $B^- = \{v \in B \mid d_B(v) \leqslant b(v) + \mu(v) - 2\}$. Note that both A^- and B^- are nonempty by the definition of $\mathscr{P}$. So for any $v \in B^-$, $d_A(v) = d_G(v) - d_B(v) \geqslant a(v) + \mu(v) - 1$, implying $|A| \geqslant 2$. Similarly, $|B| \geqslant 2$.

Claim 10. *For any $(A, B) \in \mathscr{P}$, $u \in A^-$ and $v \in B^-$, we have $A \cup \{v\}$ is not $(a-1)$-meager and every a-nice subset of $A \cup \{v\}$ contains u and v; furthermore, $B \cup \{u\}$ is not $(b-1)$-meager and every b-nice subset of $B \cup \{u\}$ contains u and v.*

Proof. Note that $\omega(A\cup\{v\}, B\setminus\{v\}) - \omega(A, B) = d_G(v) - 2d_B(v) + b(v) - a(v)$ by (2). This together with (4) and $d_B(v) \leqslant b(v)+\mu(v)-2$ implies that $\omega(A\cup\{v\}, B\setminus\{v\}) - \omega(A, B) \geqslant 1$. Thus, $(A \cup \{v\}, B \setminus \{v\})$ cannot be an $(a-1, b-1)$-meager partition by the maximality of $\omega(A, B)$. Since $B \setminus \{v\}$ is $(b-1)$-meager, $A \cup \{v\}$ cannot be $(a-1)$-meager. Similarly, $B \cup \{u\}$ is not $(b-1)$-meager. Hence there exist an a-nice subset $A' \subseteq A \cup \{v\}$ and a b-nice subset $B' \subseteq B \cup \{u\}$. Since A is $(a-1)$-meager and B is $(b-1)$-meager, we have $v \in A'$ and $u \in B'$. Now, we prove that $u \in A'$ and $v \in B'$. If $u \notin A'$ and $v \notin B'$, then (A', B') is an (a, b)-feasible pair by Proposition 6, a contradiction. Suppose by symmetry that $u \in A'$ and $v \notin B'$. Clearly, $B' \subseteq (B \cup \{u\}) \setminus \{v\}$ and $d_{B\setminus\{v\}}(u) = d_{B\cup\{u\}\setminus\{v\}}(u) \geqslant d_{B'}(u) \geqslant b(u)+\mu(u)-1$. Thus, $d_{A'}(u) \leqslant d_{A\cup\{v\}}(u) = d_G(u) - d_{B\setminus\{v\}}(u) \leqslant a(u)+\mu(u)-2$, a contradiction. □

Let $A^* \subseteq A$ such that $A^* \cap A^- \neq \emptyset$. By Claim 10, $B \cup A^*$ is not $(b-1)$-meager and there exists a b-nice subset of $B \cup A^*$, indicating that $A \setminus A^*$ is $(a-1)$-degenerate as G has no (a, b)-feasible pair. Similarly, if $B^* \subseteq B$ such that $B^* \cap B^- \neq \emptyset$, then $B \setminus B^*$ is $(b-1)$-degenerate. We point out that Claim 10 will be also used in this form frequently.

Claim 11. *For any $(A, B) \in \mathscr{P}$, every vertex in A^- is adjacent to every vertex in B^-.*

Proof. Suppose that there exist $u \in A^-$ and $v \in B^-$ such that $\mu(u, v) = 0$. By Claim 10, there is an a-nice subset $A' \subseteq A \cup \{v\}$ such that $u \in A'$, implying that $d_{A'}(u) \geqslant a(u) + \mu(u) - 1$. However, $d_{A'}(u) \leqslant d_{A\cup\{v\}}(u) = d_A(u) + \mu(u, v) \leqslant a(u) + \mu(u) - 2$, a contradiction. □

Recall that both A^- and B^- are nonempty. By Claim 11, either $|A^-| = |B^-| = 2$ or $\min\{|A^-|, |B^-|\} = 1$ as G is $K_{2,3}$-free.

Claim 12. *For any $(A, B) \in \mathscr{P}$, we have $A \setminus A^- \neq \emptyset$ and $B \setminus B^- \neq \emptyset$.*

Proof. For each $u \in A^-$, there exists a b-nice subset $B' \subseteq B \cup \{u\}$ by Claim 10. It follows that $d_{B'}(y) \geqslant b(y) + \mu(y) - 1 \geqslant \mu(y) + 1$ for each $y \in B'$, implying $|N_{B'}(y)| \geqslant 2$. If $|A^-| = |B^-| = 2$, then we let $B^- = \{v_1, v_2\}$. Since G is K_4^--free, $v_1v_2 \notin E(G)$ by Claim 11. Thus, $N_{B'}(v_1) = N_{B'}(v_2) = \{u\}$ providing that $B = B^-$. This leads to a contradiction as $v_i \in B'$ for some $i = 1, 2$, implying $B \setminus B^- \neq \emptyset$. Similarly, $A \setminus A^- \neq \emptyset$. If $\min\{|A^-|, |B^-|\} = 1$, then we assume that $A^- = \{u\}$. Clearly, $A \setminus A^- \neq \emptyset$ as $|A| \geqslant 2$. Since A is $(a-1)$-meager, there exists $x \in A \setminus \{u\}$ such that $d_{A\setminus\{u\}}(x) \leqslant a(x) + \mu(x) - 2$. Note that $d_{A\setminus\{u\}}(x) + \mu(u, x) = d_A(x) \geqslant a(x) + \mu(x) - 1$. It follows that $\mu(u, x) \geqslant 1$ and $d_A(x) \leqslant a(x) + 2\mu(x) - 2$, yielding that $ux \in E(G)$ and $d_B(x) = d_G(x) - d_A(x) \geqslant b(x) - 1 \geqslant 1$. Suppose that $B = B^-$ and $z \in N_B(x)$. Choose $v = z$ in Claim 10, implying $z \in B'$. Since $|N_{B'}(z)| \geqslant 2$, there exists $z' \in B^- \setminus \{z\}$ such that $zz' \in E(G)$. By Claim 11, $\{u, x, z, z'\}$ forms a K_4^-, a contradiction. Thus, $B \setminus B^- \neq \emptyset$. □

For any $(A, B) \in \mathscr{P}$, let $D_A = \{u \in A \mid d_A(u) \leqslant a(u) - 1\}$ and $D_B = \{v \in B \mid d_B(v) \leqslant b(v) - 1\}$. Clearly, $D_A \subseteq A^-$ and $D_B \subseteq B^-$.

