Rainbow Pancyclicity in Graph Systems

Yangyang Cheng[a,*] Guanghui Wang[a,†] Yi Zhao[b,‡]

[a]School of Mathematics,
Shandong University, 250100, Jinan, Shandong, P. R. China
[b]Department of Mathematics and Statistics,
Georgia State University, Atlanta, GA 30303, U.S.A.

Submitted: Sep 25, 2019; Accepted: Jun 9, 2021; Published: Jul 16, 2021

Abstract

Let $G_1, \dots, G_n$ be graphs on the same vertex set of size n, each graph with minimum degree $\delta(G_i) \geqslant n/2$. A recent conjecture of Aharoni asserts that there exists a rainbow Hamiltonian cycle i.e. a cycle with edge set $\{e_1, \dots, e_n\}$ such that $e_i \in E(G_i)$ for $1 \leqslant i \leqslant n$. This can be viewed as a rainbow version of the well-known Dirac theorem. In this paper, we prove this conjecture asymptotically by showing that for every $\varepsilon > 0$, there exists an integer $N > 0$, such that when $n > N$ for any graphs $G_1, \dots, G_n$ on the same vertex set of size n with $\delta(G_i) \geqslant (\frac{1}{2} + \varepsilon)n$, there exists a rainbow Hamiltonian cycle. Our main tool is the absorption technique. Additionally, we prove that with $\delta(G_i) \geqslant \frac{n+1}{2}$ for each i, one can find rainbow cycles of length $3, \dots, n-1$.

Mathematics Subject Classifications: 05C38

Keywords: Dirac theorem; rainbow Hamiltonian cycle; absorption technique; pancyclicity

1 Introduction

Let $G_1, \dots, G_t$ be t graphs on the same vertex set V of size n where t is a positive integer. We denote the edge set of G_i by $E(G_i)$ and assume that each edge in $E(G_i)$ is coloured by i for $1 \leqslant i \leqslant t$. Let S be an edge set that is a subset of $\cup_{i=1}^{t} E(G_i)$ and we say S is *rainbow* if any pair of edges in S have distinct colours. Rainbow Hamiltonian cycles

*E-mail address: mathsoul@mail.sdu.edu.cn
†Corresponding author. E-mail address: ghwang@sdu.edu.cn
‡E-mail address: yzhao6@gsu.edu

have been studied by many authors. An edge-coloured graph G is *k-bounded* if no colour appears more than k times. Erdős, Nešetřil and Rödl [7] studied the problem for which k any k-bounded K_n contains a rainbow Hamiltonian cycle and they showed that k could be any constant. Hahn and Thomassen [8] demonstrated that k could grow as fast as $n^{\frac{1}{3}}$ and conjectured that the growth of k could in fact be linear. This was confirmed by Albert, Frieze and Reed [4]. A recent result from Coulson and Perarnau [6] further strengthened this by replacing the complete graph with any *Dirac graph*. More precisely, they proved that there exists $\mu > 0$ and positive integer n_0 such that if $n \geqslant n_0$ and G is a μn-bounded edge-coloured graph on n vertices with minimum degree $\delta(G) \geqslant \frac{n}{2}$, then G contains a rainbow Hamiltonian cycle. For rainbow Hamiltonian cycles in graph systems, Aharoni et al. [3] recently gave the following elegant conjecture, which is a natural generalization of Dirac's theorem to the case of graph systems:

Conjecture 1. Given graphs $G_1, \ldots, G_n$ on the same vertex set of size n, if each graph has minimum degree at least $\frac{n}{2}$, then there exists a rainbow Hamiltonian cycle.

There have been several papers studying other rainbow structures in graph systems. For example, a well-known conjecture of Aharoni and Berger [1] asserts that if $M_1, \ldots, M_n$ are n matchings of size at least $n+1$ on the same vertex set $V = X \cup Y$ where X and Y are disjoint and all edges of M_i are between X and Y, then there exists a rainbow matching of size n. This conjecture generalizes the famous Brualdi-Stein Conjecture, which asserts that every $n \times n$ Latin square has a partial transversal of size $n-1$. The Aharoni-Berger Conjecture has been confirmed asymptotically by Pokrovskiy [12]. For more details about this topic, see [14].

