On the P_3-hull number of Kneser graphs

Luciano N. Grippo[*†**] Adrián Pastine [‡**] Pablo Torres [†§**]
Mario Valencia-Pabon [¶**] Juan C. Vera [‖**]

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Abstract

This paper considers an infection spreading in a graph; a vertex gets infected if at least two of its neighbors are infected. The P_3-hull number is the minimum size of a vertex set that eventually infects the whole graph.

In the specific case of the Kneser graph $K(n, k)$, with $n \geqslant 2k + 1$, an infection spreading on the family of k-sets of an n-set is considered. A set is infected whenever two sets disjoint from it are infected. We compute the exact value of the P_3-hull number of $K(n, k)$ for $n > 2k + 1$. For $n = 2k + 1$, using graph homomorphisms from the Knesser graph to the Hypercube, we give lower and upper bounds.

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1 Introduction

We only consider finite, simple, and undirected graphs. For a graph $G = (V, E)$, a *graph convexity* on V is a collection $\mathcal{C}$ of subsets of V such that $\varnothing, V \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The sets in $\mathcal{C}$ are called *convex sets* and the *convex hull* $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ of a set S of vertices of G is the smallest set in $\mathcal{C}$ containing S (see [7] and references therein). Some natural convexities in graphs are defined by a set $\mathcal{P}$ of paths in G, in a way that a set S of vertices of G is convex if and only if for every path $P : v_0, v_1, \ldots, v_l \in \mathcal{P}$ such that v_0 and v_l belong to S, all vertices of P belong to S (cf. [1, 8]). If we define $\mathcal{P}$ as the set of all shortest paths in G, we have the well-known *geodetic convexity* (see for example

[*]Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina.
[†]Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.
[‡]Instituto de Matemática Aplicada San Luis, Universidad Nacional de San Luis, Argentina.
[§]Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Argentina.
[¶]LIPN, Université Sorbonne Paris Nord, France.
[‖]Tilburg School of Economics and Management, The Netherlands.
[**]Email addresses: lgrippo@ungs.edu.ar (L. N. Grippo), agpastine@unsl.edu.ar (A. Pastine), ptorres@fceia.unr.edu.ar (P. Torres), valencia@lipn.univ-paris13.fr (M. Valencia-Pabon), j.c.veralizcano@tilburguniversity.edu (J. C. Vera).

[17, 11, 25]). The *monophonic convexity* is defined by considering $\mathcal{P}$ as the set of all induced paths of G [18, 15].

If we let $\mathcal{P}$ be the set of all paths of G with three vertices, we have the well-known P_3-*convexity* which will be studied in this paper. This convexity was introduced with the aim of modeling the spread of a disease in a grid [5]. Since then, many articles, in connection with this convexity, were published in the specialized literature (the reader is referred for instance to [10, 9, 2, 16, 7]).

Given a set $S \subseteq V$, the P_3-*interval* $I[S]$ of S is formed by S, together with every vertex outside S with at least two neighbors in S. If $I[S] = S$, then the set S is P_3-*convex*. The P_3-*convex hull* $H_{\mathcal{C}}(S)$ of S is the smallest P_3-convex set containing S. In what follows, we write $H(S)$ instead of $H_{\mathcal{C}}(S)$. The P_3-convex hull $H(S)$ can be formed from the sequence $I^p[S]$, where p is a nonnegative integer, $I^0[S] = S$, $I^1[S] = I[S]$, and $I^p[S] = I[I^{p-1}[S]]$, for every $p \geqslant 2$. When for some $p \in \mathbb{N}$, we have $I^q[S] = I^p[S]$, for all $q \geqslant p$, then $I^p[S]$ is a P_3-convex set. If $H(S) = V(G)$ we say that S is a P_3-*hull set* of G. The cardinality $h_{P_3}(G)$ of a minimum P_3-hull set in G is called the P_3-*hull number* of G. Centeno et al. proved that, given a graph G and an integer k, to decide whether the P_3-hull number of G is at most k is an NP-complete problem [10]. Coelho et al. [14] proved that compute the P_3-hull number is an APX-hard problem even for bipartite graphs with maximum degree four. Moreover, Chen [12] shown that the P_3-hull number of a graph is hard to approximate within a ratio $O(2^{\log^{1-\epsilon} n})$, for any $\epsilon > 0$, unless NP $\subseteq$ DTIME$(n^{\text{polylog}(n)})$. All these negative results motivate the study of the P_3-hull number on particular families of graphs.

