

Duality pairs and homomorphisms to oriented and unoriented cycles

Santiago Guzmán-Pro César Hernández-Cruz*

Facultad de Ciencias
Universidad Nacional Autónoma de México
CDMX, México

{sanguzpro, chc}@ciencias.unam.mx

Submitted: Jul 28, 2020; Accepted: Jun 22, 2021; Published: Jul 16, 2021
© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

In the homomorphism order of digraphs, a duality pair is an ordered pair of digraphs (G, H) such that for any digraph, D , $G \rightarrow D$ if and only if $D \not\rightarrow H$. The directed path on $k + 1$ vertices together with the transitive tournament on k vertices is a classic example of a duality pair.

In this work, for every undirected cycle C we find an orientation C_D and an oriented path P_C , such that (P_C, C_D) is a duality pair. As a consequence we obtain that there is a finite set, F_C , such that an undirected graph is homomorphic to C , if and only if it admits an F_C -free orientation. As a byproduct of the proposed duality pairs, we show that if T is an oriented tree of height at most 3, one can choose a dual of T of linear size with respect to the size of T .

Mathematics Subject Classifications: 05C60, 05C75

1 Introduction

We consider graphs and digraphs with neither loops nor parallel arcs. For a digraph D , we use V_D to denote its vertex set and A_D to denote its arc set. For undefined terms we refer the reader to [7]. In the homomorphism order of digraphs, a *duality pair* is an ordered pair of digraphs (G, H) such that for any digraph, D , $G \rightarrow D$ if and only if $D \not\rightarrow H$.

Our main result can be considered in three different contexts. We now present a brief introduction to each of them.

The Roy-Gallai-Hasse-Vitaver Theorem [3, 5, 14, 16] states that a graph is k -colourable if and only if it admits an orientation with no directed path on more than k vertices. This

*Supported by SEP-CONACYT A1-S-8397, CONACYT FORDECYT-PRONACES/39570/2020, and PAPIIT-DGAPA IA104521 grants.

result is a consequence of the fact that a digraph D is homomorphic to the transitive tournament on k vertices, TT_k , if and only if the directed path on $k + 1$ vertices, \vec{P}_{k+1} , is not homomorphic to D . In terms of duality pairs, (\vec{P}_{k+1}, TT_k) is a duality pair in the homomorphism order of digraphs. In [12] Nešetřil and Tardif proved that if (A, B) is a duality pair in the homomorphism order of digraphs, then A is an oriented tree. Moreover, for any oriented tree, T , there is a digraph D_T (the *dual of T*), such that (T, D_T) is a duality pair in the homomorphism order of digraphs. Their result is actually more general, dealing with relational structures, so, as other authors have done, we consider a restriction for the context of this work. In fact, in [11], the same authors consider the problem restricted to digraphs and, for a given oriented tree T , they construct a digraph D_T such that (T, D_T) is a duality pair. Their construction is simple, but of size exponential on $|V_T|$, raising the following question: Can one choose D_T to be of polynomial size with respect to $|V_T|$? (Proposed as Problem 1 in [11].) In general, this question has a negative answer [13], but there are some examples where one can pick D_T to be of polynomial size with respect to $|V_T|$. For instance, for the family of directed paths, one can choose D_T to be the corresponding dual transitive tournament, and thus D_T is of linear size when T is a directed path.

Similar notions of duality have been studied also in the context of digraph homomorphisms. In [9], Hell and Zhu defined the class of B -cycles as special orientations of cycles, and showed that for a fixed B -cycle, C , a digraph D is not homomorphic to C , if and only if there exists a path P homomorphic to D , which is not homomorphic to C . They call this notion of duality *path duality*.

For a set of oriented graphs F the class of F -graphs is the class of undirected graphs that admit an F -free orientation. In [15], Skrien found a structural characterization for the class of F -graphs when F is a set of oriented paths on 3 vertices. Some of these are proper interval graphs, proper circular-arc graphs and comparability graphs. In [4], Skrien's study of F -graphs is extended to any set of oriented graphs on 3 vertices. Two of these classes are still lacking a complete structural characterization; the so-called perfectly orientable graphs [15], and the transitive-perfectly orientable graphs [4]. In terms of F -graphs, the Roy-Gallai-Vitaver-Hasse Theorem states that, when F is the set of oriented graphs on $k + 1$ vertices with a hamiltonian directed path, the class of F -graphs is the class of k -colourable graphs. In this case, one can assume that such an orientation is also acyclic. The class of graphs that admit an acyclic F -free orientation is the class of F^* -graphs [15]. Another example of such classes are chordal graphs: when F consists of the orientation of the path on 3 vertices such that one vertex has 2 out-neighbours, the class of F^* -graphs is the class of chordal graphs. This statement follows from the fact that a graph is chordal if and only if it admits a perfect elimination ordering [2].