In this paper, we prove an asymptotic version of Conjecture 1:

Theorem 2. *For every $\varepsilon > 0$, there exists an integer $N > 0$, such that when $n > N$ for any graphs $G_1, \ldots, G_n$ on the same vertex set of size n, each graph with minimum degree $\delta(G_i) \geqslant (\frac{1}{2} + \varepsilon)n$, there exists a rainbow Hamiltonian cycle.*

After we submitted this paper, Joos and Kim in [10] proved Conjecture 1 using a method different from ours. Nevertheless, we believe that our approach is of independent interest and could be applied to attack similar problems in hypergraphs.

Furthermore, we show that given n graphs $G_1, \ldots, G_n$ with $\delta(G_i) \geqslant \frac{n+1}{2}$ for $1 \leqslant i \leqslant n$, we can find rainbow cycles with all possible lengths except a Hamiltonian one:

Theorem 3. *Given graphs $G_1, \ldots, G_n$ on the same vertex set of size n, each graph with minimum degree $\delta(G_i) \geqslant \frac{n+1}{2}$, there exist rainbow cycles of length $3, 4, \ldots, n-1$.*

Combining Theorem 3 with the result of Joos and Kim, we derive that any $G_1, \ldots, G_n$ satisfying the assumption of Theorem 3 indeed contain rainbow cycles of all possible lengths $3, \ldots, n$. The lower bound of Theorem 3 is tight because one can take n copies of $K_{\frac{n}{2},\frac{n}{2}}$ where n is even and there does not exist any odd rainbow cycle in such a system.

The main tool behind the proof of Theorem 2 is the absorbing method that was introduced by Rödl, Ruciński and Szemerédi [13]. Here we apply a rainbow version of the approach of Lo [11] by constructing a short rainbow cycle C such that for any rainbow path $P = v_1 \cdots v_p$ disjoint from C and a new colour s where the colour set of P is also disjoint from that of C, we can absorb P into C. In other words, we replace some edge u_iu_{i+1} of C by a path u_iPu_{i+1}, where u_iu_1 is coloured with s and v_pu_{i+1} is coloured with the colour of u_iu_{i+1} in C. Finally, we find a rainbow Hamiltonian path P on $V(G)\backslash V(C)$ and absorb P into C by the property of C and thus obtain a rainbow Hamiltonian cycle.

2 Preliminaries and Notation

Let $G_1, \ldots, G_n$ be n graphs on the same vertex set V where $|V| = n$. Let $\delta(G_i)$ be the minimum degree of each G_i for $1 \leqslant i \leqslant n$. By our assumption, we identify this graph system with an edge-coloured multigraph G where $E(G)$ is the disjoint union of $E(G_i)$ for $i \in [n]$ and each edge in $E(G_i)$ is coloured by i. For any subgraph H of G, let $\mathrm{Col}(H)$ be the set of colours used by the edges of H. For every vertex $v \in V(G)$ and any colour $c \in [n]$, let $N_c(v)$ be the set of neighbours of v that are adjacent to v by an edge coloured by c. Let S be any subset of V, we denote $N_c(v) \cap S$ by $N_c(v, S)$ and $|N_c(v) \cap S|$ by $d_c(v, S)$. For each pair of vertices $v_1, v_2 \in V(G)$, let $\mathrm{Col}(v_1, v_2)$ be the set of colours used for the edges between v_1 and v_2 ($\mathrm{Col}(v_1, v_2)$ is empty if there are no edges between v_1 and v_2). We will use the following version of Chernoff's bound [9].

Lemma 4. *Let X be a binomially distributed random variable and $0 < \varepsilon < \frac{3}{2}$, then*

$$P(|X - E(X)| \geqslant \varepsilon E(X)) \leqslant 2e^{-\frac{\varepsilon^2}{3}E(X)}.$$

We first prove the following useful lemma:

Lemma 5. *Let $P = v_1 \cdots v_p$ be a rainbow path and let c, c' be two colours not used on P. If $d_c(v_1, V(P)) + d_{c'}(v_p, V(P)) \geqslant p$, then there is a rainbow cycle of length p.*