Claim 13. *For any $(A, B) \in \mathscr{P}$, $u \in A^-$ and $v \in B^-$, if either $u \in D_A$ or $v \in D_B$, then $(A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) \in \mathscr{P}$. Moreover, if $u \in D_A$, then $\mu(u, v) = \mu(u)$, $d_A(u) = a(u) - 1$ and $d_B(v) = b(v) + \mu(v) - 2$; if $v \in D_B$, then $\mu(u, v) = \mu(v)$, $d_B(v) = b(v) - 1$ and $d_A(u) = a(u) + \mu(u) - 2$.*

Proof. Since every a-nice subset of $A \cup \{v\}$ contains u by Claim 10, $A \cup \{v\} \setminus \{u\}$ is $(a-1)$-meager. Similarly, $B \cup \{u\} \setminus \{v\}$ is $(b-1)$-meager. Thus, $(A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\})$ is an $(a-1, b-1)$-meager partition. By (3), $\omega(A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) - \omega(A, B) = (d_G(u) - 2d_A(u) + a(u) - b(u)) + (d_G(v) - 2d_B(v) + b(v) - a(v)) - 2\mu(u, v)$. Suppose by symmetry that $u \in D_A$. Since $d_A(u) \leqslant a(u) - 1$ and $d_B(v) \leqslant b(v) + \mu(v) - 2$, by (4), we have

$$\omega(A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) - \omega(A, B) \geqslant (2\mu(u) - 1) + 1 - 2\mu(u, v) = 2(\mu(u) - \mu(u, v)) \geqslant 0.$$

By the maximality of $\omega(A, B)$, $\omega(A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) = \omega(A, B)$. Thus, $(A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) \in \mathscr{P}$, $\mu(u, v) = \mu(u)$, $d_A(u) = a(u) - 1$ and $d_B(v) = b(v) + \mu(v) - 2$. □

By Claim 13, $D_A = \{u \in A \mid d_A(u) = a(u) - 1\}$ and $D_B = \{v \in B \mid d_B(v) = b(v) - 1\}$; in addition, $d_A(u) \geqslant a(u) - 1$ and $d_B(v) \geqslant b(v) - 1$ for each $u \in A$ and $v \in B$.

Claim 14. *For any $(A, B) \in \mathscr{P}$, we have $\min\{|A^-|, |B^-|\} = 1$.*

Proof. Suppose for a contradiction that $A^- = \{u_1, u_2\}$ and $B^- = \{v_1, v_2\}$. Since G is K_4^--free, $u_1u_2, v_1v_2 \notin E(G)$ by Claim 11. Note that $A \cup B^-$ is not $(a-1)$-meager by Claim 10. It follows that $B \setminus B^-$ is $(b-1)$-degenerate as G has no (a, b)-feasible pair and $B \setminus B^- \neq \emptyset$ by Claim 12. Thus, there exists $y \in B \setminus B^-$ such that $d_{B\setminus B^-}(y) \leqslant b(y) - 1$, implying $N_{B^-}(y) \neq \emptyset$ as $d_B(y) \geqslant b(y) + \mu(y) - 1 \geqslant b(y)$. By Claim 11, $|N_{B^-}(y)| = 1$ as G is $K_{2,3}$-free, say $N_{B^-}(y) = \{v_1\}$. By symmetry, $A \setminus A^-$ is $(a-1)$-degenerate and there exists $x_1 \in A \setminus A^-$ such that $d_{A\setminus A^-}(x_1) \leqslant a(x_1) - 1$ and $|N_{A^-}(x_1)| = 1$, say $N_{A^-}(x_1) = \{u_1\}$. Clearly, $d_{A\setminus\{u_1\}}(x_1) = d_{A\setminus A^-}(x_1) \leqslant a(x_1) - 1$ and $d_{B\setminus\{v_1\}}(y) = d_{B\setminus B^-}(y) \leqslant b(y) - 1$.

Since G has no (a,b)-feasible partition, either A is $(a-1)$-degenerate or B is $(b-1)$-degenerate. We may assume that A is $(a-1)$-degenerate. Thus, either $d_A(u_1) \leqslant a(u_1)-1$ or $d_A(u_2) \leqslant a(u_2)-1$. If $d_A(u_1) \leqslant a(u_1)-1$, then we set $u := u_1$ and $x := x_1$. If $d_A(u_1) \geqslant a(u_1)$, then $d_A(u_2) \leqslant a(u_2)-1$. Clearly, $A \setminus \{u_2\}$ is $(a-1)$-degenerate. Thus, there exists $x_2 \in A \setminus \{u_2\}$ such that $d_{A\setminus\{u_2\}}(x_2) \leqslant a(x_2)-1$. Note that $d_{A\setminus\{u_2\}}(u_1) = d_A(u_1) \geqslant a(u_1)$ as $u_1u_2 \notin E(G)$. Thus, $x_2 \neq u_1$ and $x_2 \in A \setminus A^-$. Note also that $d_A(x_2) \geqslant a(x_2)+\mu(x_2)-1 \geqslant a(x_2)$. This implies $u_2x_2 \in E(G)$. Set $u := u_2$ and $x := x_2$. In both cases, we have $ux \in E(G)$, $d_A(u) \leqslant a(u)-1$ and $d_{A\setminus\{u\}}(x) \leqslant a(x)-1$. Since G is C_5^+-free, we have $xv_1, uy \notin E(G)$. By Claim 13, $(A_0, B_0) := (A \cup \{v_1\} \setminus \{u\}, B \cup \{u\} \setminus \{v_1\}) \in \mathscr{P}$. Observe that $d_{A_0}(x) = d_{A\setminus\{u\}}(x) \leqslant a(x)-1$ and $d_{B_0}(y) = d_{B\setminus\{v_1\}}(y) \leqslant b(y)-1$. Thus, $x \in A_0^-$ and $y \in B_0^-$, yielding $xy \in E(G)$ by Claim 11. It follows that $\{u_1, u_2, v_1, v_2, x, y\}$ contains an L_3, a contradiction. □

For any $(A,B) \in \mathscr{P}$, define $A^= = \{x \in A \mid d_A(x) = a(x)+\mu(x)-1\}$ and $B^= = \{y \in B \mid d_B(y) = b(y)+\mu(y)-1\}$. A path $xuvy$ is called a *special path* with respect to (A,B), if $u \in A^-$, $v \in B^-$, $x \in A^=$ and $y \in B^=$.

Claim 15. *For any special path $xuvy$ with respect to $(A,B) \in \mathscr{P}$, if either $u \in D_A$ or $v \in D_B$, then either $vx \in E(G)$ or $uy \in E(G)$. Moreover, if $vx \in E(G)$, then $N_{A^=}(u) = \{x\}$; if $uy \in E(G)$, then $N_{B^=}(v) = \{y\}$.*

Proof. Suppose that $vx, uy \notin E(G)$. We may assume by symmetry that $u \in D_A$. By Claim 13, $(A_1, B_1) := (A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) \in \mathscr{P}$, $\mu(u,v) = \mu(u)$, $d_A(u) = a(u)-1$ and $d_B(v) = b(v)+\mu(v)-2$. This together with $d_{A_1}(v) = d_G(v)-d_B(v)-\mu(u,v)$ and $d_{B_1}(u) = d_G(u)-d_A(u)-\mu(u,v)$ implies $v \in A_1^-$ and $u \in B_1^-$. Since $x \in A^=$ and $y \in B^=$, we have $d_{A_1}(x) = d_A(x)-\mu(u,x) = a(x)+\mu(x)-1-\mu(u,x)$ and $d_{B_1}(y) = d_B(y)-\mu(v,y) = b(y)+\mu(y)-1-\mu(v,y)$, indicating $x \in A_1^-$ and $y \in B_1^-$. This contradicts Claim 14.