Even though we mainly deal with duality pairs in the homomorphism order of digraphs, the whole paper is motivated by the study of characterizations of graph classes as F -graphs, for a finite set F . For each positive integer n , $n \geq 3$, we present a finite set of oriented graphs F_n such that F_n -graphs are precisely C_n -colourable graphs, i.e., graphs that admit a homomorphism to the n -cycle. In a way similar to the Roy-Gallai-Vitaver-Hasse Theorem, we use duality pairs as a tool to find such a set F_n .

From the viewpoint of oriented cycles and path dualities, it turns out that our result yields another class of cycles, in addition to the B -cycles studied in [9], having path duality. The class we propose, AC -cycles, is somewhat more restrictive, but the result can be strengthened: for any AC -cycle, C , we obtain an oriented path P_C , such that a digraph D is not homomorphic to C , if and only if P_C is homomorphic to D .

The class of AC -cycles corresponds to the family of duals, D_P , for oriented paths P in a special set, for the moment the set \mathcal{Q} . Moreover, these AC -cycles are duals of linear size with respect to their corresponding path P . In [8] Hell and Nešetřil showed that the core of any oriented tree of height 3 is a path in \mathcal{Q} . Hence, we conclude that for any tree T of height at most 3, one can choose a dual D_T of linear size with respect to the core of T , and thus of linear size with respect to T .

The rest of this work is structured as follows. In Section 2, we introduce basic notation, concepts and results needed for later developments. Our main result is stated and proved in Section 3. Finally, in Section 4 we consider the different interpretations of our main result in the three contexts introduced above. Conclusions are briefly presented in Section 5.

2 Preliminary results

When G and H are graphs, we write $G \rightarrow H$ to denote that G is homomorphic to H . When x and y are vertices of a digraph D , we write $x \rightarrow y$ to denote that (x, y) is an arc of D . It should always be clear from the context to which interpretation of the symbol \rightarrow we are referring to. Nonetheless, when speaking of homomorphisms, we will use capital letters for digraphs, and when dealing with arcs in a digraph, we will use small-case letter for vertices.

An *oriented path* P is a sequence of distinct vertices (p_0, \dots, p_n) such that, for each $i \in \{0, \dots, n-1\}$, either $(p_i, p_{i+1}) \in A_P$, or $(p_{i+1}, p_i) \in A_P$ (but not both), and P has no more arcs. If $p_i \rightarrow p_{i+1}$ we say that (p_i, p_{i+1}) is a *forward arc*; if $p_{i+1} \rightarrow p_i$ the arc (p_{i+1}, p_i) is a *backward arc*. The direction in which P is traversed is emphasized by saying that the *initial vertex* of P is p_0 and the *terminal vertex* of P is p_n . If all arcs in P are forward (backward) arcs, we say that P is a *directed path*, with forward (backward) direction and denote it by \overrightarrow{P}_{n+1} (\overleftarrow{P}_{n+1}), where n is the number of arcs of P . An oriented path is *alternating* if every two successive arcs are oppositely oriented. We denote by A_n , the alternating path on n vertices that begins with a forward arc, if $n = 1$, then A_n denotes the single vertex with no arcs. A *semi-walk* on a digraph D , is a sequence $v_1 a_1 v_2 a_2 \dots a_{n-1} v_n$, where $v_i \in V_D$ for $i \in \{1, \dots, n\}$, and a_i is an arc with endpoints v_i and v_{i+1} , for $i \in \{1, \dots, n-1\}$. An arc a_i in a semi-walk is a *forward arc* if $a_i = (v_i, v_{i+1})$; otherwise, we say it is a *backward arc*. A semi-walk is *closed* if $v_1 = v_n$. The *pattern* of the semi-walk $v_1 a_1 v_2 a_2 \dots v_n$, is a sequence $l_1 \dots l_{n-1}$ of symbols in $\{\rightarrow, \leftarrow\}$, where $l_i = \rightarrow$ if a_i is a forward arc; $l_i = \leftarrow$ otherwise.

An *oriented cycle* C is an oriented graph obtained by identifying the initial and terminal vertex of an oriented path P . If all arcs have the same direction, we speak of a *directed cycle*, and denote it by \overrightarrow{C}_n .

The *net length* $\ell(X)$ of an oriented path or oriented cycle, X , is the number of forward arcs minus the number of backward arcs in X , note that the net length may be negative. The following statement is proved in [6], but we use the restatement found in [10] for its simplicity.