In this paper we deal with the problem of computing the P_3-hull number of Kneser graphs $\mathrm{K}(n, k)$. Kneser graphs have a very nice structure. For an overview on this relevant family of graphs we refer the reader to [20]. Many graph theoretic parameters have been computed for Kneser graphs $\mathrm{K}(n, k)$. Some examples are the independence number [19], the chromatic number [22], the diameter [28].

The aim of this work is twofold, first to contribute to the knowledge of Kneser graphs; second to obtain new formulas for the hull number within a family of graphs having nice structure.

This article is organized as follows. In Section 2 we present some preliminaries definitions and concepts. Section 3 is devoted to our results. Finally, we give some concluding remarks in Section 4.

Related work

Infection problems appear in the literature under many different names and were studied by researches of various fields [13]. An infection problem already studied on Kneser graphs is zero forcing (see [6]). The zero forcing problem follows the infection rule where an infected vertex v will infect one of its neighbors w if all neighbors of v except for w are already infected. The zero forcing number of G is the size of a smallest set S of initially infected vertices that forces the whole graph to become infected. Another infection problem is the *bootstrap percolation* on a graph (see for example, [4, 3, 23, 24, 26, 27] and references therein): an infection spreads over the vertices of a connected graph G following a deterministic spreading rule in such a way that an infected vertex will remain infected forever. Given a set $S \subseteq V(G)$ of initially infected vertices, we can build a sequence $S_0 = S, S_1, S_2, \ldots$ in which S_{i+1} is obtained from S_i using such a spreading rule. Under the r-neighbor bootstrap percolation on a graph G, the spreading rule is a threshold rule in which S_{i+1} is obtained from S_i by adding to it the vertices of G which have at least r neighbors in S_i. The set S_0 is a *percolating set* of G if there exists a t such that $S_t = V(G)$. Let $t_r(S)$ be the minimum t such that $S_t = V(G)$. The *percolation time* of G is defined as $t_r(G) = \max\{t_r(S) : S \text{ percolates } G\}$. Notice that this infection problem is related to graph convexities. In fact, the 2-neighbor bootstrap percolation problem on graphs is very close to the P_3-convexity on graphs. The 2-neighbor bootstrap percolation problem has been studied by several authors. For example, the maximum percolation time of the 2-neighbor bootstrap percolation problem has been studied by Benevides et al. [4], Marcilon et al. [23] and Przykucki [26]. The smallest or largest size of a percolating set with a given property has been studied by Benevides et al. [3] and Morris [24]. Moreover, Przykucki [26] and Riedl [27] studied some problems concerning the size of 2-percolating sets. Notice that the problem of finding a minimum size 2-percolating set on a graph is equivalent to determining the P_3-hull number of such graph. As we have mentioned previously, the problem of computing the P_3-hull number of a graph is a very hard problem, even for bipartite graphs. Therefore, it is interesting to find infinite graph families where such parameter can be easily determined in polynomial time.

2 Preliminaries

Given a graph G, $N_G(u)$ stands for the neighborhood of u in G. Let A and B be two sets. Given an integer a such that $0 \leqslant a \leqslant |A|$, $\binom{A}{a}$ stands for the set whose elements are the a-element subsets of A, and $\binom{A}{a}\binom{B}{b}$ the set whose elements are the subsets of $A \cup B$ with a elements in A and b elements in B. Notice that $\binom{A}{0} = \{\varnothing\}$, $\binom{A}{0}\binom{B}{b} = \binom{B}{b}$, and $\binom{A}{a}\binom{B}{0} = \binom{A}{a}$.

