The Erdős-Hajnal property for graphs with no fixed cycle as a pivot-minor

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Abstract

We prove that for every integer $k$, there exists $\varepsilon > 0$ such that for every $n$-vertex graph $G$ with no pivot-minor isomorphic to $C_k$, there exist disjoint sets $A, B \subseteq V(G)$ such that $|A|, |B| \geq \varepsilon n$, and $A$ is either complete or anticomplete to $B$. This proves the analog of the Erdős-Hajnal conjecture for the class of graphs with no pivot-minor isomorphic to $C_k$.

Mathematics Subject Classifications: 05C55, 05C75

1 Introduction

In this paper all graphs are simple, having no loops and no parallel edges. For a graph $G$, let $\omega(G)$ be the maximum size of a clique, that is a set of pairwise adjacent vertices and let $\alpha(G)$ be the maximum size of an independent set, that is a set of pairwise non-adjacent vertices. Erdős and Hajnal [9] proposed the following conjecture in 1989.

Conjecture 1 (Erdős and Hajnal [9]). For every graph $H$, there is $\varepsilon > 0$ such that all graphs $G$ with no induced subgraph isomorphic to $H$ satisfies

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^\varepsilon.$$
This conjecture still remains open. See [5] for a survey on this conjecture. We can ask the same question for weaker containment relations. Recently Chudnovsky and Oum [6] proved that this conjecture holds if we replace “induced subgraphs” with “vertex-minors” as follows. This is weaker in the sense that every induced subgraph $G$ is a vertex-minor of $G$ but not every vertex-minor of $G$ is an induced subgraph of $G$.

**Theorem 2** (Chudnovsky and Oum [6]). *For every graph $H$, there exists $\varepsilon > 0$ such that every graph with no vertex-minors isomorphic to $H$ satisfies*

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^{\varepsilon}.$$

We ask whether Conjecture 1 holds if we replace “induced subgraphs” with “pivot-minors” as follows.

**Conjecture 3.** *For every graph $H$, there exists $\varepsilon > 0$ such that every graph $G$ with no pivot-minor isomorphic to $H$ satisfies*

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^{\varepsilon}.$$

The detailed definition of pivot-minors will be presented in Section 3. For now, note that the analog for vertex-minors is weakest, the analog for pivot-minors is weaker than that for induced subgraphs but stronger than that for vertex-minors. This is because every induced subgraph of $G$ is a pivot-minor of $G$, and every pivot-minor of $G$ is a vertex-minor of $G$. In other words, Conjecture 1 implies Conjecture 3 and Conjecture 3 implies Theorem 2. We verify Conjecture 3 for $H = C_k$, the cycle graph on $k$ vertices as follows.

**Theorem 4.** *For every $k \geq 3$, there exists $\varepsilon > 0$ such that every graph with no pivot-minor isomorphic to $C_k$ satisfies*

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^{\varepsilon}.$$

We actually prove a stronger property, as Chudnovsky and Oum [6] did. Before stating this property, let us first state a few terminologies. A class $\mathcal{G}$ of graphs closed under taking induced subgraphs is said to have the *Erdős-Hajnal property* if there exists $\varepsilon > 0$ such that every graph $G$ in $\mathcal{G}$ satisfies

$$\max(\alpha(G), \omega(G)) \geq |V(G)|^{\varepsilon}.$$

A class $\mathcal{G}$ of graphs closed under taking induced subgraphs is said to have the *strong Erdős-Hajnal property* if there exists $\varepsilon > 0$ such that every $n$-vertex graph in $\mathcal{G}$ with $n > 1$ has disjoint sets $A, B$ of vertices such that $|A|, |B| \geq \varepsilon n$ and $A$ is either complete or anti-complete to $B$. It is an easy exercise to show that the strong Erdős-Hajnal property implies the Erdős-Hajnal property, see [1, 10].

Chudnovsky and Oum [6] proved that the class of graphs with no vertex-minors isomorphic to $H$ for a fixed graph $H$ has the strong Erdős-Hajnal property, implying Theorem 2. We propose its analog for pivot-minors as a conjecture, which implies the theorem...
of Chudnovsky and Oum [6]. Note that this conjecture is not true if we replace the pivot-minor with induced graphs. For example, the class of triangle-free graphs does not have the strong Erdős-Hajnal property [10].

**Conjecture 5.** For every graph $H$, there exists $\varepsilon > 0$ such that for all $n > 1$, every $n$-vertex graph with no pivot-minor isomorphic to $H$ has two disjoint sets $A, B$ of vertices such that $|A|, |B| \geq \varepsilon n$ and $A$ is complete or anti-complete to $B$.

We prove that this conjecture holds if $H = C_k$. In other words, the class of graphs with no pivot-minor isomorphic to $C_k$ has the strong Erdős-Hajnal property as follows. This implies Theorem 4.

**Theorem 6.** For every integer $k \geq 3$, there exists $\varepsilon > 0$ such that for all $n > 1$, every $n$-vertex graph with no pivot-minor isomorphic to $C_k$ has two disjoint sets $A, B$ of vertices such that $|A|, |B| \geq \varepsilon n$ and $A$ is complete or anti-complete to $B$.

This paper is organized as follows. In Section 2, we will introduce basic definitions and review necessary theorems of Rödl [20] and Bonamy, Bousquet, and Thomassé [2]. In Section 3, we will present several tools to find a pivot-minor isomorphic to $C_k$. In particular, it proves that a long anti-hole contains $C_k$ as a pivot-minor. In Section 4, we will present the proof of the main theorem, Theorem 6. In Section 5, we will relate our theorem to the problem on $\chi$-boundedness, and discuss known results and open problems related to polynomial $\chi$-boundedness and the Erdős-Hajnal property.