Proof. If $\{c, c'\} \cap \mathrm{Col}(v_1, v_p) \neq \emptyset$, then $C = v_1 \cdots v_pv_1$ is a rainbow cycle by choosing the colour of v_1v_p to be an element in $\{c, c'\} \cap \mathrm{Col}(v_1, v_p)$. So we assume that $\{c, c'\} \cap \mathrm{Col}(v_1, v_p) = \emptyset$. Suppose that there exists no rainbow cycle of length p. For each vertex v_i with $v_{i+1} \in N_c(v_1, V(P))$ where $2 \leqslant i \leqslant p-2$, we get that $v_i \notin N_{c'}(v_p, V(P))$ since otherwise the cycle $v_1v_2 \cdots v_iv_pv_{p-1} \cdots v_{i+1}v_1$ must be a rainbow cycle where the colours of v_1v_{i+1} and v_pv_i are chosen to be c and c'. Thus we get $d_c(v_1, V(P)) - 1 + d_{c'}(v_p, V(P)) \leqslant p - 2$, which implies $p \leqslant p - 1$, a contradiction. □

Our first result shows that a rainbow Hamiltonian path exists under a slightly weaker condition than that of Conjecture 1:

Proposition 6. *Given graphs $G_1, \ldots, G_n$ on the same vertex set V of size n, where $\delta(G_i) \geqslant \frac{n-1}{2}$ for $i \in [n]$, then there exists a rainbow Hamiltonian path.*

Proof. Suppose not, let $P = v_1 \cdots v_k$, where $k \leqslant n-1$, be a rainbow path with the maximum length. Thus there exist at least two colors c, c' that are not used by the edges in P. Now consider the neighbourhood $N_c(v_1)$ and $N_{c'}(v_k)$, we have

$$d_c(v_1) + d_{c'}(v_k) \geqslant \frac{n-1}{2} + \frac{n-1}{2} = n - 1.$$

For each vertex $u \in V - V(P)$, we have $u \notin N_c(v_1) \cup N_{c'}(v_k)$ and otherwise we can extend P into a larger rainbow path, a contradiction. Thus we get $N_c(v_1), N_{c'}(v_k) \subseteq V(P)$. However, since $|V(P)| \leqslant n-1$ and $d_c(v_1, V(P)) + d_{c'}(v_k, V(P)) \geqslant n-1 \geqslant |V(P)|$, by Lemma 5 we get a rainbow cycle C of length k. Suppose that the colour c'' is not used by this cycle. Since the monochromatic graph coloured by c'' is connected, at least one edge e_0 coloured by c'' is between $V(C)$ and $V - V(C)$. Therefore, $V(C) \cup \{e_0\}$ contains a rainbow path with length $k+1$, a contradiction. $\square$

The lower bound here is best possible. One can take n copies of $K_{\frac{n}{2}-1,\frac{n}{2}+1}$ where n is even and there does not exist a rainbow Hamiltonian path since $K_{\frac{n}{2}-1,\frac{n}{2}+1}$ does not contain a Hamiltonian path.

3 Proof of Theorem 3

In this section, we give a proof of Theorem 3. Let $G = (V, \bigcup_{i=1}^{n} E(G_i))$ be the edge-colored multigraph with G_i as the graph of color i. We first find a rainbow cycle of length $n-1$ by following a classical proof of Dirac theorem. Then we obtain a rainbow cycle of length $n-2$ or $n-3$ and use it to build cycles of other lengths.

Claim 7. *G contains a rainbow cycle of length $n-1$.*

Proof. By Proposition 6, we first find a rainbow Hamiltonian path $P = v_1 v_2 \cdots v_n$. Without loss of generality, suppose the colour of edge $v_i v_{i+1}$ is i for $1 \leqslant i \leqslant n-1$ and the only colour that does not appear in P is n. Now consider the subpath $P' = v_1 v_2 \cdots v_{n-1}$. Since $|N_{n-1}(v_1) \backslash \{v_n\}| \geqslant \frac{n-1}{2}$ and $|N_n(v_{n-1}) \backslash \{v_n\}| \geqslant \frac{n-1}{2}$, we get $d_{n-1}(v_1, V(P')) + d_n(v_n, V(P')) \geqslant n-1 = |V(P')|$. By Lemma 5, we can find a rainbow cycle of length $n-1$. $\square$

Claim 8. *G contains either a rainbow cycle of length $n-2$ or a rainbow cycle of length $n-3$.*