Let n be a positive integer. We denote by $[n]$ the set $\{1, \cdots, n\}$. For positive integers n and k such that $n \geqslant 2k$, the *Kneser graph*, denoted $\mathrm{K}(n,k)$, has as vertex set $\binom{[n]}{k}$ and two vertices are adjacent if they have empty intersection.

We introduce two more graphs in order to study the P_3-hull number of the Kneser graph $\mathrm{K}(2k+1,k)$, the *n-cube* and *middle levels graph.* For any $n \in \mathbb{Z}^+$, the *n-dimensional hypercube* (or n-cube), denoted Q_n, is the graph in which the vertices are all binary n-tuples of length n (i.e., the set $\{0,1\}^n$), and two vertices are adjacent if and only if they differ in exactly one position. For any $i \in \{0, \ldots, n\}$ we denote by Q_n^i the ith-layer of Q_n, that is, the subgraph of Q_n induced by all the vertices having exactly i ones.

The middle levels graph M_{2k+1} is the graph whose vertices are all k-element and all $(k+1)$-element subsets of $\{1, 2, \ldots, 2k+1\}$, with an edge between any pair of sets where one is a proper subset of the other. The name middle levels graph for M_{2k+1} comes from the fact that it is isomorphic to the subgraph of the hypercube Q_{2k+1} induced by all the vertices in the middle two layers Q_{2k+1}^k and Q_{2k+1}^{k+1}. It is not difficult to see that M_{2k+1} is a bipartite connected graph of order $2\binom{2k+1}{k}$. Johnson and Kierstead [21] provide a natural 2-to-1 graph homomorphism ϕ from M_{2k+1} to $\mathrm{K}(2k+1,k)$ defined by:

$$\phi(X) = \begin{cases} X, & \text{if } |X| = k; \\ \{1,\cdots,2k+1\} \setminus X, & \text{if } |X| = k+1. \end{cases}$$

3 Hull number of Kneser graphs

Let $k \geqslant 1$ and $n \geqslant 2k+1$. For $i \in \{0,\ldots,k\}$, let $\mathcal{F}_i = \binom{[k+1]}{i}\binom{[n]\setminus[k+1]}{k-i}$. Then $\{\mathcal{F}_i : i = 0,\ldots,k\}$ is a partition of the vertex set of $\mathrm{K}(n,k)$.

Lemma 1. *Let $k \geqslant 1$ and $n \geqslant 2k+1$. Let $i,j \leqslant k$ be such that $i \leqslant j+1 \leqslant i+n-2k$ and $(i,j) \notin \{(1,0),(3k+1-n,k)\}$. Then, $\mathcal{F}_i \subseteq I[\mathcal{F}_{k-j}]$.*

Proof. Let $0 \leqslant i,j \leqslant k$. Any vertex in $\mathcal{F}_i$ has exactly $d_{i,j} := \binom{k+1-i}{k-j}\binom{n-2k+i-1}{j}$ neighbors in $\mathcal{F}_{k-j}$. Thus $\mathcal{F}_i \subseteq I[\mathcal{F}_{k-j}]$ if and only if $d_{i,j} \geqslant 2$. As $d_{i,j} \geqslant 0$ we analyze when it is equal to zero or one. Notice that $d_{i,j} = 0$ if and only if $j+1 < i$ or $j+1 > i+n-2k$. Also, $d_{i,j} = 1$ if and only if $\binom{k+1-i}{k-j} = 1$ and $\binom{n-2k+i-1}{j} = 1$. That is when $j = k$ or $j+1 = i$, and $j = 0$ or $j+1 = i+n-2k$. □

Lemma 2. *Let $k \geqslant 1$ and $n \geqslant 2k+1$. Then $\mathcal{F}_1$ is a hull set of* $\mathrm{K}(n,k)$.