Suppose that $vx \in E(G)$ and there exists $x' \in N_{A^=}(u) \setminus \{x\}$. Clearly, $x'uvy$ forms another special path with respect to (A,B). It follows that either $uy \in E(G)$ or $vx' \in E(G)$. In both cases, we can find a K_4^-, a contradiction. Similarly, if $uy \in E(G)$, then $N_{B^=}(v) = \{y\}$. □

Claim 16. *For any $(A,B) \in \mathscr{P}$, let $u \in A^-$ and $v \in B^-$. If $u \in D_A$ and $x \in N_{A^=}(u)$ with $vx \notin E(G)$, then $(A \cup \{v\} \setminus \{x\}, B \cup \{x\} \setminus \{v\}) \in \mathscr{P}$; if $v \in D_B$ and $y \in N_{B^=}(v)$ with $uy \notin E(G)$, then $(A \cup \{y\} \setminus \{u\}, B \cup \{u\} \setminus \{y\}) \in \mathscr{P}$.*

Proof. Assume that $u \in D_A$ and $x \in N_{A^=}(u)$ with $vx \notin E(G)$. We first show that $B \cup \{x\} \setminus \{v\}$ is $(b-1)$-meager. If not, then there is a b-nice subset $B' \subseteq B \cup \{x\} \setminus \{v\}$. This implies that $x \in B'$ as B is $(b-1)$-meager. Since $vx \notin E(G)$ and $x \in A^=$, $d_{B'}(x) \leqslant d_{B\cup\{x\}\setminus\{v\}}(x) = d_B(x) = d_G(x)-d_A(x) = b(x)+\mu(x)-2$, contradicting with $x \in B'$. Now, we prove that $A \cup \{v\} \setminus \{x\}$ is $(a-1)$-meager. Otherwise, there is an a-nice subset $A' \subseteq A \cup \{v\} \setminus \{x\}$. Since A is $(a-1)$-meager, we have $v \in A'$ and $d_{A'}(v) \geqslant a(v)+\mu(v)-1$. Note that $d_B(v) = b(v)+\mu(v)-2$ by Claim 13 as $u \in D_A$. It follows that $d_{A'}(v) \leqslant d_{A\cup\{v\}\setminus\{x\}}(v) = d_A(v) = a(v)+\mu(v)-1$ as $vx \notin E(G)$. Thus, $d_{A'}(v) = d_A(v)$, implying $u \in A'$ as $uv \in E(G)$. The fact $d_{A'}(u) \leqslant d_{A\cup\{v\}\setminus\{x\}}(u) =$

$d_A(u) + \mu(u,v) - \mu(u,x) \leqslant a(u) + \mu(u) - 2$ also indicates that $u \notin A'$, a contradiction. Therefore, $(A \cup \{v\} \setminus \{x\}, B \cup \{x\} \setminus \{v\})$ is an $(a-1, b-1)$-meager partition. With simple calculations, we have $\omega((A \cup \{v\} \setminus \{x\}, B \cup \{x\} \setminus \{v\})) = \omega(A, B)$ in view of (3) and (4). Thus, $(A \cup \{v\} \setminus \{x\}, B \cup \{x\} \setminus \{v\}) \in \mathscr{P}$. Similarly, if $v \in D_B$ and $y \in N_{B^=}(v)$ with $uy \notin E(G)$, then $(A \cup \{y\} \setminus \{u\}, B \cup \{u\} \setminus \{y\}) \in \mathscr{P}$. □

Fix a partition $(A, B) \in \mathscr{P}$. By Claim 14, we may assume by symmetry that

$$A^- = \{u\} \text{ and } |B^-| \geqslant |A^-|.$$

By Claim 10, $B \cup \{u\}$ is not $(b-1)$-meager. Since G has no (a, b)-feasible pair, $A \setminus \{u\}$ is $(a-1)$-degenerate, implying that there exists $x_1 \in A \setminus \{u\}$ such that $d_{A\setminus\{u\}}(x_1) \leqslant a(x_1) - 1$. Note that $d_A(x_1) \geqslant a(x_1) + \mu(x_1) - 1$ as $x_1 \in A \setminus A^-$ and $d_{A\setminus\{u\}}(x_1) = d_A(x_1) - \mu(u, x_1)$. It follows that $\mu(u, x_1) = \mu(x_1)$, $d_{A\setminus\{u\}}(x_1) = a(x_1) - 1$ and $d_A(x_1) = a(x_1) + \mu(x_1) - 1$. Hence,

$$x_1 \in N_{A^=}(u).$$

Recall that either A is $(a-1)$-degenerate or B is $(b-1)$-degenerate. It follows that either $D_A \neq \emptyset$ or $D_B \neq \emptyset$. In what follows, we may assume that

$$D_B \neq \emptyset. \tag{5}$$

Otherwise, let $D_B = \emptyset$. Clearly, B is b-feasible and A is $(a-1)$-degenerate. Thus, $D_A = \{u\}$. If $|B^-| = 1$, then the case can be reduced to (5) by symmetry as $D_A \neq \emptyset$. Suppose that $|B^-| \geqslant 2$ and $v_1, v_2 \in B^-$. Since G is K_4^--free, either $x_1v_1 \notin E(G)$ or $x_1v_2 \notin E(G)$ by Claim 11. By symmetry, assume that $x_1v_1 \notin E(G)$. Clearly, $(A_2, B_2) := (A \cup \{v_1\} \setminus \{u\}, B \cup \{u\} \setminus \{v_1\}) \in \mathscr{P}$, $\mu(u, v) = \mu(u)$ and $d_B(v) = b(v) + \mu(v) - 2$ for each $v \in B^-$ by Claim 13. It is easy to check that $v_1 \in A_2^-$, $x_1 \in D_{A_2} \subseteq A_2^-$ and $u \in B_2^-$. Thus, $B_2^- = \{u\}$ by Claim 14. Again, this can be reduced to (5) as $|B_2^-| = 1$ and $D_{A_2} \neq \emptyset$.

For each $v \in D_B$ and the fixed vertex x_1, let $A_v = A \cup \{v\} \setminus \{x_1\}$ and $B_v = B \cup \{x_1\} \setminus \{v\}$.

Claim 17. *For each $v \in D_B$, if $x_1v \notin E(G)$, then* (i) $\mu(v) = 1$; (ii) $(A_v, B_v) \in \mathscr{P}$, *$u \in A_v^-$, $v \in A_v^=$ and $x_1 \in B_v^-$.*

Proof. (i) By Claim 13, $(A_3, B_3) := (A \cup \{v\} \setminus \{u\}, B \cup \{u\} \setminus \{v\}) \in \mathscr{P}$, $\mu(v) = \mu(u, v)$ and $d_A(u) = a(u) + \mu(u) - 2$ as $v \in D_B$. Recall that $d_{A\setminus\{u\}}(x_1) = a(x_1) - 1$. Thus, $d_{A_3}(x_1) = d_{A\setminus\{u\}}(x_1) = a(x_1) - 1$ as $x_1v \notin E(G)$, yielding $x_1 \in D_{A_3}$. Note that $d_{B_3}(u) = d_G(u) - d_A(u) - \mu(u, v) = b(u) + \mu(u) - 1 - \mu(u, v)$. This implies $u \in B_3^-$ as $\mu(u, v) \geqslant 1$. Applying Claim 13 with $(A_3, B_3) \in \mathscr{P}$, $x_1 \in D_{A_3}$ and $u \in B_3^-$, we have $d_{B_3}(u) = b(u) + \mu(u) - 2$. It follows that $\mu(u, v) = 1$, implying $\mu(v) = 1$.