## 2 Preliminaries

Let $\mathbb{N}$ be the set of positive integers and for each $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$. For a graph $G = (V, E)$, let $\overline{G} = (V, \binom{V}{2} - E)$ be the complement of $G$. We write $\Delta(G)$ and $\delta(G)$ to denote the maximum degree of $G$ and the minimum degree of $G$ respectively.

Let $T$ be a tree rooted at a specified node $v_r$, called the root. If the path from $v_r$ to a node $y$ in $T$ contains $x \in V(T) - \{y\}$, we say that $x$ is an ancestor of $y$, and $y$ is a descendant of $x$. If one of $x$ and $y$ is an ancestor of the other, we say that $x, y$ are related. We say that two disjoint sets $X$ and $Y$ of nodes of $T$ are unrelated if no pairs of $x \in X$ and $y \in Y$ are related.

For disjoint vertex sets $X$ and $Y$, we say $X$ is complete to $Y$ if every vertex of $X$ is adjacent to all vertices of $Y$. We say $X$ is anti-complete to $Y$ if every vertex of $X$ is non-adjacent to $Y$. A pure pair of a graph $G$ is a pair $(A, B)$ of disjoint subsets of $V(G)$ such that $A$ is complete or anti-complete to $B$.

For a vertex $u$, let $N_G(u)$ denote the set of neighbors of $u$ in $G$. For each $U \subseteq V(G)$, we write

$$N_G(U) := \bigcup_{u \in U} N_G(u) - U.$$  

The following lemma is proved in Section 2 of [2].
Lemma 7 (Bonamy, Bousquet, and Thomassé [2]). For every connected graph \( G \) and a vertex \( v_r \in V(G) \), there exist an induced subtree \( T \) of \( G \) rooted at \( v_r \) and a function \( r : V(G) \to V(T) \) satisfying the following.

(T1) \( r(v_r) = v_r \), and for each \( u \in V(G) - \{v_r\} \), the vertex \( r(u) \) is a neighbor of \( u \). In particular, \( T \) is a dominating tree of \( G \).

(T2) If \( r(x) \) and \( r(y) \) are not related, then \( xy \notin E(G) \).

Rödl [20] proved the following theorem. Its weaker version was later proved by Fox and Sudakov [11] without using the regularity lemma. A set \( U \) of vertices of a graph \( G \) is an \( \varepsilon \)-stable set of a graph \( G \) if \( G[U] \) has at most \( \varepsilon \binom{|U|}{2} \) edges. Similarly, \( U \) is an \( \varepsilon \)-clique of a graph \( G \) if \( G[U] \) has at least \( (1 - \varepsilon)\binom{|U|}{2} \) edges.

Theorem 8 (Rödl [20]). For all \( \varepsilon > 0 \) and a graph \( H \), there exists \( \delta > 0 \) such that every \( n \)-vertex graph \( G \) with no induced subgraph isomorphic to \( H \) has an \( \varepsilon \)-stable set or an \( \varepsilon \)-clique of size at least \( \delta n \).

We will use the following simple lemma. We present its proof for completeness.

Lemma 9. Let \( G \) be a graph. Every \( \varepsilon \)-stable set \( U \) of \( G \) has a subset \( U' \) of size at least \( |U|/2 \) with \( \Delta(G[U']) \leq 4\varepsilon |U'| \).

Proof. Let \( U' \) be the set of vertices of degree of at most \( 2\varepsilon |U| \) in \( G[U] \). Because

\[
\sum_{v \in U} \deg_{G[U]}(v) < \varepsilon |U|^2,
\]

we have \( |U'| \geq |U|/2 \). Moreover, for each vertex \( v \in U' \), we have \( \deg_{G[U']}(v) \leq 2\varepsilon |U| \leq 4\varepsilon |U'| \). \( \square \)

Using Lemma 9, we can deduce the following corollary of Theorem 8.

Corollary 10. For all \( \alpha > 0 \) and a graph \( H \), there exists \( \delta > 0 \) such that every graph \( G \) with no induced subgraph isomorphic to \( H \) has a set \( U \subseteq V(G) \) with \( |U| \geq \delta |V(G)| \) such that either \( \Delta(G[U]) \leq \alpha |U| \) or \( \Delta(G'[U]) \leq \alpha |U| \).

The following easy lemma will be used to find a connected induced subgraph inside the output of Corollary 10. We omit its easy proof.

Lemma 11. A graph \( G \) has a pure pair \((A, B)\) such that \( |A|, |B| \geq |V(G)|/3 \) or has a connected induced subgraph \( H \) such that \( |V(H)| \geq |V(G)|/3 \).

Lemma 12 (Bonamy, Bousquet, and Thomassé [2, Lemma 3]). Let \( T \) be a tree rooted at \( v_r \) and \( w : V(T) \to \mathbb{R} \) be a non-negative weight function on \( V(T) \) with \( \sum_{x \in V(T)} w(x) = 1 \). Then there exists either a path \( P \) from \( v_r \) with weight at least \( 1/4 \) or two unrelated sets \( A \) and \( B \) both with weight at least \( 1/4 \).
A hole is an induced cycle of length at least 5.

