Turán Density of 2-edge-colored Bipartite Graphs with Application on $\{2, 3\}$-Hypergraphs

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Abstract

We consider the Turán problems of 2-edge-colored graphs. A 2-edge-colored graph $H = (V, E_r, E_b)$ is a triple consisting of the vertex set V, the set of red edges E_r and the set of blue edges E_b where E_r and E_b do not have to be disjoint. The Turán density $\pi(H)$ of H is defined to be $\lim_{n\to\infty} \max_{G_n} h_n(G_n)$, where G_n is chosen among all possible 2-edge-colored graphs on n vertices containing no H as a subgraph and $h_n(G_n) = (|E_r(G)| + |E_b(G)|)/\binom{n}{2}$ is the formula to measure the edge density of G_n. We will determine the Turán densities of all 2-edge-colored bipartite graphs. We also give an important application on the Turán problems of $\{2, 3\}$-hypergraphs.

Mathematics Subject Classifications: 5D05, 05C65, 05D40

1 Introduction

Given a graph H, the Turán problem asks for the maximum possible number of edges (denoted as $ex(n, H)$) in a graph G on n vertices without a copy of H as a subgraph. The Mantel's theorem [13] states that any graph on n vertices with no triangle contains at most $\lfloor n^2/4 \rfloor$ edges. Turán [16] proved that the maximal number of edges in a k-clique free graph on n vertices is at most $(k-2)n^2/(2k-2)$. The famed Erdős-Stone-Simonovits Theorem [7, 8] proved that the Turán density of any graph H is $\pi(H) = 1 - \frac{1}{\mathcal{X}(H)-1}$, where $\mathcal{X}(H)$ is the chromatic number of H. For hypergraphs the extremal problems are harder, see Keevash [12] for a complete survey of some results and methods on uniform hypergraphs. Although Turán type problems for graphs and hypergraphs have been actively studied

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for decades, there are only few results on non-uniform hypergraphs, see [14, 15, 10] for related work. Motivated by the study of non-uniform Turán problems [3], in this paper we study a Turán-type problem on edge-colored graphs and show an application on Turán problems of non-uniform hypergraphs of edge size 2 or 3.

A hypergraph $H = (V, E)$ consists of a vertex set V and an edge set $E \subseteq 2^V$. An r-uniform hypergraph is a hypergraph such that all its hyperedges have size r. Given positive integers $k \geqslant r \geqslant 2$, and a set of colors C, with $|C| = k$, a k-edge-colored r-uniform hypergraph H (for short, k-colored r-graph) is an r-uniform hypergraph that allows k different colors on each hyperedge. We express H as $H = (V, E_1, E_2, \dots, E_k)$ where E_i denotes the set of hyperedges colored by ith color in C, note $E_1, E_2, \dots, E_k$ do not have to be disjoint. We say H' is a subgraph of H, denoted by $H' \subseteq H$, if $V(H') \subseteq V(H)$, $E_i(H') \subseteq E_i(H)$ for every i. Given a family of k-colored r-graphs $\mathcal{H}$, we say G is *$\mathcal{H}$-free* if it doesn't contain any member of $\mathcal{H}$ as a subgraph. To measure the edge density of G of size n, we use $h_n(G)$, which is defined by

$$h_n(G) := \sum_{i=1}^{k} \frac{|E_i(G)|}{\binom{n}{r}},$$

where $n = |V(G)|$. Then we define the Turán density of $\mathcal{H}$ as

$$\pi(\mathcal{H}) := \lim_{n\to\infty} \pi_n(\mathcal{H}) = \lim_{n\to\infty} \max_{G_n} h_n(G_n),$$

where the maximum is taken over all $\mathcal{H}$-free k-colored r-graphs G_n on n vertices.

By a simple average argument of Katona-Nemetz-Simonovits [11], this limit always exists.

Theorem 1. *For any fixed family $\mathcal{H}$ of k-colored r-graphs, $\pi(\mathcal{H})$ is well-defined, i.e. $\lim_{n\to\infty} \pi_n(\mathcal{H})$ exists.*

When $\mathcal{H} = \{H\}$, we simply write $\pi(\{H\})$ as $\pi(H)$. Note that $\pi(\mathcal{H})$ agrees with the definition of

$$\pi(\mathcal{H}) = \frac{ex(n, \mathcal{H})}{\binom{n}{r}},$$

where $ex(n, \mathcal{H})$ is the maximum number of hyperedges in an n-vertex $\mathcal{H}$-free k-colored r-graph.