Theorem 1. [10] *For $n \geq 1$, an oriented graph G is homomorphic to \vec{P}_n , if and only if, every oriented path homomorphic to G has net length at most n .*

Theorem 1 shows that directed paths have path duality. Now we introduce another family of oriented graphs, proposed by Hell and Zhu in [9], that have path duality. An oriented path P is *minimal* if it contains no proper oriented path P' such that $\ell(P') = \ell(P)$. An oriented cycle $C = (c_0, \dots, c_n, \dots, c_{m-1}, c_0)$ is a *B-cycle*, if (c_0, \dots, c_n) is a forward directed path, and $(c_0, c_{m-1}, \dots, c_n)$ is a minimal oriented path of net length $n - 1$. As mentioned in Section 1, *B-cycles* have path duality.

Theorem 2. [9] *Let C be a B-cycle. A digraph D is homomorphic to C if and only if every oriented path homomorphic to D is also homomorphic to C .*

A digraph G is balanced if every oriented cycle in G has net length zero. A digraph on n vertices is balanced if and only if $D \rightarrow \vec{P}_{n-1}$ (see [7]). Since every directed cycle has non-null net length, every balanced digraph must be acyclic, and thus there is at least one vertex with no in-neighbours. Let G be a connected balanced digraph and $x \in V_G$ such that $d^-(x) = 0$. We define the *level* of vertex $v \in V_G$ as the net length of any oriented path from x to v . The fact that the level of every vertex is well-defined follows from the choice of G , i.e. connected and balanced. The maximum level of the vertices in G is called the *height* of G . For two digraphs G and H , the *interval* $[G, H]$ consists on all digraphs M such that $G \rightarrow M \rightarrow H$. The following statement is a useful and well-known result about the homomorphism order of digraphs.

Proposition 3. [8] *If G is a balanced digraph of height 3, then $G \in [\vec{P}_3, \vec{P}_4]$.*

Two oriented graphs G and H are *homomorphically equivalent*, if and only if $G \rightarrow H$, and $H \rightarrow G$. Thus, it follows that G and H are homomorphically equivalent, if and only if, for any digraphs L and R , $L \rightarrow G$ if and only if $L \rightarrow H$, and, $G \rightarrow R$ if and only if $H \rightarrow R$. An ordered pair of digraphs (G, H) is a *duality pair*, if for any digraph L , $G \not\rightarrow L$ if and only if $L \rightarrow H$. In this case, we say that H is a *dual* of G . It is not hard to notice that if such a dual exists, then it is unique up to homomorphic equivalence. The transitive tournament on n vertices is denoted by TT_n . A classic example of a family of duality pairs, is given by the following theorem.

Theorem 4. [1] *For $n \geq 2$, an oriented graph G is homomorphic to TT_n if and only if \vec{P}_{n+1} is not homomorphic to G , i.e., for every $n \geq 2$, (\vec{P}_{n+1}, TT_n) is a duality pair.*

We conclude this section with the following straightforward observation that we will use more than once in this work.

Observation 5. *Let G, H and R, S be pairs of homomorphically equivalent oriented graphs, then (G, R) is a duality pair if and only if (H, S) is a duality pair.*

3 Main results

We first define the family of oriented paths for which we will find a family of duals. For $n \geq 3$ we denote by Q_n the oriented path on n vertices (q_0, \dots, q_{n-1}) with the following properties: the first two arcs are forward arcs, the suboriented path (q_1, \dots, q_{n-2}) is an alternating path, and the two final arcs have the same direction. Note that, by the first two conditions, $(q_1, \dots, q_{n-k}) = A_{n-(k+1)}$ for $1 \leq k \leq (n-3)$, and $(q_{n-3}, q_{n-2}, q_{n-1}) = \vec{P}_3$ or $(q_{n-3}, q_{n-2}, q_{n-1}) = \overleftarrow{P}_3$, depending on the parity of n . This is illustrated in Figure 1. In particular, Q_3 and Q_4 are the directed paths on 3 and 4 vertices respectively.

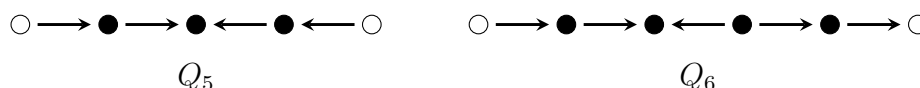


Figure 1: The oriented paths Q_{n+2} with the vertices of their mid-section, A_n , coloured black ($n \in \{3, 4\}$). In Q_5 the three final vertices induce a directed path with all arcs backward, while in Q_6 the three final vertices induce a directed path with all arcs forward.

Observation 6. For every integer n , $n \geq 5$, Q_n is homomorphic to Q_{n-2} . In particular, if n is even then $Q_n \rightarrow \vec{P}_4$, and if n is odd then $Q_n \rightarrow \vec{P}_3$.

Proof. Let n be an integer, $n \geq 5$, and let $Q_n = (q_0, \dots, q_n)$. By identifying q_3 with q_1 , and q_4 with q_2 , we obtain a homomorphism from Q_n to Q_{n-2} . \square

It is also straightforward to calculate the net length of the oriented paths Q_n .