Lemma 13 (Bonamy, Bousquet, and Thomassé [2, Lemma 4]). For given \( k \geq 3 \), there exist \( \alpha = \alpha(k) > 0 \) and \( \varepsilon = \varepsilon(k) > 0 \) such that for any \( n \)-vertex graph \( G \) with \( n \geq 2 \) and \( \Delta(G) \leq \alpha n \), if \( G \) has no holes of length at least \( k \) and has a dominating induced path, then \( G \) contains a pair \( (A, B) \) of disjoint vertex sets such that \( |A|, |B| \geq \varepsilon n \).

3 Finding a cycle as a pivot-minor

For a given graph \( G \) and an edge \( uv \), a graph \( G \wedge uv \) obtained from \( G \) by pivoting \( uv \) is defined as follows. Let \( V_1 = N_G(u) \cap N_G(v) \), \( V_2 = N_G(u) - N_G(v) \), \( V_3 = N_G(v) - N_G(u) \). Then \( G \wedge uv \) is the graph obtained from \( G \) by complementing adjacency between vertices between \( V_i \) and \( V_j \) for all \( 1 \leq i < j \leq 3 \) and swapping the label of \( u \) and \( v \). See Figure 1 for an illustration. We say that \( H \) is a pivot-minor of \( G \) if \( H \) can be obtained from \( G \) by deleting vertices and pivoting edges. For this paper, we will also say that \( H \) is a pivot-minor of \( G \), when \( G \) has a pivot-minor isomorphic to \( H \). A pivot-minor \( H \) of \( G \) is proper if \( |V(H)| < |V(G)| \).

We describe several scenarios for constructing \( C_k \) as a pivot-minor. The following proposition is an easy one; One can obtain a desired pivot-minor from a longer cycle of the same parity.

Proposition 14. For \( m \geq k \geq 3 \) with \( m \equiv k \pmod{2} \), the cycle \( C_m \) has a pivot-minor isomorphic to \( C_k \).

Proof. We proceed by induction on \( m - k \). We may assume that \( m > k \). Let \( xy \) be an edge of \( C_m \). Then \( (C_m \wedge xy) - x - y \) is isomorphic to \( C_{m-2} \), which contains a pivot-minor isomorphic to \( C_k \) by the induction hypothesis.

Proposition 15. For integers \( k \geq 3 \) and \( m \geq \frac{3}{2}k + 6 \), the graph \( \overline{C_m} \) has a pivot-minor isomorphic to \( C_k \).

Before proving Proposition 15, we present a simple lemma on partial complements of the cycle graph. The partial complement\(^1 \) \( G \oplus S \) of a graph \( G \) by a set \( S \) of vertices is a

\(^1\)We found this concept in a paper by Kamiński, Lozin, and Milanič [14], though it may have been studied previously, as it is a natural concept.
Figure 2: Obtaining an \((s - 2, 3)\)-cycle from an \((s, 9)\)-cycle when \(s > 9\) in the proof of Lemma 16.

graph obtained from \(G\) by changing all edges within \(S\) to non-edges and non-edges within \(S\) to edges.

For \(s \geq t \geq 0\), we say that \(G\) is an \((s, t)\)-cycle if \(G\) is isomorphic to a graph \(C_s \oplus X\) for a set \(X\) of \(t\) consecutive vertices in the cycle \(C_s\).

**Lemma 16.** Let \(s \geq t \geq 6\). An \((s, t)\)-cycle contains a pivot-minor isomorphic to an \((s - 2, t - 6)\)-cycle.

**Proof.** Let \(v_1, \ldots, v_s\) be the vertices of \(C_s\) in the cyclic order where \(X = \{v_1, \ldots, v_t\}\). Then it is easy to check that \((C_s \oplus X) \land v_2v_{t-1} - \{v_2, v_{t-1}\}\) is isomorphic to \(C_{s-2} \oplus X'\) where \(X'\) consists of \(t - 6\) consecutive vertices on the cycle. See Figure 2.

**Proof of Proposition 15.** As \(C_m\) is an \((m, m)\)-cycle, by Lemma 16, \(C_m\) contains a pivot-minor isomorphic to an \((m - 2i, m - 6i)\)-cycle for all \(i \leq m/6\).

Let us fix \(i = \lceil (k - 2)/4 \rceil\). Then \(m - 6i \geq m - 6 \cdot (k + 1)/4 \geq 9/2\) and therefore \(C_m\) contains a pivot-minor \(H\) isomorphic to an \((m - 2i, m - 6i)\)-cycle and \(m - 6i \geq 5\). We may assume that \(H = C_{m-2i} \oplus X\) where \(C_{m-2i} = v_1 \cdots v_{m-2i}\) and \(X = \{v_{4i+1}, \ldots, v_{m-2i}\}\).

Note that \(H\) contains an induced cycle \(C = v_1 \cdots v_{4i}v_{4i+1}v_{m-2i}v_1\) of length \(4i + 2\geq k\). If \(k\) is even, then by Proposition 14, \(H\) contains a pivot-minor isomorphic to \(C_k\). So we may assume that \(k\) is odd and therefore \(|V(C)| = 4i + 2 \geq k + 1\).

Let \(x = v_{m-2i}, y = v_{4i+1}\) be the two vertices in \(V(C) \cap X\). Since \(m - 6i \geq 5\), there is a common neighbor \(z\) of \(x\) and \(y\) in \(X\). Then \(z\) has exactly two neighbors \(x\) and \(y\) in \(V(C)\). Then \(H[V(C) \cup \{z\}] \land yz - y - z\) is a cycle of length \(4i + 1\). Since \(4i + 1 \geq k\), by Proposition 14, it contains a pivot-minor isomorphic to \(C_k\).