In this paper, we let $k = 2$. A 2-edge-colored graph is a simple graph (without loops) where each edge is colored either red or blue, or both. We call an edge a double-colored edge if it is colored with both colors. For short, we call the 2-edge-colored graphs simply as 2-colored graphs. A 2-colored graph H can be written as a triple $H = (V, E_r, E_b)$ where V is the vertex set, $E_r \subseteq \binom{V}{2}$ is the set of red edges and $E_b \subseteq \binom{V}{2}$ is the set of blue edges. Denote $|E_r|$ and $|E_b|$ as the size of each set, denote H_r, H_b as the induced subgraphs of H generated by all the red edges and all the blue edges respectively. A graph can be considered as a special 2-colored graph with only one color. We say H is *proper* if there exists at least one edge in each class E_r and E_b. Throughout the paper, we consider the proper 2-colored graphs. The results in this paper were finished in year 2018 and recently we noticed that our study is similar but different to a Turán problem on edge-colored graphs defined by Diwan and Mubayi [4] in which the authors ask for the minimum m, such that the 2-colored graph G, if both its red and blue edges are at least $m+1$, contains a given 2-colored graph F? What we do differently in this paper is the study of the Turán density defined above for 2-colored graphs.

It is easy to see that $\pi(H) \geqslant 1$ for any proper 2-colored graph H, since we can take a complete graph with all edges a single color that does not contain a copy of H.

Definition 2. A 2-colored graph H is called bipartite if H does not contain an odd cycle of length $l \geqslant 3$ with all edges colored by the same color.

For a 2-colored graph H, we say H is *degenerate* if $\pi(H) = 1$. Note that if H is degenerate, then it must be bipartite. Otherwise, say $H_b = (V, E_b)$ is not a bipartite graph, one may consider the union of the red complete graph and an extremal graph respect to H_b, then the resulting graph is a H-free 2-colored graph with edge density at least $1 + \pi(H_b) > 1$, a contradiction.

In this paper, we will determine the Turán densities of all 2-colored bipartite graphs and characterize the 2-colored graphs achieving these Turán values. The notation $[n]$ is the set of $\{1, \dots, n\}$. For convenience, we represent an edge $\{a, b\}$ by ab.

Definition 3. Given two k-colored r-graphs G and H, a *graph homomporhism* is a map $f\colon V(G) \to V(H)$ which keeps the colored edges, that is, $f(e) \in E_i(H)$ whenever $e \in E_i(G)$ for $i \in [k]$. We say G is H*-colorable* if there is a graph homomorphism from G to H.

Theorem 4. *The Turán densities of all bipartite* 2*-colored graphs are in the set* $\left\{1, \frac{4}{3}, \frac{3}{2}\right\}$.

1. *A* 2*-colored graph H is degenerate if and only if it is T-colorable, where T is the* 2*-colored graph with vertices* $[4]$ *and red edges* $\{12, 13, 34\}$, *blue edges* $\{12, 23, 34\}$.

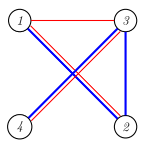
2. *A* 2*-colored graph H satisfies $\pi(H) = \frac{4}{3}$, then H must be H_8-colorable but not T-colorable, where H_8 is the* 2*-colored graph with vertices* $[8]$, *red edges are*

$$E_r(H_8) = \{12, 13, 24, 34, 16, 37, 48, 25, 35, 18, 46, 27\},$$

blue edges are

$$E_b(H_8) = \{56, 57, 68, 78, 26, 15, 47, 38, 35, 18, 46, 27\}.$$

3. *A* 2*-colored bipartite graph H satisfies $\pi(H) = \frac{3}{2}$, then H is not H_8-colorable.*



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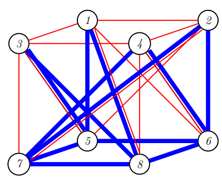


H_8

Our consideration on 2-colored graphs is motivated by the study of Turán density of non-uniform hypergraphs, which was first introduced by Johnston and Lu [10], then studied by us [3]. We refer a non-uniform hypergraph H as R-graph, where R is the set of all the cardinalities of edges in H. For example, H is a hypergraph on vertices $\{1, 2, 3, 4\}$ with edges $\{1\}, \{2, 3\}$ and $\{1, 2, 4\}$, then the edge type of H is $R(H) = \{1, 2, 3\}$ as the cardinalities of all edges are $1, 2, 3$. Given a hypergraph H with edge type $R(H)$, the Turán density of H is defined as:

$$\pi(H) = \lim_{n\to\infty} \max\{ \sum_{e\in E(G)} \frac{1}{\binom{n}{|e|}} \},$$

where the maximum is taken over all H-free hypergraphs G on n vertices satisfying $R(G_n) \subseteq R(H)$.