Observation 7. For every positive integer n , $n \geq 3$, the net length of Q_n is described by the equations

$$\ell(Q_n) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd and greater than 3.} \end{cases}$$

It is clear that \vec{P}_3 maps to Q_n for all n at least 3. Now, if n is even, by Observation 7, $\ell(Q_n) = 3$ and since $\ell(\vec{P}_3) = 2$, Theorem 1 ensures that Q_n does not map to \vec{P}_3 . On the other hand, if n is odd, we know that $Q_n \rightarrow \vec{P}_3$ by Observation 6. So the following statement holds.

Lemma 8. For every integer n , $n \geq 4$, Q_n is homomorphically equivalent to \vec{P}_3 if and only if n is odd.

For $n \geq 3$, we denote by AC_n the oriented cycle obtained from identifying the initial and terminal vertices of the alternating path A_{n+1} . For this work we will denote the

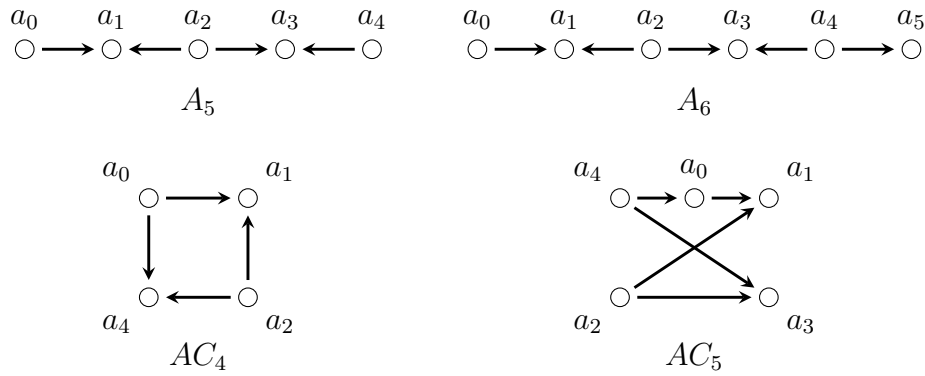


Figure 2: The oriented paths A_{n+1} and oriented cycles AC_n for $n \in \{4, 5\}$.

vertices of AC_n as $(a_0, a_1, \dots, a_{n-1}, a_0)$. Note that, if n is even, every two consecutive arcs have opposite direction, and if n is odd, every pair of consecutive arcs, except for $a_{n-1} \rightarrow a_0 \rightarrow a_1$, have opposite direction. In Figure 2 we illustrate A_4 , AC_4 , A_5 and AC_5 .

Lemma 9. *For every integer n , $n \geq 4$, the cycle AC_n is homomorphically equivalent to \vec{P}_2 if and only if n is even.*

Proof. Since \vec{P}_2 is an asymmetric arc, then \vec{P}_2 is homomorphic to any non-trivial oriented graph. If n is even, every two consecutive arcs of AC_n have opposite direction, thus, the largest directed path of AC_n is \vec{P}_2 . But if n is odd, there is a copy of \vec{P}_3 contained in AC_n . Therefore, by Theorem 4, AC_n is homomorphic to P_2 if and only if n is even. \square

In order to avoid a very long proof for our main result, we attempt to break it down in a reasonable amount of statements. We start with the following one.

Proposition 10. *For every integer n , $n \geq 3$, the oriented path Q_{n+1} is not homomorphic to AC_n .*

Proof. First note that when $n = 3$ then $AC_3 \cong TT_3$ and $Q_4 \cong \vec{P}_4$. So by Theorem 4, $Q_4 \not\cong AC_3$. For the remaining cases, i.e., $n \geq 4$, we will proceed by contradiction. If n is even, notice that \vec{P}_3 maps to Q_{n+1} . Thus, if Q_{n+1} maps to AC_n we obtain a mapping of \vec{P}_3 to AC_n . But as n is even, by Lemma 9 AC_n maps to \vec{P}_2 , and since $\vec{P}_2 \cong TT_2$, this contradicts the fact that (\vec{P}_3, TT_2) is a duality pair.