A **generalized fan** is a graph \(G\) with a specified vertex \(c\), called the **center**, such that \(G - c\) is an induced path of length at least 1, called the **main path** of \(G\) and both ends of the main path are adjacent to \(c\). If \(c\) is adjacent to all vertices of \(G - c\), then \(G\) is called a **fan**.
An interval of a generalized fan with a center $c$ is a maximal subpath of the main path having no internal vertex adjacent to $c$. The length of an interval is its number of edges. A generalized fan is an $(a_1, \ldots, a_s)$-fan if the lengths of intervals are $a_1, \ldots, a_s$ in order. Note that an $(a_1, \ldots, a_s)$-fan is also an $(a_s, \ldots, a_1)$-fan. An $(a_1, \ldots, a_s)$-fan is a $k$-good fan if $a_1 \geq k - 2$ or $a_s \geq k - 2$. An $(a_1, \ldots, a_s)$-fan is a strongly $k$-good fan if $s \geq 2$ and either $a_1 \geq k - 2$ and $a_s$ is odd, or $a_s \geq k - 2$ and $a_1$ is odd. It is easy to observe that every $k$-good fan has a hole of length at least $k$. However, that does not necessarily lead to a pivot-minor isomorphic to $C_k$ because of the parity issues. In the next proposition, we show that every strongly $k$-good fan has a pivot-minor isomorphic to $C_k$.

**Proposition 17.** Let $k \geq 5$ be an integer. Every strongly $k$-good fan has a pivot-minor isomorphic to $C_k$.

**Proof.** Let $G$ be an $(a_1, \ldots, a_s)$-fan such that $s \geq 2$, $a_1 \geq k - 2$, and $a_s$ is odd. We proceed by the induction on $|V(G)|$. We may assume that $G$ has no proper pivot-minor that is a strongly $k$-good fan. Note that $C_{a_i+2}$ is an induced subgraph of $G$, hence if $a_1 \equiv k \pmod{2}$, then $C_k$ is isomorphic to a pivot-minor of $G$ by Proposition 14. Thus we may assume that $a_1 \not\equiv k \pmod{2}$ and so $a_1 \geq k - 1$.

If $a_i$ is odd for some $1 < i < s$, then $G$ contains a smaller strongly $k$-good fan by taking the first $i$ intervals, contradicting our assumption. Thus $a_i$ is even for all $1 < i < s$. If $a_i \geq 3$ for some $i > 1$, then let $uv$ be an internal edge of the $i$-th interval. Then $G \backslash uv - u - v$ is a strongly $k$-good fan, contradicting our assumption. Thus, we may assume that $a_i \leq 2$ for all $i > 1$ and so $G$ is an $(a_1, 2, \ldots, 2, 1)$-fan.

Let $xy$ be the last interval of $G$ with length 1. Then $G \backslash xy - x - y$ is a $(a_1, 2, \ldots, 2, 1)$-fan with $s - 1$ intervals. By the assumption, we may assume that $s = 2$ and $G \backslash xy - x - y$ is an $(a_1 - 1)$-fan with one interval, which is a cycle with $a_1 + 1$ edges. As $a_1 + 1 \geq k$ and $a_1 + 1 \equiv k \pmod{2}$, Proposition 14 implies that $G$ contains a pivot-minor isomorphic to $C_k$. \qed

### 4 Proof of Theorem 6

First we choose $\alpha > 0$ and $\varepsilon_0 > 0$ so that

$$4\alpha \leq \alpha(\lceil \frac{3}{2}k + 6 \rceil) \quad \text{and} \quad \varepsilon_0 = \varepsilon(\lceil \frac{3}{2}k + 6 \rceil) \quad \text{where} \quad \alpha(\cdot), \varepsilon(\cdot) \quad \text{are specified in Lemma 13} \quad (1)$$

and in addition $\alpha < 1/(8k)$ as well. Let $\delta > 0$ be a constant obtained by applying Corollary 10 with $\alpha/3$ as $\alpha$ and $C_k$ as $H$. Choose $\varepsilon > 0$ so that

$$\varepsilon < \min \left( \frac{\delta}{12}, (1 - 4(k + 3)\alpha) \frac{\delta}{240}, \frac{\varepsilon_0 \delta}{12} \right).$$

Let $n > 1$ be an integer and $G$ be an $n$-vertex graph with no pivot-minor isomorphic to $C_k$. In particular, $G$ does not have $C_k$ as an induced subgraph. To derive a contradiction, we assume that $G$ contains no pure pair $(A, B)$ with $|A|, |B| \geq \varepsilon n$. We may assume that $\varepsilon n > 1$, because otherwise an edge or a non-edge of $G$ gives a pure pair.
By Corollary 10, there exists a subset \( U \) of \( V(G) \) such that \(|U| \geq \delta|V(G)|\) and \( \Delta(G^0)[U] \leq (\alpha/3)|U| \) for some \( G^0 \in \{G, \overline{G}\} \). By the assumption on \( G \), \( G^0[U] \) has no pure pair \((A, B)\) with \(|A|, |B| \geq (\varepsilon/\delta)|U|\). As \( \varepsilon/\delta < 1/3 \), by Lemma 11, \( G^0[U] \) has a connected induced subgraph \( G' \) such that \(|V(G')| \geq |U|/3\). Let \( n' = |V(G')|\).

