The Erdős-Hajnal property for graphs with no fixed cycle as a pivot-minor

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Abstract

We prove that for every integer k, there exists $\varepsilon > 0$ such that for every n-vertex graph G with no pivot-minor isomorphic to C_k, there exist disjoint sets $A, B \subseteq V(G)$ such that $|A|, |B| \geqslant \varepsilon n$, and A is either complete or anticomplete to B. This proves the analog of the Erdős-Hajnal conjecture for the class of graphs with no pivot-minor isomorphic to C_k.

Mathematics Subject Classifications: 05CC55, 05C75

1 Introduction

In this paper all graphs are simple, having no loops and no parallel edges. For a graph G, let $\omega(G)$ be the maximum size of a clique, that is a set of pairwise adjacent vertices and let $\alpha(G)$ be the maximum size of an independent set, that is a set of pairwise non-adjacent vertices. Erdős and Hajnal [9] proposed the following conjecture in 1989.

Conjecture 1 (Erdős and Hajnal [9]). For every graph H, there is $\varepsilon > 0$ such that all graphs G with no induced subgraph isomorphic to H satisfies

$$\max(\alpha(G), \omega(G)) \geqslant |V(G)|^{\varepsilon}.$$

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This conjecture still remains open. See [5] for a survey on this conjecture. We can ask the same question for weaker containment relations. Recently Chudnovsky and Oum [6] proved that this conjecture holds if we replace "induced subgraphs" with "vertex-minors" as follows. This is weaker in the sense that every induced subgraph G is a vertex-minor of G but not every vertex-minor of G is an induced subgraph of G.

Theorem 2 (Chudnovsky and Oum [6])**.** *For every graph H, there exists $\varepsilon > 0$ such that every graph with no vertex-minors isomorphic to H satisfies*

$$\max(\alpha(G), \omega(G)) \geqslant |V(G)|^{\varepsilon}.$$

We ask whether Conjecture 1 holds if we replace "induced subgraphs" with "pivot-minors" as follows.

Conjecture 3. For every graph H, there exists $\varepsilon > 0$ such that every graph G with no pivot-minor isomorphic to H satisfies

$$\max(\alpha(G), \omega(G)) \geqslant |V(G)|^{\varepsilon}.$$

The detailed definition of pivot-minors will be presented in Section 3. For now, note that the analog for vertex-minors is weakest, the analog for pivot-minors is weaker than that for induced subgraphs but stronger than that for vertex-minors. This is because every induced subgraph of G is a pivot-minor of G, and every pivot-minor of G is a vertex-minor of G. In other words, Conjecture 1 implies Conjecture 3 and Conjecture 3 implies Theorem 2. We verify Conjecture 3 for $H = C_k$, the cycle graph on k vertices as follows.

Theorem 4. *For every $k \geqslant 3$, there exists $\varepsilon > 0$ such that every graph with no pivot-minor isomorphic to C_k satisfies*

$$\max(\alpha(G), \omega(G)) \geqslant |V(G)|^{\varepsilon}.$$

We actually prove a stronger property, as Chudnovsky and Oum [6] did. Before stating this property, let us first state a few terminologies. A class $\mathcal{G}$ of graphs closed under taking induced subgraphs is said to have *the Erdős-Hajnal property* if there exists $\varepsilon > 0$ such that every graph G in $\mathcal{G}$ satisfies

$$\max(\alpha(G), \omega(G)) \geqslant |V(G)|^{\varepsilon}.$$

A class $\mathcal{G}$ of graphs closed under taking induced subgraphs is said to have *the strong Erdős-Hajnal property* if there exists $\varepsilon > 0$ such that every n-vertex graph in $\mathcal{G}$ with $n > 1$ has disjoint sets A, B of vertices such that $|A|, |B| \geqslant \varepsilon n$ and A is either complete or anti-complete to B. It is an easy exercise to show that the strong Erdős-Hajnal property implies the Erdős-Hajnal property, see [1, 10].

Chudnovsky and Oum [6] proved that the class of graphs with no vertex-minors isomorphic to H for a fixed graph H has the strong Erdős-Hajnal property, implying Theorem 2. We propose its analog for pivot-minors as a conjecture, which implies the theorem of Chudnovsky and Oum [6]. Note that this conjecture is not true if we replace the pivot-minor with induced graphs. For example, the class of triangle-free graphs does not have the strong Erdős-Hajnal property [10].

Conjecture 5. For every graph H, there exists $\varepsilon > 0$ such that for all $n > 1$, every n-vertex graph with no pivot-minor isomorphic to H has two disjoint sets A, B of vertices such that $|A|, |B| \geqslant \varepsilon n$ and A is complete or anti-complete to B.

We prove that this conjecture holds if $H = C_k$. In other words, the class of graphs with no pivot-minor isomorphic to C_k has the strong Erdős-Hajnal property as follows. This implies Theorem 4.

Theorem 6. *For every integer $k \geqslant 3$, there exists $\varepsilon > 0$ such that for all $n > 1$, every n-vertex graph with no pivot-minor isomorphic to C_k has two disjoint sets A, B of vertices such that $|A|, |B| \geqslant \varepsilon n$ and A is complete or anti-complete to B.*

This paper is organized as follows. In Section 2, we will introduce basic definitions and review necessary theorems of Rödl [20] and Bonamy, Bousquet, and Thomassé [2]. In Section 3, we will present several tools to find a pivot-minor isomorphic to C_k. In particular, it proves that a long anti-hole contains C_k as a pivot-minor. In Section 4, we will present the proof of the main theorem, Theorem 6. In Section 5, we will relate our theorem to the problem on χ-boundedness, and discuss known results and open problems related to polynomial χ-boundedness and the Erdős-Hajnal property.