Now suppose that there is an odd integer $n \geq 5$ and a homomorphism $\varphi: Q_{n+1} \rightarrow AC_n$. We first show that φ is not a surjective mapping. Recall that the only vertex in AC_n with in-degree and out-degree greater than 0 is a_0 . Note that the only vertices in $Q_{n+1} = (q_0 \dots q_n)$ with in-degree and out-degree greater than 0, are q_1 and q_{n-1} . Thus, $\varphi(q_1) = a_0 = \varphi(q_{n-1})$, and hence $\varphi(q_0) = a_{n-1} = \varphi(q_{n-2})$ and $\varphi(q_2) = a_1 = \varphi(q_n)$. Since $|V_{Q_n} - \{q_0, q_1, q_2, q_{n-2}, q_{n-1}, q_n\}| = n - 5$, and $(n - 5) + 3 < n = |V_{AC_n}|$, φ is not surjective. Now we observe that the existence of such a non-surjective homomorphism leads to a contradiction. First, it is not hard to verify that $AC_n - a_i$ is homomorphic

to \vec{P}_3 for any $i \in \{2, \dots, n-2\}$. Since φ is not surjective, and by previous arguments $\{a_{n-1}, a_0, a_1\} \subseteq \varphi[V_{Q_{n+1}}]$, then by composing homomorphisms, Q_{n+1} is homomorphic to \vec{P}_3 . This contradicts the fact that $\ell(Q_{n+1}) = 3$ (Observation 7) and Theorem 1. \square

Proposition 10 implies that if D is a digraph and $Q_{n+1} \rightarrow D$ then $D \not\rightarrow AC_n$. The following lemma asserts that the converse implication holds when n is odd.

Lemma 11. *For any odd integer n , $n \geq 3$, and any digraph D , if $Q_{n+1} \not\rightarrow D$ then $D \rightarrow AC_n$.*

Proof. Let D be a digraph such that $Q_{n+1} \not\rightarrow D$ and let G be the underlying graph of D . Since our goal is to construct a homomorphism $D \rightarrow AC_n$ we may assume that D is connected (and so is G).

Let X be the set of vertices of D having positive in- and out-degrees. If X is an empty set, then $D \rightarrow TT_2$ and so D trivially maps to AC_n . From now on we will assume that X is a non-empty set. The remaining vertices of D can be partitioned into two sets: the set of vertices I having only in-neighbours and the set of vertices O having only out-neighbours. It is straightforward to verify that I and O are independent sets. Moreover, X is also an independent set; otherwise there is a pair of vertices $x_1, x_2 \in X$ such that $(x_1, x_2) \in A_D$ and since x_1 has an in-neighbour u and x_2 had an out-neighbour v , then $\vec{P}_4 \rightarrow D$. Recall that by Observation 6, $Q_{n+1} \rightarrow \vec{P}_4$ and thus $Q_{n+1} \rightarrow D$ which is a contradiction. We now refine our partition (X, I, O) with respect to the distance in G of vertices to X . For $i \in \{1, \dots, \text{diam}(G)\}$ we define $I_i = \{v \in I : d(v, X) = i\}$ and $O_i = \{v \in O : d(v, X) = i\}$.

By construction of the sets X, I_i and O_i for $i \in \{1, \dots, \text{diam}(G)\}$, it is not a hard task to verify that for every arc $(u, v) \in A_D$ one of the following statements holds:

- $u \in O_1$ and $v \in X$ or $u \in X$ and $v \in I_1$,
- there is a positive integer i such that $u \in O_i$ and $v \in I_{i-1} \cup I_{i+1}$, or
- there is a positive integer i such that $u \in O_i$ and $v \in I_i$.

If the third case does not hold, then the overall structure of D can be depicted as follows

$$\dots I_4 \leftarrow O_3 \rightarrow I_2 \leftarrow O_1 \rightarrow X \rightarrow I_1 \leftarrow O_2 \rightarrow I_3 \leftarrow O_4 \dots$$

Noticing that in this case $D \rightarrow \vec{P}_3$ is no more than routine work. Since $\vec{P}_3 \rightarrow AC_n$, then $D \rightarrow AC_n$.

Now suppose that there is a positive integer i such that there is an arc from O_i to I_i in D , and let i_0 be the minimum of all such integers. We depict the structure of D with respect to the previously defined partition in Figure 3.

Note that if $i_0 \geq \frac{n-1}{2}$ we can map D to AC_n . Indeed, first consider the relation $x \sim y$ defined by the following conditions:

- $x, y \in X$,
- $x, y \in I_i$ or $x, y \in O_i$ for some $i \in \{1, \dots, \frac{n-1}{2} - 1\}$, or

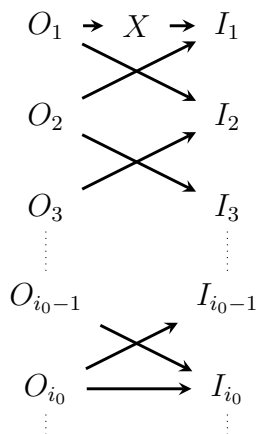


Figure 3: The overall structure of D with respect to the proposed partition.

- $x \in I_i$ and $y \in I_j$ for some $i, j \in \{\frac{n-1}{2}, \dots, \text{diam}(G)\}$ or $x \in O_i$ and $y \in O_j$ for some $i, j \in \{\frac{n-1}{2}, \dots, \text{diam}(G)\}$.