A *degenerate* R-graph H has the smallest Turán density, $|R| - 1$, where $|R|$ is the size of set R. For a history of degenerate extremal graph problems, see [9]. Let $r \geqslant 3$, for r-uniform hypergraphs the r-partite hypergraphs are degenerate and they generalize the bipartite graphs. An interesting problem is what the degenerate non-uniform hypergraph look like? In [3], we prove that except for the case $R \neq \{1, 2\}$, there always exist non-trivial degenerate R-graphs for any set R of two distinct positive integers. The degenerate $\{1, 3\}$-graphs are characterized in [3], what about the the degenerate $\{2, 3\}$-graphs? In the last section of this paper, we will apply the 2-colored graphs to bound the Turán density of some $\{2, 3\}$-graphs.

The paper is organized as follows: in Section 2, we show some lemmas on the k-colored r-uniform hypergraphs; in Section 3, we classify the Turán densities of all 2-colored bipartite graphs; in Section 4, we give an application of the Turán density of 2-colored graphs on $\{2, 3\}$-graphs.

2 Lemmas on k-colored r-graphs

2.1 Supersaturation and Blowing-up

In this section, we give some definitions and lemmas related to the k-colored r-graphs for $k \geqslant r \geqslant 2$. These are natural generalizations from the Turán theory of graphs. We first define the *blow-up* of a k-colored r-graph.

Definition 1 (*Blow-up Families)*. For any k-colored r-graph H on n vertices and positive integers $s_1, s_2, \dots, s_n$, the *blow-up* of H is a new k-colored r-graph, denoted by $H(s_1, s_2, \dots, s_n) = (V,\ E_1, \dots,\ E_k)$, satisfying

- $V := \bigsqcup_{i=1}^{n} V_i$, where $|V_i| = s_i$,
- $E_j = \bigcup_{F\in E_j(H)} \prod_{i\in F} V_i$, for each $j \in [k]$.

When $s_1 = s_2 = \cdots = s_n = s$, we simply write it as $H(s)$.

Lemma 5 (Supersaturation)**.** *For any k-colored r-graph H and $a > 0$, then there are $b, n_0 > 0$ so that if G is a k-colored r-graph on $n > n_0$ vertices with $h_n(G) > \pi(H) + a$ then G contains at least $b\binom{n}{v(H)}$ copies of H.*

Proof. Since we have $\lim_{n\to\infty} \pi_n(H) = \pi(H)$, there exists an $n_0 > 0$ so that if $t > n_0$ then $\pi_t(H) < \pi(H) + \frac{a}{r}$. Suppose $n > t$, and G is a k-colored r-graph on n vertices with $h_n(G) > \pi(H) + a$. Let T represent any t-set, then G must contain at least $\frac{a}{2}\binom{n}{t}$ t-sets $T \subseteq V(G)$ satisfying $h_t(G[T]) > (\pi(H) + \frac{a}{2})$. Otherwise, we would have

$$\begin{aligned}\sum_T h_t(G[T]) &\leqslant \binom{n}{t}(\pi(H) + \frac{a}{2}) + \frac{a}{2}\binom{n}{t} \\ &= (\pi(H) + a)\binom{n}{t}.\end{aligned}$$

But we also have

$$\begin{aligned}\binom{t}{r}\sum_T h_t(G[T]) &= \binom{n-r}{t-r}\binom{n}{r}h_n(G) \\ &> \binom{n-r}{t-r}\binom{n}{r}(\pi(H) + a) \\ &= (\pi(H) + a)\binom{t}{r}\binom{n}{t}.\end{aligned}$$

A contradiction. Since $t > n_0$, it follows that each of the $\frac{a}{2}\binom{n}{t}$ t-sets $T \subseteq V(G)$ satisfying $h_t(G[T]) > (\pi(H) + \frac{a}{r})$ contains a copy of H, so the number of copies of H in G is at least $\frac{a}{2}\binom{n}{t}/\binom{n-v(H)}{t-v(H)} = \frac{a}{2}\binom{n}{v(H)}/\binom{t}{v(H)}$. Let $b = \frac{a}{2}/\binom{t}{v(H)}$, the result follows. ☐

The 'blow-up' does not change the Turán density of k-colored r-graphs. The following result and proof are natural generalization of results on uniform hypergraphs, see [12].

Lemma 6. *For any $s > 1$ and any k-colored r-graph H, $\pi(H(s)) = \pi(H)$.*

Proof. First, since any H-free r-graph G is also $H(s)$-free, we have $\pi(H) \leqslant \pi(H(s))$. We will show that for any $a > 0$, $\pi(H(s)) < \pi(H) + a$.