2 Preliminaries

Let $\mathbb{N}$ be the set of positive integers and for each $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$. For a graph $G = (V, E)$, let $\overline{G} = (V, \binom{V}{2} - E)$ be the complement of G. We write $\Delta(G)$ and $\delta(G)$ to denote the maximum degree of G and the minimum degree of G respectively.

Let T be a tree rooted at a specified node v_r, called the *root*. If the path from v_r to a node y in T contains $x \in V(T) - \{y\}$, we say that x is an *ancestor* of y, and y is a *descendant* of x. If one of x and y is an ancestor of the other, we say that x, y are *related*. We say that two disjoint sets X and Y of nodes of T are *unrelated* if no pairs of $x \in X$ and $y \in Y$ are related.

For disjoint vertex sets X and Y, we say X is *complete* to Y if every vertex of X is adjacent to all vertices of Y. We say X is *anti-complete* to Y if every vertex of X is non-adjacent to Y. A *pure pair* of a graph G is a pair (A, B) of disjoint subsets of $V(G)$ such that A is complete or anticomplete to B.

For a vertex u, let $N_G(u)$ denote the set of neighbors of u in G. For each $U \subseteq V(G)$, we write

$$N_G(U) := \bigcup_{u \in U} N_G(u) - U.$$

The following lemma is proved in Section 2 of [2].

Lemma 7 (Bonamy, Bousquet, and Thomassé [2])**.** *For every connected graph G and a vertex $v_r \in V(G)$, there exist an induced subtree T of G rooted at v_r and a function $r : V(G) \to V(T)$ satisfying the following.*

(T1) *$r(v_r) = v_r$ and for each $u \in V(G) - \{v_r\}$, the vertex $r(u)$ is a neighbor of u. In particular, T is a dominating tree of G.*

(T2) *If $r(x)$ and $r(y)$ are not related, then $xy \notin E(G)$.*

Rödl [20] proved the following theorem. Its weaker version was later proved by Fox and Sudakov [11] without using the regularity lemma. A set U of vertices of G is an *ε-stable set* of a graph G if $G[U]$ has at most $\varepsilon\binom{|U|}{2}$ edges. Similarly, U is an *ε-clique* of a graph G if $G[U]$ has at least $(1-\varepsilon)\binom{|U|}{2}$ edges.

Theorem 8 (Rödl [20])**.** *For all $\varepsilon > 0$ and a graph H, there exists $\delta > 0$ such that every n-vertex graph G with no induced subgraph isomorphic to H has an ε-stable set or an ε-clique of size at least δn.*

We will use the following simple lemma. We present its proof for completeness.

Lemma 9. *Let G be a graph. Every ε-stable set U of G has a subset U' of size at least $|U|/2$ with $\Delta(G[U']) \leqslant 4\varepsilon|U'|$.*

Proof. Let U' be the set of vertices of degree at most $2\varepsilon|U|$ in $G[U]$. Because

$$\sum_{v\in U} \deg_{G[U]}(v) < \varepsilon|U|^2,$$

we have $|U'| \geqslant |U|/2$. Moreover, for each vertex $v \in U'$, we have $\deg_{G[U']}(v) \leqslant 2\varepsilon|U| \leqslant 4\varepsilon|U'|$. □

Using Lemma 9, we can deduce the following corollary of Theorem 8.

Corollary 10. *For all $\alpha > 0$ and a graph H, there exists $\delta > 0$ such that every graph G with no induced subgraph isomorphic to H has a set $U \subseteq V(G)$ with $|U| \geqslant \delta|V(G)|$ such that either $\Delta(G[U]) \leqslant \alpha|U|$ or $\Delta(\overline{G}[U]) \leqslant \alpha|U|$.*

The following easy lemma will be used to find a connected induced subgraph inside the output of Corollary 10. We omit its easy proof.

Lemma 11. *A graph G has a pure pair (A, B) such that $|A|, |B| \geqslant |V(G)|/3$ or has a connected induced subgraph H such that $|V(H)| \geqslant |V(G)|/3$.*

Lemma 12 (Bonamy, Bousquet, and Thomassé [2, Lemma 3])**.** *Let T be a tree rooted at v_r and $w : V(T) \to \mathbb{R}$ be a non-negative weight function on $V(T)$ with $\sum_{x\in V(T)} w(x) = 1$. Then there exists either a path P from v_r with weight at least $1/4$ or two unrelated sets A and B both with weight at least $1/4$.*

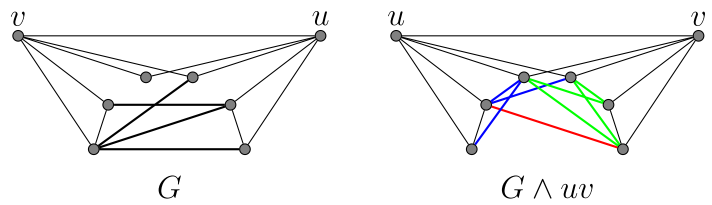

Figure 1: Pivoting uv.

A *hole* is an induced cycle of length at least 5.

Lemma 13 (Bonamy, Bousquet, and Thomassé [2, Lemma 4]). *For given $k \geqslant 3$, there exist $\alpha = \alpha(k) > 0$ and $\varepsilon = \varepsilon(k) > 0$ such that for any n-vertex graph G with $n \geqslant 2$ and $\Delta(G) \leqslant \alpha n$, if G has no holes of length at least k and has a dominating induced path, then G contains a pair (A, B) of disjoint vertex sets such that A is anticomplete to B and $|A|, |B| \geqslant \varepsilon n$.*

3 Finding a cycle as a pivot-minor

For a given graph G and an edge uv, a graph $G \wedge uv$ obtained from G by *pivoting* uv is defined as follows. Let $V_1 = N_G(u) \cap N_G(v)$, $V_2 = N_G(u) - N_G(v)$, $V_3 = N_G(v) - N_G(u)$. Then $G \wedge uv$ is the graph obtained from G by complementing adjacency between vertices between V_i and V_j for all $1 \leqslant i < j \leqslant 3$ and swapping the label of u and v. See Figure 1 for an illustration. We say that H is a *pivot-minor* of G if H can be obtained from G by deleting vertices and pivoting edges. For this paper, we will also say that H is a pivot-minor of G, when G has a pivot-minor isomorphic to H. A pivot-minor H of G is *proper* if $|V(H)| < |V(G)|$.

