Representing graphs as the intersection of cographs and threshold graphs

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Abstract

A graph G is said to be the intersection of graphs $G_1, G_2, \ldots, G_k$ if $V(G) = V(G_1) = V(G_2) = \cdots = V(G_k)$ and $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$. For a graph G, $\dim_{COG}(G)$ (resp. $\dim_{TH}(G)$) denotes the minimum number of cographs (resp. threshold graphs) whose intersection gives G. We present several new bounds on these parameters for general graphs as well as some special classes of graphs. It is shown that for any graph G: (a) $\dim_{COG}(G) \leqslant \mathrm{tw}(G) + 2$, (b) $\dim_{TH}(G) \leqslant \mathrm{pw}(G) + 1$, and (c) $\dim_{TH}(G) \leqslant \chi(G) \cdot \mathrm{box}(G)$, where $\mathrm{tw}(G)$, $\mathrm{pw}(G)$, $\chi(G)$ and $\mathrm{box}(G)$ denote respectively the treewidth, pathwidth, chromatic number and boxicity of the graph G. We also derive the exact values for these parameters for cycles and show that every forest is the intersection of two cographs. These results allow us to derive improved bounds on $\dim_{COG}(G)$ and $\dim_{TH}(G)$ when G belongs to some special graph classes.

Mathematics Subject Classifications: 05C70, 05C62

1 Introduction

All graphs in this paper are simple, finite and undirected, unless otherwise mentioned. Let $G(V, E)$ be a graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G.

Given graphs $G_1, G_2, \ldots, G_k$ such that $V(G) = V(G_1) = V(G_2) = \cdots = V(G_k)$, we say that $G = G_1 \cap G_2 \cap \cdots \cap G_k$ if $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$; in this case we say that "G is the intersection of the graphs $G_1, G_2, \ldots, G_k$", or that "G can be represented as the intersection of k graphs $G_1, G_2, \ldots, G_k$".

Let $\mathcal{A}$ be a class of graphs. We are concerned with the question of representing a graph G as the intersection of a small number of graphs from $\mathcal{A}$. Kratochvíl and Tuza [26] defined the *intersection dimension* of a graph G with respect to a graph class $\mathcal{A}$, denoted by $\dim_{\mathcal{A}}(G)$, as the smallest number of graphs from $\mathcal{A}$ whose intersection gives G. This is formally defined below.

Definition 1 (Intersection dimension of graph [26]). Given a class $\mathcal{A}$ of graphs and a graph $G(V, E)$, the intersection dimension of G with respect to $\mathcal{A}$ is defined as:

$$\dim_{\mathcal{A}}(G) = \min\{k \colon \exists G_1, G_2, \ldots, G_k \in \mathcal{A} \text{ such that } G = \bigcap_{i=1}^{k} G_i\}$$

Kratochvíl and Tuza also note that for a graph class $\mathcal{A}$, $\dim_{\mathcal{A}}(G)$ exists for every graph G if and only if $\mathcal{A}$ contains all complete graphs and all graphs that can be obtained by removing an edge from a complete graph. The notion of intersection dimension was introduced as a generalization of some well-studied notions like boxicity, circular dimension and overlap dimension of graphs (see [26]).

A "complement reducible graph" or *cograph* is a graph that can be recursively constructed from copies of K_1 (the graph containing one vertex and no edges) using the disjoint union and complementation operations. Cographs turn out to be exactly those graphs that do not contain P_4—a path on four vertices—as an induced subgraph [12].

A *split graph G* is a graph whose vertices can be partitioned into two sets, one of which is an independent set in G and the other a clique in G. Cographs that are also split graphs form the class of graphs known as *threshold graphs* [8]. Threshold graphs have been widely studied in the literature and have several different equivalent definitions (see [28, 22]). For example, they are exactly the graphs that do not contain an induced subgraph isomorphic to P_4, $2K_2$ (the graph having four vertices and two disjoint edges) or C_4 (the cycle on four vertices) [11]. In fact, this characterization follows from the fact that split graphs are exactly those graphs that do not contain $2K_2$, C_4 or C_5 (the cycle on five vertices) as an induced subgraph [18].

In this paper, we use COG and TH to denote the class of cographs and the class of threshold graphs respectively. Thus, $\dim_{COG}(G)$ is the intersection dimension of a graph G with respect to cographs, and for short, we call this the "cograph dimension of G". Similarly, we shall call $\dim_{TH}(G)$ the "threshold dimension of G".

2   Related Work

The notion of representing a given graph as the intersection of graphs from a special class of graphs appears frequently in the literature. For example, the *boxicity* of a graph G, denoted as $\mathrm{box}(G)$, is the smallest integer d such that G is the intersection of d-dimensional boxes (such a box is just the Cartesian product of d closed intervals of the

real line). It is well-known that box(G) is equal to the smallest integer k such that G is the intersection of k interval graphs [30]; or in other words, box(G) = $\dim_{INT}(G)$, where INT denotes the class of interval graphs. The representation of planar graphs as the intersection of graphs from a specific class of graphs has also received much attention in the literature. Thomassen's [35] proof that every planar graph has boxicity at most 3 showed that every planar graph is the intersection of at most 3 interval graphs. Shahrokhi introduced the *clique cover width* (refer [34]) as a generalization of the bandwidth of a graph. He showed that every graph G having clique cover width ccw(G) is the intersection of $\lceil \log(\text{ccw}(G)) \rceil + 1$ co-bipartite graphs and a unit-interval graph. As a consequence, he obtains the result that every planar graph is the intersection of one chordal graph, 4 co-bipartite graphs, and one unit-interval graph. We would like to remark here that it follows from the Four Colour Theorem [31] that every planar graph is the intersection of 3 co-bipartite graphs and a complete 4-partite graph (which is a cograph).

The threshold dimension of graphs is closely related to the "threshold cover", which is a parameter that has been widely studied. We say that a graph G "is covered by" graphs $G_1, G_2, \ldots, G_k$, if $V(G) = V(G_1) = V(G_2) = \cdots = V(G_k)$ and $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$. Note that since cographs are closed under complementation, $\dim_{COG}(\overline{G})$ is the smallest integer k such that the graph G can be covered by k cographs. Similarly, since threshold graphs are closed under complementation, $\dim_{TH}(\overline{G})$ is the smallest number of threshold graphs using which a graph G can be covered. Chvátal and Hammer introduced the parameter $t(G)$, defined as the smallest number of threshold graphs required to cover a graph G [11], the study of which has resulted in several influential papers [37, 13]. The parameter $t(G)$ has been called the "threshold dimension" of G due to the equivalent definition of this parameter as the smallest number of linear inequalities on $|V(G)|$ variables such that every inequality is satisfied by a vector in $\{0, 1\}^{|V(G)|}$ if and only if it is the characteristic vector of an independent set in G (refer [29] for details). The parameter $t(G)$ is also known as the *threshold cover* of G [13]. In this paper, we shall refer to $t(G)$ exclusively as the "threshold cover" of G. We reserve the term "threshold dimension of G" for $\dim_{TH}(G)$. Since $t(G) = \dim_{TH}(\overline{G})$, our results about $\dim_{TH}(G)$ can also be thought of as results about $t(\overline{G})$. Thus $\dim_{TH}(G)$ is the minimum integer k such that there exist k linear inequalities

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leqslant t_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leqslant t_2 \\
&\vdots \\
a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &\leqslant t_k
\end{aligned}
$$

on the variables $x_1, x_2, \ldots, x_n$, where $n = |V(G)|$, such that the characteristic vector of a set $S \subseteq V(G)$ satisfies all the inequalities if and only if S is a clique in G. In other words, it is the minimum number of halfspaces in $\mathbb{R}^n$ whose intersection contains exactly those corners of the n-dimensional hypercube that correspond to cliques in G (the corners of the n-dimensional hypercube are the points in $\{0, 1\}^n$).