By the supersaturation lemma, for any $a > 0$, there are $b, n_0 > 0$ so that if G is a k-colored r-graph on $n > n_0$ vertices with $h_n(G) > \pi(H) + a$ then G contains at least $b\binom{n}{v(H)}$ copies of H. Consider an auxiliary $v(H)$-graph U on the same vertex set as G such that the edges of U correspond to copies of H in G. Note that U contains at least $b\binom{n}{v(H)}$ edges. For any $S > 0$, if n is large enough we can find a copy K of $K_{v(H)}^{v(H)}(S)$ in U. Note that K is the complete $v(H)$-partite v(H)-graph with S vertices in each part, then $\pi(K) = 0$. Fix one such K in U. Color each edge of K with one of the $v(H)!$ colors corresponding to the possible orderings with which the vertices of H are mapped into the parts of K. By Ramsey theory, one of the color classes contains at least $S^v/v!$ edges. For large enough S (such that $S^v/v! \geqslant s$) it follows that U contains a monochromatic copy of $K_{v(H)}^{v(H)}(s)$, which gives a copy of $H(s)$ in G. Thus $\pi(H(s)) < \pi(H) + a$. ☐

Note when we say G is H-colorable, it is equivalent to say G is a subgraph of a blow-up of H. It is easy to prove the following lemmas.

Lemma 7. *Let $\mathcal{H}$ be a family of k-colored r-graphs. If G is H-colorable for any $H \in \mathcal{H}$, then $\pi(G) \leqslant \pi(\mathcal{H})$.*

Definition 2. Given two k-colored r-graphs G_1 and G_2 with vertices set V_1 and V_2, we define the product of G_1 and G_2, denoted by $G_1 \times G_2 = (V_1 \times V_2,\ E_1, \dots, E_k)$, where for any $i \in [k]$,

$$E_i = E_i(G_1) \times E_i(G_2) = \{e \times f \mid e \in E_i(G_1), f \in E_i(G_2)\},$$

where $e \times f$ is defined through the following way: denote $e = \{v_1, \dots, v_r\} \in E_i(G_1)$, $f = \{u_1, \dots, u_r\} \in E_i(G_2)$, then $e \times f = \cup_{\sigma \in S_r}\{(v_1, u_{\sigma(1)}), \dots, (v_r, u_{\sigma(r)})\}$, where $\sigma = (\sigma(1), \cdots, \sigma(r))$ takes over all permutations of $[r]$.

Lemma 8. *A k-colored r-graph G is G_1 and G_2 colorable, then it's $(G_1 \times G_2)$-colorable.*

Proof. There exist two graph homomorphisms $f_1 : V(G) \mapsto V(G_1)$ and $f_2 : V(G) \mapsto V(G_2)$ such that for any edge $e = \{v_1, \dots, v_r\} \in E(G)$, without loss of generality, let $e \in E_1(G)$, we have

$$f_1(e) = \{f_1(v_1), \dots, f_1(v_r)\} \in E_1(G_1),$$

and

$$f_2(e) = \{f_2(v_1), \dots, f_2(v_r)\} \in E_1(G_2).$$

Define a map $f := f_1 \times f_2$ from $V(G)$ to $V(G_1) \times V(G_2)$, such that $f(v) = (f_1(v), f_2(v))$ for any $v \in V(G)$. Then we have

$$f(e) = \{(f_1(v_1), f_2(v_1)), \dots, (f_1(v_r), f_2(v_r))\} \in f_1(e) \times f_2(e) \subseteq E_1(G_1 \times G_2).$$

Thus the map f is a graph homomorphism. Hence G is $(G_1 \times G_2)$-colorable. ☐

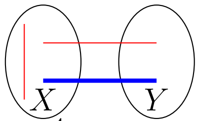
2.2 Construction of 2-colored graphs

To compute the lower bound of $\pi(H)$, we need to construct a family of H-free 2-colored graphs G_n with $h_n(G_n)$ as large as possible. Here are three useful constructions.