We describe several scenarios for constructing C_k as a pivot-minor. The following proposition is an easy one; One can obtain a desired pivot-minor from a longer cycle of the same parity.

Proposition 14. *For $m \geqslant k \geqslant 3$ with $m \equiv k \pmod 2$, the cycle C_m has a pivot-minor isomorphic to C_k.*

Proof. We proceed by induction on $m - k$. We may assume that $m > k$. Let xy be an edge of C_m. Then $(C_m \wedge xy) - x - y$ is isomorphic to C_{m-2}, which contains a pivot-minor isomorphic to C_k by the induction hypothesis. □

Proposition 15. *For integers $k \geqslant 3$ and $m \geqslant \frac{3}{2}k + 6$, the graph $\overline{C_m}$ has a pivot-minor isomorphic to C_k.*

Before proving Proposition 15, we present a simple lemma on partial complements of the cycle graph. The *partial complement*[1] $G \oplus S$ of a graph G by a set S of vertices is a

[1]We found this concept in a paper by Kamiński, Lozin, and Milanič [14], though it may have been studied previously, as it is a natural concept.

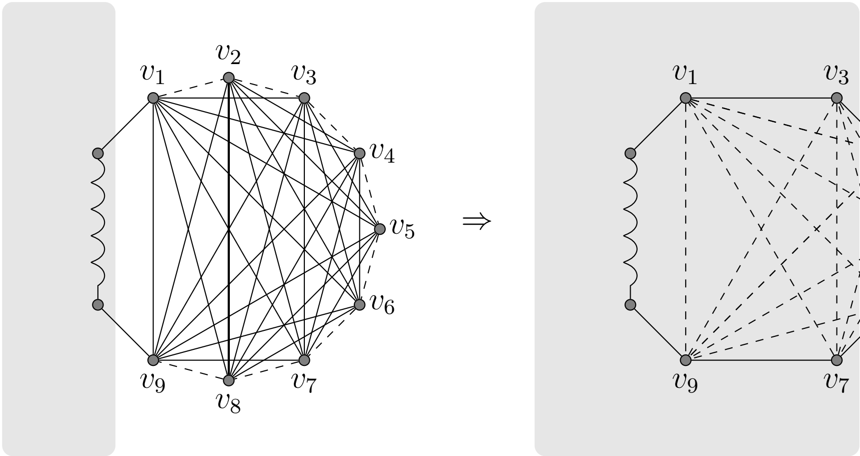

Figure 2: Obtaining an $(s-2,3)$-cycle from an $(s,9)$-cycle when $s>9$ in the proof of Lemma 16.

graph obtained from G by changing all edges within S to non-edges and non-edges within S to edges.

For $s \geqslant t \geqslant 0$, we say that G is an (s,t)-*cycle* if G is isomorphic to a graph $C_s \oplus X$ for a set X of t consecutive vertices in the cycle C_s.

Lemma 16. *Let $s \geqslant t \geqslant 6$. An (s,t)-cycle contains a pivot-minor isomorphic to an $(s-2,t-6)$-cycle.*

Proof. Let $v_1, \ldots, v_s$ be the vertices of C_s in the cyclic order where $X = \{v_1, \ldots, v_t\}$. Then it is easy to check that $(C_s \oplus X) \wedge v_2v_{t-1} - \{v_2, v_{t-1}\}$ is isomorphic to $C_{s-2} \oplus X'$ where X consists of $t-6$ consecutive vertices on the cycle. See Figure 2. □

Proof of Proposition 15. As $\overline{C_m}$ is an (m,m)-cycle, by Lemma 16, $\overline{C_m}$ contains a pivot-minor isomorphic to an $(m-2i, m-6i)$-cycle for all $i \leqslant m/6$.

Let us fix $i = \lceil (k-2)/4 \rceil$. Then $m-6i \geqslant m-6\cdot(k+1)/4 \geqslant 9/2$ and therefore $\overline{C_m}$ contains a pivot-minor H isomorphic to an $(m-2i,m-6i)$-cycle and $m-6i \geqslant 5$. We may assume that $H = C_{m-2i} \oplus X$ where $C_{m-2i} = v_1 \cdots v_{m-2i}$ and $X = \{v_{4i+1}, \ldots, v_{m-2i}\}$.

Note that H contains an induced cycle $C = v_1 \cdots v_{4i}v_{4i+1}v_{m-2i}v_1$ of length $4i+2 \geqslant k$. If k is even, then by Proposition 14, H contains a pivot-minor isomorphic to C_k. So we may assume that k is odd and therefore $|V(C)| = 4i+2 \geqslant k+1$.

Let $x = v_{m-2i}$, $y = v_{4i+1}$ be the two vertices in $V(C) \cap X$. Since $m-6i \geqslant 5$, there is a common neighbor z of x and y in X. Then z has exactly two neighbors x and y in $V(C)$. Then $H[V(C) \cup \{z\}] \wedge yz - y - z$ is a cycle of length $4i+1$. Since $4i+1 \geqslant k$, by Proposition 14, it contains a pivot-minor isomorphic to C_k. □

A *generalized fan* is a graph G with a specified vertex c, called the *center*, such that $G-c$ is an induced path of length at least 1, called the *main path* of G and both ends of the main path are adjacent to c. If c is adjacent to all vertices of $G-c$, then G is called a *fan*.