Then, by identifying the vertices in the same \sim -equivalence class we obtain an oriented graph isomorphic to AC_n .

In order to conclude our proof we show that if $i_0 < \frac{n-1}{2}$, then there is a homomorphism from Q_{n+1} to D which contradicts the choice of D , and so the claim of this lemma will follow. Consider an arc (u, v) from O_{i_0} to I_{i_0} . By construction of our partition, there are two alternating paths of length i_0 , one from u to some vertex $x_u \in X$ and another one from v to some vertex $x_v \in X$. By the choice of X , x_u has an in-neighbour y_u and x_v an out-neighbour y_v . Since the previously described uy_u - and vy_v -paths can be concatenated with the arc (u, v) , we obtain a path in D isomorphic to Q_{2i_0+4} . Recall that $i_0 \leq \frac{n-1}{2} - 1$ so $2i_0 + 4 \leq n - 1$ and both quantities are even. Hence, by Observation 6, Q_{n+1} maps to Q_{2i_0+4} and thus to D . We have reached the anticipated contradiction and thus the lemma follows. \square

It is straightforward to verify that if a digraph D has a symmetric arc, then every oriented tree is homomorphic to D . Also, if G is an oriented graph, then D is not homomorphic to G . So if T is an oriented tree, and D_T is any of its duals, then D_T is an oriented graph, $T \rightarrow D$, and $D \not\rightarrow D_T$. For this reason, we state and prove the following theorem for oriented graphs only, but clearly it also holds for general digraphs.

Theorem 12. *For any integer n , $n \geq 3$, an oriented graph G is homomorphic to AC_n if and only if Q_n is not homomorphic to G . In other words, the ordered pair (Q_{n+1}, AC_n) is a duality pair.*

Proof. If n is even, by Lemma 8, Q_{n+1} is homomorphically equivalent to \vec{P}_3 , and by Lemma 9, AC_n is homomorphically equivalent to \vec{P}_2 . Thus, by Observation 5, if n is even, (Q_{n+1}, AC_n) is a duality pair if and only if (\vec{P}_3, \vec{P}_2) is a duality pair. The latter

statement holds since $\vec{P}_2 \cong TT_2$, and (\vec{P}_3, TT_2) is duality pair (Theorem 4). Finally, if n is odd, by Proposition 10 we know that if $Q_{n+1} \rightarrow D$ then $D \not\rightarrow AC_n$. On the other hand, if $Q_{n+1} \not\rightarrow D$, by Lemma 11 then $D \rightarrow AC_n$. Therefore, (Q_{n+1}, AC_n) is a duality pair for every positive integer $n, n \geq 3$. \square

In Figure 4 we exhibit two of the duality pairs described in Theorem 12.

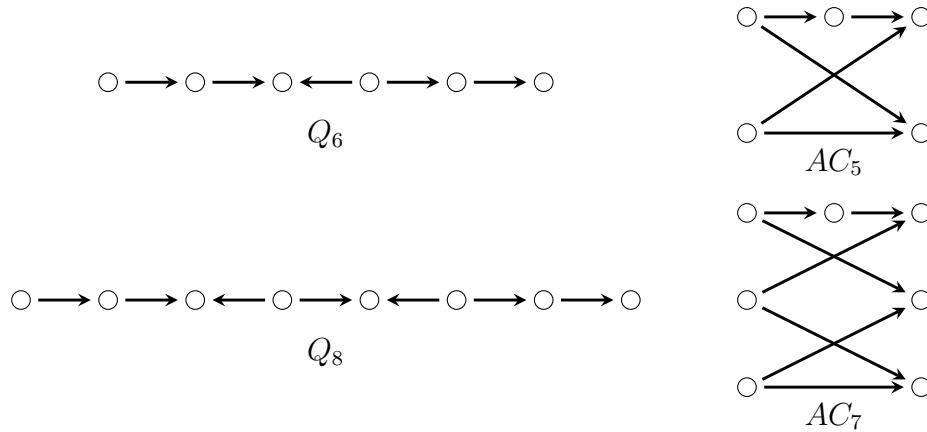


Figure 4: Two duality pairs (Q_6, AC_5) and (Q_8, AC_7) .

4 Implications

We say that an oriented cycle C is an AC -cycle if $C \cong AC_n$ for some positive integer n . The following result is a weaker version of Theorem 12.

Corollary 13. *Let C be an AC -cycle. A digraph D is homomorphic to C , if and only if every oriented path homomorphic to D is also homomorphic to C .*

Thus, in terms of path dualities, we can extend Theorem 2 with this corollary as follows.

Theorem 14. *Any oriented cycle C that is a B -cycle or an AC -cycle, has path duality, i.e., a digraph G is homomorphic to C , if and only if every path homomorphic to G is also homomorphic to C .*

Recall that a digraph is a *core*, if and only if it is not homomorphic to any proper subgraph. The following statement is a well-known result in homomorphism order of digraphs.