Proof. Suppose that G neither contains a cycle of size $n-3$ nor $n-2$. By Proposition 6, we can find a rainbow path $P_1 = v_1 v_2 \cdots v_{n-3}$ whose order is $n-3$ and, without loss of generality, suppose the colour of edge $v_i v_{i+1}$ is i for $1 \leqslant i \leqslant n-4$ and the set of colours that are not used in P_1 is $S = \{n-3, n-2, n-1, n\}$. We can deduce that $N_{n-1}(v_1) \cap N_n(v_{n-3}) \cap (V(G) \setminus V(P_1)) = \emptyset$ since otherwise we already find a rainbow cycle of length $n-2$, a contradiction. Now we get $d_{n-1}(v_1, V(P_1)) + d_n(v_{n-3}, V(P_1)) \geqslant n-2 \geqslant |V(P_1)|$. By Lemma 5, we can find a rainbow cycle of length $n-3$, a contradiction. $\square$

Let $C = v_1 \cdots v_p$ be a rainbow cycle where $p = n-2$ or $n-3$. In the remainder of the proof, we let $v_i = v_{i-p}$ for $i > p$. We will use the following claim as a tool to analyse the structure of G when it does not contain rainbow cycles of all length $3, \ldots, p+1$.

Claim 9. *Let c, c' be two colours not used on C and $x \in V \setminus V(C)$. If $d_c(x, V(C)) + d_{c'}(x, V(C)) \geqslant p$, then one of the following properties is true:*

(1) there exist rainbow cycles of length $3, \ldots, p+1$*;*

(2) $d_c(x, V(C)) + d_{c'}(x, V(C)) = p$ *and we can partition* $V(C)$ *into disjoint sets* S_1 *and* S_2*, where* $S_1 = \{v_{i+j-2} : v_j \in N_c(x, V(C))\}$ *and* $S_2 = N_{c'}(x, V(C))$ *for some* $3 \leqslant i \leqslant p+1$.

Proof. Suppose that $d_c(x, V(C)) + d_{c'}(x, V(C)) \geqslant p$ and there is no rainbow cycle of length i for some $3 \leqslant i \leqslant p+1$, thus for each vertex $v_j \in N_c(x, V(C))$ we have $v_{i+j-2} \notin N_{c'}(x, V(C))$ since otherwise the cycle $xv_jv_{j+1}\cdots v_{j+i-2}x$ is a rainbow cycle of length i by choosing the colours of xv_j and xv_{j+i-2} to be c and c'. Therefore, we get $S_1 \cap S_2 = \emptyset$ by definition. However, since $|S_1| + |S_2| \geqslant p$ and $S_1 \cup S_2 \subseteq V(C)$ it follows that $V(C)$ is partitioned into S_1 and S_2 and $|S_1| + |S_2| = p$, which implies $d_c(x, V(C)) + d_{c'}(x, V(C)) = p$. □

Case 1. $p = n-2$.

Suppose $V(G) \backslash V(C) = \{v', v''\}$ and the colours not used by C are $n-1$ and n. Suppose that for some $3 \leqslant j \leqslant n-1$, there does not exist a rainbow cycle of size j in G. By Claim 9, we conclude that $d_{n-1}(v', V(C)) + d_n(v', V(C)) \leqslant n-2$, which implies that $d_{n-1}(v', v'') + d_n(v', v'') \geqslant n+1-(n-2) = 3$. This is a contradiction. Therefore, G contains rainbow cycles of all sizes $3, \ldots, n-1$.

Case 2. $p = n-3$.

Suppose $V(G) \backslash V(C) = \{v'_1, v'_2, v'_3\}$ and the colours not used by C are $n-2, n-1$ and n. Suppose that for some $3 \leqslant i \leqslant n-2$, there does not exist a rainbow cycle of size i in G. We know that $d_{n-1}(v'_3, V(C)) + d_n(v'_3, V(C)) \geqslant 2(\frac{n+1}{2} - 2) \geqslant n-3$, thus by Claim 9 we get $d_{n-1}(v'_3, V(C)) + d_n(v'_3, V(C)) = n-3$. This implies that $d_{n-1}(v'_3, \{v'_1, v'_2\}) = d_n(v'_3, \{v'_1, v'_2\}) = 2$ and hence $\{n-1, n\} \subseteq \mathrm{Col}(v'_3v'_1) \cap \mathrm{Col}(v'_3v'_2)$.