Hellmuth and Wieseke [24] showed that the problem of determining whether the edge

set of an input graph can be written as the union of the edge sets of 2 cographs is NP-complete. As cographs are closed under complementation, this implies that the problem of determining whether $\dim_{COG}(G) \leqslant 2$ for an input graph G is NP-complete. Yannakakis [37] showed that the problem of determining if an input graph G has $t(G) \leqslant k$ is NP-complete for every fixed $k \geqslant 3$. As $\dim_{TH}(G) = t(\overline{G})$, this means that it is NP-hard to determine if the threshold dimension of a graph is at most k for every fixed $k \geqslant 3$. Raschle and Simon [29] showed that it can be decided in polynomial-time whether $t(G) \leqslant 2$ for an input graph G; thus the problem of checking whether an input graph has threshold dimension at most 2 is solvable in polynomial time. Gimbel and Nešetřil [21] discuss the problem of deciding whether the vertex set of an input graph G can be partitioned into k parts such that the subgraph induced in G by each part is a cograph and show that this problem is NP-complete for every fixed $k \geqslant 2$.

3   Our results and organization of the paper

The paper is organized as follows. Section 4 contains some preliminary observations and definitions which will be used later sections. We study the threshold and cograph dimensions of forests in Section 5, in which it is shown that every forest is the intersection of at most two cographs and that there exist trees with threshold dimension at least 3. The cograph and threshold dimensions of cycles are studied in Section 6, where a proof is given that shows that cycles on more than 6 vertices cannot be represented as the intersection of two cographs. From this fact, we can deduce the exact values for the cograph dimension and threshold dimension of every cycle. In Sections 7 and 8, we derive the following upper bounds on the cograph dimension and threshold dimension of a graph:

- $\dim_{COG}(G) \leqslant \mathrm{tw}(G) + 2$,

- $\dim_{TH}(G) \leqslant \mathrm{pw}(G) + 2$, and

- $\dim_{TH}(G) \leqslant \chi(G) \cdot \mathrm{box}(G)$.

Here $\mathrm{tw}(G)$, $\mathrm{pw}(G)$, $\chi(G)$ and $\mathrm{box}(G)$ denote respectively the treewidth, pathwidth, chromatic number and boxicity of the graph G. Note that the upper bound of $\mathrm{tw}(G) + 2$ on the cograph dimension of any graph G is equal to the upper bound on boxicity proved in [10]. This is interesting considering that the boxicity of a graph G is nothing but $\dim_{INT}(G)$ where INT is the class of interval graphs. Kratochvíl and Tuza [26] showed that $\dim_{PER}(G) \leqslant 4$ for every planar bipartite graph G. Our bound on threshold dimension in terms of boxicity implies the stronger result that $\dim_{TH}(G) \leqslant 4$ for every planar bipartite graph G.

In Section 9, we focus on the cograph dimension of planar graphs and their subclasses. Kratochvíl and Tuza [26] show that $\dim_{PER}(G) \leqslant 12$ (here, PER denotes the class of permutation graphs) when G is a planar graph and ask whether this bound can be improved. Since cographs and threshold graphs are subclasses of permutation graphs, any upper bound on $\dim_{COG}(G)$ or $\dim_{TH}(G)$ is also an upper bound on $\dim_{PER}(G)$. We

show that if G is a planar graph, then $\dim_{TH}(G) \leqslant 12$ and $\dim_{COG}(G) \leqslant 10$. For the latter result, we use a (slightly improved) upper bound on cograph dimension in terms of the acyclic chromatic number that was first obtained by Aravind and Subramaniam [3], and the upper bound on cograph dimension of forests obtained in Section 5. We use a similar upper bound on the cograph dimension in terms of the star chromatic number (again observed in [3]) to show upper bounds on the cograph dimension of planar graphs with lower bounds on girth. The bounds derived in Sections 8 and 7 allow us to obtain new upper bounds on the cograph dimension and threshold dimensions of partial 2-trees and outerplanar graphs. These results are summarized in Table 1.

Graph G is	$\dim_{COG}(G) \leqslant$	$\dim_{TH}(G) \leqslant$
Planar	10 (Corollary 34)	12 (Corollary 23)
girth $\geqslant 4$	9 (Corollary 24)	9 (Corollary 24)
$8 \geqslant$ (girth g) $\geqslant 5$	$13 - g$ (Corollary 36)	9 (Corollary 24)
girth $\geqslant 13$	3 (Corollary 36)	9 (Corollary 24)
Partial 2-tree	4 (Corollary 29)	9 (Corollary 26)
Outerplanar	4 (Corollary 29)	6 (Corollary 25)
weak dual is a path	3 (Corollary 31)	3 (Corollary 31)
Planar bipartite	4 (Corollary 22)	4 (Corollary 22)
Forest	2 (Corollary 9)	4 (Corollary 22)

Table 1: Upper bounds on the cograph and threshold dimensions of some subclasses of planar graphs.

In order to show that the technique using the star chromatic number cannot yield a bound on the cograph dimension of outerplanar graphs that is better than the one obtained using treewidth, we present an outerplanar graph whose star chromatic number can be proved to be at least 6. This graph, which contains 20 vertices and 33 edges, is in our opinion simpler than the earlier known example, which contains 41 vertices and 71 edges.

4 Preliminaries

Given a graph $G(V, E)$, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. For a vertex $u \in V(G)$, $N(u)$ denotes the set of neighbours of u and $N[u] = N(u) \cup \{u\}$. Given a set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced in G by the vertices in S. An *induced P_4 (resp. $2K_2$, C_4)* in G is an induced subgraph of G that is isomorphic to P_4 (resp. $2K_2$, C_4).

Observation 2. *Let $\mathcal{A}$ be a class of graphs and let $G, G_1, G_2, \ldots, G_k$ be graphs such that $G = G_1 \cap G_2 \cap \cdots \cap G_k$. Then $\dim_{\mathcal{A}}(G) \leqslant \dim_{\mathcal{A}}(G_1) + \dim_{\mathcal{A}}(G_2) + \cdots + \dim_{\mathcal{A}}(G_k)$.*

Proof. For each $i \in \{1, 2, \ldots, k\}$, there exists a collection $\mathcal{H}_i \subseteq \mathcal{A}$ of at most $\dim_{\mathcal{A}}(G_i)$ graphs such that $G_i = \bigcap_{H \in \mathcal{H}_i} H$. Since $G = \bigcap_{i=1}^{k} G_i = \bigcap_{i=1}^{k} \bigcap_{H \in \mathcal{H}_i} H$, we now have that $\dim_{\mathcal{A}}(G) \leqslant \sum_{i=1}^{k} \dim_{\mathcal{A}}(G_i)$. $\qquad\square$

Definition 3 (the join operation). The *join* of two vertex-disjoint graphs G_1 and G_2, denoted as $G_1 + G_2$, is the graph G having $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv \colon u \in E(G_1), v \in E(G_2)\}$.

The join operation is called the "Zykov sum" operation in [26].

Definition 4 (the disjoint union operation). The *disjoint union* of two vertex-disjoint graphs G_1 and G_2, denoted as $G_1 \uplus G_2$, is the graph G having $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

A class of graphs $\mathcal{A}$ is said to be "closed" under the join operation (resp. the disjoint union operation) if for any $G_1, G_2 \in \mathcal{A}$, we have $G_1 + G_2 \in \mathcal{A}$ (resp. $G_1 \uplus G_2 \in \mathcal{A}$). The following observation is easy to see.

Observation 5. *Let $\mathcal{A}$ be a class of graphs that is closed under the join operation (resp. disjoint union operation). Then for any two graphs G_1 and G_2, $\dim_{\mathcal{A}}(G_1 + G_2)$ (resp. $\dim_{\mathcal{A}}(G_1 \uplus G_2)) \leqslant \max\{\dim_{\mathcal{A}}(G_1), \dim_{\mathcal{A}}(G_2)\}$.*

Note that the class of cographs is closed under both the join and disjoint union operations whereas the class of threshold graphs is not closed under either operation. A class of graphs is said to be *hereditary* if it is closed under taking induced subgraphs; i.e. for any graph G in the class, every induced subgraph of G is also in the class.

Partial 2-trees are exactly the graphs that have treewidth at most 2 [4]. Outerplanar graphs are the planar graphs that have a planar embedding in which every vertex is on the boundary of the outer face. They form a subclass of partial 2-trees [4]. For any terminology or notation that is not defined herein, please refer [15].

5   Cograph and Threshold Dimensions of Forests

In this section, we shall show that every forest is the intersection of at most two cographs and then, using a known characterization of graphs that have a threshold cover of size 2, we show that there exist trees that are not the intersection of two threshold graphs.

We first show a construction using which given any forest F, two cographs whose intersection gives F can be constructed. We describe the construction for a tree, and the construction for forests as an extension to it.