Proof. First we show by induction that $\mathcal{F}_t \cup \mathcal{F}_{k-t} \subset H(\mathcal{F}_1)$ for $t = 1,\ldots,\lfloor k/2 \rfloor$. To do this notice that taking $i = j = k-1$ in Lemma 1 we obtain the base case $t = 1$. Now assume the statement is true for $t \geqslant 1$. Taking $i = t+1$ and $j = t$ in Lemma 1 we obtain $\mathcal{F}_{t+1} \subset I[\mathcal{F}_{k-t}] \subset H(\mathcal{F}_1)$. Also, taking $i = j = k-t-1$ we obtain $\mathcal{F}_{k-t-1} \subset I[\mathcal{F}_{t+1}]$ completing the induction. To finish the proof, notice that taking $i = k$ and $j = k-1$ in Lemma 1 we obtain $\mathcal{F}_k \subset I[\mathcal{F}_1]$ and taking $i = j = 0$ in Lemma 1 we obtain $\mathcal{F}_0 \subset I[\mathcal{F}_k]$. □

Theorem 3. *Let $k \geqslant 1$ and $n \geqslant 2k+3$. Then $h_{P_3}(\mathrm{K}(n,k)) = 2$.*

Proof. Let $A_1 = [k]$ and $A_2 = [k+1] \setminus \{k\}$ and define $\mathcal{S} = \{A_1, A_2\}$. We will show that $\{A_1, A_2\}$ is a P_3-hull set of $\mathrm{K}(n,k)$.

Notice that A_1 and A_2 are neighbors of all the vertices in $\mathcal{F}_0$. Hence $\mathcal{F}_0 \subset H(\mathcal{S})$. Taking $i = j = k$ in Lemma 1 we obtain $\mathcal{F}_k \subset I[\mathcal{F}_0] \subset H(\mathcal{S})$. Taking $i = k-1$ and $j = k$ in Lemma 1 we obtain $\mathcal{F}_{k-1} \subset I[\mathcal{F}_0] \subset H(\mathcal{S})$. Also, taking $i = j = 1$ we obtain $\mathcal{F}_1 \subset I[\mathcal{F}_{k-1}] \subset H(\mathcal{S})$. The statement follows by Lemma 2. □

Theorem 4. $h_{P_3}(\mathrm{K}(2k+2,k)) = 3$, *for every $k \geqslant 3$.*

Proof. First, we will prove that $h_{P_3}(\mathrm{K}(2k+2,k)) > 2$. Let $\mathcal{S} = \{S_1, S_2\} \subseteq \mathrm{K}(2k+2,k)$ and let $A = S_1 \cup S_2$. We split the proof into the only two possible cases for $|S_1 \cap S_2|$.

Case 1: $|S_1 \cap S_2| = k-1$.

Since $|A| = k+1$, each vertex in $\binom{\overline{A}}{k}$ is adjacent to S_1 and S_2 and thus $\binom{\overline{A}}{k} \subseteq I[\mathcal{S}]$. Symmetrically, since $|\overline{A}| = k+1$ and $\binom{\overline{A}}{k} \subseteq I[\mathcal{S}]$, we conclude that $\binom{A}{k} \subseteq I^2[\mathcal{S}]$. Let C be any vertex in $\mathrm{K}(2k+2,k)$. Since $|C| = k \geqslant 3$, either $|C \cap A| \geqslant 2$ or $|C \cap \overline{A}| \geqslant 2$. Assume, without loosing generality, that $|C \cap A| \geqslant 2$. If $C \notin \binom{A}{k} \cup \binom{\overline{A}}{k}$, then $|C \cap \overline{A}| \geqslant 1$. Hence C has no neighbors in $\binom{A}{k}$ and it has at most one neighbor in $\binom{\overline{A}}{k}$ which implies that $C \notin H\left(\binom{A}{k} \cup \binom{\overline{A}}{k}\right)$. Therefore, $H(\mathcal{S}) = I^2[\mathcal{S}] = \binom{A}{k} \cup \binom{\overline{A}}{k}$.

Case 2: $|S_1 \cap S_2| = k-2$.