By symmetry we now suppose that $\mathrm{Col}(v'_av'_b) = \{n-2, n-1, n\}$ for every $1 \leqslant a < b \leqslant 3$. Let $T_1 = \{v_{j+i-3} \mid v_j \in N_{n-2}(v'_1, V(C))\}$ and $T_2 = N_{n-1}(v'_2, V(C))$, by an analogy to the proof of Claim 9, we find that $T_1 \cap T_2 = \emptyset$, otherwise suppose that $v_{j+i-3} \in T_2$ for some j, we thus have $v'_2v'_1v_jv_{j+1}\cdots v_{j+i-3}v'_2$ is a rainbow cycle with length i by choosing the colours of $v'_2v'_1$, v'_1v_j and $v_{j+i-3}v'_2$ to be n, $n-2$ and $n-1$, which is a contradiction. We actually get:

$$n-3 \leqslant \frac{n+1}{2} - 2 + \frac{n+1}{2} - 2 \leqslant d_{n-2}(v'_1, V(C)) + d_{n-1}(v'_2, V(C)) \leqslant |V(C)| = n-3,$$

thus all the inequalities above must be equalities and we get

$$|T_1| + d_{n-1}(v'_2, V(C)) = n-3,$$

which implies that $V(C)$ is partitioned into T_1 and T_2. Since all colours and vertices are symmetric, the similar conclusion follows by considering T_1 and $N_a(v'_b, V(C))$ for any $n-1 \leqslant a \leqslant n$ and $2 \leqslant b \leqslant 3$. Thus, we finally obtain $T_2 = N_a(v'_b, V(C))$ for any $n-1 \leqslant a \leqslant n$ and $2 \leqslant b \leqslant 3$. Now let $T'_1 = \{v_{j+i-3} \mid v_j \in N_n(v'_1, V(C))\}$ and recall that $T_2 = N_{n-1}(v'_2, V(C))$. Therefore, we reach the similar conclusion that $T_2 = N_a(v'_b, V(C))$ for any $n-2 \leqslant a \leqslant n-1$ and $2 \leqslant b \leqslant 3$ by considering T'_1 and T_2. This implies

$T_2 = N_a(v'_b, V(C))$ for any $n-2 \leqslant a \leqslant n$ and $2 \leqslant b \leqslant 3$. By symmetry, we actually get $T_2 = N_a(v'_b, V(C))$ for any $n-2 \leqslant a \leqslant n$ and $1 \leqslant b \leqslant 3$.

Now we claim that there exists some $v_{j_0} \in T_2$ such that $v_{j_0-1} \notin T_2$. Suppose not, for each $v_j \in T_2$ we have $v_{j-1} \in T_2$. This implies $T_2 = V(C)$ but this is impossible since $T_1 \neq \emptyset$. Now since $v_{j_0-1} \notin T_2$ we get $v_{j_0-1} \in T_1$ and there exists some $v_{j_1} \in N_{n-2}(v'_1, V(C))$ such that $j_0 - 1 = j_1 + i - 3$, which implies $j_0 = j_1 + i - 2$. However, in this case the cycle $v'_1 v_{j_1} v_{j_1+1} \cdots v_{j_0} v'_1$ is a rainbow cycle of length i by choosing the colours of $v'_1 v_{j_1}$ and $v'_1 v_{j_0}$ to be $n-2$ and $n-1$, a contradiction. This concludes the proof. $\square$

4 Proof of Theorem 2

In this section, we give a proof of Theorem 2 by proving a rainbow type of absorbing lemma. By $0 < \alpha \ll \beta$ we mean that there exists an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that the subsequent argument is valid for any $0 < \alpha \leqslant f(\beta)$. We first introduce the absorbing lemma for Theorem 2:

Lemma 10. *Let n, μ, ε be such that $\frac{1}{n} \ll \mu \ll \varepsilon$. For any graphs $G_1, \dots, G_n$ on the same vertex set of size n, each graph having minimum degree at least $(\frac{1}{2} + \varepsilon)n$, there exists a rainbow cycle C with length at most μn such that for every rainbow path P with $V(P) \cap V(C) = \emptyset$ and $Col(P) \cap Col(C) = \emptyset$, if s is a colour that is not used by C and P, then there exists a rainbow cycle C' with*

(i) $V(C') = V(C) \cup V(P)$;

(ii) $Col(P \cup C) \cup \{s\} = Col(C')$.