Then \( n' \geq (\delta/3)n \) and \( \Delta(G') \leq (\alpha/3)|U| \leq \alpha n' \). By the assumption on \( G \),

\[
G' \text{ contains no pure pair } (A, B) \text{ with } |A|, |B| \geq (3\varepsilon/\delta)n'.
\]

By applying Lemma 7 with \( G' \), we obtain a dominating induced tree \( T \) and \( r: V(G') \to V(T) \) satisfying Lemma 7 (T1)–(T2) with \( G' \). For each \( u \in V(T) \), let

\[
w(u) := \frac{|r^{-1}(\{u\})|}{n'}
\]

be the weight of \( u \). By applying Lemma 12 with the weight \( w \), we obtain either an induced path \( P \) of \( T \) with weight at least \( 1/4 \) or two unrelated sets \( A \) and \( B \) both with weight at least \( 1/4 \).

In the latter case, Lemma 7 (T2) implies that \( r^{-1}(A) \) is anticomplete to \( r^{-1}(B) \) in \( G' \) and \(|r^{-1}(A)|, |r^{-1}(B)| \geq n'/4 \geq (3\varepsilon/\delta)n' \), contradicting (2).

Hence, there exists an induced path \( P \) in \( G' \) with \(|V(P) \cup N_{G'}(V(P))| \geq n'/4 \). Let \( W := V(P) \cup N_{G'}(V(P)) \). Note that \( n'/4 \geq \delta n/12 > \varepsilon n > 1 \) and so \( |W| \geq 2 \).

Suppose that \( G' \) is an induced subgraph of \( \overline{G} \). Using (1), we apply Lemma 13 to \( G'[W] \) with \( 4\alpha \) as \( \alpha \) and \( \lfloor \frac{3}{2}k + 6 \rfloor \) as \( k \). Then we can deduce from (2) and \( \varepsilon|W| \geq \frac{12\varepsilon n'}{\delta} = (3\varepsilon/\delta)n' \) that the graph \( G'[W] \) contains an induced cycle \( C_m \) with \( m \geq \lfloor \frac{3}{2}k + 6 \rfloor \) and by Proposition 15, \( \overline{G'} \) contains a pivot-minor isomorphic to \( C_k \), and so does \( G \), a contradiction.

Thus \( G' \) is an induced subgraph of \( G \). Let \( G^* := G'[W] \) and let \( n^* = |W| \). Then \( G^* \) has no pivot-minor isomorphic to \( C_k \), \( n^* \geq n'/4 \), and \( \Delta(G^*) \leq 4\alpha n^* \). By (2), \( G^* \) contains no pure pair \((A, B)\) with \(|A|, |B| \geq (12\varepsilon/\delta)n^* \). Now the theorem follows from applying the following lemma with \( G^*, n^*, 4\alpha, 12\varepsilon/\delta \) playing the roles of \( G, n, \alpha, \varepsilon \) respectively in the statement of the lemma.

**Lemma 18.** Let \( k \geq 3 \) be an integer. Let \( 0 < \alpha < 1/(2k) \), \( 0 < \varepsilon \leq (1 - (k + 3)\alpha)/20 \). Let \( G \) be a graph on \( n \geq 2 \) vertices such that \( \Delta(G) \leq \alpha n \) and \( G \) has no pure pair \((A, B)\) with \(|A|, |B| \geq \varepsilon n \). If \( G \) has a dominating induced path \( P \), then \( G \) has a pivot-minor isomorphic to \( C_k \).

**Proof.** Suppose that \( G \) has no pivot-minor isomorphic to \( C_k \). Note that \( \varepsilon n > 1 \) as otherwise we have a pure pair on two vertices since \( n \geq 2 \). Let us label vertices of \( P \) by \( 1, 2, \ldots, s \) in the order.

As \( P \) is a dominating path of \( G \) and \( 1 \leq \Delta(G) \leq \alpha n \), we have \( 2\alpha ns \geq (\alpha n + 1)s \geq n \) and therefore

\[
s \geq 1/(2\alpha).
\]

Note that \( s - k > 0 \) because \( \alpha < \frac{1}{2k} \). As \( P \) is an induced path, it contains a pure pair \((A, B)\) with \(|A|, |B| \geq \lfloor \frac{s - 1}{2} \rfloor \) and so \( \frac{s^2}{2} \leq \lfloor \frac{s^2}{2} \rfloor < \varepsilon n \). Because \( \varepsilon n > 1 \), we have
For each \( i \in [s - k + 1] \), let
\[
U_i^- := \{1, \ldots, i - 1\}, \quad U_i^0 := \{i, \ldots, i + k - 1\}, \quad \text{and} \quad U_i^+ := \{i + k, \ldots, s\}.
\]

In other words, this partitions \( P \) into three (possibly empty) subpaths. Furthermore, for all \( i \in [s - k + 1] \) and \( u \in N_G(U_i^-) - V(P) \), let
\[
m_i^-(u) := \max(N_G(u) \cap U_i^-)
\]
and for all \( i \in [s - k + 1] \) and \( u \in N_G(U_i^+) - V(P) \), let
\[
m_i^+(u) := \min(N_G(u) \cap U_i^+),
\]
indicating the largest neighbor of \( u \) in \( U_i^- \) and the smallest neighbor of \( u \) in \( U_i^+ \) respectively. For each \( i \in [s - k + 1] \), let
\[
A_i := N_G(U_i^0) - V(P) \quad \text{and} \quad B_i := (N_G(U_i^-) \cap N_G(U_i^+)) - (A_i \cup V(P)).
\]

Note that for each \( u \in B_i \), we have
\[
m_i^+(u) - m_i^-(u) \not\equiv k \pmod{2},
\]
because otherwise \((u, m_i^-(u), m_i^-(u) + 1, \ldots, m_i^+(u), u)\) forms an induced cycle of length at least \( k \) and Proposition 14 implies that \( G \) contains a pivot-minor isomorphic to \( C_k \), a contradiction.