$\boldsymbol{G_A}$**:** A 2-colored graph G_A on n vertices is generated by partitioning the vertex set into two parts such that $V(G_A) = X \cup Y$ and the red edges either meet two vertices in X or meet one vertex in X plus the other in Y, the blue edges meet one vertex in X plus the other in Y. In other words, the red edges $E_r(G_A) = \{\binom{X}{2}\} \cup \{\binom{X}{1} \times \binom{Y}{1}\}$ and blue edges $E_b(G_A) = \{\binom{X}{1} \times \binom{Y}{1}\}$. Let $|V(G_A)| = n$, $|X| = xn$ and $|Y| = (1-x)n$ for some real number $x \in (0, 1)$. We have

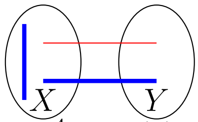
$$\begin{aligned} h_n(G_A) &= \frac{\binom{|X|}{2} + 2\binom{|X|}{1}\binom{|Y|}{1}}{\binom{n}{2}} \\ &= 4x - 3x^2 + o_n(1), \end{aligned}$$

which reaches the maximum $\frac{4}{3}$ at $x = \frac{2}{3}$.



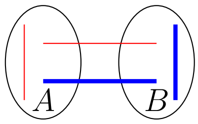
G_A: $h_n(G_A) = \frac{4}{3} + o_n(1)$ at $|X| = \frac{2}{3}n$.

G_B: It is obtained from G_A by simply exchanging red edges with blue edges. In other words, the red edges $E_r(G_B) = \{\binom{X}{1} \times \binom{Y}{1}\}$ and blue edges $E_b(G_B) = \{\binom{X}{2}\} \cup \{\binom{X}{1} \times \binom{Y}{1}\}$.



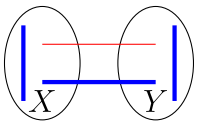
G_B: $h_n(G_B) = \frac{4}{3} + o_n(1)$ at $|X| = \frac{2}{3}n$.

G_C: A 2-colored graph G_C on n vertices is generated by partitioning the vertex set into two parts such that $V(G_C) = A \cup B$ and the red edges either meet two vertices in A or meet one vertex in A plus the other in B, the blue edges either meet two vertices in B or meet one vertex in A plus the other in B. In other words, the red edges $E_r(G_C) = \{\binom{A}{2}\} \cup \{\binom{A}{1} \times \binom{B}{1}\}$ and blue edges $E_b(G_C) = \{\binom{A}{1} \times \binom{B}{1}\} \cup \{\binom{B}{2}\}$.

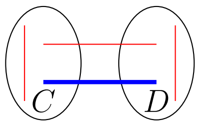


G_C: $h_n(G_C) = \frac{3}{2} + o_n(1)$ at $|A| = \frac{1}{2}n$.

G_D and G_E: Two variations of G_C are the following constructions:



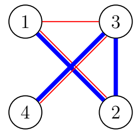
G_D: $h_n(G_D) = \frac{3}{2} + o_n(1)$.



G_E: $h_n(G_E) = \frac{3}{2} + o_n(1)$.

Following a similar description of above constructions, the red/blue edges of G_D are in the sets $E_r(G_D) = \{\binom{X}{1} \times \binom{Y}{1}\}$ and $E_b(G_D) = \{\binom{V(G_D)}{2}\} \setminus E_r(G_D)$ respectively; the blue/red edges of G_E are in the sets $E_b(G_E) = \{\binom{C}{1} \times \binom{D}{1}\}$ and $E_r(G_E) = \{\binom{V(G_E)}{2}\} \setminus E_b(G_E)$ respectively.

Example 1. The product of G_A and G_B is a blow-up of T, where $V(T) = [4]$, the red edges $\{12, 13, 34\}$ and the blue edges $\{12, 23, 34\}$:



T

We define a map $f : V(H) \to \{1, 2, 3, 4\}$ as follows:

1. If v appears in X of G_A and in Y of G_B, set $f(v) = 1$.
2. If v appears in Y of G_A and in X of G_B, set $f(v) = 2$.
3. If v appears in X of G_A and in X of G_B, set $f(v) = 3$.
4. If v appears in Y of G_A and in Y of G_B, set $f(v) = 4$.

One can check f is a graph homomorphism from the product $G_A \times G_B$ to T.

3 Turán density of bipartite 2-colored graphs

In this section, we will prove results in Theorem 4. We first give a boundary to divide the Turán densities of 2-colored non-bipartite graphs and 2-colored bipartite graphs.

Lemma 9.

1. *For any 2-colored non-bipartite graph H, $\pi(H) \geqslant \frac{3}{2}$.*
2. *For any 2-colored bipartite graph H, $\pi(H) \leqslant \frac{3}{2}$.*

Before proceeding to the proof, we see several important 2-colored graphs whose Turán density achieves value $\frac{3}{2}$, and we will use these results to prove Lemma 9. The following lemma will be used in the proof of Lemma 12 which is useful to prove item 2 of Lemma 9.