Proposition 15. [8] *Let G be a digraph in $[\vec{P}_3, \vec{P}_4]$, then G is homomorphically equivalent to Q_n for some even integer $n \geq 4$. Moreover for every even integer, $n \geq 4$, the path Q_n is a core.*

Now, we give a partial answer to the problem of determining if one can choose a dual, D_T , of an oriented tree, T , of polynomial size with respect to $|V_T|$.

Theorem 16. *Let T be an oriented tree of positive height at most 3, and P_T its core. One can choose a dual D_T of T of linear size with respect to $|V_{P_T}|$. Since $|V_{P_T}| \leq |V_T|$, then D_T is of linear size with respect to $|V_T|$.*

Proof. If T_1 is a tree of height 1, then \vec{P}_2 is homomorphically equivalent to T_1 . When T_2 is a tree of height 2, then T_2 is homomorphically equivalent to \vec{P}_3 . So by Theorem 4, (T_1, TT_1) and (T_2, TT_2) are duality pairs.

If T_3 is a tree of height 3, then by Propositions 3 and 15, T_3 is homomorphically equivalent to a path P , and $P \cong Q_{n+1}$ for an odd integer n , $n \geq 3$. Thus, by Theorem 12, (T_3, AC_n) is a duality pair. For a tree T in any of these cases, the size of the chosen dual is linear with respect to the core of T . \square

Finally, we connect our result to the study of hereditary graph properties characterized as the class of F -graphs for a finite set F . For $n \geq 4$ an even integer, denote by F_n , the set of surjective homomorphic images of Q_n . Clearly, F_n is a finite set since the order of any oriented graph in F_n is bounded by n . Notice that the last three results of this section deal with (undirected) graphs.

Theorem 17. *Let G be a graph and let n be an even integer. If $n \geq 4$, then G is C_{n-1} -colourable if and only if G is an F_n -graph. That is, there is an orientation of G with no induced oriented graph in F_n .*

By Observation 6, the directed path on 4 vertices belongs to F_n for an even integer $n \geq 4$. Thus, it is straightforward to notice that the directed 3- and 4-cycles also belong to F_n . Thus, from the previous corollary we obtain the following one.

Corollary 18. *Let G be a graph and let n be an even integer. If $n \geq 4$, then G is C_{n-1} -colourable if and only if G is an $(F_n - \{\vec{C}_3, \vec{C}_4\})^*$ -graph. That is, there is an acyclic orientation of G with no induced oriented graph in $F_n - \{\vec{C}_3, \vec{C}_4\}$.*

In particular, F_6 consists of the eight oriented graphs depicted in Figure 5.

Corollary 19. *The following statements are equivalent for a graph G .*

- G is homomorphic to the 5-cycle,
- G admits an orientation that has no semi-walk with pattern $\rightarrow\rightarrow\leftarrow\rightarrow\rightarrow$,
- G admits an $\{\vec{C}_3, TT_3, \vec{P}_4, \vec{C}_4, C'_4, Q_6, D_5, \overleftarrow{D}_5\}$ -free orientation, and
- G admits an acyclic $\{TT_3, \vec{P}_4, C'_4, Q_6, D_5, \overleftarrow{D}_5\}$ -free orientation.

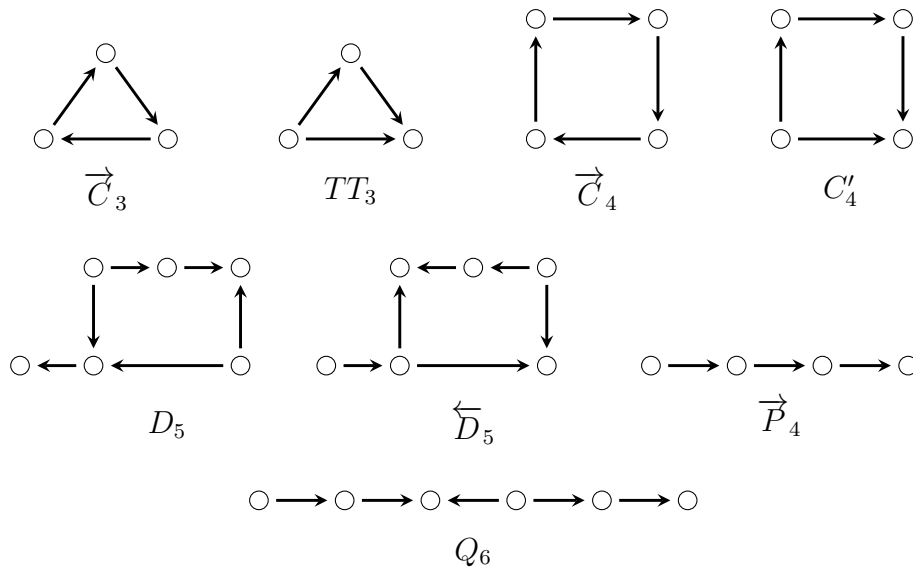


Figure 5: The eight oriented graphs in F_6 .