For each \( i \in [s - k + 1] \), let
\[
C_i^1 := \{u \in N_G(U_i^-) - (A_i \cup B_i \cup V(P)) : m_i^-(u) \equiv 1 \pmod{2}\},
\]
\[
C_i^2 := \{u \in N_G(U_i^-) - (A_i \cup B_i \cup V(P)) : m_i^-(u) \equiv 0 \pmod{2}\},
\]
\[
D_i^1 := \{u \in N_G(U_i^+) - (A_i \cup B_i \cup V(P)) : m_i^+(u) \equiv k \pmod{2}\}, \quad \text{and}
\]
\[
D_i^2 := \{u \in N_G(U_i^+) - (A_i \cup B_i \cup V(P)) : m_i^+(u) \equiv k + 1 \pmod{2}\}.
\]

Recall that \( P \) is dominating. Hence, for each \( i \), the sets \( \{A_i, B_i, C_i^1, C_i^2, D_i^1, D_i^2, V(P)\} \) forms a partition of \( V(G) \) into 7 possibly empty sets.

If there exists an edge between \( u \in C_i^j \) and \( v \in D_i^j \) for some \( j \in [2] \), then we obtain an induced cycle \((u, m_i^-(u), m_i^-(u) + 1, \ldots, m_i^+(v), v, u)\) having length \( m_i^+(v) - m_i^-(u) + 3 > k \) and \( m_i^+(v) - m_i^-(u) + 3 \equiv k \pmod{2} \), contradicting our assumption that \( G \) has no pivot-minor isomorphic to \( C_k \) by Proposition 14. Thus \( C_i^j \) is anticomplete to \( D_i^j \). Hence,
\[
\min\{|C_i^j|, |D_i^j|\} < \varepsilon n.
\]
for all \( i \in [s - k + 1] \) and \( j \in [2] \). Furthermore, we prove the following.
Figure 3: $s' = i + k - 1$. Bold lines indicate $m_i^-(u)$ and $m_i^+(u)$.

Claim 19. Let $i \in [s-k+1]$. For each $v \in B_i$, all integers in $N_G(v) \cap U_i^-$ have the same parity and all integers in $N_G(v) \cap U_i^+$ have the same parity.

Proof of Claim 19. If $N_G(v) \cap U_i^+$ has two integers $a < b$ of the different parity, then $G$ contains a strongly $k$-good generalized fan by taking a subpath of $P$ from $m_i^-(v)$ to $b$ as its main path and $v$ as its center. Then by Proposition 17, $G$ contains a pivot-minor isomorphic to $C_k$, contradicting the assumption. Thus all integers in $N_G(v) \cap U_i^+$ have the same parity and similarly all integers in $N_G(v) \cap U_i^-$ have the same parity.

Claim 20. For all $i \in [s-k+1]$, $|B_i| < 2(\alpha + 2\varepsilon)n$.

Proof of Claim 20. Suppose $|B_i| \geq 2(\alpha + 2\varepsilon)n$ for some $i \in [s-k+1]$. Then there exists $r_B \in \{0, 1\}$ such that

$$B' := \{u \in B_i : m_i^+(u) \equiv r_B \pmod{2}\}$$

has size at least $(\alpha + 2\varepsilon)n$. By (4), $m_i^+(u) \equiv k + r_B + 1 \pmod{2}$ for all $u \in B'$.

We claim that if $uv$ is an edge in $G[B']$, then $(m_i^-(u), m_i^+(u)) = (m_i^-(v), m_i^+(v))$. Suppose not. Without loss of generality, we may assume that $m_i^+(u) < m_i^+(v)$, because otherwise we may reverse the ordering of $P$ to ensure that $m_i^-(u) \neq m_i^-(v)$ and swap $u$ and $v$ if necessary.

If $m_i^+(u) \geq m_i^+(v)$, then by Claim 19, $\{m_i^-(v), m_i^-(v) + 1, \ldots, m_i^+(u), u, v\}$ induces a strongly $k$-good generalized fan with $v$ as a center and $(m_i^-(v), m_i^-(v) + 1, \ldots, m_i^+(u), u)$ as its main path. This implies that $G$ has a pivot-minor isomorphic to $C_k$ by Proposition 17, contradicting our assumption.
If \( m_i^+(u) < m_i^+(v) \), then \((m_i^-(v), m_i^+(v) + 1, \ldots, m_i^+(u), u, v)\) is an induced cycle of length \( m_i^+(u) - m_i^-(v) + 3 \geq k \), and \( m_i^+(u) - m_i^-(v) + 3 \equiv (k + r_B + 1) - r_B + 3 \equiv k \) (mod 2), a contradiction by Proposition 14.