Let T be a tree in which one vertex r has been arbitrarily selected to be the root. The ancestor-descendant and parent-child relations on $V(T)$ are then defined in the usual way (i.e., for $x, y \in V(T)$, x is an ancestor of y if and only if x lies on the path between r and y in T; x is the parent of y if and only if x is an ancestor of y and $xy \in E(T)$). The vertices that are at an even distance from the root r are called "even vertices" and those that are at an odd distance from the root r are called "odd vertices". In the following, for any vertex $v \in V(T)$, we denote by $p(v)$ its parent vertex. We let $p(r) = r$.

Let T_o (resp. T_e) denote the graph with vertex set $V(T_o)$ (resp. $V(T_e)) = V(T)$ and $E(T_o)$ (resp. $E(T_e)) = E(T) \cup \{uv \colon u$ is an odd (resp. even) vertex and v is a descendant of $p(u)\}$.

Lemma 6. *Let T be a tree with a root. Then both T_o and T_e are cographs.*

Proof. We shall first prove that T_o is a cograph using induction on $|V(T)|$. Clearly, T_o is a cograph when $|V(T)| = 1$, since K_1 is a cograph. Let $T^1, T^2, \ldots, T^k$ be the trees which form the components of $T - N[r]$. It is easy to see that for each $i \in \{1, 2, \ldots, k\}$, there exists exactly one vertex r_i in T^i such that in T, $p(p(r_i)) = r$. Choose r_i to be the root of T^i, for each $i \in \{1, 2, \ldots, k\}$. Observe that $T_o = (T[\{r\}] \uplus T_o^1 \uplus T_o^2 \uplus \cdots \uplus T_o^k) + \sum_{u \in N(r)} T[\{u\}]$ (here, for a vertex $v \in V(T)$, $T[\{v\}]$ refers to the subgraph of T isomorphic to K_1 that contains just the vertex v). By the induction hypothesis, $T_o^1, T_o^2, \ldots, T_o^k$ are all cographs. Since K_1 is a cograph and cographs are closed under the join and disjoint union operations, we have that T_o is a cograph.

Next let us prove that T_e is a cograph. Let $T^1, T^2, \ldots, T^k$ be the trees that form the components of $T - r$. For each $i \in \{1, 2, \ldots, k\}$, let r_i denote the unique vertex in T^i such that in T, $p(r_i) = r$. Choose r_i as the root of T^i, for each $i \in \{1, 2, \ldots, k\}$. Observe that $T_e = (T_o^1 \uplus T_o^2 \uplus \cdots \uplus T_o^k) + T[\{r\}]$. By our earlier observation, we know that each of $T_o^1, T_o^2, \ldots, T_o^k$ are cographs. Therefore, it follows that T_e is a cograph. $\square$

Lemma 7. *Let T be a tree with a root. Then $T = T_o \cap T_e$.*

Proof. Since each of T_o and T_e are supergraphs of T, we only need to show that $E(T_o) \cap E(T_e) \subseteq E(T)$. Suppose for the sake of contradiction that there exists an edge $xy \in E(T_o) \cap E(T_e)$ such that $xy \notin E(T)$. As $xy \in E(T_o) \cap E(T_e)$, we may assume without loss of generality that y is a descendant of $p(x)$ in T. Since $xy \notin E(T)$, we have that $xy \notin E(T_o)$ if x is even and $xy \notin E(T_e)$ if x is odd. This contradicts our assumption that $xy \in E(T_o) \cap E(T_e)$. $\square$

Combining Lemma 6 and Lemma 7, we have the following theorem.

Theorem 8. *For any tree T, $\dim_{COG}(T) \leqslant 2$.*

Since cographs are closed under the disjoint union operation, we can now deduce from Observation 5 that the cograph dimension of every forest is at most 2.

Corollary 9. *Let F be a forest then $\dim_{COG}(F) \leqslant 2$.*

Clearly, there are trees, even paths, that are not cographs, and therefore this bound is tight. Note that when T is a path, then we can choose one of its endpoints as the root so that the graphs T_o and T_e are split graphs. Since T_o and T_e are also cographs by Lemma 6, we get that each of them is a threshold graph. We thus have the following from Lemma 7.

Corollary 10. *For every path P, $\dim_{TH}(P) \leqslant 2$.*

Next, we show that there are trees having threshold dimension at least 3. Chvátal and Hammer [11] defined the auxiliary graph G^* corresponding to a graph G as follows: $V(G^*) = E(G)$ and two vertices uv and xy of $V(G^*) = E(G)$ are adjacent in G^* if and only if $ux, vy \notin E(G)$. They asked whether $t(G) = \chi(G^*)$ for every graph G. Although Cozzens and Leibowitz [13] gave a negative answer to this question, Raschle and Simon [29] proved the following theorem (a shorter proof was recently given in [19]).

Theorem 11 (Raschle-Simon). *A graph G has a threshold cover of size 2 if and only if G^* is bipartite.*

This theorem can be used to prove that the tree T shown in Figure 1 has threshold dimension at least 3. Consider the graph $\overline{T}^*$. Note that $(ci)(dj)(ei)(fh)(gd)(fc)(eb)(da)(hb)$ (ic) is an odd cycle in $\overline{T}^*$. By Theorem 11, we then have that $t(\overline{T}) > 2$. This implies that $\dim_{TH}(T) > 2$.
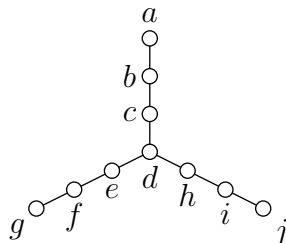

Figure 1: A tree that is not the intersection of two threshold graphs.

6   Cograph and Threshold Dimensions of Cycles

We shall now turn our attention to cycles, and show that cycles on more than 6 vertices cannot be represented as the intersection of two cographs.

Let C_n denote the cycle on n vertices having $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{v_n v_1\}$. In the following, we shall let v_{n+t} denote v_t and v_{1-t} denote v_{n+1-t}, for $1 \leqslant t \leqslant n$. We prove that for $n \geqslant 7$, there do not exist cographs A and B such that $C_n = A \cap B$. We will use the well-known fact that the diameter of any induced subgraph of a cograph is at most 2; i.e. if G' is an induced subgraph of a cograph G, then there cannot be two vertices that are at a distance of 3 or more in G' (since the shortest path between them in G' will contain a P_4 that is an induced subgraph of G', and therefore also an induced subgraph of G).

First we derive a lemma that is true for any cycle that is the intersection of two cographs, provided that the cycle contains at least 6 vertices.

Lemma 12. *Let A and B be two cographs such that $C_n = A \cap B$ for some $n \geqslant 6$. Then there does not exist $i, j \in \{1, 2, \ldots, n\}$ such that $v_i v_j, v_i v_{j+1}, v_i v_{j+2} \notin E(A)$.*

Proof. Suppose there exist $i, j \in \{1, 2, \ldots, n\}$ such that $v_i v_j, v_i v_{j+1}, v_i v_{j+2} \notin E(A)$. Since $E(C_n) \subseteq E(A)$, we have that $v_i \notin \{v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$. Consider the graph A. Let v_x be the last neighbour of v_i in the sequence $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ and let v_y be the first neighbour of v_i in the sequence $v_{j+3}, v_{j+4}, \ldots, v_{i-1}$. Clearly, the vertices v_x, v_y, v_i are pairwise distinct. We can see that $v_i v_x v_{x+1} \ldots v_j v_{j+1} v_{j+2} \ldots v_{y-1} v_y v_i$ forms a cycle with at least 6 vertices in the cograph A. Note that v_x and v_y are the only neighbours of v_i in this cycle. Consider the subgraph A' induced in A by the vertices in $\{v_{x+1}, v_{x+2}, \ldots, v_y, v_i\}$.

Since A' is clearly a connected induced subgraph of the cograph A, there should be a path of length at most 2 in A' between v_i and every other vertex in A'. As v_y is the only neighbour of v_i in A', each vertex in $\{v_{x+1}, v_{x+2}, \ldots, v_{y-1}\}$ is adjacent to v_y in A' and therefore $v_y v_{x+1}, v_y v_{x+2}, \ldots, v_y v_{y-1} \in E(A)$. By applying the same arguments on the subgraph induced in A by the vertices in $\{v_i, v_x, v_{x+1}, \ldots, v_{y-1}\}$, we can show that $v_x v_{x+1}, v_x v_{x+2}, \ldots, v_x v_{y-1} \in E(A)$.