An *interval* of a generalized fan with a center c is a maximal subpath of the main path having no internal vertex adjacent to c. The *length* of an interval is its number of edges. A generalized fan is an $(a_1, \dots, a_s)$-*fan* if the lengths of intervals are $a_1, \dots, a_s$ in order. Note that an $(a_1, \dots, a_s)$-fan is also an $(a_s, \dots, a_1)$-fan. An $(a_1, \dots, a_s)$-fan is a k-*good fan* if $a_1 \geqslant k-2$ or $a_s \geqslant k-2$. An $(a_1, \dots, a_s)$-fan is a *strongly k-good fan* if $s \geqslant 2$ and either $a_1 \geqslant k-2$ and a_s is odd, or $a_s \geqslant k-2$ and a_1 is odd. It is easy to observe that every k-good fan has a hole of length at least k. However, that does not necessarily lead to a pivot-minor isomorphic to C_k because of the parity issues. In the next proposition, we show that every strongly k-good fan has a pivot-minor isomorphic to C_k.

Proposition 17. *Let $k \geqslant 5$ be an integer. Every strongly k-good fan has a pivot-minor isomorphic to C_k.*

Proof. Let G be an $(a_1, \dots, a_s)$-fan such that $s \geqslant 2$, $a_1 \geqslant k-2$, and a_s is odd. We proceed by the induction on $|V(G)|$. We may assume that G has no proper pivot-minor that is a strongly k-good fan. Note that C_{a_1+2} is an induced subgraph of G, hence if $a_1 \equiv k \pmod 2$, then C_k is isomorphic to a pivot-minor of G by Proposition 14. Thus we may assume that $a_1 \not\equiv k \pmod 2$ and so $a_1 \geqslant k-1$.

If a_i is odd for some $1 < i < s$, then G contains a smaller strongly k-good fan by taking the first i intervals, contradicting our assumption. Thus a_i is even for all $1 < i < s$. If $a_i \geqslant 3$ for some $i > 1$, then let uv be an internal edge of the i-th interval. Then $G \wedge uv - u - v$ is a strongly k-good fan, contradicting our assumption. Thus, we may assume that $a_i \leqslant 2$ for all $i > 1$ and so G is an $(a_1, 2, \dots, 2, 1)$-fan.

Let xy be the last interval of G with length 1. Then $G \wedge xy - x - y$ is a $(a_1, 2, \dots, 2, 1)$-fan with $s-1$ intervals. By the assumption, we may assume that $s = 2$ and $G \wedge xy - x - y$ is an $(a_1 - 1)$-fan with one interval, which is a cycle with $a_1 + 1$ edges. As $a_1 + 1 \geqslant k$ and $a_1 + 1 \equiv k \pmod 2$, Proposition 14 implies that G contains a pivot-minor isomorphic to C_k. □

4 Proof of Theorem 6

First we choose $\alpha > 0$ and $\varepsilon_0 > 0$ so that

$$4\alpha \leqslant \alpha(\lceil \tfrac{3}{2}k + 6 \rceil) \text{ and } \varepsilon_0 = \varepsilon(\lceil \tfrac{3}{2}k + 6 \rceil) \text{ where } \alpha(\cdot), \varepsilon(\cdot) \text{ are specified in Lemma 13} \tag{1}$$

and in addition $\alpha < 1/(8k)$ as well. Let $\delta > 0$ be a constant obtained by applying Corollary 10 with $\alpha/3$ as α and C_k as H. Choose $\varepsilon > 0$ so that

$$\varepsilon < \min\left(\frac{\delta}{12}, (1 - 4(k+3)\alpha)\frac{\delta}{240}, \frac{\varepsilon_0 \delta}{12}\right).$$

Let $n > 1$ be an integer and G be an n-vertex graph with no pivot-minor isomorphic to C_k. In particular, G does not have C_k as an induced subgraph. To derive a contradiction, we assume that G contains no pure pair (A, B) with $|A|, |B| \geqslant \varepsilon n$. We may assume that $\varepsilon n > 1$, because otherwise an edge or a non-edge of G gives a pure pair.

By Corollary 10, there exists a subset U of $V(G)$ such that $|U| \geqslant \delta|V(G)|$ and $\Delta(G^0)[U]) \leqslant (\alpha/3)|U|$ for some $G^0 \in \{G, \overline{G}\}$. By the assumption on G, $G^0[U]$ has no pure pair (A, B) with $|A|, |B| \geqslant (\varepsilon/\delta)|U|$. As $\varepsilon/\delta < 1/3$, by Lemma 11, $G^0[U]$ has a connected induced subgraph G' such that $|V(G')| \geqslant |U|/3$. Let $n' = |V(G')|$.

Then $n' \geqslant (\delta/3)n$ and $\Delta(G') \leqslant (\alpha/3)|U| \leqslant \alpha n'$. By the assumption on G,

$$G' \text{ contains no pure pair } (A, B) \text{ with } |A|, |B| \geqslant (3\varepsilon/\delta)n'. \tag{2}$$

By applying Lemma 7 with G', we obtain a dominating induced tree T and $r : V(G') \to V(T)$ satisfying Lemma 7 (T1)–(T2) with G'. For each $u \in V(T)$, let

$$w(u) := \frac{|r^{-1}(\{u\})|}{n'}$$

be the *weight* of u. By applying Lemma 12 with the weight w, we obtain either an induced path P of T with weight at least $1/4$ or two unrelated sets A and B both with weight at least $1/4$.

In the latter case, Lemma 7 (T2) implies that $r^{-1}(A)$ is anticomplete to $r^{-1}(B)$ in G' and $|r^{-1}(A)|, |r^{-1}(B)| \geqslant n'/4 \geqslant (3\varepsilon/\delta)n'$, contradicting (2).

Hence, there exists an induced path P in G' with $|V(P) \cup N_{G'}(V(P))| \geqslant n'/4$. Let $W := V(P) \cup N_{G'}(V(P))$. Note that $n'/4 \geqslant \delta n/12 > \varepsilon n > 1$ and so $|W| \geqslant 2$.