5 Conclusions

Consider an odd integer n , $n \geq 5$, and an oriented graph G . Note that a polynomial-time certifying algorithm that determines if an oriented graph G is homomorphic to AC_n can be obtained from the proof of Lemma 11. The yes-certificate is the refined partition $X, \{O_i\}, \{I_i\}$ that induces a homomorphism $\varphi: G \rightarrow AC_n$, and the no-certificate is the semi-walk constructed when $i_0 < \frac{n-1}{2}$.

As a nice consequence, we obtain that graphs admitting a homomorphism to an odd cycle can be characterized as those graphs having an orientation avoiding a well defined finite set of oriented graphs.

As previously mentioned, Nešetřil and Tardif exhibited a family of oriented paths P_n such that any dual of P_n is of exponential size with respect to $|V_{P_n}|$ [13]. It is not hard to observe that the net length of P_n increases as n increases. Theorem 16 shows that for every oriented tree T of height at most 3 there is a dual of polynomial size with respect to $|V_T|$. We finish this work by proposing the following questions.

Question 20. Is it true that for any positive integer k , there is a polynomial $p_k(x)$ such that for any oriented tree T , if $\ell(T) \leq k$ then there is a dual D_T of T such that its size is bounded by $p_k(|V_T|)$?

In the event that the answer to Question 20 results negative, from the results obtained in the present work, the following question still makes sense.

Question 21. Which is the maximum integer k , for which one can find a polynomial $p_k(x)$ such that for any oriented tree T , if $\ell(T) \leq k$ then there is a dual D_T of T such that its size is bounded by $p_k(|V_T|)$?

Acknowledgements

We thank the anonymous referees for their very thorough revision of the manuscript; their suggestions helped to greatly improve the quality of the present work.

References

- [1] G. S. Bloom and S. A. Burr, On unavoidable digraphs in orientations of graphs, *J. Graph Theory* 11(4) (1987) 453–462.
- [2] D. R. Fulkerson and O. A. Gross, Incidence matrices and interval graphs, *Pacific J. Math.* 15(3) (1965) 835–855.
- [3] T. Gallai, On directed paths and circuits, *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, Academic Press, New York, 1968, 115–118
- [4] S. Guzmán-Pro and C. Hernández-Cruz, Orientations without forbidden patterns on three vertices, [arXiv:2003.05605](https://arxiv.org/abs/2003.05605).
- [5] M. Hasse, Zur algebraischen Begründung der Graphentheorie, I, *Math. Nachr.* 28 (1964/1965) 275–290.
- [6] R. Häggkvist, P. Hell, D. J. Miller, and V. Neumann-Lara, On multiplicative graphs and the product conjecture, *Combinatorica* 8 (1988) 63–74.
- [7] P. Hell, and J. Nešetřil, *Graphs and Homomorphisms*, volume 28 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press 2004.
- [8] P. Hell, and J. Nešetřil, Images of Rigid Digraphs, *European J. Combin.* 12(1) (1991) 33–42.
- [9] P. Hell, H. Zhou, and X. Zhu, Homomorphisms to Oriented Cycles, *Combinatorica* 13(4) (1993) 421–433.
- [10] P. Hell, X. Zhu, The Existence of Homomorphisms to Oriented Cycles, *SIAM J. Discrete Math.* 8(2) (1995) 208–222.
- [11] J. Nešetřil, and C. Tardif, A dualistic approach to bounding the chromatic number of a graph, *European J. Combin.* 29 (2008) 254–260.
- [12] J. Nešetřil, and C. Tardif, Duality theorems for finite structures (characterizing gaps and good characterizations), *J. Combin. Theory Ser. B* 80 (2000) 80–97.
- [13] J. Nešetřil, and C. Tardif, Short Answers to Exponentially Long Questions: Extremal Aspects of Homomorphism Duality, *SIAM J. Discrete Math.* 19(4) (2005) 914–920.
- [14] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, *Rev. Fr. Inform. Rech. Oper.* 1 (1967) 129–132.
- [15] D. J. Skrien, A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular-arc graphs, and nested interval graphs, *J. Graph Theory* 6(3) (1982) 309–316

- [16] L. M. Vitaver, Determination of minimal colouring of vertices of a graph by means of Boolean powers of the incidence matrix, Dokl. Akad. Nauk SSSR 147 (1962) 758–759 (in Russian).
- [17] X. Zhu, A Polynomial Algorithm for Homomorphisms to Oriented Cycles, J. Algorithms 19(3) (1995) 333–345.