Let $A = S_1 \cup S_2$. Hence, $|A| = k+2$ and $|\overline{A}| = k$. Thus, $\binom{\overline{A}}{k} = \{\overline{A}\}$ and $I[\mathcal{S}] = \{\overline{A}, S_1, S_2\}$. In addition, for each $i \in \{1,2\}$, $C \cap S_i \neq \varnothing$ for every $C \in \binom{A}{k}$. Therefore, $H(\mathcal{S}) = \{S_1, S_2, \overline{A}\}$.

Since in both cases $H(\{S_1, S_2\})$ is properly contained in $\mathrm{K}(2k+2,k)$, we conclude that $h_{P_3}(\mathrm{K}(2k+2,k)) \geqslant 3$.

To show $h_{P_3}(\mathrm{K}(2k+2,k)) \leqslant 3$, let $\mathcal{S} = \{A_1, A_2, A_3\}$, where $A_1 = [k]$, $A_2 = [k+1] \setminus \{k\}$ and $A_3 = \{3, \dots, k+2\}$. We will prove that $\mathcal{S}$ is a hull set of $\mathrm{K}(2k+2,k)$. As in the proof of Theorem 3, A_1 and A_2 are neighbors of all the vertices in $\mathcal{F}_0$ and hence $\mathcal{F}_0 \subset I^1(\{A_1, A_2\})$. Taking $i = j = k$ in Lemma 1 we obtain $\mathcal{F}_k \subseteq H[\{A_1, A_2\}]$. We have, then, $\{A_1, A_2\} \subset \mathcal{F}_0 \cup \mathcal{F}_k \subseteq H[\{A_1, A_2\}]$. It is not difficult to see that $\mathcal{F}_0 \cup \mathcal{F}_k = H[\{A_1, A_2\}]$, as $I[\mathcal{F}_0 \cup \mathcal{F}_k] = \mathcal{F}_0 \cup \mathcal{F}_k$. Indeed, for any $B \in \mathcal{F}_j$ with $0 < j < k$ we have B connected to $\mathcal{F}_0 \cup \mathcal{F}_k$ only when $j = 1$ or $j = k-1$. If $j = 1$, B has exactly one neighbor in $\mathcal{F}_k$ and none in $\mathcal{F}_0$. Similarly, when $j = k-1$, B has exactly one neighbor in $\mathcal{F}_0$ and none in $\mathcal{F}_k$.

Let $\mathcal{S}_1 = \binom{\{1,2\}}{1}\binom{[2k+2]\setminus[k+2]}{k-1}$. We have $\mathcal{S}_1 = N(A_3) \cap \mathcal{F}_1$. As $\mathcal{F}_1 \subset N(H(\{A_1, A_2\}))$ and $A_3 \notin H(\{A_1, A_2\})$ we have $\mathcal{S}_1 \subset H(\mathcal{S})$. Now let $\mathcal{S}_2 = \{A \in \mathcal{F}_{k-1} : A \cap \{1,2\} = 1\}$. Every element in $\mathcal{S}_2$ has a neighbor in $\mathcal{S}_1$; to see this, let a be the only element in $A \cap \{1,2\}$ and let b be the only element in $A \setminus [k+1]$. Then, there exists $Y \subseteq \{a, k+3, \cdots, 2k+2\} \setminus \{b\}$ with $Y \in N(A) \cap \mathcal{S}_1$. Also, $\mathcal{S}_2 \subset \mathcal{F}_{k-1} \subset N(\mathcal{F}_0)$. As $\mathcal{F}_0 \cap \mathcal{S}_1 \subset \mathcal{F}_0 \cap \mathcal{F}_1 = \varnothing$, we have $\mathcal{S}_2 \subset H(\mathcal{S})$.