(ii) Recall that $d_A(u) = a(u) + \mu(u) - 2$ and $\mu(u, v) = \mu(v) = 1$. Since $v \in D_B$ and $x_1 \in A^=$, we have $d_{A_v}(u) = d_A(u) + \mu(u, v) - \mu(u, x_1) = a(u) + \mu(u) - 1 - \mu(u, x_1)$, $d_{A_v}(v) = d_G(v) - d_B(v) = a(v)$ and $d_{B_v}(x_1) = d_G(x_1) - d_A(x_1) = b(x_1) + \mu(x_1) - 2$. Now, we show that B_v is $(b-1)$-meager. If not, then there exists a b-nice subset $B' \subseteq B_v$. Since B is $(b-1)$-meager, we have $x_1 \in B'$ and $d_{B_v}(x_1) \geqslant d_{B'}(x_1) \geqslant b(x_1) + \mu(x_1) - 1$, a contradiction. Next, we prove that A_v is $(a-1)$-meager. Otherwise, there is an a-nice subset $A' \subseteq A_v$. Since A is $(a-1)$-meager, we have $v \in A'$ and $d_{A_v}(v) \geqslant d_{A'}(v) \geqslant a(v)+\mu(v)-1 = a(v)$. This implies that $d_{A_v}(v) = d_{A'}(v)$. Thus, $u \in A'$ as $uv \in E(G)$. It follows that $d_{A_v}(u) \geqslant d_{A'}(u) \geqslant a(u)+\mu(u)-1$, a contradiction. Therefore, (A_v, B_v) is an $(a-1, b-1)$-meager partition. Simple calculations together with (3) and (4) show that $\omega(A_v, B_v) = \omega(A, B)$, implying $(A_v, B_v) \in \mathscr{P}$. Moreover, $u \in A_v^-$, $v \in A_v^=$ and $x_1 \in B_v^-$ by noting that $\mu(u, x_1) \geqslant 1$ and $\mu(v) = 1$. □

Now, we conclude that D_B is an independent set. Otherwise, there is an edge vv' contained in $G[D_B]$. Since G is K_4^--free, we have $x_1v, x_1v' \notin E(G)$. By Claim 17, $\mu(v) = 1$ and $(A_v, B_v) \in \mathscr{P}$. It follows that $d_{B_v}(v') = d_B(v') - \mu(v, v') = b(v') - 2$, contradicting Claim 13.

Note that $B \setminus D_B$ is $(b-1)$-degenerate by Claim 10 as $B \setminus D_B \neq \emptyset$ by Claim 12. Thus, there exists $y \in B \setminus D_B$ such that $d_{B\setminus D_B}(y) \leqslant b(y) - 1$.

Claim 18. *For each $y \in B \setminus D_B$ satisfying $d_{B\setminus D_B}(y) \leqslant b(y) - 1$, we have $|N_{D_B}(y)| = 1$.*

Proof. Note that $d_B(y) = d_{B\setminus D_B}(y) + d_{D_B}(y) \geqslant b(y)$ as $y \in B \setminus D_B$. It follows that $d_{D_B}(y) \geqslant 1$. This together with Claim 11 yields that $1 \leqslant |N_{D_B}(y)| \leqslant 2$ as G is $K_{2,3}$-free. Suppose that $N_{D_B}(y) = \{v_1, v_2\}$ and $v_1v_2 \notin E(G)$ as D_B is independent. Clearly, $d_B(y) = d_{B\setminus D_B}(y) + d_{D_B}(y) \leqslant b(y) - 1 + \mu(v_1, y) + \mu(v_2, y)$. Since G is $\{C_5^+, K_{2,3}\}$-free, $x_1v_1, x_1v_2, x_1y \notin E(G)$. By Claim 17, $(A_{v_1}, B_{v_1}) \in \mathscr{P}$, $u \in A_{v_1}^-$ and $v_1 \in A_{v_1}^=$. Note also that $v_2 \in D_{B_{v_1}}$ as $d_{B_{v_1}}(v_2) = d_B(v_2) = b(v_2) - 1$. Since $d_{B_{v_1}}(y) = d_B(y) - \mu(v_1, y) \leqslant b(y) - 1 + \mu(v_2, y) \leqslant b(y) + \mu(y) - 1$, we have either $y \in B_{v_1}^-$ or $y \in B_{v_1}^=$. If $y \in B_{v_1}^-$, then $uy \in E(G)$ by Claim 11; if $y \in B_{v_1}^=$, then v_1uv_2y forms a special path with respect to (A_{v_1}, B_{v_1}), indicating that either $uy \in E(G)$ or $v_1v_2 \in E(G)$ by Claim 15. In both cases, $\{u, v_1, v_2, y\}$ contains a K_4^-, a contradiction. □

By Claim 18, we can fix such a vertex $y \in B \setminus D_B$ and assume that

$$N_{D_B}(y) = \{v_1\}$$

for some vertex $v_1 \in D_B$. It follows that $d_B(y) = d_{B\setminus D_B}(y) + d_{D_B}(y) \leqslant b(y) - 1 + \mu(v_1, y) \leqslant b(y) + \mu(y) - 1$, thus either $y \in B^- \setminus D_B$ or $y \in B^=$. If $y \in B^- \setminus D_B$, then $uy \in E(G)$ by Claim 11. If $y \in B^=$, then x_1uv_1y forms a special path with respect to (A, B). Since $v_1 \in D_B$, we have either $x_1v_1 \in E(G)$ or $uy \in E(G)$ by Claim 15. Hence, we conclude

$$\text{either } x_1v_1 \in E(G) \text{ or } uy \in E(G). \tag{6}$$

Claim 19. *If $uy \in E(G)$, then $\mu(x_1) = 1$; if $x_1v_1 \in E(G)$, then $y \in B^=$, $\mu(v_1, y) = \mu(y) = 1$, $d_B(y) = b(y)$ and $d_{B\setminus D_B}(y) = b(y) - 1$.*

Proof. If $uy \in E(G)$, then $x_1v_1, x_1y \notin E(G)$ as G is K_4^--free. By Claim 17, $(A_{v_1}, B_{v_1}) \in \mathscr{P}$, $u \in A_{v_1}^-$ and $d_{A_{v_1}}(u) = a(u) + \mu(u) - 1 - \mu(u, x_1)$. Note that $y \in D_{B_{v_1}}$ as $d_{B_{v_1}}(y) = d_{B\setminus D_B}(y) \leqslant b(y) - 1$. It follows that $d_{A_{v_1}}(u) = a(u) + \mu(u) - 2$ by Claim 13, implying $\mu(u, x_1) = 1$. The desired result follows by noting that $\mu(x_1) = \mu(u, x_1)$.