Combining Lemma 10 and Proposition 6, we immediately reach a proof of Theorem 2 as follows:

Proof of Theorem 2. Let C be the absorbing cycle we given in Lemma 10. Now for each i, let $G'_i = G_i - V(C)$, the subgraph of G_i induced on $V(G_i) \setminus V(C)$. Then $\delta(G'_i) \geqslant \frac{1+\varepsilon}{2}|V(G'_i)|$. Let W be the set of colours that do not appear on any edge of C. One can thus construct a rainbow Hamiltonian path $P_1 = v_0 v_1 \cdots v_t$ using exactly $|W| - 1$ colours of W by Proposition 6. Suppose s_1 is the unique colour in W that is not used by P_1. By Lemma 10 there exists a rainbow cycle C' with $V(C') = V(C) \cup V(P)$ and $\mathrm{Col}(P \cup C) \cup \{s_1\} = \mathrm{Col}(C')$, which implies C' is a rainbow Hamiltonian cycle and thus we conclude the proof. $\square$

In the remaining part of this section, we give a proof of Lemma 10. First we introduce some notation and basic results. For any pair of two not necessarily distinct vertices $x_1, x_2 \in V(G)$ and four distinct colours $1 \leqslant s, i, j, k \leqslant n$, we define the set of absorbing paths $A_{s,i,j,k}(x_1, x_2)$ to be the family of edge-coloured 3-paths P that satisfy the following conditions:

(i) $P = v_1 v_2 v_3 v_4$ where $\{v_1, v_2, v_3, v_4\} \cap \{x_1, x_2\} = \emptyset$;

(ii) the edges $v_1 v_2, v_2 v_3, v_3 v_4$ are coloured respectively by i, j, k;

(iii) $s \in \mathrm{Col}(x_1v_2)$ and $j \in \mathrm{Col}(x_2v_3)$.

For every path P in $A_{s,i,j,k}(x_1,x_2)$, we say that P is an *absorbing path* for (x_1,x_2) with colour pattern (s,i,j,k). In practice, we always choose x_1 and x_2 to be two endpoints of some path Q.

Claim 11. *For each pair (x_1,x_2) and four distinct colours s,i,j,k, we have $|A_{s,i,j,k}(x_1,x_2)| \geqslant \frac{\varepsilon n^4}{8}$ when n is sufficiently large.*

Proof. First, choose a vertex $v_2 \in N_s(x_1)\backslash\{x_2\}$. Pick another vertex $v_3 \in (N_j(v_2)\cap N_j(x_2))\backslash\{x_1\}$. The total number of such v_2, v_3 is at least $(\frac{n}{2}+\varepsilon n-1)(2\varepsilon n-1)$. Now fix v_2 and v_3. Choose $v_1 \in N_i(v_2)\backslash\{x_1,x_2,v_3\}$. Choose another vertex $v_4 \in N_k(v_3)\backslash\{x_1,x_2,v_1,v_2\}$. Note that the total number of such v_1, v_4 is at least $(\frac{n}{2}+\varepsilon n-3)(\frac{n}{2}+\varepsilon n-4)$ and hence we derive that there exist at least

$$\left(\frac{n}{2}+\varepsilon n-1\right)(2\varepsilon n-1)\left(\frac{n}{2}+\varepsilon n-3\right)\left(\frac{n}{2}+\varepsilon n-4\right) \geqslant \frac{\varepsilon n^4}{8}$$

absorbing paths for (x_1,x_2) when n is sufficiently large. □

Proof of Lemma 10. Let $\mu_1 = \mu/5$ and ℓ be new constant such that $\lceil \mu_1 n\rceil - 1 \leqslant \ell \leqslant \lceil \mu_1 n\rceil + 1$ and ℓ is divisible by 3. For simplicity, we assume that $\ell = \mu_1 n$. We fix $\ell/3$ groups of colours $C_i = \{3i-2, 3i-1, 3i\}$ where $i = 1,\dots,\ell/3$. Let $\mathcal{P}_{C_i}$ be the set of all the paths $P = v_0v_1v_2v_3$ in G where the colours of v_0v_1, v_1v_2, v_2v_3 are $3i-2, 3i-1, 3i$ for all $1 \leqslant i \leqslant \ell/3$.