Suppose that G' is an induced subgraph of $\overline{G}$. Using (1), we apply Lemma 13 to $G'[W]$ with 4α as α and $\lceil \frac{3}{2}k + 6 \rceil$ as k. Then we can deduce from (2) and $\varepsilon'|W| \geqslant \frac{12\varepsilon}{\delta}\frac{n'}{4} = (3\varepsilon/\delta)n'$ that the graph $G'[W]$ contains an induced cycle C_m with $m \geqslant \lceil \frac{3}{2}k + 6 \rceil$ and by Proposition 15, $\overline{G'}$ contains a pivot-minor isomorphic to C_k, and so does G, a contradiction.

Thus G' is an induced subgraph of G. Let $G^* := G'[W]$ and let $n^* = |W|$. Then G^* has no pivot-minor isomorphic to C_k, $n^* \geqslant n'/4$, and $\Delta(G^*) \leqslant 4\alpha n^*$. By (2), G^* contains no pure pair (A, B) with $|A|, |B| \geqslant (12\varepsilon/\delta)n^*$. Now the theorem follows from applying the following lemma with $G^*, n^*, 4\alpha, 12\varepsilon/\delta$ playing the roles of $G, n, \alpha, \varepsilon$ respectively in the statement of the lemma.

Lemma 18. *Let $k \geqslant 3$ be an integer. Let $0 < \alpha < 1/(2k)$, $0 < \varepsilon \leqslant (1 - (k + 3)\alpha)/20$. Let G be a graph on $n \geqslant 2$ vertices such that $\Delta(G) \leqslant \alpha n$ and G has no pure pair (A, B) with $|A|, |B| \geqslant \varepsilon n$. If G has a dominating induced path P, then G has a pivot-minor isomorphic to C_k.*

Proof. Suppose that G has no pivot-minor isomorphic to C_k. Note that $\varepsilon n > 1$ as otherwise we have a pure pair on two vertices since $n \geqslant 2$. Let us label vertices of P by 1, 2, $\ldots$, s in the order.

As P is a dominating path of G and $1 \leqslant \Delta(G) \leqslant \alpha n$, we have $2\alpha n s \geqslant (\alpha n + 1)s \geqslant n$ and therefore

$$s \geqslant 1/(2\alpha).$$

Note that $s - k > 0$ because $\alpha < \frac{1}{2k}$. As P is an induced path, it contains a pure pair (A, B) with $|A|, |B| \geqslant \lfloor \frac{s-1}{2} \rfloor$ and so $\frac{s-2}{2} \leqslant \lfloor \frac{s-1}{2} \rfloor < \varepsilon n$. Because $\varepsilon n > 1$, we have

$2\varepsilon n + 2 < 4\varepsilon n$ and so

$$s < 2\varepsilon n + 2 < 4\varepsilon n. \tag{3}$$

Now, for each $i \in [s-k+1]$, let

$$U_i^- := \{1, \ldots, i-1\},\ \ U_i^0 := \{i, \ldots, i+k-1\},\ \text{ and }\ U_i^+ := \{i+k, \ldots, s\}.$$

In other words, this partitions P into three (possibly empty) subpaths. Furthermore, for all $i \in [s-k+1]$ and $u \in N_G(U_i^-) - V(P)$, let

$$m_i^-(u) := \max(N_G(u) \cap U_i^-)$$

and for all $i \in [s-k+1]$ and $u \in N_G(U_i^+) - V(P)$, let

$$m_i^+(u) := \min(N_G(u) \cap U_i^+),$$

indicating the largest neighbor of u in U_i^- and the smallest neighbor of u in U_i^+ respectively. For each $i \in [s-k+1]$, let

$$\begin{aligned} A_i &:= N_G(U_i^0) - V(P) \text{ and} \\ B_i &:= (N_G(U_i^-) \cap N_G(U_i^+)) - (A_i \cup V(P)). \end{aligned}$$

Note that for each $u \in B_i$, we have

$$m_i^+(u) - m_i^-(u) \not\equiv k \pmod 2, \tag{4}$$

because otherwise $(u, m_i^-(u), m_i^-(u)+1, \ldots, m_i^+(u), u)$ forms an induced cycle of length at least k and Proposition 14 implies that G contains a pivot-minor isomorphic to C_k, a contradiction.

For each $i \in [s-k+1]$, let

$$\begin{aligned} C_i^1 &:= \{u \in N_G(U_i^-) - (A_i \cup B_i \cup V(P)) : m_i^-(u) \equiv 1 \pmod 2\}, \\ C_i^2 &:= \{u \in N_G(U_i^-) - (A_i \cup B_i \cup V(P)) : m_i^-(u) \equiv 0 \pmod 2\}, \\ D_i^1 &:= \{u \in N_G(U_i^+) - (A_i \cup B_i \cup V(P)) : m_i^+(u) \equiv k \pmod 2\}, \text{ and} \\ D_i^2 &:= \{u \in N_G(U_i^+) - (A_i \cup B_i \cup V(P)) : m_i^+(u) \equiv k+1 \pmod 2\}. \end{aligned}$$

Recall that P is dominating. Hence, for each i, the sets $\{A_i, B_i, C_i^1, C_i^2, D_i^1, D_i^2, V(P)\}$ forms a partition of $V(G)$ into 7 possibly empty sets.

If there exists an edge between $u \in C_i^j$ and $v \in D_i^j$ for some $j \in [2]$, then we obtain an induced cycle $(u, m_i^-(u), m_i^-(u)+1, \ldots, m_i^+(v), v, u)$ having length $m_i^+(v) - m_i^-(u) + 3 > k$ and $m_i^+(v) - m_i^-(u) + 3 \equiv k \pmod 2$, contradicting our assumption that G has no pivot-minor isomorphic to C_k by Proposition 14. Thus C_i^j is anticomplete to D_i^j. Hence,

$$\min\{|C_i^j|, |D_i^j|\} < \varepsilon n. \tag{5}$$

for all $i \in [s-k+1]$ and $j \in [2]$. Furthermore, we prove the following.