If $x_1v_1 \in E(G)$, then $uy, x_1y \notin E(G)$ as G is K_4^--free. Clearly, $y \in B^=$, $\mu(y) = \mu(v_1, y)$ and $d_{B\setminus D_B}(y) = b(y) - 1$. By Claim 16, $(A_4, B_4) := (A \cup \{y\} \setminus \{u\}, B \cup \{u\} \setminus \{y\}) \in \mathscr{P}$. Note that $d_{A_4}(x_1) = d_{A\setminus\{u\}}(x_1) = a(x_1) - 1$ and $d_{B_4}(v_1) = d_B(v_1) + \mu(u, v_1) - \mu(v_1, y) \leqslant b(v_1) + \mu(v_1) - 2$. Thus, $x_1 \in D_{A_4}$ and $v_1 \in B_4^-$. By Claim 13, $d_{B_4}(v_1) = b(v_1) + \mu(v_1) - 2$, indicating $\mu(v_1, y) = 1$. Thus, $\mu(y) = \mu(v_1, y) = 1$, $d_B(y) = b(y)$ and $d_{B\setminus D_B}(y) = b(y) - 1$. □

Now, we may further assume that

$$|D_B| \geqslant 2. \tag{7}$$

Otherwise, $D_B = \{v_1\}$ as $v_1 \in D_B$. If $uy \in E(G)$, then $u \in A_{v_1}^-$ and $x_1, y \in D_{B_{v_1}}$ by Claim 17 and the proof of Claim 19. Thus, $A_{v_1}^- = \{u\}$ by Claim 14 and $|D_{B_{v_1}}| \geqslant 2$. If $x_1v_1 \in E(G)$, then $v_1 \in B_4^-$ and $x_1, y \in D_{A_4}$ by the proof of Claim 19. Again, $B_4^- = \{v_1\}$ by Claim 14 and $|D_{A_4}| \geqslant 2$. Thus, we can reduce both cases to (7), as desired.

Let $D = D_B \cup \{y\}$. It follows from (6) and (7) that $N_D(v) = \emptyset$ for each $v \in D_B \setminus \{v_1\}$ as G is $\{K_4^-, C_5^+\}$-free and D_B is independent. This implies that $d_{B\setminus D}(v) = d_B(v) = b(v) - 1 \geqslant 1$, i.e., $B \setminus D \neq \emptyset$. By Claim 10, $B \setminus D$ is $(b-1)$-degenerate. Thus, there exists $z \in B \setminus D$ such that $d_{B\setminus D}(z) \leqslant b(z) - 1$. This together with $d_B(z) \geqslant b(z)$ gives that $N_D(z) \neq \emptyset$ and

$$d_B(z) = d_{B\setminus D}(z) + d_D(z) \leqslant b(z) - 1 + \sum_{x\in N_D(z)} \mu(x, z). \tag{8}$$

In what follows, we proceed our proof by considering $N_D(z)$ according to (6).

Case 1. $x_1v_1 \in E(G)$. By Claim 19, we have $y \in B^=$, $\mu(y) = 1$, $d_B(y) = b(y)$ and $d_{B\setminus D_B}(y) = b(y) - 1$. We first establish the following easy but useful claim.

Claim 20. (i) *There exists $w \in N_{A^=}(x_1)$ such that $uw \notin E(G)$, $\mu(x_1, w) = \mu(w)$ and $d_{A\setminus\{u,x_1\}}(w) = a(w) - 1$.* (ii) *If there exists $y' \in N_{B^=}(y)$, then $v_1y' \in E(G)$.*

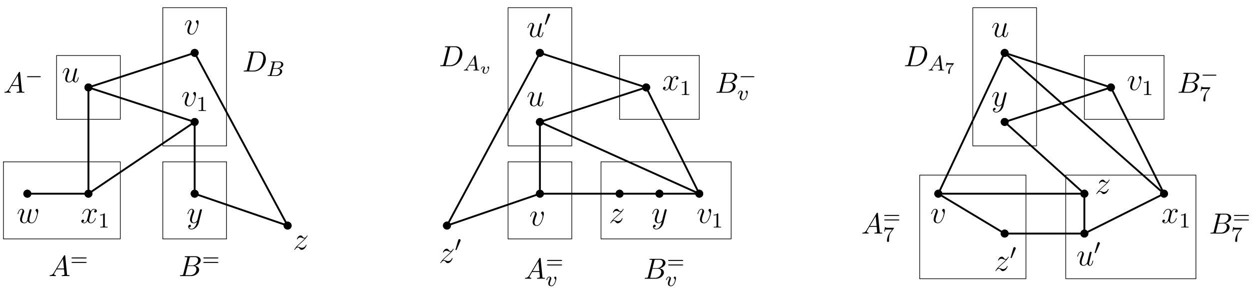
Proof. (i) Let $U = \{u, x_1\}$. Clearly, $A \setminus U \neq \emptyset$ as $d_{A\setminus U}(x_1) = d_{A\setminus\{u\}}(x_1) = a(x_1) - 1 \geqslant 1$. By Claim 10, $A \setminus U$ is $(a-1)$-degenerate, implying that there exists $w \in A \setminus U$ such that $d_{A\setminus U}(w) \leqslant a(w) - 1$. It follows that $d_U(w) = d_A(w) - d_{A\setminus U}(w) \geqslant a(w) + \mu(w) - 1 - (a(w) - 1) = \mu(w) \geqslant 1$, i.e., $N_U(w) \neq \emptyset$. Thus, $|N_U(w)| = 1$ as G is K_4^--free, implying $d_U(w) \leqslant \mu(w)$. Then $d_U(w) = \mu(w)$, $d_A(w) = a(w) + \mu(w) - 1$ and $d_{A\setminus U}(w) = a(w) - 1$. Since $w \in A^=$ and $N_{A^=}(u) = \{x_1\}$ by Claim 15, we have $uw \notin E(G)$, $x_1w \in E(G)$ and $\mu(x_1, w) = \mu(w)$.

(ii) Suppose that $y' \in N_{B^=}(y)$ such that $v_1y' \notin E(G)$. Since G is $\{K_4^-, C_5^+\}$-free, we have $x_1y, uy, uy' \notin E(G)$. By Claim 13, we have $(A_5, B_5) := (A \cup \{v_1\} \setminus \{u\}, B \cup \{u\} \setminus \{v_1\}) \in \mathscr{P}$ together with the following formulas: (i) $d_{A_5}(v_1) = d_A(v_1) - \mu(u, v_1) = a(v_1) + \mu(v_1) - 2$; (ii) $d_{B_5}(u) = d_B(u) - \mu(u, v_1) \leqslant b(u) + \mu(u) - 2$; (iii) $d_{A_5}(x_1) = d_A(x_1) + \mu(v_1, x_1) - \mu(u, x_1) \leqslant a(x_1) + \mu(x_1) - 1$; (iv) $d_{B_5}(y) = d_B(y) - \mu(v_1, y) = b(y) - 1$; (v) $d_{B_5}(y') = d_B(y') = b(y') + \mu(y') - 1$. It follows that $v_1 \in A_5^-$, $u \in B_5^-$, $x_1 \in A_5^- \cup A_5^=$, $y \in D_{B_5} \subseteq B_5^-$ and $y' \in B_5^=$. By Claim 14, $A_5^- = \{v_1\}$, implying $x_1 \in A_5^=$. Thus, x_1v_1yy' forms a special path with respect to (A_5, B_5). By Claim 15, either $x_1y \in E(G)$ or $v_1y' \in E(G)$ as $y \in D_{B_5}$, a contradiction. □