Hence, \((m_i^-(v), m_i^+(u)) = (m_i^-(v), m_i^+(v))\) for all \( uv \in E(G[B']) \). Let \( C_1, \ldots, C_t \) be the connected components of \( G[B'] \). By the above observation, for each \( j \in [t] \), there exist \( a_j \in U_i^- \) and \( b_j \in U_i^+ \) such that \( V(C_j) \subseteq N_G(a_j) \cap N_G(b_j) \). So, \( |V(C_j)| \leq \alpha n \). As |\( B' \)\| \( \geq (\alpha + 2\varepsilon)n \), there exists a set \( I \subseteq \{1, 2, \ldots, t\} \) such that \( \varepsilon n \leq \sum_{i \in I} |V(C_i)| \leq (\alpha + \varepsilon)n \). Let \( A := \bigcup_{i \in I} V(C_i) \) and \( B := B' - A \). Then \((A, B)\) is a pure pair of \( G \) with \(|A|, |B| \geq \varepsilon n\), a contradiction.

**Claim 21.** There exist \( i_* \in [s - k + 1] \) and \( j_* \in [2] \) such that

\[ |C_{i_*}^{\beta_1}|, |D_{i_*}^{3-j_*}| \geq 3\varepsilon n. \]

**Proof of Claim 21.** First, since \( \Delta(G) \leq \alpha n \), \(|A_i| \leq k\alpha n\) for each \( i \in [s - k + 1] \).

Let \( f(i) := |C_i^1| + |C_i^2| \). Then

\[
\begin{align*}
  f(1) & = 0, \\
  f(s - k + 1) & = n - |A_{s-k+1}| - s \quad \text{because } U_{s-k+1}^+ = D_{s-k+1}^1 = D_{s-k+1}^2 = B_{s-k+1} = \emptyset, \\
  & \geq n - k\alpha n - 4\varepsilon n \quad \text{by (3) and the assumption that } \Delta(G) \leq \alpha n, \\
  & = (1 - k\alpha - 4\varepsilon)n \geq 6\varepsilon n,
\end{align*}
\]

and for each \( i \in [s - k] \), we have

\[ f(i + 1) - f(i) \leq \deg_G(i) \leq \alpha n. \]

Hence, there exists \( i_* \in [s - k + 1] \) such that \( 6\varepsilon n \leq f(i_*) < (6\varepsilon + \alpha)n \). As \(|B_{i_*}| < 2(\alpha + 2\varepsilon)n\), we have

\[
\begin{align*}
  |D_{i_*}^1| + |D_{i_*}^2| & = n - |A_{i_*}| - |B_{i_*}| - (|C_{i_*}^1| + |C_{i_*}^2|) - |V(P)| \\
  & \geq n - k\alpha n - 2(\alpha + 2\varepsilon)n - (6\varepsilon + \alpha)n - 4\varepsilon n \\
  & = (1 - (k + 3)\alpha - 14\varepsilon)n \geq 6\varepsilon n.
\end{align*}
\]

So, there exist \( a, b \in \{1, 2\} \) such that \(|C_{i_*}^{\alpha}|, |D_{i_*}^{j_*}| \geq 3\varepsilon n\). By (5), \( a \neq b \) and so we take \( j_* := a \). This proves the claim.

**Claim 22.** For each component \( C \) of \( G[C_{i_*}^\alpha] \) and each component \( D \) of \( G[D_{i_*}^{3-j_*}] \), \((C, D)\) is a pure pair of \( G \).

**Proof of Claim 22.** Assume not. By symmetry, we may assume that \( C \) has a vertex \( u \) having both a neighbor and a non-neighbor in \( D \), because otherwise we swap \( C \) and \( D \) by reversing the order of \( P \). As \( D \) is connected, there exist \( v, v' \in V(D) \) such that \( uv, vv' \in E(G) \) and \( uv' \notin E(G) \).
Note that \( m_i^+(v) \equiv m_i^+(v') \pmod{2} \) and
\[
\text{for every neighbor } \ell \in N_G(v) \cap U_i^+, \text{ the number } \ell - m_i^+(v) \text{ is even,}
\] (6)
because otherwise for the minimum \( \ell \in N_G(v) \cap U_i^+ \) with odd \( \ell - m_i^+(v) \), a vertex set \( \{v, m_i^+(u), m_i^-(u) + 1, \ldots, \ell, u\} \) induces a strongly \( k \)-good generalized fan with \( v \) as its center, a contradiction by Proposition 17.

If \( m_i^+(v) \leq m_i^+(v') \), then \( \{v, u, m_i^+(u), m_i^-(u) + 1, \ldots, m_i^+(v'), v'\} \) induces a strongly \( k \)-good generalized fan with \( v \) as a center by (6).

If \( m_i^+(v) > m_i^+(v') \), then simply \( \{u, m_i^+(u), m_i^-(u) + 1, \ldots, m_i^+(v'), v', v, u\} \) is an induced cycle whose length is at least \( k \) and is of the same parity with \( k \). Hence Proposition 14 implies a contradiction.

By Claim 22, there exists \( S \in \{C_i^+, D_i^3\} \) such that every component of \( G[S] \) has less than \( \varepsilon n \) vertices. By Claim 21, we can greedily find a set of components of \( G[S] \) covering at least \( \varepsilon n \) vertices and at most \( 2\varepsilon n \) vertices. Since \( |S| \geq 3\varepsilon n \), the vertices of \( S \) covered by this set of components with the vertices of \( S \) not covered by this set of components give a pure pair \( (A, B) \) with \( |A|, |B| \geq \varepsilon n \), a contradiction. This proves the lemma. \( \square \)

5 Discussions

For a graph \( G \), we write \( \chi(G) \) to denote its chromatic number and \( \omega(G) \) to denote its clique number, that is the maximum size of a clique. A class \( \mathcal{G} \) of graphs is called \( \chi \)-bounded if there exists a function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) such that for every induced subgraph \( H \) of a graph in \( \mathcal{G} \), \( \chi(H) \leq f(\omega(H)) \). In addition, we say \( \mathcal{G} \) is polynomially \( \chi \)-bounded if \( f \) can be taken as a polynomial.