Lemma 10. *Let K_3 be a triangle with three double-colored edges, i.e.*

$$K_3 = ([3], \{12, 13, 23\}, \{12, 13, 23\}).$$

Then

$$ex(n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor.$$

In particular, $\pi(K_3) = \frac{3}{2}$.

Proof. Observe that K_3 is not contained in G_C, thus $\pi(K_3) \geqslant \frac{3}{2}$. Now we prove the other direction. Let n be a positive integer and G be any K_3-free 2-colored graph on n vertices. Construct an auxillary graph F on the same vertex set $V(G)$ and with the edge sets consisting of all double-colored edges in G. Let $H = E_r(F)$ consisting of all red colored edges of F. Notice that H is triangle-free. By Mantel's theorem, we have

$$|E(H)| \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Note that H is a subgraph of G and the number of the rest of edges in G is at most $\binom{n}{2}$. Therefore, we have

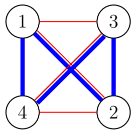
$$|E(G)| \leqslant \binom{n}{2} + |E(H)| \leqslant \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{3}{2} + o(1)\right)\binom{n}{2}.$$

This implies that $\pi(K_3) = \frac{3}{2}$. □

Corollary 11. *Let* $K_3^- = ([3], \{12, 13, 23\}, \{12, 13\})$*, then* $\pi(K_3^-) = \frac{3}{2}$.

Proof. Since K_3^- is a subgraph of K_3, then $\pi(K_3^-) \leqslant \frac{3}{2}$. By Lemma 9, $\pi(K_3^-) \geqslant \frac{3}{2}$. The result follows. □

Except the 2-colored non-bipartite graph, some bipartite graphs also achieves $\pi(H) = \frac{3}{2}$. See the following 2-colored graph on four vertices $\{1, 2, 3, 4\}$:



T_1

Lemma 12. $T_1 = ([4], \{12, 34, 13, 24\}, \{12, 34, 14, 23\}$. *Then*

$$ex(n, T_1) = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor \text{ for any } n \neq 3$$

and $ex(3, T_1) = 6$. *In particular, we have* $\pi(T_1) = \frac{3}{2}$.

Proof. When $n \leqslant 3$, the complete 2-colored graph does not contain T_1. Thus $ex(n, T_1) = 0, 0, 2, 6$ when $n = 0, 1, 2, 3$, respectively. The assertion holds for $n \leqslant 3$. It is sufficient to prove for $n \geqslant 4$. Since T_1 is not contained in G_C, we have

$$ex(n, T_1) \geqslant \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor.$$

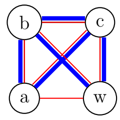
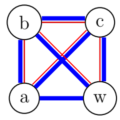
Now we prove the other direction by induction. We may assume $n \geqslant 4$. Let n be a positive integer and G be any T_1-free 2-colored graph on n vertices.

Note K_3 is referring to a triangle with 3 double colored edges.

Case 1: G doesn't contain K_3 as a subgraph, by Lemma 10, we have

$$|E(G)| \leqslant ex(n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Case 2: G contains a copy of K_3, let $V_1 = \{a, b, c\}$ be the vertices of this triangle and $V_2 = V(G) \setminus V_1$. Then there are at most 4 edges from any vertex in V_2 to V_1. To see this, suppose there are 5 edges from the vertex $w \in V_2$ to V_1, then there are only two possible graphs on $V_1 \cup \{w\}$ and each of them contains a copy of T_1. A contradiction.

Applying the inductive hypothesis to $G[V_2]$, we have

$$|E(G[V_2])| \leqslant \binom{n-3}{2} + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + \epsilon.$$

Here $\epsilon = 1$ if $n = 6$ and 0 otherwise.

Then the number of edges in G is:

if $n \neq 6$,

$$\begin{aligned}
|E(G)| &= |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)| \\
&\leqslant 6 + \binom{n-3}{2} + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 4(n-3) \\
&= \binom{n}{2} + n + \left\lfloor \frac{n^2 - 6n + 9}{4} \right\rfloor \\
&= \binom{n}{2} + \left\lfloor \frac{n^2 - 2n + 9}{4} \right\rfloor \\
&= \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor .
\end{aligned}$$

if $n = 6$,

$$\begin{aligned}
|E(G)| &= |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)| \\
&\leqslant 6 + \binom{n-3}{2} + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + \epsilon + 4(n-3) \\
&= 24 \\
&= \binom{6}{2} + \left\lfloor \frac{6^2}{4} \right\rfloor .
\end{aligned}$$

The induction step is finished. It follows that $h_n(G) \leqslant \frac{3}{2}$. Therefore, $\pi(T_1) = \frac{3}{2}$. □

Proof of Lemma 9. For Item 1, let H be a 2-colored non-bipartite graph, without loss of generality, assume H contains an odd cycle with red edges. For any n, let G be a 2-colored graph generated by construction G_D, then H can not be contained in G. Similarly, if H contains an odd cycle with blue edges, then it is not contained in any 2-colored graph generated by construction G_E. Thus $\pi(H) \geqslant \frac{3}{2}$.