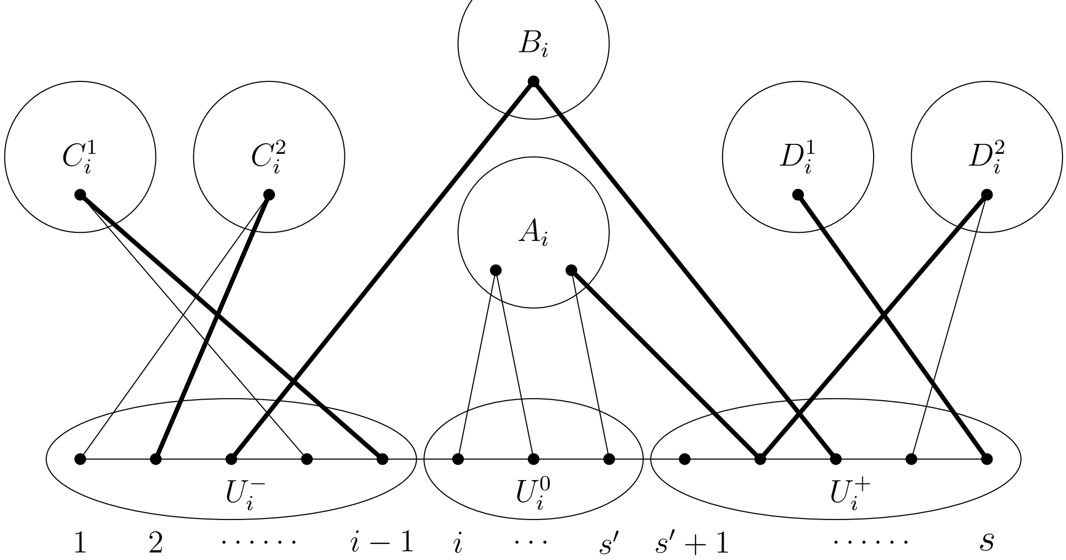

Figure 3: $s' = i + k - 1$. Bold lines indicate $m_i^-(u)$ and $m_i^+(u)$.

Claim 19. *Let $i \in [s-k+1]$. For each $v \in B_i$, all integers in $N_G(v) \cap U_i^-$ have the same parity and all integers in $N_G(v) \cap U_i^+$ have the same parity.*

Proof of Claim 19. If $N_G(v) \cap U_i^+$ has two integers $a < b$ of the different parity, then G contains a strongly k-good generalized fan by taking a subpath of P from $m_i^-(v)$ to b as its main path and v as its center. Then by Proposition 17, G contains a pivot-minor isomorphic to C_k, contradicting the assumption. Thus all integers in $N_G(v) \cap U_i^+$ have the same parity and similarly all integers in $N_G(v) \cap U_i^-$ have the same parity. ■

Claim 20. *For all $i \in [s-k+1]$, $|B_i| < 2(\alpha + 2\varepsilon)n$.*

Proof of Claim 20. Suppose $|B_i| \geqslant 2(\alpha + 2\varepsilon)n$ for some $i \in [s-k+1]$. Then there exists $r_B \in \{0, 1\}$ such that

$$B' := \{u \in B_i : m_i^-(u) \equiv r_B \pmod 2\}$$

has size at least $(\alpha + 2\varepsilon)n$. By (4), $m_i^+(u) \equiv k + r_B + 1 \pmod 2$ for all $u \in B'$.

We claim that if uv is an edge in $G[B']$, then $(m_i^-(u), m_i^+(u)) = (m_i^-(v), m_i^+(v))$. Suppose not. Without loss of generality, we may assume that $m_i^-(u) < m_i^-(v)$, because otherwise we may reverse the ordering of P to ensure that $m_i^-(u) \neq m^-(v)$ and swap u and v if necessary.

If $m_i^+(u) \geqslant m_i^+(v)$, then by Claim 19, $\{m_i^-(v), m_i^-(v)+1, \ldots, m_i^+(u), u, v\}$ induces a strongly k-good generalized fan with v as a center and $(m_i^-(v), m_i^-(v)+1, \ldots, m_i^+(u), u)$ as its main path. This implies that G has a pivot-minor isomorphic to C_k by Proposition 17, contradicting our assumption.

If $m_i^+(u) < m_i^+(v)$, then $(m_i^-(v), m_i^-(v)+1, \ldots, m_i^+(u), u, v)$ is an induced cycle of length $m_i^+(u) - m_i^-(v) + 3 \geqslant k$, and $m_i^+(u) - m_i^-(v) + 3 \equiv (k + r_B + 1) - r_B + 3 \equiv k \pmod 2$, a contradiction by Proposition 14.

Hence, $(m_i^-(u), m_i^+(u)) = (m_i^-(v), m_i^+(v))$ for all $uv \in E(G[B'])$. Let $C_1, \ldots, C_t$ be the connected components of $G[B']$. By the above observation, for each $j \in [t]$, there exist $a_j \in U_i^-$ and $b_j \in U_i^+$ such that $V(C_j) \subseteq N_G(a_j) \cap N_G(b_j)$. So, $|V(C_j)| \leqslant \alpha n$. As $|B'| \geqslant (\alpha+2\varepsilon)n$, there exists a set $I \subseteq \{1, 2, \ldots, t\}$ such that $\varepsilon n \leqslant |\bigcup_{i\in I} V(C_i)| \leqslant (\alpha+\varepsilon)n$. Let $A := \bigcup_{i\in I} V(C_i)$ and $B := B' - A$. Then (A, B) is a pure pair of G with $|A|, |B| \geqslant \varepsilon n$, a contradiction. ■

Claim 21. *There exist $i_* \in [s-k+1]$ and $j_* \in [2]$ such that*

$$|C_{i_*}^{j_*}|, |D_{i_*}^{3-j_*}| \geqslant 3\varepsilon n.$$

Proof of Claim 21. First, since $\Delta(G) \leqslant \alpha n$, $|A_i| \leqslant k\alpha n$ for each $i \in [s-k+1]$.