Now consider a random set W by selecting an element from each $\mathcal{P}_{C_i} (i \in [\ell/3])$ independently where every element in $\mathcal{P}_{C_i}$ is chosen with probability $1/|\mathcal{P}_{C_i}|$. For any colour s and any pair (x_1,x_2), set $A_s(x_1,x_2) = \bigcup_{i=1}^{\ell/3}(A_{s,3i-2,3i-1,3i}(x_1,x_2)\cap W)$. Now for each $i \in [\ell/3]$, let the random variable X_i be the indicative variable of the event that $W \cap A_{s,3i-2,3i-1,3i}(x_1,x_2) \neq \emptyset$. Hence we get $|A_s(x_1,x_2)| = \sum_{i=1}^{\ell/3} X_i$ and all X_i's are independent. Using Claim 11, we get

$$P(X_i = 1) = |A_{s,3i-2,3i-1,3i}(x_1,x_2)|/|\mathcal{P}_{C_i}| \geqslant \frac{\varepsilon n^4}{8}/n^4 \geqslant \frac{\varepsilon}{8}$$

for $i \in [\ell/3]$ and hence

$$E(|A_s(x_1,x_2)|) = \sum_{i=1}^{\ell/3} E(X_i) \geqslant \frac{\varepsilon \ell}{24}.$$

By Lemma 4 with $\varepsilon = 1/2$, we see that

$$P\left(|A_s(x_1,x_2)| < \frac{\varepsilon\ell}{48}\right) \leqslant 2e^{-\frac{\varepsilon\ell}{288}} \leqslant 2e^{-\frac{\varepsilon\mu_1 n}{10^3}}.$$

Now let Y be the number of pairs of 3-paths in W that intersect with each other. For some distinct $1 \leqslant i,j \leqslant \ell/3$, let $Y_{i,j}$ be the indicative variable of the event that the path we choose in A_{C_i} intersects with the path we choose in A_{C_j}. Thus we have $Y = \sum_{i,j} Y_{i,j}$.

We claim that the size of set $\{\{P_1, P_2\} \mid P_1 \in A_{C_i}, P_2 \in A_{C_j}, P_1, P_2$ are intersecting with each other$\}$ is at most $16n^7$ for fixed i, j. Since the number of P_1 is at most n^4, and when P_1 is fixed, the number of P_2 that we can choose is at most $16n^3$. Besides, it is obvious that when P_1, P_2 are fixed, the probability that we have chosen P_1, P_2 together is $\frac{1}{|A(C_i)||A(C_j)|} \leqslant \frac{1}{(n^4/8)^2}$ because $|A_{C_i}|, |A_{C_j}| \geqslant \frac{n^4}{8}$ when n is sufficiently large. Therefore, we get

$$E(Y) \leqslant \binom{\ell/3}{2} \cdot 16 \cdot n^7 \cdot \frac{1}{(n^4/8)^2} \leqslant 10^3 \mu_1^2 n \leqslant \frac{\varepsilon \mu_1 n}{200}.$$

Using Markov's inequality, we get that

$$P\left(Y \geqslant \frac{\varepsilon \mu_1 n}{100}\right) \leqslant \frac{1}{2}.$$

Now choose sufficiently large n such that

$$2n^3 e^{-\frac{\varepsilon \mu_1 n}{10^3}} + \frac{1}{2} < 1.$$

Thus by the union bound, with positive possibility, for each s and any pair (x_1, x_2) we have (i) $|A_s(x_1, x_2)| \geqslant \frac{\varepsilon \ell}{48} \geqslant \frac{\varepsilon \mu_1 n}{48}$, and (ii) $Y < \frac{\varepsilon \mu_1 n}{100}$.

Fix such W, we delete one 3-paths in each intersecting pair of W. Suppose that the remaining path family is W'. Thus W' is a family containing mutually disjoint 3-paths and for every s and any pair (x_1, x_2) we get that

$$\left| \bigcup_{i=1}^{\ell/3} (A_{s,3i-2,3i-1,3i}(x_1, x_2) \cap W') \right| \geqslant \frac{\varepsilon \mu_1 n}{48} - \frac{\varepsilon \mu_1 n}{100} \geqslant \frac{\varepsilon \mu_1 n}{100}.$$