Now we claim that $\mathcal{F}_1 \subset N(\mathcal{S}_2)$. This implies that $\mathcal{F}_1 \subset H(\mathcal{S})$, as $\mathcal{S}_2 \cap \mathcal{F}_k \subset \mathcal{F}_{k-1} \cap \mathcal{F}_k = \varnothing$ and $\mathcal{F}_1 \subset N(\mathcal{F}_k)$. Notice that by Lemma 2 we obtain $H(\mathcal{S}) = \mathrm{K}(2k+1,k)$. To show $\mathcal{F}_1 \subset N(\mathcal{S}_2)$, let $A \in \mathcal{F}_1$ and let $c \in A \cap [k+1]$. If $c \in \{1,2\}$, let a be the integer in $\{1,2\} \setminus \{c\}$ and let X be any $(k-2)$-set in $\binom{[k+1]\setminus\{1,2\}}{k-2}$. Otherwise, let $a = 1$ and let X be the only $(k-2)$-set in $[k+1] \setminus \{1,2,c\}$. Let $b \notin A \setminus [k+1]$. Then, $Y = \{a\} \cup X \cup \{b\}$ is a vertex in $\mathcal{S}_2$ having A as a neighbor. Therefore, $Y \in N(A) \cap \mathcal{S}_2$ and so, $A \in N(\mathcal{S}_2)$. □

Remark 5. $h_{P_3}(\mathrm{K}(6,2)) = 2$.

Proof. Let $A = \{1,2,3\}$. Let $S_i = A \setminus \{i\}$ for each $i \in \{1,2\}$. Since $C \cap S_i = \varnothing$, for every $C \in \binom{\overline{A}}{2}$ and for each $i \in \{1,2\}$, $\binom{\overline{A}}{2} \subseteq H(\{S_1, S_2\})$. Hence $\binom{A}{2} \subseteq H(\{S_1, S_2\})$. If $C \notin \binom{A}{2} \cup \binom{\overline{A}}{2}$, then $|C \cap A| = |C \cap \overline{A}| = 1$. Hence C is adjacent to $A \setminus C \in \binom{A}{2}$ and $\overline{A} \setminus C \in \binom{\overline{A}}{2}$ and thus $C \in H(\{S_1, S_2\})$. Therefore, $\{S_1, S_2\}$ is a hull set of $\mathrm{K}(6,2)$. □

Theorem 6. $h_{P_3}(\mathrm{K}(2k+1,k)) \leqslant k^2+k$.

Proof. From Lemma 2, we have $h_{P_3}(\mathrm{K}(2k+1,k)) \leqslant |\,\mathcal{F}_1\,| = k^2+k$. $\square$

3.1 Preservation of P_3 convexity under homomorphisms and its inverses

Let $G=(V,E)$ a graph. For any vertex $u \in V$, let $N_G(u)$ denote the subset of neighbor vertices of u in G, that is, the set $\{v \in V : uv \in E\}$. Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be graphs. A graph *homomorphism* between graphs G_1 and G_2, denoted by $\phi : G_1 \to G_2$, is a mapping ϕ from $V(G_1)$ to $V(G_2)$ such that $\phi(u)$ and $\phi(v)$ are adjacent in G_2 whenever u and v are adjacent in G_1. A graph homomorphism $\phi : G_1 \to G_2$ is called *locally bijective* if for all $u \in G_1$ the restriction of ϕ to $N_{G_1}(u)$ is a bijection between $N_{G_1}(u)$ and $N_{G_2}(\phi(u))$.

Lemma 7. *Let $\phi : G_1 \to G_2$ be a locally bijective graph homomorphism. Let $S \subseteq G_2$. Then, $\phi^{-1}(H(S)) \subseteq H(\phi^{-1}(S))$.*