For Item 2, it is sufficient to prove that any 2-colored bipartite graph H is T_1-colorable. For any 2-colored bipartite graph H, the subgraph H_r can be partitioned into two disjoint parts $V_1(H_r)$ and $V_2(H_r)$ such that the red edges form a bipartite graph between $V_1(H_r)$ and $V_2(H_r)$. Similarly for the subgraph H_b, the blue edges form a bipartite graph between

$V_1(H_b)$ and $V_2(H_b)$. Let S be the set of vertices incidents to double colored edges, then S can be divided into four classes: $V_1(H_r) \cap V_1(H_b), V_1(H_r) \cap V_2(H_b), V_2(H_r) \cap V_1(H_b)$ and $V_2(H_r) \cap V_2(H_b)$. We define a map $f : V(H) \to \{1, 2, 3, 4\}$ as follows:

1. If $v \in V_1(H_r) \cap V_1(H_b)$, set $f(v) = 1$.
2. If $v \in V_1(H_r) \cap V_2(H_b)$, set $f(v) = 4$.
3. If $v \in V_2(H_r) \cap V_1(H_b)$, set $f(v) = 3$.
4. If $v \in V_2(H_r) \cap V_2(H_b)$, set $f(v) = 2$.
5. If $uv \in E_r(H) \setminus E_b(H)$, set $f(u) = 1, f(v) = 2$.
6. If $uv \in E_b(H) \setminus E_r(H)$, set $f(u) = 3, f(v) = 4$.

One can verify that this map f is a graph homomorphism from H to T_1. By Lemma 12, we have $\pi(H) \leqslant \frac{3}{2}$. □

3.1 The degenerate 2-colored graphs

In this part, we will determine the degenerate 2-colored graphs. We will see that the 2-colored bipartite graph $T = ([4], \{12, 13, 34\}, \{12, 23, 34\})$ shown in Example 1 plays an important role.

Lemma 13. *Let n be a positive integer, for any T-free 2-colored graph G on n vertices, G has at most $\binom{n+1}{2}$ edges. Thus T is degenerate.*

Proof. We will prove this lemma by induction on n. It is trivial for $n = 1, 2, 3, 4$. Assume $n \geqslant 5$. We assume that the statement holds for any T-free 2-colored graphs on less than n vertices.

Let $G = (V, E_r, E_b)$ be a T-free 2-colored graph on n vertices. We also assume G contains at least one double-colored edge uv, or else $|E_r(G)| + |E_b(G)| \leqslant \binom{n}{2} < \binom{n+1}{2}$. Then G is one of the following cases.

Case 1: There exists a vertex w so that both uw and vw are double-colored edges. Since G is T-free, there is no double-colored edges from u, v, w to the rest of the vertices. By inductive hypothesis, when G is restricted to the complement set of $\{u, v, w\}$, the number of edges of $G[V \setminus \{u, v, w\}]$ is at most $\binom{n-2}{2}$ edges. Thus, G has at most

$$6 + 3(n-3) + \binom{n-2}{2} = \binom{n+1}{2}.$$

Case 2: Now we assume no such w exists. Let $X = \{x \in V : |E(\{x\}, \{u, v\})| \geqslant 3\}$. That is, for each vertex $x \in X$, x has exactly 3 edges connecting to u and v. Since G is T-free, for each $x \in X$, x has no double-colored edges to any vertex not in $\{u, v, x\}$. In particular, the induced subgraph $G[X]$ of G has no double-colored edge. Let

$V_1 = \{u, v\} \cup X$ and V_2 be the complement set. Then the induced subgraph $G[V_1]$ has at most

$$2 + 3|X| + \binom{|X|}{2} < \binom{|X|+3}{2} = \binom{|V_1|+1}{2}$$

edges. Applying the inductive hypothesis to $G[V_2]$, then $G[V_2]$ has at most $\binom{|V_2|+1}{2}$ edges. Note that all edges from X to V_2 are single colored and the number of edges from $\{u, v\}$ to each vertex in V_2 is at most 2. Thus the total number of edges from V_1 to V_2 is at most $|V_1||V_2|$ edges. Combining these facts together, we have G has at most N edges, where

$$N = \binom{|V_1|+1}{2} + |V_1||V_2| + \binom{|V_2|+1}{2} = \binom{|V|+1}{2}.$$

We finish the inductive step. Then we have

$$\pi(T) = \lim_{n\to\infty} \max_{G_n} h_n(G_n) \leqslant \lim_{n\to\infty} \frac{\binom{n+1}{2}}{\binom{n}{2}} = 1,$$