Now, we consider $N_D(z)$ and assert that $v_1 \notin N_D(z)$. Otherwise, let $v_1z \in E(G)$. Clearly, $uw, uy, uz, wy, x_1y, wv_1, x_1z \notin E(G)$ and $N_{D_B}(z) = \{v_1\}$ as G is $\{K_4^-, C_5^+\}$-free. We focus on the partition $(A_4, B_4) = (A \cup \{y\} \setminus \{u\}, B \cup \{u\} \setminus \{y\}) \in \mathscr{P}$ defined in the second part of the proof of Claim 19. Clearly, $x_1, y \in D_{A_4} \subseteq A_4^-$, $v_1 \in B_4^-$ and $w \in A_4^=$ as $d_{A_4}(w) = d_A(w) = a(w) + \mu(w) - 1$. Note that $d_{B_4}(z) = d_B(z) - \mu(y,z) \leqslant b(z) - 1 + \sum_{x \in N_{D_B}(z)} \mu(x,z)$ by (8). It follows that $z \in B_4^=$ as $N_{D_B}(z) = \{v_1\}$ and $z \notin B_4^-$ by Claim 14. Then wx_1v_1z forms a special path with respect to (A_4, B_4). By Claim 15, either $wv_1 \in E(G)$ or $x_1z \in E(G)$ as $x_1 \in D_{A_4}$, a contradiction. We further show that there exists $v \in D_B \setminus \{v_1\}$ such that $v \in N_D(z)$. Otherwise, $N_D(z) = \{y\}$. In view of (8), we know $z \in B^- \cup B^=$. If $z \in B^-$, then $\{u, v_1, x_1, y, z\}$ contains a C_5^+ as $uz \in E(G)$ by Claim 11. Thus, $z \in N_{B^=}(y)$, implying $v_1 \in N_D(z)$ by Claim 20(ii), a contradiction.

Claim 21. *$N_D(z) = \{v, y\}$ with $\mu(z) = 1$ and $d_B(z) = b(z) + 1$.*

Proof. Note that $1 \leqslant |N_{D_B}(z)| \leqslant 2$ as G is $K_{2,3}$-free. Note that $x_1v, x_1y, x_1z, wv, v_1v, vy \notin E(G)$ as G is $\{K_4^-, C_5^+\}$-free. By Claim 17, $\mu(v) = \mu(u,v) = 1$ and $(A_v, B_v) \in \mathscr{P}$; moreover, $u \in A_v^-$ and $x_1 \in B_v^-$. Note also that $d_{A_v}(w) = d_A(w) - \mu(x_1, w) = d_{A \setminus \{u, x_1\}}(w) = a(w) - 1$. Thus, $u, w \in A_v^-$ and $x_1 \in B_v^-$, implying $B_v^- = \{x_1\}$ by Claim 14. If $|N_{D_B}(z)| = 2$, then there exists $v' \in D_B \setminus \{v_1, v\}$ such that $x_1v', vv' \notin E(G)$ as G is K_4^--free. Note that $d_{B_v}(v') = d_B(v') = b(v') - 1$, indicating $v' \in D_{B_v} \subseteq B_v^-$, a contradiction. Hence, $N_{D_B}(z) = \{v\}$. This implies that $1 \leqslant |N_D(z)| \leqslant 2$. If $|N_D(z)| = 1$, then $d_{B_v}(z) = d_B(z) - \mu(v,z) = d_{B \setminus D}(z) \leqslant b(z) - 1$, thus $z \in D_{B_v} \subseteq B_v^-$, a contradiction. Thus, we conclude that $N_D(z) = \{v, y\}$. Observe that $z \in B \setminus B^-$; otherwise, $\{u, v_1, x_1, y, z\}$ contains a C_5^+ as $uz \in E(G)$ by Claim 11. Note that $\mu(v) = \mu(y) = 1$ by Claims 17 and 19 as $x_1v, uy \notin E(G)$. Hence, $b(z) + \mu(z) - 1 \leqslant d_B(z) \leqslant b(z) + 1$ by (8), giving that $\mu(z) \leqslant 2$. If $\mu(z) = 2$, then $d_B(z) = b(z) + 1$ and $z \in B^=$. It follows that $z \in N_{B^=}(y)$, implying $v_1z \in E(G)$ by Claim 20(ii), a contradiction. Hence, $\mu(z) = 1$ and $z \notin B^=$, indicating $d_B(z) = b(z) + 1$. $\square$



(A, B) (A_v, B_v) (A_7, B_7)

Figure 1: Partitions in $\mathscr{P}$

Note that $(A_v, B_v) \in \mathscr{P}$ by Claim 17; additionally, $u \in A_v^-$, $v \in A_v^=$ and $x_1 \in B_v^-$. In what follows, we show that $B_v^- = \{x_1\}$, $u, w \in D_{A_v}$, $v_1 \in N_{B_v^=}(x_1)$ with $d_{B_v \setminus \{x_1\}}(v_1) = b(v_1) - 1$, $y \in N_{B_v^=}(v_1)$ with $d_{B_v \setminus \{x_1, v_1\}}(y) = b(y) - 1$, and $v \in N_{A_v^=}(u)$ with $d_{A_v \setminus D_{A_v}}(v) =$

$a(v)-1$. If so, we may view B_v, A_v as the new parts A, B by the symmetry between the functions a,b, and make sure that we are still in Case 1 as $v_1u \in E(G)$.

Recall that $\mu(v)=\mu(y)=1$. Since G is $\{K_4^-, C_5^+\}$-free, we have $x_1v, x_1y, vy, uy \notin E(G)$. Note that $d_{A_v}(w) = d_{A\setminus\{u,x_1\}}(w) = a(w)-1$ and $d_{B_v}(v_1) = d_B(v_1)+\mu(x_1,v_1) = b(v_1)-1+\mu(x_1,v_1) \leqslant b(v_1)+\mu(v_1)-1$. It follows that $w \in D_{A_v}$ and $v_1 \in B_v^- \cup B_v^=$. Since $u,w \in A_v^-$ and $x_1 \in B_v^-$, we have $B_v^- = \{x_1\}$ and $v_1 \in B_v^=$ by Claim 14. Thus, $d_{B_v}(v_1) = b(v_1)+\mu(v_1)-1$ and $\mu(x_1,v_1)=\mu(v_1)$. This implies that $d_{B_v\setminus\{x_1\}}(v_1) = d_{B_v}(v_1)-\mu(x_1,v_1) = b(v_1)-1$ and $d_{B_v\setminus\{x_1,v_1\}}(y) = d_B(y)-\mu(v_1,y) = b(y)-1$. In addition, $N_{A_v^-}(v)=\{u\}$ as G is C_5^+-free and $d_{A_v\setminus D_{A_v}}(v) = d_{A_v}(v)-\mu(u,v) = a(v)-1$. It remains to show that $u \in D_{A_v}$. By Claim 10, $A_v \setminus D_{A_v}$ is $(a-1)$-degenerate. Thus, there exists $w' \in A_v \setminus D_{A_v}$ such that $d_{A_v\setminus D_{A_v}}(w') \leqslant a(w')-1$ and $|N_{D_{A_v}}(w')|=1$ by Claim 18. We may assume that $N_{D_{A_v}}(w')=\{u_1\}$ and $u \notin D_{A_v}$. Clearly, $u_1v_1 \notin E(G)$ and $w' \neq u$ as G is K_4^--free. Now, we may view B_v, A_v as the new parts A, B by the symmetry between the functions a,b, and x_1,u_1,v_1 play the roles in (B_v,A_v) that u,v,x_1 occupied in the original partition (A,B), respectively. Let $A_6 = A_v \cup \{v_1\} \setminus \{u_1\}$ and $B_6 = B_v \cup \{u_1\} \setminus \{v_1\}$. By Claim 17, we have $\mu(u_1)=1$, $(A_6,B_6) \in \mathscr{P}$, $v_1 \in A_6^-$ and $x_1 \in B_6^-$. Note that $d_{A_6}(w') = d_{A_v\setminus D_{A_v}}(w') \leqslant a(w')-1$ and $d_{B_6}(y) = d_{B_v}(y)-\mu(v_1,y) = b(y)-1$. Thus, $v_1, w' \in A_6^-$ and $x_1, y \in B_6^-$. This contradicts Claim 14. Hence, $u \in D_{A_v}$.