Proof. We prove by induction on i that $\phi^{-1}(I^i[S]) \subseteq I^i[\phi^{-1}(S)]$ for all $i \geqslant 0$. In the base case $i=0$ we actually have equality. Now assume the statement is true for $i>0$. Let $u \in \phi^{-1}[I^{i+1}(S)]$, that is $\phi(u) \in I^{i+1}(S)$. If $\phi(u) \in I^i[S]$, then by induction $u \in I^i[\phi^{-1}(S)] \subseteq I^{i+1}[\phi^{-1}(S)]$. Thus, assume $\phi(u) \notin I^i[S]$. Then there are two neighbors v and w of $\phi(u)$ in $I^i(S)$. By assumption, $\phi^{-1}(v) \cup \phi^{-1}(w) \subseteq I^i[\phi^{-1}(S)]$. As ϕ is locally bijective, $N_{G_1}(u) \cap \phi^{-1}(v) = \{v'\}$ and $N_{G_1}(u) \cap \phi^{-1}(w) = \{w'\}$, for some $v', w' \in H$. As $v', w' \in I^i[\phi^{-1}(S)]$ we have $u \in I^{i+1}[\phi^{-1}(S)]$. $\square$

Lemma 8. *Let $\phi : G_1 \to G_2$ be a locally bijective graph homomorphism. Let $S \subset G_1$. Then, $\phi(H(S)) \subseteq H(\phi(S))$.*

Proof. We prove by induction on i that $\phi(I^i[S]) \subseteq I^i[\phi(S)]$ for all $i \geqslant 0$. In the base case $i=0$ we actually have equality. Now assume the statement is true for $i>0$. Let $u \in I^{i+1}(S)$, we want to show $\phi(u) \in I^{i+1}[\phi(S)]$. If $u \in I^i[S]$, then by induction $\phi(u) \in I^i[\phi(S)] \subseteq I^{i+1}[\phi(S)]$. Thus, assume $u \notin I^i[S]$. Then $|N_{G_1}(u) \cap I^i[S]| \geqslant 2$. As ϕ is locally bijective, $|N_{G_2}(\phi(u)) \cap \phi(I^i[S])| \geqslant 2$ also, and thus $u \in I^{i+1}[S]$. $\square$

Theorem 9. *Let $\phi : G_1 \to G_2$ be a surjective, locally bijective graph homomorphism. Then $h_{P_3}(G_2) \leqslant h_{P_3}(G_1) \leqslant \max\{|\phi^{-1}(u)| : u \in G_2\} h_{P_3}(G_2)$.*

Proof. Let S_1 be hull set for G_1. From Lemma 8 we obtain $H(\phi(S_1)) \supseteq \phi(H(S_1)) = \phi(G_1) = G_2$. Thus $h_{P_3}(G_1) = |S_1| \geqslant |\phi(S_1)| \geqslant h_{P_3}(G_2)$. Let S_2 be hull set for G_2. From Lemma 7, $H(\phi^{-1}(S_2)) \supseteq \phi^{-1}(H(S_2)) = \phi^{-1}(G_2) = G_1$. Thus $h_{P_3}(G_1) \leqslant |\phi^{-1}(S_2)| \leqslant \max\{|\phi^{-1}(u)| : u \in G_2\}|S_2| = \max\{|\phi^{-1}(u)| : u \in G_2\} h_{P_3}(G_2)$. $\square$

Corollary 10. *Let $k \geqslant 1$ be an integer. Then, $h_{P_3}(\mathrm{K}(2k+1,k)) \leqslant h_{P_3}(M_{2k+1}) \leqslant 2h_{P_3}(\mathrm{K}(2k+1,k))$.*

Proof. The result follows from Theorem 9 by noticing that the 2-to-1 graph homomorphism from M_{2k+1} to $\mathrm{K}(2k+1,k)$ defined at the end of Section 2 is surjective and locally bijective. $\square$

3.2 Lower bound of $h_{P_3}(\mathrm{K}(2k+1,k))$

In order to deduce a lower bound for $h_{P_3}(\mathrm{K}(2k+1,k))$, we need the following preliminary results.