Every polynomially \( \chi \)-bounded class of graphs has the strong Erdős-Hajnal property, but the converse does not hold; see the survey paper by Scott and Seymour [22]. So it is natural to ask whether the class of graphs with no pivot-minor isomorphic to \( C_k \) is polynomially \( \chi \)-bounded, which is still open. So far Choi, Kwon, and Oum [4] showed that it is \( \chi \)-bounded.

**Theorem 23** (Choi, Kwon, and Oum [4, Theorem 4.1]). For each \( k \geq 3 \), the class of graphs with no pivot-minor isomorphic to \( C_k \) is \( \chi \)-bounded.

They showed that \( \chi(G) \leq 2(6k^3 - 26k^2 + 25k - 1)^{\omega(G) - 1} \) holds for graphs \( G \) having no pivot-minor isomorphic to \( C_k \), far from being a polynomial. Theorem 23 is now implied by a recent theorem of Scott and Seymour [21], solving three conjectures of Gyárfás [13] on \( \chi \)-boundedness all at once.

**Theorem 24** (Scott and Seymour [21]). For all \( k \geq 0 \) and \( \ell > 0 \), the class of all graphs having no induced cycle of length \( k \) modulo \( \ell \) is \( \chi \)-bounded.

To see why Theorem 24 implies Theorem 23, take \( \ell := 2[k/2] \) and apply Proposition 14. Still the bound obtained from Theorem 24 is far from being a polynomial.

And yet no one was able to answer the following problem of Esperet.
**Problem 25** (Esperet; see [15]). Is it true that every \( \chi \)-bounded class of graphs polynomially \( \chi \)-bounded?

Thus it is natural to pose the following conjecture.

**Conjecture 26.** For every graph \( H \), the class of graphs with no pivot-minor isomorphic to \( H \) is polynomially \( \chi \)-bounded.

It is open whether Conjecture 26 holds when \( H = C_k \). Conjecture 26 implies not only Conjectures 3, 5 but also the following conjecture of Geelen (see [8]) proposed in 2009 at the DIMACS workshop on graph colouring and structure held at Princeton University.

**Conjecture 27** (Geelen; see [8]). For every graph \( H \), the class of graphs with no vertex-minor isomorphic to \( H \) is \( \chi \)-bounded.

Of course it is natural to pose the following conjecture, weaker than Conjecture 26 but stronger than Conjecture 27.

**Conjecture 28** (Kim, Kwon, Oum, and Sivaraman [16]). For every graph \( H \), the class of graphs with no vertex-minor isomorphic to \( H \) is polynomially \( \chi \)-bounded.

For vertex-minors, more results are known. Kim, Kwon, Oum, and Sivaraman [16] proved that for each \( k \geq 3 \), the class of graphs with no vertex-minor isomorphic to \( C_k \) is polynomially \( \chi \)-bounded. Their theorem is now implied by the following two recent theorems. To describe these theorems, we first have to introduce a few terms. A *circle graph* is the intersection graph of chords in a circle. In particular, \( C_k \) is a circle graph. The *rank-width* of a graph is one of the width parameters of graphs, measuring how easy it is to decompose a graph into a tree-like structure while keeping every cut to have a small ‘rank’. Rank-width was introduced by Oum and Seymour [19]. We will omit the definition of the rank-width.

**Theorem 29** (Geelen, Kwon, McCarty, and Wollan [12]). For each circle graph \( H \), there is an integer \( r(H) \) such that every graph with no vertex-minor isomorphic to \( H \) has rank-width at most \( r(H) \).

**Theorem 30** (Bonamy and Pilipczuk [3]). For each \( k \), the class of graphs of rank-width at most \( k \) is polynomially \( \chi \)-bounded.

As noted in [6], it is easy to prove directly that the class of graphs of bounded rank-width has the strong Erdős-Hajnal property, without using Theorem 30. See Figure 4 for a diagram showing the containment relations between these properties.

So, one may wonder whether the class of graphs with no pivot-minor isomorphic to \( C_k \) has bounded rank-width. Unfortunately, if \( k \) is odd, then it is not true, because all bipartite graphs have no pivot-minor isomorphic to \( C_k \) for odd \( k \) and yet have unbounded rank-width, see [17]. If \( k \) is even, then it would be true if the following conjecture hold.

**Conjecture 31** (Oum [18]). For every bipartite circle graph \( H \), there is an integer \( r(H) \) such that every graph with no pivot-minor isomorphic to \( H \) has rank-width at most \( r(H) \).
Note.

Chudnovsky, Scott, Seymour, and Spirkl [7] proved that for every graph $H$, the class of graphs $G$ such that neither $G$ nor $\overline{G}$ has any subdivision of $H$ as an induced subgraph has the strong Erdős-Hajnal property. This implies that when $k$ is even, the class of graphs with no induced even hole of length at least $k$ and no induced even anti-hole of length at least $k$ has the strong Erdős-Hajnal property. This is because every subdivision of a large theta graph\(^2\) contains a large even hole. This implies Theorem 4 for even $k$ but not for odd $k$ by Propositions 14 and 15. The authors would like to thank the authors of [7] to share this observation.

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References


\(^2\)A *theta graph* is a graph consisting of three internally disjoint paths of length at least 1 joining two fixed vertices.