Now let us consider the graph B. As $C_n = A \cap B$, we can conclude from the above observations that $v_x v_{x+2}, v_x v_{x+3}, \ldots, v_x v_{y-1}, v_y v_{x+1}, v_y v_{x+2}, \ldots, v_y v_{y-2} \notin E(B)$. Note that this means that there is no vertex in $\{v_{x+1}, v_{x+2}, \ldots, v_{y-1}\}$ that is adjacent to both v_x and v_y. Consider the subgraph B' induced in B by the vertices in $\{v_x, v_{x+1}, \ldots, v_y\}$. As B' is a connected induced subgraph of the cograph B, there must be a path of distance at most 2 between v_x and v_y in B'. Since there is no vertex in $\{v_{x+1}, v_{x+2}, \ldots, v_y\}$ that is adjacent to both v_x and v_y, this implies that $v_x v_y \in E(B')$, and therefore $v_x v_y \in E(B)$. But then, $v_x v_y v_{y-1} v_{y-2}$ is an induced P_4 in B, contradicting the fact that B is a cograph. $\square$

Lemma 13. *Let A and B be two cographs such that $C_n = A \cap B$ for some $n \geqslant 7$. Then there does not exist $i, j \in \{1, 2, \ldots, n\}$ such that $v_i v_j, v_i v_{j+1} \notin E(A)$ and $v_i \notin \{v_{j-2}, v_{j+3}\}$.*

Proof. Let us assume that such i and j exist. Note that since $v_i v_j, v_i v_{j+1} \notin E(A) \supseteq E(C_n)$ and $v_i \notin \{v_{j-2}, v_{j+3}\}$, we can conclude that $v_i \notin \{v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$. By Lemma 12, $v_i v_{j-1}$ and $v_i v_{j+2} \in E(A)$. Note that $v_j v_{j+2} \in E(A)$, as otherwise $v_i v_{j+2} v_{j+1} v_j$ will be an induced P_4 in A. Similarly, $v_{j-1} v_{j+1} \in E(A)$, as otherwise $v_i v_{j-1} v_j v_{j+1}$ will form an induced P_4 in A. As $C_n = A \cap B$, we now have that $v_i v_{j-1}, v_i v_{j+2}, v_{j-1} v_{j+1}, v_j v_{j+2} \notin E(B)$. This tells us that $v_{j-1} v_{j+2} \in E(B)$, as otherwise, $v_{j-1} v_j v_{j+1} v_{j+2}$ will be an induced P_4 in B. Further, using Lemma 12 on B, we can infer that at least one of the edges $v_i v_j, v_i v_{j+1}$ must be present in $E(B)$. If $v_i v_j \in E(B)$, then $v_i v_j v_{j-1} v_{j+2}$ will be an induced P_4 in B and if $v_i v_{j+1} \in E(B)$, then $v_i v_{j+1} v_{j+2} v_{j-1}$ will be an induced P_4 in B. In either case, we have a contradiction to the fact that B is a cograph. $\square$

Corollary 14. *Let $C_n = A \cap B$, for some $n \geqslant 7$, where A and B are cographs, and let $i \in \{1, 2, \ldots, n\}$. Then:*

- *exactly one of the edges in $\{v_i v_{i+3}, v_i v_{i+4}\}$ is in $E(A)$ and the other is in $E(B)$, and*

- *exactly one of the edges in $\{v_i v_{i-3}, v_i v_{i-4}\}$ is in $E(A)$ and the other is in $E(B)$.*

Proof. By Lemma 13, at least one of $v_i v_{i+3}, v_i v_{i+4}$ is in $E(A)$. Similarly, by applying Lemma 13 on B, at least one of these two edges is also in $E(B)$. Since $v_i v_{i+3}, v_i v_{i+4} \notin E(C_n) = E(A) \cap E(B)$, we can conclude that exactly one of them is in A and the other in B. By symmetry, we have that exactly one of the edges in $\{v_i v_{i-3}, v_i v_{i-4}\}$ is in $E(A)$ and the other is in $E(B)$. $\square$

Lemma 15. *Let $C_n = A \cap B$, for some $n \geqslant 7$, where A and B are cographs. If for some $i \in \{1, 2, \ldots, n\}$, the edge $v_i v_{i+4} \in E(A)$, then $v_{i+1} v_{i+5}$ is in $E(A)$.*

Proof. Suppose that $v_i v_{i+4} \in E(A)$. From Corollary 14, exactly one of the edges in $\{v_{i+1}v_{i+4}, v_{i+1}v_{i+5}\} \in E(A)$. If $v_{i+1}v_{i+4} \in E(A)$, then the vertex v_{i+4} is adjacent to both the vertices $v_{(i+4)-3}$ and $v_{(i+4)-4}$ in A, which contradicts Corollary 14. Thus, $v_{i+1}v_{i+4} \notin E(A)$, which by Corollary 14 means that $v_{i+1}v_{i+5} \in E(A)$. $\qquad\square$

Theorem 16. *There does not exist two cographs A and B such that $A \cap B = C_n$, when $n \geqslant 7$. In other words, $\dim_{COG}(C_n) > 2$, when $n \geqslant 7$.*

Proof. Suppose such cographs A and B exist. From Corollary 14, we may assume without loss of generality that $v_1 v_5 \in E(A)$. Then by applying Lemma 15 repeatedly, we can conclude that for every $i \in \{1, 2, \ldots, n\}$, $v_i v_{i+4} \in E(A)$. By Corollary 14, this implies that for every $i \in \{1, 2, \ldots, n\}$, $v_i v_{i+3} \notin E(A)$. If $v_i v_{i+2} \notin E(A)$ for some $i \in \{1, 2, \ldots, n\}$, then $v_i v_{i-1} v_{i+3} v_{i+2}$ will form an induced P_4 in A, contradicting the fact that A is a cograph. Therefore, we can conclude that for every $i \in \{1, 2, \ldots, n\}$, $v_i v_{i+2} \in E(A)$. Let us consider the path $v_1 v_3 v_4 v_6$ in A. Since $v_1 v_5 \in E(A)$, we have $v_1 v_5 \notin E(B)$. If $n \geqslant 8$, then we can apply Lemma 13 on B to get that $v_1 v_6 \in E(B)$, which implies that $v_1 v_6 \notin E(A)$. Thus, if $n \geqslant 8$, we have the induced P_4 $v_1 v_3 v_4 v_6$ in A, which contradicts the fact that A is a cograph. We can therefore conclude that $n = 7$. Then since $v_i v_{i+4} \in E(A)$ for each $i \in \{1, 2, \ldots, n\}$, we have $v_1 v_5, v_4 v_1 \in E(A)$. This contradicts Corollary 14. $\qquad\square$

Figure 2 shows that every cycle on less than 7 vertices is the intersection of at most two cographs (note that cycles on less than 5 vertices are themselves cographs).
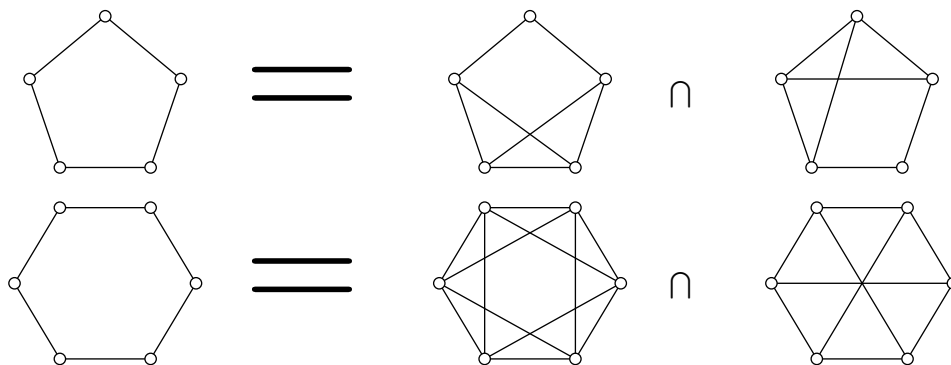

Figure 2: Cycles with less than 7 vertices are the intersection of at most two cographs.

Remark 17. $\dim_{COG}(C_n) \leqslant 2$, when $n \leqslant 6$.