Let $f(i) := |C_i^1| + |C_i^2|$. Then

$$\begin{aligned} f(1) &= 0, \\ f(s-k+1) &= n - |A_{s-k+1}| - s \quad \text{because } U_{s-k+1}^+ = D_{s-k+1}^1 = D_{s-k+1}^2 = B_{s-k+1} = \varnothing, \\ &\geqslant n - k\alpha n - 4\varepsilon n \quad \text{by (3) and the assumption that } \Delta(G) \leqslant \alpha n, \\ &= (1 - k\alpha - 4\varepsilon)n \geqslant 6\varepsilon n, \end{aligned}$$

and for each $i \in [s-k]$, we have

$$f(i+1) - f(i) \leqslant \deg_G(i) \leqslant \alpha n.$$

Hence, there exists $i_* \in [s-k+1]$ such that $6\varepsilon n \leqslant f(i_*) < (6\varepsilon+\alpha)n$. As $|B_{i_*}| < 2(\alpha+2\varepsilon)n$, we have

$$\begin{aligned} |D_{i_*}^1| + |D_{i_*}^2| &= n - |A_{i_*}| - |B_{i_*}| - (|C_{i_*}^1| + |C_{i_*}^2|) - |V(P)| \\ &\geqslant n - k\alpha n - 2(\alpha+2\varepsilon)n - (6\varepsilon+\alpha)n - 4\varepsilon n \\ &= (1 - (k+3)\alpha - 14\varepsilon)n \geqslant 6\varepsilon n. \end{aligned}$$

So, there exist $a, b \in \{1, 2\}$ such that $|C_{i_*}^a|, |D_{i_*}^b| \geqslant 3\varepsilon n$. By (5), $a \neq b$ and so we take $j_* := a$. This proves the claim. ■

Claim 22. *For each component C of $G[C_{i_*}^{j_*}]$ and each component D of $G[D_{i_*}^{3-j_*}]$, (C, D) is a pure pair of G.*

Proof of Claim 22. Assume not. By symmetry, we may assume that C has a vertex u having both a neighbor and a non-neighbor in D, because otherwise we swap C and D by reversing the order of P. As D is connected, there exist $v, v' \in V(D)$ such that $uv, vv' \in E(G)$ and $uv' \notin E(G)$.

Note that $m^+_{i_*}(v) \equiv m^+_{i_*}(v') \pmod 2$ and

$$\text{for every neighbor } \ell \in N_G(v) \cap U^+_{i_*}, \text{ the number } \ell - m^+_{i_*}(v) \text{ is even,} \tag{6}$$

because otherwise for the minimum $\ell \in N_G(v) \cap U^+_{i^*}$ with odd $\ell - m^+_{i_*}(v)$, a vertex set $\{v, m^-_{i_*}(u), m^-_{i_*}(u)+1, \ldots, \ell, u\}$ induces a strongly k-good generalized fan with v as its center, a contradiction by Proposition 17.

If $m^+_{i_*}(v) \leqslant m^+_{i_*}(v')$, then $\{v, u, m^-_{i_*}(u), m^-_{i_*}(u)+1, \ldots, m^+_{i_*}(v'), v'\}$ induces a strongly k-good generalized fan with v as a center by (6).

If $m^+_{i_*}(v) > m^+_{i_*}(v')$, then simply $(u, m^-_{i_*}(u), m^-_{i_*}(u)+1, \ldots, m^+_{i_*}(v'), v', v, u)$ is an induced cycle whose length is at least k and is of the same parity with k. Hence Proposition 14 implies a contradiction. ■

By Claim 22, there exists $S \in \{C^{j_*}_{i_*}, D^{3-j_*}_{i_*}\}$ such that every component of $G[S]$ has less than εn vertices. By Claim 21, we can greedily find a set of components of $G[S]$ covering at least εn vertices and at most $2\varepsilon n$ vertices. Since $|S| \geqslant 3\varepsilon n$, the vertices of S covered by this set of components with the vertices of S not covered by this set of components give a pure pair (A, B) with $|A|, |B| \geqslant \varepsilon n$, a contradiction. This proves the lemma. □

5 Discussions

For a graph G, we write $\chi(G)$ to denote its chromatic number and $\omega(G)$ to denote its clique number, that is the maximum size of a clique. A class $\mathcal{G}$ of graphs is called *χ-bounded* if there exists a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every induced subgraph H of a graph in $\mathcal{G}$, $\chi(H) \leqslant f(\omega(H))$. In addition, we say $\mathcal{G}$ is *polynomially χ-bounded* if f can be taken as a polynomial.

Every polynomially χ-bounded class of graphs has the strong Erdős-Hajnal property, but the converse does not hold; see the survey paper by Scott and Seymour [22]. So it is natural to ask whether the class of graphs with no pivot-minor isomorphic to C_k is polynomially χ-bounded, which is still open. So far Choi, Kwon, and Oum [4] showed that it is χ-bounded.

Theorem 23 (Choi, Kwon, and Oum [4, Theorem 4.1]). *For each $k \geqslant 3$, the class of graphs with no pivot-minor isomorphic to C_k is χ-bounded.*

They showed that $\chi(G) \leqslant 2(6k^3 - 26k^2 + 25k - 1)^{\omega(G)-1}$ holds for graphs G having no pivot-minor isomorphic to C_k, far from being a polynomial. Theorem 23 is now implied by a recent theorem of Scott and Seymour [21], solving three conjectures of Gyárfás [13] on χ-boundedness all at once.