Let $W' = \{P_1, \ldots, P_t\}$ be the path family we found before and let S be the set of the colours that do not appear in any path in W'. Let $V' = V(G) \setminus \bigcup_{i=1}^t V(P_i)$. Without loss of generality, we suppose that $P_i = v_1^{(i)} v_2^{(i)} v_3^{(i)} v_4^{(i)}$ for $1 \leqslant i \leqslant t$. Now for P_1, P_2, it is obvious that we can find a vertex $u_1 \in V'$ such that $u_1 v_4^{(1)}$ and $u_1 v_1^{(2)}$ are two edges coloured with distinct colours in S. Delete these two colours from S and the vertex u_1 from V'. Repeat the above process for the path pair $\{P_2, P_3\}, \ldots, \{P_t, P_1\}$, and at last we find $u_1, \ldots, u_t$ and a rainbow cycle C with size at most $5\ell = \mu n$ that contains all the vertices in $\bigcup_{i=1}^t V(P_i)$ and those u_i where $1 \leqslant i \leqslant t$. For every rainbow path $P \subseteq V(G) - V(C)$ such that the colour set of P is disjoint with the colour set of C, if x_1, x_2 are two endpoints of P and s is a colour that does not appear in C and P, then the pair (x_1, x_2) has at least one absorbing path $P_0 = u_1 u_2 u_3 u_4$ in C with colour pattern $(s, 3i-1, 3i-2, 3i)$ for some $i \in [\ell/3]$ since $\frac{\varepsilon \mu_1 n}{100} \geqslant 1$ when n is sufficiently large. Therefore, we insert the path P into the cycle C to get a rainbow cycle $\{C - u_2 u_3\} \cup u_2 P u_3$ where $x_1 u_2$ is coloured by s and $x_2 u_3$ is coloured by $3i-2$, which completes our proof. □

Acknowledgements

The first two authors are supported by the National Natural Science Foundation of China (11631014, 11871311) and Shandong University multidisciplinary research and innovation team of young scholars. The third author is partially supported by NSF grant DMS 1700622. The authors thank referees for thorough comments that improve the presentation of this paper.

References

[1] Ron Aharoni and Eli Berger. Rainbow matchings in r-partite r-graphs. *The Electronic Journal of Combinatorics*, 16(1):#R119, 2009.

[2] Ron Aharoni, Eli Berger, Maria Chudnovsky, David Howard, and Paul Seymour. Large rainbow matchings in general graphs. *European Journal of Combinatorics*, 79:222–227, 2019.

[3] Ron Aharoni, Matt DeVos, Sebastián González Hermosillo de la Maza, Amanda Montejano, and Robert Šámal. A rainbow version of Mantel's theorem. *Advances in Combinatorics*, 2, 2020.

[4] Michael Albert, Alan Frieze, and Bruce Reed. Multicoloured Hamiltonian cycles. *The Electronic Journal of Combinatorics*, 2(1):#R10, 1995.

[5] János Barát, András Gyárfás, and Gábor N Sárközy. Rainbow matchings in bipartite multigraphs. *Periodica Mathematica Hungarica*, 74(1):108–111, 2017.

[6] Matthew Coulson and Guillem Perarnau. A rainbow Dirac's theorem. *SIAM Journal on Discrete Mathematics*, 34(3):1670–1692, 2020.

[7] Paul Erdős, Jaroslav Nešetřil, and Vojtěch Rödl. Some problems related to partitions of edges of a graph. *Graphs and Other Combinatorial Topics, Teubner, Leipzig*, 54-63, 1983.

[8] Geňa Hahn and Carsten Thomassen. Path and cycle sub-ramsey numbers and an edge-colouring conjecture. *Discrete Mathematics*, 62(1):29–33, 1986.

[9] Svante Janson, Tomasz Luczak, and Andrzej Ruciński. *Random Graphs*, volume 45. John Wiley & Sons, 2011.

[10] Felix Joos and Jaehoon Kim. On a rainbow version of Dirac's theorem. *Bulletin of the London Mathematical Society*, 52(3):498–504, 2020.

[11] Allan Lo. An edge-colored version of Dirac's theorem. *SIAM Journal on Discrete Mathematics*, 28(1):18–36, 2014.

[12] Alexey Pokrovskiy. An approximate version of a conjecture of Aharoni and Berger. *Advances in Mathematics*, 333:1197–1241, 2018.

[13] Vojtěch Rödl, Endre Szemerédi, and Andrzej Ruciński. An approximate Dirac-type theorem for k-uniform hypergraphs. *Combinatorica*, 28(2):229–260, 2008.

[14] Ian M. Wanless. Transversals in Latin squares: a survey. In *Surveys in Combinatorics 2011*, pages 403–437. Cambridge University Press, 2011.