It is easy to see that any cycle C is the intersection of at most three threshold graphs as follows. Let v be any vertex on the cycle. Then $C - v$ is a path and therefore, by Corollary 10, there exist two threshold graphs G_1 and G_2 such that $G_1 \cap G_2 = C - v$. Let G'_1 and G'_2 be obtained from G_1 and G_2 respectively by adding v as a universal vertex. Since the graph obtained by adding a universal vertex to any threshold graph is again a threshold graph, G'_1 and G'_2 are threshold graphs. Let G_3 be the graph obtained from C by making every pair of vertices from $V(C) \setminus \{v\}$ adjacent to each other (thus,

$V(C) \setminus \{v\}$ is a clique in G_3). It is easy to verify that G_3 is a threshold graph and that $C = G_1' \cap G_2' \cap G_3$. Therefore, we have the following result.

Remark 18. For any cycle C, $\dim_{COG}(C) \leqslant \dim_{TH}(C) \leqslant 3$.

It is easy to see that a C_4 is a cograph but not a threshold graph. However, it can be represented as the intersection of 2 threshold graphs as shown in Figure 3. The graph C_5 can be seen to have threshold dimension at least 3 as follows. Suppose for the sake of contradiction that $\dim_{TH}(C_5) \leqslant 2$. Since $\overline{C_5}$ is isomorphic to C_5, we have $t(C_5) = t(\overline{C_5}) \leqslant 2$, which means that the edges of a C_5 can be covered by at most two threshold graphs. Let H_1 and H_2 be two threshold graphs that cover the edges of a C_5. If H_i, for some $i \in \{1, 2\}$, contains three edges, then it contains either an induced P_4 or an induced $2K_2$, which is a contradiction to the fact that H_i is a threshold graph. So each of H_1 and H_2 can cover at most two edges of the C_5, which contradicts the fact that every edge of the C_5 is contained in at least one of H_1 or H_2.
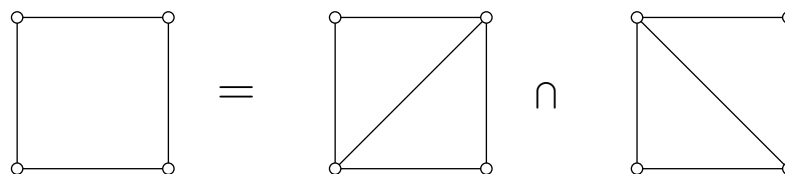

Figure 3: The threshold dimension of C_4 is 2.

We can see that the threshold dimension of a C_6 is also at least 3 as follows. Suppose for the sake of contradiction that $\dim_{TH}(C_6) \leqslant 2$. Consider the cycle $C = v_1 v_2 \ldots v_6 v_1$ on 6 vertices. Let $G = \overline{C}$. From our assumption, we have $t(G) \leqslant 2$. Thus there exist two threshold graphs H_1 and H_2 such that $E(G) = E(H_1) \cup E(H_2)$. Now consider the edges $v_1 v_4$, $v_2 v_5$ and $v_3 v_6$ in $E(G)$. If any two of them belong to H_i, for some $i \in \{1, 2\}$, then H_i would contain an induced $2K_2$, P_4 or C_4, which is a contradiction (to see this, note that for any two of these edges, their endpoints induce a C_4 in G). Therefore at least one of the edges $v_1 v_4, v_2 v_5, v_3 v_6$ is not contained in either H_1 or H_2, contradicting the fact that $E(G) = E(H_1) \cup E(H_2)$.

Note that we could also have used Theorem 11 to show that C_5 and C_6 are not the intersection of two threshold graphs. We thus get the values shown in Table 2 for the cograph dimension and threshold dimension of cycles of every size.

n	$\dim_{COG}(C_n)$	$\dim_{TH}(C_n)$
3	1	1
4	1	2
5	2	3
6	2	3
$\geqslant 7$	3	3

Table 2: Cograph and threshold dimensions of cycles

7 Threshold dimension, boxicity and chromatic number

A k-dimensional box, or k-box for short, is defined as the Cartesian product of k closed intervals in $\mathbb{R}$. The *boxicity* of a graph G, denoted by $\text{box}(G)$, is the minimum integer k for which there is an assignment of k-boxes to the vertices of G such that for $u, v \in V(G)$, $uv \in E(G)$ if and only if the k-boxes corresponding to u and v intersect. It is known that for any graph G, $\text{box}(G) = \dim_{INT}(G)$, where INT denotes the class of interval graphs [14]. The *chromatic number* of a graph G, denoted as $\chi(G)$, is the minimum number of colours required to colour the vertices of G such that no two adjacent vertices receive the same colour.

We shall now prove the main result of this section.

Theorem 19. *Let G be any graph. Then,* $\dim_{TH}(G) \leqslant \chi(G)\text{box}(G)$.

Proof. Let $\text{box}(G) = k$. Then there exists a collection of k-boxes $\{B_u\}_{u \in V(G)}$ such that for $u, v \in V(G)$, $uv \in E(G) \Leftrightarrow B_u \cap B_v \neq \emptyset$. For $u \in V(G)$, let $B_u = [l_1(u), r_1(u)] \times [l_2(u), r_2(u)] \times \cdots \times [l_k(u), r_k(u)]$. Note that this means that $uv \notin E(G)$ if and only if there exists $j \in \{1, 2, \ldots, k\}$ such that $r_j(u) < l_j(v)$ or $r_j(v) < l_j(u)$.

Let $c : V(G) \to \{1, 2, \ldots, \chi(G)\}$ be a proper vertex colouring of G.

Now, for each $i \in \{1, 2, \ldots, \chi(G)\}$ and $j \in \{1, 2, \ldots, k\}$, we construct a graph G_{ij} on vertex set $V(G)$ as follows. Define $E(G_{ij}) = \{uv \mid c(u) \neq i, c(v) \neq i\} \cup \{uv \mid c(u) = i, c(v) \neq i \text{ and } r_j(u) \geqslant l_j(v)\}$. Note that each G_{ij} is a split graph (the vertices with colour i form an independent set and all other vertices form a clique). We shall now show that

$$G = \bigcap_{\substack{1 \leqslant i \leqslant \chi(G) \\ 1 \leqslant j \leqslant k}} G_{ij}$$

If $uv \in E(G)$, then clearly $c(u) \neq c(v)$ and for each $j \in \{1, 2, \ldots, k\}$, we have $r_j(u) \geqslant l_j(v)$ and $r_j(v) \geqslant l_j(u)$. It follows that each G_{ij} is a supergraph of G. So we only need to show that for each $u, v \in V(G)$ such that $uv \notin E(G)$, there exists $i \in \{1, 2, \ldots, \chi(G)\}$ and $j \in \{1, 2, \ldots, k\}$ such that $uv \notin E(G_{ij})$. Consider $uv \notin E(G)$ (where $u \neq v$). If $c(u) = c(v) = i$, then clearly, $uv \notin E(G_{ij})$, for all $1 \leqslant j \leqslant k$. So let us assume that $c(u) \neq c(v)$. As $uv \notin E(G)$, we can assume without loss of generality that there exists $j \in \{1, 2, \ldots, k\}$ such that $r_j(u) < l_j(v)$. Let $c(u) = i$, which means that $c(v) \neq i$. It now follows from the definition of G_{ij} that $uv \notin E(G_{ij})$.

To complete the proof, we shall show that each G_{ij} is a cograph, which would imply that it is also a threshold graph, as every split graph that is also a cograph is a threshold graph [8]. In particular, we show that no G_{ij} contains an induced P_4. Suppose for the sake of contradiction that there exists $i \in \{1, 2, \ldots, \chi(G)\}$ and $j \in \{1, 2, \ldots, k\}$ such that G_{ij} contains an induced path $wxyz$. As $\{u \in V(G_{ij}) \mid c(u) = i\}$ forms an independent set in G_{ij} and $\{u \in V(G_{ij}) \mid c(u) \neq i\}$ forms a clique in G_{ij}, we can conclude that $c(w) = c(z) = i$, $c(x) \neq i$ and $c(y) \neq i$. Since $wx \in E(G_{ij})$ and $wy \notin E(G_{ij})$, we get $l_j(y) > r_j(w) \geqslant l_j(x)$. Similarly, since $yz \in E(G_{ij})$ and $xz \notin E(G_{ij})$, we get $l_j(x) > r_j(z) \geqslant l_j(y)$. We now have both $l_j(y) > l_j(x)$ and $l_j(x) > l_j(y)$, which is a contradiction. $\qquad\square$

Since threshold graphs form a subclass of cographs, we have the following corollary.