Lemma 11. *Let $n > 1$ and $1 \leqslant i \leqslant n-1$ be integers. Let S be the set of vertices in the ith-layer Q_n^i of the hypercube Q_n. Then, S is a P_3-hull set of Q_n.*

Proof. Let $x = (x_1, \cdots, x_n)$ be any vertex in Q_n^{i-1}. Clearly, there exist two coordinates x_p, x_q in x, with $1 \leqslant p < q \leqslant n$, such that $x_p = x_q = 0$. The vertices $y = (x_1, \cdots, x_{p-1}, 1, x_{p+1}, \cdots, x_n)$ and $z = (x_1, \cdots, x_{q-1}, 1, x_{q+1}, \cdots, x_n)$ are vertices in S adjacent to x. In the same way, for any vertex w in Q_n^{i+1} we can pick two different coordinates w_p and w_q such that $w_p = w_q = 1$. Then we can find two vertices u and v in S adjacent to w, where u (resp. v) is equal to w except in the pth (resp. qth) coordinate which is equal to 0. Thus, w has at least two neighbors in S. As this property holds for any $1 \leqslant i \leqslant n-1$ then, we conclude that S is a P_3-hull set of the hypercube Q_n. □

Concerning with the P_3-hull number of the n-dimensional hypercube Q_n, the following result has been obtained recently by Brešar and Valencia-Pabon [7].

Theorem 12 ([7])**.** *For any $n \geqslant 1$, $h_{P_3}(Q_n) = \lceil \frac{n}{2} \rceil + 1$.*

Lemma 13. *Let $k \geqslant 1$ be an integer. Then, $h_{P_3}(M_{2k+1}) \geqslant k+2$.*

Proof. Let S be a P_3-hull set of M_{2k+1}. For any vertex $w \in S$ let $\tilde{w}$ be a vertex in the hypercube Q_{2k+1} such that $\tilde{w}_j = 1$ if $j \in w$, and $\tilde{w}_j = 0$ otherwise, for $1 \leqslant j \leqslant 2k+1$. As M_{2k+1} is isomorphic to the subgraph of Q_{2k+1} induced by the vertices in the two middle layers Q_{2k+1}^k and Q_{2k+1}^{k+1} then, by Lemma 11, the set $S' = \{\tilde{w} : w \in S\}$ is a P_3-hull set of Q_{2k+1}. Therefore, by Theorem 12, $|S'| \geqslant \lceil \frac{2k+1}{2} \rceil + 1 = k+2$. □

Finally, by Lemma 13 and Corollary 10, we have the following theorem.

Theorem 14. *Let $k \geqslant 1$ be an integer. Then, $h_{P_3}(K(2k+1,k)) \geqslant \lceil \frac{k}{2} \rceil + 1$.*

4 Discussion

Corollary 10 gives an upper bound for the P_3-hull number of M_{2k+1} in terms of the P_3-hull number of $\mathrm{K}(2k+1,k)$. Exact values for $h_{P_3}(\mathrm{K}(2k+1,k))$ and $h_{P_3}(M_{2k+1})$, calculated with the aid of a computer, are shown in Table 1.

So we have the following conjecture.

Conjecture 15. $\left\lceil \frac{h_{P_3}(M_{2k+1})}{2} \right\rceil = h_{P_3}(\mathrm{K}(2k+1,k))$, for any integer $k \geqslant 1$.

The lower bound for the P_3-hull number of $\mathrm{K}(2k+1,k)$ obtained in Theorem 14 seems to be far from being tight. In addition to results given in Table 1, we also have computational evidence showing that $h_{P_3}(\mathrm{K}(2k+1,k))$ is at most equal to 11, 16 and 23 for $k = 5, 6$ and 7, respectively. Notice that $h_{P_3}(\mathrm{K}(2k+1,k))$ seems to be equal to $\frac{k(k-1)}{2} + c$, being c a constant, with $c \leqslant 2$. So we have the following conjecture.

Conjecture 16. $h_{P_3}(\mathrm{K}(2k+1,k)) = \Theta(k^2)$.

k	$h_{P_3}(\mathrm{K}(2k+1,k))$	$h_{P_3}(M_{2k+1})$
1	2	3
2	3	6
3	5	9
4	8	$\leqslant 15$

Table 1: Some exact results.

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