Now, we consider the partition (B_v,A_v), which satisfies all the conditions of Case 1 by the above argument. We mention that x_1,u,v_1,v,y play the roles in (B_v,A_v) that u,v_1,x_1,y,w occupied in the original partition (A,B), respectively. By Claim 21, we may assume that there exist $u' \in D_{A_v}\setminus\{u\}$ and $z' \in A_v \setminus (D_{A_v}\cup\{v\})$ such that $N_{D_{A_v}\cup\{u\}}(z') = \{v,u'\}$, $\mu(u')=\mu(z')=1$ and $d_{A_v}(z') = a(z')+1$.

Let $A_7 = A_v \cup \{y\} \setminus \{u'\}$ and $B_7 = B_v \cup \{u'\} \setminus \{y\}$. Since G is $\{K_4^-, C_5^+\}$-free, we know that $u'y, u'u, u'v_1, u'v, x_1y, uy, vy \notin E(G)$. Then we have the following equalities: (i) $d_{A_7}(y) = d_{A_v}(y) = d_G(y)-d_{B_v}(y) = a(y)-1$; (ii) $d_{A_7}(u) = d_{A_v}(u) = a(u)-1$; (iii) $d_{A_7}(v) = d_{A_v}(v) = a(v)$; (iv) $d_{B_7}(u') = d_{B_v}(u') = d_G(u')-d_{A_v}(u') = b(u')$; (v) $d_{B_7}(x_1) = d_{B_v}(x_1)+\mu(u',x_1) = b(x_1)+\mu(x_1)-1$; (vi) $d_{B_7}(v_1) = d_{B_v}(v_1)-\mu(v_1,y) = b(v_1)+\mu(v_1)-2$. We claim that $(A_7,B_7) \in \mathscr{P}$. Clearly, A_7 is $(a-1)$-meager. If not, then there is an a-nice subset $A' \subseteq A_7$. Since A_v is $(a-1)$-meager, we have $y \in A'$ and $d_{A_7}(y) \geqslant d_{A'}(y) \geqslant a(y)+\mu(y)-1 = a(y)$, a contradiction. Now we prove that B_7 is $(b-1)$-meager. If not, then there is a b-nice subset $B' \subseteq B_7$. Since B_v is $(b-1)$-meager, we have $u' \in B'$ and $d_{B_7}(u') \geqslant d_{B'}(u') \geqslant b(u')+\mu(u')-1 = b(u')$. Thus, $d_{B_7}(u') = d_{B'}(u') = b(u')$, implying $x_1 \in B'$ as $x_1u \in E(G)$. Then, $d_{B_7}(x_1) \geqslant d_{B'}(x_1) \geqslant b(x_1)+\mu(x_1)-1$. It follows that $d_{B_7}(x_1) = d_{B'}(x_1) = b(x_1)+\mu(x_1)-1$, implying $v_1 \in B'$ as $v_1x_1 \in E(G)$. Hence, $d_{B_7}(v_1) \geqslant d_{B'}(v_1) \geqslant b(v_1)+\mu(v_1)-1$, a contradiction. Thus, (A_7,B_7) is an $(a-1,b-1)$-meager partition. By (3) and (4), $\omega(A_7,B_7)=\omega(A,B)$, as claimed.

Note that $u,y \in D_{A_7}$, $v \in A_7^=$, $v_1 \in B_7^-$ and $u',x_1 \in B_7^=$. In what follows, we prove that $B_7^- = \{v_1\}$, $x_1 \in N_{B_7^=}(v_1)$ with $d_{B_7\setminus\{v_1\}}(x_1) = b(x_1)-1$, and $v \in N_{A_7^=}(u)$ with $d_{A_7\setminus D_{A_7}}(v) = a(v)-1$, If so, we may view B_7, A_7 as the new parts A, B by the symmetry between the functions a,b, and again we are still in Case 1 as $x_1u \in E(G)$.

By Claim 14, $B_7^- = \{v_1\}$. Now, we show that $d_{B_7\setminus\{v_1\}}(x_1) = b(x_1)-1$. Note that $d_{B_7\setminus\{v_1\}}(x_1) = d_{B_7}(x_1)-\mu(v_1,x_1) = b(x_1)+\mu(x_1)-1-\mu(v_1,x_1) \geqslant b(x_1)-1$. It suffices to prove that $d_{B_7\setminus\{v_1\}}(x_1) \leqslant b(x_1)-1$. Suppose for a contradiction that $d_{B_7\setminus\{v_1\}}(x_1) > b(x_1)$. By Claim 10, $B_7 \setminus \{v_1\}$ is $(b-1)$-degenerate as G has no (a,b)-feasible pair. This implies that there exists $y'' \in B_7 \setminus \{v_1\}$ such that $d_{B_7\setminus\{v_1\}}(y'') \leqslant b(y'') - 1$. Clearly, $y'' \neq x_1$ and $d_{B_7}(y'') \geqslant b(y'') + \mu(y'') - 1$. Note also that $d_{B_7}(y'') = d_{B_7\setminus\{v_1\}}(y'') + \mu(v_1, y'') \leqslant b(y'') - 1 + \mu(y'')$. Thus, $d_{B_7}(y'') = b(y'') + \mu(y'') - 1$ and $y'' \in B_7^=$. Then vuv_1y'' forms a special path with respect to (A_7, B_7). By Claim 15, either $v_1v \in E(G)$ or $uy'' \in E(G)$ as $u \in D_{A_7}$. In either case, we have a K_4^-, a contradiction. It remains to prove that $d_{A_7\setminus D_{A_7}}(v) = a(v) - 1$. By Claim 11, we have $N_{D_{A_7}}(v) = \{u\}$ as G is C_5^+-free. Thus, $d_{A_7\setminus D_{A_7}}(v) = d_{A_7}(v) - \mu(u, v) = a(v) - 1$ (by noting that $\mu(v) = 1$), as desired.