Corollary 20. *Let G be any graph. Then, $\dim_{COG}(G) \leqslant \dim_{TH}(G) \leqslant \chi(G)\mathrm{box}(G)$.*

A *grid intersection graph* is a graph whose vertices can be mapped to horizontal and vertical line segments in the plane in such a way that two vertices are adjacent in the graph if and only if the line segments corresponding to them intersect and no two parallel line segments intersect. Clearly, every grid intersection graph has boxicity at most 2, since horizontal and vertical line segments are just degenerate 2-boxes. We shall use the following theorem of Hartman et al. [23].

Theorem 21 (Hartman et al.)**.** *Every planar bipartite graph is a grid intersection graph.*

From Theorem 21 and Theorem 19, we now have the following corollary.

Corollary 22. *For any planar bipartite graph G, $\dim_{COG}(G) \leqslant \dim_{TH}(G) \leqslant 4$.*

Note that the above corollary is a strengthening of Theorem 3.10 of [26] which states that for every planar bipartite graph G, $\dim_{PER}(G) \leqslant 4$, where PER denotes the class of permutation graphs (note that cographs, and therefore threshold graphs, form a subclass of permutation graphs). Using the Four Colour Theorem [31] and the fact that the boxicity of any planar graph is at most 3 [35], Theorem 19 also implies the following strengthening of Corollary 3.11 of [26] which states that $\dim_{PER}(G) \leqslant 12$ for every planar graph G.

Corollary 23. *For any planar graph G, $\dim_{TH}(G) \leqslant 12$.*

Since every triangle-free planar graph has a 3-colouring (Grötzsch's Theorem), we also get the following.

Corollary 24. *For any triangle-free planar graph G, $\dim_{TH}(G) \leqslant 9$.*

Since for any outerplanar graph G, we have $\chi(G) \leqslant 3$ (folklore) and $\mathrm{box}(G) \leqslant 2$ [33], we get the following result as a corollary of Theorem 19.

Corollary 25. *For any outerplanar graph G, $\dim_{TH}(G) \leqslant 6$.*

Partial 2-trees are also 3-colourable, but there exist partial 2-trees that have boxicity 3 [5]. Thus we have the following.

Corollary 26. *For any partial 2-tree G, $\dim_{TH}(G) \leqslant 9$.*

Adiga, Bhowmick and Chandran [1] showed that the boxicity of any graph with maximum degree Δ is $O(\Delta \log^2 \Delta)$. This means, by Theorem 19 and the fact that $\chi(G) \leqslant \Delta+1$, that $\dim_{TH}(G)$ is $O(\Delta^2 \log^2 \Delta)$ for any graph G with maximum degree Δ.

8 Upper bounds using treewidth and pathwidth

A tree decomposition of a graph G is a pair (T, f), where $f : V(T) \to 2^{V(G)}$ is an assignment of subsets of $V(G)$ to the vertices of a tree T, satisfying the following properties:

1. For every vertex $u \in V(G)$, the subgraph induced in T by the set $\{x \in V(T) \colon u \in f(x)\}$ is connected (note that this implies that every vertex $u \in V(G)$ is contained in $f(x)$ for some $x \in V(T)$). Observe that this means that if $u \in f(x) \cap f(y)$, for some $x, y \in V(T)$, then for every $z \in V(T)$ that lies on the path between x and y in T, $u \in f(z)$.

2. For every edge $uv \in E(G)$, there exists $x \in V(T)$ such that $u, v \in f(x)$.

The width of a tree decomposition (T, f) of a graph G is defined as $\max_{x \in V(T)}\{|f(x)| - 1\}$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, the minimum width of a tree decomposition of G.

Chandran and Sivadasan [10] showed that for any graph G, $\mathrm{box}(G) \leqslant \mathrm{tw}(G) + 2$. Combining this with Theorem 19 and the fact that $\chi(G) \leqslant \mathrm{tw}(G) + 1$ (folklore), we get that for every graph G, $\dim_{TH}(G) \leqslant \chi(G)(\mathrm{tw}(G) + 2) \leqslant (\mathrm{tw}(G) + 1)(\mathrm{tw}(G) + 2)$. As shown in [3], the fact that the star chromatic number of a graph G is at most $\frac{(\mathrm{tw}(G)+1)(\mathrm{tw}(G)+2)}{2}$ [17] can be combined with Theorem 35 from Section 9 to deduce that $\dim_{COG}(G) \leqslant \frac{(\mathrm{tw}(G)+1)(\mathrm{tw}(G)+2)}{2}$ for any graph G. In this section we show that this upper bound can be improved.

We would like to note here that though the general strategy used in the theorem below is similar to that in the proof of Theorem 14 in [10], the details in the two proofs are very different.

Theorem 27. *Let (T, f) be a tree decomposition of a graph G having width $k - 1$. Then there exist cographs $G_0, G_1, \ldots, G_k$ such that $G = G_0 \cap G_1 \cap \cdots \cap G_k$.*

Proof. As (T, f) is a tree decomposition of G having width $k - 1$, we have that for each $x \in V(T)$, $|f(x)| \leqslant k$.

Let G' be the graph with $V(G') = V(G)$ and $E(G') = \{uv \colon \exists x \in V(T) \text{ such that } u, v \in f(x)\}$. Note that G' is a supergraph of G and also that G' is a chordal graph (folklore; to see this, note that the intersection graph of subtrees of a tree is a chordal graph [20]). Note that (T, f) is a tree decomposition of G' as well. Since G' is a chordal graph, $\chi(G') = \omega(G') = \mathrm{tw}(G') + 1$ [22, 32]. Therefore, there exists a proper vertex colouring $c : V(G') \to \{1, 2, \ldots, k\}$ of G'. It is easy to see that c is proper vertex colouring of G too. In particular, we have the stronger property that if $u, v \in V(G)$ such that there exists $x \in V(T)$ having $u, v \in f(x)$, then $c(u) \neq c(v)$.

Choose an arbitrary vertex $r \in V(T)$ as the root of T and define the ancestor-descendant relation among the vertices of T in the usual way (i.e., for $x, y \in V(T)$, x is an ancestor of y if and only if x lies on the path between r and y in T). For every vertex $u \in V(G)$, define $h(u)$ to be that vertex in $f^{-1}(u) = \{x \in V(T) \colon u \in f(x)\}$ that is an ancestor of every other vertex in $f^{-1}(u)$ (it is easy to see, using property 1 of tree decompositions, that there is always exactly one such vertex). We shall now construct a

binary relation R on $V(G)$ as follows. Let $<$ be an arbitrarily chosen linear ordering of the vertices in $V(G)$. For two vertices $u, v \in V(G)$ such that $h(u) \neq h(v)$, we have uRv if and only if $h(u)$ is an ancestor of $h(v)$ in T. If $h(u) = h(v)$, then we have uRv if and only if $u < v$. Note that R is a partial order on $V(G)$.

We now construct the $k+1$ cographs $G_0, G_1, \ldots, G_k$ such that $G = \bigcap_{i=0}^{k} G_i$ as follows.

Define $V(G_0) = V(G)$ and $E(G_0) = \{uv \colon uRv\}$. By property 2 of tree decompositions, we have that for any edge $uv \in E(G)$, one of $h(u)$ or $h(v)$ is an ancestor of the other (this includes the case $h(u) = h(v)$). Therefore, G_0 is a supergraph of G.

For each $i \in \{1, 2, \ldots, k\}$, define $V(G_i) = V(G)$ and $E(G_i) = E(G) \cup \{uv \colon (u, v), (v, u) \notin R\} \cup \{uv \colon uRv \text{ and } c(u) \neq i\}$. Clearly, G_i is a supergraph of G.

First, we shall show that $G = \bigcap_{i=0}^{k} G_i$. For this, we only need to show that for any $uv \notin E(G)$, there exists $i \in \{0, 1, \ldots, k\}$ such that $uv \notin E(G_i)$. Let $uv \notin E(G)$. If $(u, v), (v, u) \notin R$, then $uv \notin E(G_0)$. So let us assume without loss of generality that uRv. Then from the definition of $G_{c(u)}$, it is clear that $uv \notin E(G_{c(u)})$.

It only remains to be proven that each G_i, $0 \leqslant i \leqslant k$, is a cograph.