Theorem 24 (Scott and Seymour [21]). *For all $k \geqslant 0$ and $\ell > 0$, the class of all graphs having no induced cycle of length k modulo ℓ is χ-bounded.*

To see why Theorem 24 implies Theorem 23, take $\ell := 2\lceil k/2 \rceil$ and apply Proposition 14. Still the bound obtained from Theorem 24 is far from being a polynomial.

And yet no one was able to answer the following problem of Esperet.

Problem 25 (Esperet; see [15])**.** Is it true that every χ-bounded class of graphs polynomially χ-bounded?

Thus it is natural to pose the following conjecture.

Conjecture 26. For every graph H, the class of graphs with no pivot-minor isomorphic to H is polynomially χ-bounded.

It is open whether Conjecture 26 holds when $H = C_k$. Conjecture 26 implies not only Conjectures 3, 5 but also the following conjecture of Geelen (see [8]) proposed in 2009 at the DIMACS workshop on graph colouring and structure held at Princeton University.

Conjecture 27 (Geelen; see [8])**.** For every graph H, the class of graphs with no vertex-minor isomorphic to H is χ-bounded.

Of course it is natural to pose the following conjecture, weaker than Conjecture 26 but stronger than Conjecture 27.

Conjecture 28 (Kim, Kwon, Oum, and Sivaraman [16])**.** For every graph H, the class of graphs with no vertex-minor isomorphic to H is polynomially χ-bounded.

For vertex-minors, more results are known. Kim, Kwon, Oum, and Sivaraman [16] proved that for each $k \geqslant 3$, the class of graphs with no vertex-minor isomorphic to C_k is polynomially χ-bounded. Their theorem is now implied by the following two recent theorems. To describe these theorems, we first have to introduce a few terms. A *circle graph* is the intersection graph of chords in a circle. In particular, C_k is a circle graph. The *rank-width* of a graph is one of the width parameters of graphs, measuring how easy it is to decompose a graph into a tree-like structure while keeping every cut to have a small 'rank'. Rank-width was introduced by Oum and Seymour [19]. We will omit the definition of the rank-width.

Theorem 29 (Geelen, Kwon, McCarty, and Wollan [12])**.** *For each circle graph H, there is an integer $r(H)$ such that every graph with no vertex-minor isomorphic to H has rank-width at most $r(H)$.*

Theorem 30 (Bonamy and Pilipczuk [3])**.** *For each k, the class of graphs of rank-width at most k is polynomially χ-bounded.*

As noted in [6], it is easy to prove directly that the class of graphs of bounded rank-width has the strong Erdős-Hajnal property, without using Theorem 30. See Figure 4 for a diagram showing the containment relations between these properties.

So, one may wonder whether the class of graphs with no pivot-minor isomorphic to C_k has bounded rank-width. Unfortunately, if k is odd, then it is not true, because all bipartite graphs have no pivot-minor isomorphic to C_k for odd k and yet have unbounded rank-width, see [17]. If k is even, then it would be true if the following conjecture hold.

Conjecture 31 (Oum [18])**.** For every bipartite circle graph H, there is an integer $r(H)$ such that every graph with no pivot-minor isomorphic to H has rank-width at most $r(H)$.

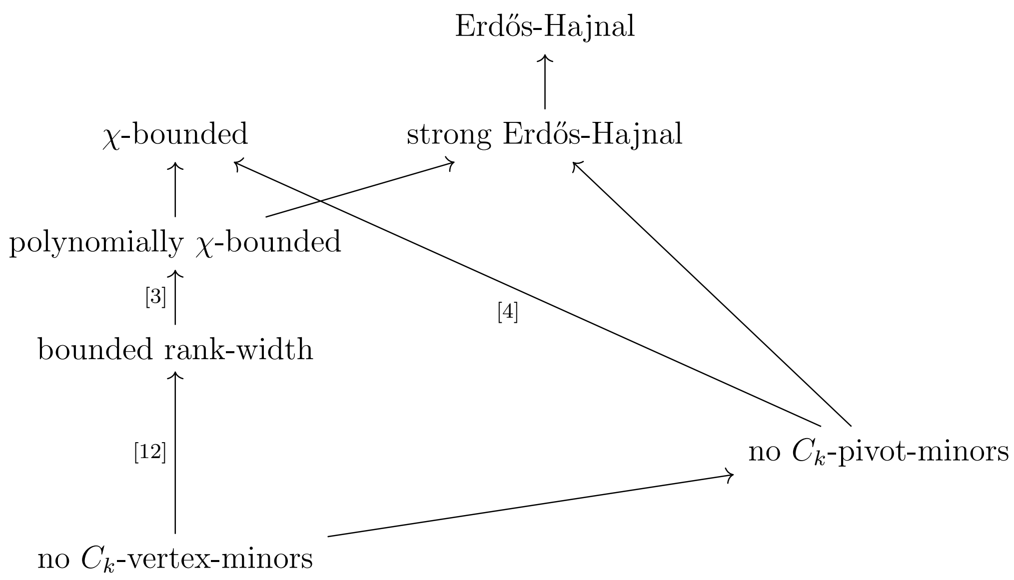

Figure 4: Known implications between properties of classes of graphs.

Note.

Chudnovsky, Scott, Seymour, and Spirkl [7] proved that for every graph H, the class of graphs G such that neither G nor $\overline{G}$ has any subdivision of H as an induced subgraph has the strong Erdős-Hajnal property. This implies that when k is even, the class of graphs with no induced even hole of length at least k and no induced even anti-hole of length at least k has the strong Erdős-Hajnal property. This is because every subdivision of a large theta graph[2] contains a large even hole. This implies Theorem 4 for even k but not for odd k by Propositions 14 and 15. The authors would like to thank the authors of [7] to share this observation.

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[2] A *theta graph* is a graph consisting of three internally disjoint paths of length at least 1 joining two fixed vertices.

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