Now, we consider the partition (B_7, A_7), and v_1, u, x_1, v play the roles in (B_7, A_7) that u, v_1, x_1, y occupied in the original partition (A, B), respectively. We show that $u'z, uz' \in E(G)$; if so, then $\{u, v, z, u', z'\}$ contains a C_5^+, a contradiction. Recall that $\mu(z) = 1$ and $d_B(z) = b(z)+1$ by Claim 21. If $u'z \notin E(G)$, then $d_{B_7}(z) = d_{B_v}(z) - \mu(y, z) = d_B(z) - \mu(v, z) - \mu(y, z) = b(z) - 1$, implying $z \in D_{B_7}$. Thus, $u, y \in A_7^-$ and $v_1, z \in B_7^-$, contradicting Claim 14. Next, we show that $uz' \in E(G)$. Since G is $K_{2,3}$-free, $yz' \notin E(G)$. Note that $\mu(z') = 1$ and $d_{A_v}(z') = a(z') + 1$. Thus, $d_{A_7}(z') = d_{A_v}(z') - \mu(u', z') = a(z')$, implying $z' \in A_7^=$. By Claim 20(ii), $uz' \in E(G)$ as $z' \in N_{A_7^=}(v)$. Thus, we complete the proof of Case 1.

Case 2. $uy \in E(G)$. Clearly, $x_1v_1 \notin E(G)$ and $N_{D_B}(y) = \{v_1\}$. By Claims 17 and 19, $\mu(v_1) = \mu(x_1) = 1$. Note that $1 \leqslant |N_D(z)| \leqslant 2$ as G is $K_{2,3}$-free. If $|N_D(z)| = 2$, then $yz \in E(G)$; otherwise, we have $z \in B \setminus D_B$ such that $d_{B\setminus D_B}(z) \leqslant b(z) - 1$, implying $|N_{D_B}(z)| = 1$ by Claim 18, a contradiction. It follows that $v_1z \notin E(G)$ as G is K_4^--free. Thus, there exists $v \in D_B \setminus \{v_1\}$ such that $vz \in E(G)$ and $\{u, v, v_1, y, z\}$ contains a C_5^+, a contradiction. Hence, $|N_D(z)| = 1$ and $d_B(z) \leqslant b(z) - 1 + \mu(z)$ by (8).

Claim 22. *$N_D(z) = \{v_2\}$ for some $v_2 \in D_B \setminus \{v_1\}$.*

Proof. Suppose not. Clearly, $z \in B^=$ as G is K_4^--free. It follows that $d_{B\setminus D}(z) = b(z) - 1$ and $d_D(z) = \mu(z)$. If $N_D(z) = \{v_1\}$, then x_1uv_1z forms a special path with respect to (A, B). Since $v_1 \in D_B$, either $x_1v_1 \in E(G)$ or $uz \in E(G)$ by Claim 15, implying a K_4^- in both cases, a contradiction. If $N_D(z) = \{y\}$, then $d_B(z) = b(z) + \mu(z) - 1$ and $\mu(y, z) = \mu(z)$. Since G is $\{K_4^-, C_5^+\}$-free, we have $x_1v_1, x_1y, x_1z, v_1z \notin E(G)$. By Claim 17, $(A_{v_1}, B_{v_1}) \in \mathscr{P}$, $u \in A_{v_1}^-$, $v_1 \in A_{v_1}^=$ and $x_1 \in D_{B_{v_1}} \subseteq B_{v_1}^-$. Note that $d_{B_{v_1}}(y) = d_B(y) - \mu(v_1, y) = d_{B\setminus D_B}(y) \leqslant b(y) - 1$. It follows that $y \in D_{B_{v_1}} \subseteq B_{v_1}^-$. Thus, $A_{v_1}^- = \{u\}$ by Claim 14. Since G is C_5^+-free, we have $N_{D_{B_{v_1}}}(z) = \{y\}$. Thus, $d_{B_{v_1}\setminus D_{B_{v_1}}}(z) = d_{B_{v_1}}(z) - \mu(y, z) = d_B(z) - \mu(y, z) = b(z) - 1$. Moreover, $v_1 \in A_{v_1}^=$ with $d_{A_{v_1}\setminus\{u\}}(v_1) = d_{A_{v_1}}(v_1) - \mu(u, v_1) = a(v_1) - 1$. Now, we view A_{v_1} B_{v_1} as the new parts A, B and the case can be reduced to Case 1 as $v_1y \in E(G)$. In fact, v_1, u, y, z play the roles in (A_{v_1}, B_{v_1}) that x_1, u, v_1, y occupied in the original partition (A, B) of Case 1, respectively. □

Let $Z := \{z^* \in B \setminus D : d_{B\setminus D}(z^*) \leqslant b(z^*) - 1\}$. Clearly, $z \in Z \subseteq B^- \cup B^=$. By Claim 22, for each $z^* \in Z$, we may assume that $N_D(z^*) = \{v^*\}$ for some $v^* \in D_B \setminus \{v_1\}$. Now, we show that $uz^* \in E(G)$ for each $z^* \in Z$. If $z^* \in B^-$, then we're done by Claim 11.

Thus, $z^* \in B^=$ and $x_1uv^*z^*$ forms a special path with respect to (A, B). By Claim 15, either $x_1v^* \in E(G)$ or $uz^* \in E(G)$. If $x_1v^* \in E(G)$, then the case can be reduced to Case 1, where z^* and v^* play the roles of y and v_1. Thus, we conclude that $uz^* \in E(G)$ for each $z^* \in Z$.

Note that $N_{D\cup Z}(y) = N_{D_B}(y)$ as $yz^* \notin E(G)$ for each $z^* \in Z$. Thus, $d_{B\setminus(D\cup Z)}(y) = d_{B\setminus D_B}(y) = b(y) - 1 \geqslant 1$, i.e., $B \setminus (D \cup Z) \neq \emptyset$. By Claim 10, $B \setminus (D \cup Z)$ is $(b-1)$-degenerate. Hence, there exists $z' \in B\setminus(D\cup Z)$ such that $d_{B\setminus(D\cup Z)}(z') \leqslant b(z')-1$, implying $|N_{D\cup Z}(z')| \geqslant 1$ by noting that $d_B(z') \geqslant b(z')$. Since u is adjacent to each vertex in $D\cup Z$, we have $|N_{D\cup Z}(z')| \leqslant 2$ as G is $K_{2,3}$-free. If $|N_{D\cup Z}(z')| = 2$, then $N_{D\cup Z}(z') \nsubseteq D_B$ by Claim 18. It is easy to check that G contains a K_4^- or C_5^+, a contradiction. Let $N_{D\cup Z}(z') = \{y'\}$. If $y' \in D$, then $d_{B\setminus D}(z') = d_{B\setminus(D\cup Z)}(z') \leqslant b(z') - 1$, indicating $z' \in Z$, a contradiction. Thus, $y' \in Z$ and $d_{B\setminus(D_B\cup\{y'\})}(z') = d_{B\setminus(D\cup Z)}(z') \leqslant b(z') - 1$. Now, we may view y', z' and $D_B \cup \{y'\}$ as the new y, z and D, respectively. Since $uy' \in E(G)$, we are still in Case 2. By Claim 22, we have $N_{D_B\cup\{y'\}}(z') \subseteq D_B$. This leads to a contradiction as $y' \notin D_B$, completing the proof of Case 2. Thus, we complete the proof of Theorem 5.

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