Suppose for the sake of contradiction that G_0 is not a cograph. Then there exists an induced path (on four vertices) $P = abcd$ in G_0. From the definition of G_0, $uv \in E(G_0)$ if and only if either uRv or vRu. Let us orient the edges of P such that an edge uv of P gets oriented from u to v if and only if uRv. Let $\hat{P}$ be the path P together with the orientations on its edges. Suppose that there is a directed path of length 2 in $\hat{P}$, which we will assume without loss of generality to be $a \to b \to c$. Recalling that R is a partial order, aRb and bRc together imply aRc, which contradicts the fact that $ac \notin E(G_0)$. Thus, there is no directed path of length 2 in $\hat{P}$. We can therefore assume without loss of generality that $\hat{P}$ is the path $a \to b \leftarrow c \to d$. We then have both aRb and cRb, which implies by the definition of R that either aRc or cRa. But then $ac \in E(G_0)$, which is a contradiction.

Now let us suppose for the sake of contradiction that G_i is not a cograph, for some $i \in \{1, 2, \ldots, k\}$. Then there exists an induced path (on four vertices) $P = abcd$ in G_i. Let Q be the path $bdac$ in $\overline{G_i}$. From the definition of G_i, it is clear that if $uv \in E(\overline{G_i})$, then either uRv and $c(u) = i$ or vRu and $c(v) = i$. Now let us orient the edges of Q such that an edge $uv \in E(Q)$ gets oriented from u to v if and only if uRv. Let $\hat{Q}$ be the path Q together with the orientations on its edges. Suppose that there is a directed path of length 2 in $\hat{Q}$, which we will assume without loss of generality to be $b \to d \to a$. By our observation above, we have that bRd, dRa, and $c(b) = c(d) = i$. This means that bRa and that $h(d)$ lies on the path between $h(b)$ and $h(a)$ in T. Since $ab \in E(G)$, the former and the definition of h together implies that $b \in f(h(a))$. By property 1 of tree decompositions, the latter now implies that $b \in f(h(d))$. Since $d \in f(h(d))$, this contradicts the fact that $c(b) = c(d)$. Therefore, we can assume that there is no directed path of length 2 in $\hat{Q}$. Then we can assume without loss of generality that $\hat{Q}$ is $b \to d \leftarrow a \to c$. This means that we have bRd, aRd, and $c(b) = c(a) = i$. By the definition of R, we then have that either bRa or aRb. Since $c(a) = c(b)$, we have $ab \notin E(G)$. As $c(a) = c(b) = i$, we further have $ab \notin E(G_i)$, which is a contradiction. $\qquad \square$

Corollary 28. *For any graph G, $\dim_{COG}(G) \leqslant \mathrm{tw}(G) + 2$.*

Corollary 29. *For every partial 2-tree G, $\dim_{COG}(G) \leqslant 4$.*

Since outerplanar graphs are partial 2-trees, this means that every outerplanar graph is the intersection of at most 4 cographs.

A tree decomposition (T, f) of a graph G is said to be a *path decomposition* of G if T is a path. The pathwidth of G, denoted by $\mathrm{pw}(G)$, is defined as the minimum width of a path decomposition of G. Clearly, for any graph G, $\mathrm{tw}(G) \leqslant \mathrm{pw}(G)$.

Theorem 30. *For every graph G, $\dim_{TH}(G) \leqslant \mathrm{pw}(G) + 1$.*

Proof. Let (T, f) be a path decomposition of G of width $\mathrm{pw}(G)$. Following the proof of Theorem 27, we can select an endvertex of the path T to be the root r and construct the graphs $G_0, G_1, \ldots, G_{\mathrm{pw}(G)+1}$. Then, the relation R defined on $V(G)$ has the property that for any two vertices $u, v \in V(G)$, we have either uRv or vRu. Therefore, the graphs $G_0, G_1, \ldots, G_{\mathrm{pw}(G)+1}$ have the following properties:

  (a) G_0 is a complete graph, and

  (b) for each $i \in \{1, 2, \ldots, \mathrm{pw}(G) + 1\}$, G_i is a split graph (the vertices in $\{u \colon c(u) = i\}$ form an independent set and the rest form a clique in G_i), and therefore a threshold graph.

From (a), we have that $G = G_1 \cap G_2 \cap \cdots \cap G_{\mathrm{pw}(G)+1}$ and so by (b), we conclude that the graph G can be represented as the intersection of $\mathrm{pw}(G) + 1$ threshold graphs. Thus, we have the theorem. $\square$

Although Corollary 25 says that every outerplanar graph is the intersection of at most 6 threshold graphs, we are not aware of any outerplanar graph that has threshold dimension more than 3 (cycles of more than 6 vertices have threshold dimension equal to 3 as shown in Section 6). Using Theorem 30, we can get upper bounds better than 6 for the threshold dimension for a special kind of outerplanar graph.

The weak dual G^* of an outerplanar graph G, given some planar embedding of G, is its dual graph with the vertex corresponding to the outer face removed. That is, $V(G^*)$ is the set of internal faces of G and there is an edge between two internal faces f and f' in G^* if and only if they share an edge in G. Let G be a 2-connected outerplanar graph whose weak dual is a path. Construct another outerplanar graph G' by adding edges to G such that every internal face of G' is a triangle and the weak dual of G' is also a path. It can be seen that if we arrange the internal faces of G' in the order in which they appear on this path, then every vertex appears in a consecutive set of faces in that order. It then follows that G' is an interval graph (see Theorem 8.1 in [22]) with no clique of size more than 3 (as G' is outerplanar). This implies that $\mathrm{pw}(G) \leqslant \mathrm{pw}(G') \leqslant 2$ (see Theorem 29 in [4]). Then we can use Theorem 30 to get the following result, which is a generalization of Remark 18.

Corollary 31. *If G is a 2-connected outerplanar graph whose weak dual is a path, then $\dim_{COG}(G) \leqslant \dim_{TH}(G) \leqslant 3$.*

9 Upper bounds for cograph dimension using vertex partitions

In this section, we borrow a technique from [26] which can be used for graphs whose vertex set can be partitioned in such a way that each pair of parts induces a subgraph with a bounded cograph dimension. The following lemma that we use is a slight variation of a lemma that appears in [26]. This technique and its role in connecting certain intersection dimensions of a graph with its acyclic and star chromatic numbers also appears in [3].

Recall that $G_1 + G_2$ denotes the join of two graphs G_1 and G_2.

Let $\alpha : \mathbb{N}^+ \to \mathbb{N}^+$ be the function $\alpha(x) = \begin{cases} x & \text{if } x \text{ is odd} \\ x - 1 & \text{if } x \text{ is even} \end{cases}$.

Lemma 32. *Let $\mathcal{H}$ be a hereditary class of graphs which is closed under the join operation. If G is a graph whose vertices can be partitioned into k parts $V_1, V_2, \ldots, V_k$ in such a way that for any $i, j \in \{1, 2, \ldots, k\}$, $\dim_{\mathcal{H}}(G[V_i \cup V_j]) \leqslant t$, then $\dim_{\mathcal{H}}(G) \leqslant \alpha(k)t$.*

Proof. We begin by noting that as $\mathcal{H}$ is hereditary, for any $i \in \{1, 2, \ldots, k\}$, $\dim_{\mathcal{H}}(G[V_i]) \leqslant t$. Consider the complete graph K_k whose vertices are labelled from c_1 to c_k. For an edge $e = c_i c_j$ in this complete graph, we denote by G_e the graph $G[V_i \cup V_j]$. For a matching M in the complete graph, let U_M denote the set of vertices of the complete graph that are not matched by M, i.e, the set of vertices of the complete graph that have no edge of M incident on them. We also define $G_M = \Sigma_{e \in M} G_e + \Sigma_{c_i \in U_M} G[V_i]$. Since $\mathcal{H}$ is closed under the join operation, it follows by Observation 5 that $\dim_{\mathcal{H}}(G_M) \leqslant t$. It is also easy to see that G_M is a supergraph of G. Now consider a proper edge colouring of the complete graph K_k using $\chi'(K_k)$ colours. This colouring can be seen as a partitioning of the edge set of the complete graph into matchings $M_1, M_2, \ldots, M_{\chi'(K_k)}$. Consider $uv \notin E(G)$. If there exists i such that $u, v \in V_i$, then $uv \notin E(G_{M_j})$ for any j. If there is no such i, then surely there exists i, j such that $u \in V_i$ and $v \in V_j$. Let M_r be the matching that contains the edge $c_i c_j$ of the complete graph. Then, it can be seen that $uv \notin E(G_{M_r})$. This allows us to conclude that $G = \bigcap_{1 \leqslant i \leqslant \chi'(K_k)} G_{M_i}$. This implies that $\dim_{\mathcal{H}}(G) \leqslant \chi'(K_k)t$. The proof is completed by noting the well known fact that $\chi'(K_k) = \alpha(k)$. □

The *acyclic chromatic number* of a graph G, denoted by $\chi_a(G)$, is the minimum number of colours required to properly vertex colour G such that the union of any two colour classes induces a forest in G.

We would like to note that some generalized variants of Theorems 33 and 35 appear in the work of Aravind and Subramanian [3]. Theorem 9 and Corollary 11(b) of that paper give upper bounds on the intersection dimension of a graph G with respect to any hereditary class that is closed under the disjoint union and join operations. We note here that any such class $\mathcal{A}$ has to contain the class of cographs and therefore any upper bound on the cograph dimension of a graph G is also an upper bound on $\dim_{\mathcal{A}}(G)$. The following theorem essentially follows from Theorem 9 of [3] and Corollary 9 in this paper.

Theorem 33. *For any graph G, $\dim_{COG}(G) \leqslant 2 \cdot \alpha(\chi_a(G))$.*

Proof. Let $k = \chi_a(G)$. Consider an acyclic vertex colouring of G using k colours. Let $V_1, V_2, \ldots, V_k$ denote the colour classes into which $V(G)$ gets partitioned by the colouring.

Note that since for any $i, j \in \{1, 2, \ldots, k\}$, $G[V_i \cup V_j]$ is a forest, we have from Corollary 9 that $\dim_{COG}(G[V_i \cup V_j]) \leqslant 2$. Now from Lemma 32, we have the theorem. $\quad\square$

By Borodin's Theorem [6], the acyclic chromatic number of any planar graph is at most 5. Therefore, we have the following corollary.

Corollary 34. *For any planar graph G, $\dim_{COG}(G) \leqslant 10$.*

A *star colouring* of a graph G is a proper vertex colouring of G such that the union of any two colour classes induces in G a forest whose every component is a star (such a forest is called a "star forest"). In other words, it is a proper vertex colouring of G such that every path on 4 vertices in G needs at least three colours. The minimum number of colours required in any star colouring of a graph G is called its *star chromatic number*, denoted by $\chi_s(G)$. It follows from Corollary 11(b) of [3] that for any graph G, $\dim_{COG}(G) \leqslant \chi_s(G)$. The following theorem essentially states this, with the small improvement that $\chi_s(G)$ is replaced with $\alpha(\chi_s(G))$.

Theorem 35. *For any graph G, $\dim_{COG}(G) \leqslant \alpha(\chi_s(G))$.*

Proof. Let $k = \chi_s(G)$. Therefore, $V(G)$ can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that for any $i, j \in \{1, 2, \ldots, k\}$, $G[V_i \cup V_j]$ is a star forest. Since star forests are cographs, we have that for any $i, j \in \{1, 2, \ldots, k\}$, $\dim_{COG}(G[V_i \cup V_j]) = 1$. Now from Lemma 32, we have the theorem. $\quad\square$

We note here that every planar graph G has $\chi_s(G) \leqslant 20$ and there exists a planar graph having star chromatic number 10 [2]. For planar graphs with lower bounds on girth, better bounds on the acyclic vertex colouring number and star chromatic number are known. Every planar graph of girth at least 5 and 7 can be acyclically vertex coloured with 4 and 3 colours respectively [7], implying that the cograph dimension of these graphs is at most 8 and 6 respectively. Further, every planar graph with girth at least 6, 7, 8, 9 and 13 can be star coloured with 8, 7, 6, 5 and 4 colours respectively [36, 27, 9]. We therefore can use Theorems 33 and 35 to get the following.

Corollary 36. *For any planar graph G with girth g:*

(i) *if $g \geqslant 5$, then $\dim_{COG}(G) \leqslant 8$*

(ii) *if $g \geqslant 6$, then $\dim_{COG}(G) \leqslant 7$,*

(iii) *if $g \geqslant 7$, then $\dim_{COG}(G) \leqslant 6$,*

(iv) *if $g \geqslant 8$, then $\dim_{COG}(G) \leqslant 5$, and*

(v) *if $g \geqslant 13$, then $\dim_{COG}(G) \leqslant 3$.*

As there exist planar bipartite graphs with star chromatic number at least 8 [25], we cannot hope to use this technique to get an improvement over the bound given in Corollary 22.

A number of upper bounds are known for the acyclic chromatic number and star chromatic number of various special classes of graphs. These bounds can be used with Theorems 33 and 35 to establish upper bounds on the cograph dimension of graphs belonging to these classes.

A note on the tightness of the upper bound of Fertin et al. The results of Fertin, Raspaud and Reed [17] and also those of Albertson et al. [2] imply that for outerplanar graph (in fact, any partial 2-tree) G, $\chi_s(G) \leqslant 6$. Thus Theorem 35 does not give us a bound better than the one given by Corollary 29. An upper bound better than 6 for the star chromatic number of outerplanar graphs would have been helpful, but there are outerplanar graphs with star chromatic number equal to 6, and an example is given in [17]. However, that example contains 48 vertices and 93 edges, and was shown to have star chromatic number 6 using a computer check. Later, Albertson et al. [2] presented a construction using which graphs with treewidth t and star chromatic number $\binom{t+1}{2}$ can be constructed for any $t \geqslant 1$. For $t = 3$, their construction gives an outerplanar graph on 41 vertices and 71 edges having star chromatic number at least 6. We give an outerplanar graph with 20 vertices and 33 edges for which it can be proven that the star chromatic number is at least 6.
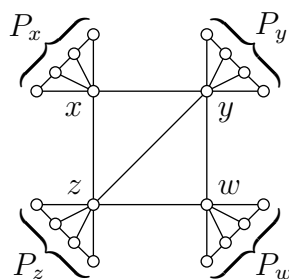

Figure 4: The graph G.

Theorem 37. *The outerplanar graph G shown in Figure 4 has star chromatic number at least 6.*

Proof. Suppose for the sake of contradiction that G has a star colouring c using the colours 1, 2, 3, 4 and 5. Since every star coloring is also a proper vertex coloring, we can assume without loss of generality that $c(x) = 1$, $c(y) = 2$ and $c(z) = 3$, and further that no vertex in P_x has color 1. As c is a star coloring, the vertices in P_x receive at least 3 different colors, implying that at least one of the colors 2 or 3 is present in the path P_x. Let us assume without loss of generality that one of the vertices in P_x, which we shall denote by x_2, is colored 2. We now get that $c(w) \neq 1$, as otherwise, x_2xyw would be a bicolored P_4.

So we can assume without loss of generality that $c(w) = 4$. If a vertex y_1 in P_y is colored 1, then we would have the bicolored P_4 x_2xyy_1. Since P_y also cannot be bicolored, we get that there is a vertex y_i in P_y that is colored i, for each $i \in \{3, 4, 5\}$. Now if there is a vertex w_2 in P_w that is colored 2, then we will have a bicolored P_4 y_4yww_2, which is a contradiction. This implies that there is a vertex w_3 in P_w that is colored 3. Similarly, since if there is a vertex z_2 in P_z that is colored 2, there will be the bicolored P_4 y_3yzz_2, we get that there is a vertex z_4 in P_z that is colored 4. Now the path w_3wzz_4 is a bicolored P_4, contradicting the fact that c is a star coloring. $\square$

10 Conclusion

We end with some open questions.

Question. Can the upper bounds shown for the cograph dimension and threshold dimension for any of the classes of graphs studied be improved?

Question. Does there exist a linear upper bound on the threshold dimension in terms of the treewidth of the graph?

Question. Does there exist a planar graph whose cograph dimension is more than 3?

Dujmović et al. [16] formally defined the notion of *layered treewidth* (this idea also appears implicitly in the work of Shahrokhi [34]) and showed that the layered treewidth of every planar graph is at most 3. It could be possible that an upper bound on the cograph dimension of a graph in terms of its layered treewidth can be obtained, and this could lead to an improvement to the upper bound on the cograph dimension of planar graphs.

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