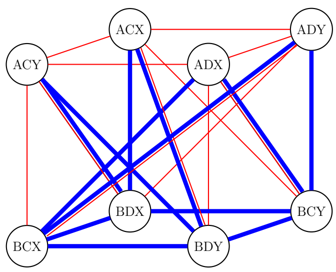
implying $\pi(T) = 1$. T is degenerate. □

Proof of Item 1 of Theorem 4. Assume H is a degenerate 2-colored graph, then it must be G_A and G_B-colorable. By Lemma 8, it must be $G_A \times G_B$-colorable. Note that the product of these two graphs is T-colorable. Thus H is T-colorable, see Example 1. By Lemma 13, the result follows. □

Remark 14. Note both $h_n(G_A)$ and $h_n(G_B)$ are equal to $\frac{4}{3} + o_n(1)$, then any 2-colored graph H with $\pi(H) < \frac{4}{3}$ is G_A and G_B-colorable, from above proof, H is then T-colorable, thus further implies $\pi(H) = 1$.

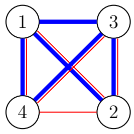
3.2 Non-degenerate 2-colored bipartite graphs

In this part, we will further classify the non-degenerate 2-colored bipartite graphs. By Lemma 9, the largest possible Turán density of a 2-colored bipartite graph H is $\frac{3}{2}$, so if $\pi(H) < \frac{3}{2}$, it must be contained in the construction G_c and its variations G_D, G_E, thus it must be colored by the product of these constructions. While the product of graphs generated by the three constructions is a blow-up of following graph H_8. Let ACX stand for the vertex in $A \times C \times X$, similar for other labels:



H_8

To compute the Turán density of H_8, we need the following 2-colored graph $T_2 = ([4], \{12, 14, 23, 24, 34\}, \{12, 13, 14, 23, 34\})$. T_2 is not contained in a variation of G_C, thus $\pi(T_2) \geqslant \frac{3}{2}$.



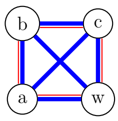
T_2

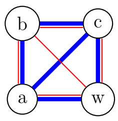
Lemma 15. *For any positive integer n, let G be a $\{T_1, T_2\}$-free 2-colored graph on n vertices. Then $|E(G)| \leqslant \binom{n}{2} + \left\lfloor \frac{n^2+3n}{6} \right\rfloor$. Thus $\pi(\{T_1, T_2\}) \leqslant \frac{4}{3}$.*

Proof. It is not hard to check the cases for $n \leqslant 3$. Let $n \geqslant 4$, by induction on n we assume the statement holds for any $\{T_1, T_2\}$-free graph on less than n vertices. Note if G contains no double-colored edge, the result is trivial. Thus we assume G contains at least one double-colored edge. Then G is one of the following cases.

Case 1: G contains a triangle consisting of three double-colored edges, let $V_1 = \{a, b, c\}$ be the vertices of this triangle and $V_2 = V(G) \setminus V_1$. By Lemma 12 "Case 2", for any vertex $w \in V_2$, there are at most 4 edges from w to V_1.

Case 2: G contains $V_1 = \{a, b, c\}$ such that $|E(G[V_1])| = 5$, without loss of generality, let ab, bc be double colored edges, and ac is blue colored edge. Let $V_2 = V(G) \setminus V_1$. For any vertex $w \in V_2$, there are at most 4 edges to V_1. If there are 5 edges from w to V_1, then the following graphs include all of the possibilities and they contain T_1, T_2 as subgraph respectively.





Case 3: G contains two incident double-colored edges ab and bc, but no edge connecting a and c. Let $V_1 = \{a, b, c\}$, $V_2 = V(G) \setminus V_1$. Then there cannot be 5 edges from any vertex $w \in V_2$ to V_1, otherwise, G is a graph either in Case 1 or in Case 2. Thus there are at most 4 edges from any vertex in V_2 to V_1.

Case 4: If G is not the above three cases, then for any double-colored edge connecting a and b, there are at most 2 edges from any other vertex to $\{a, b\}$.

Applying the inductive hypothesis to $G[V_2]$, we have

$$|E(G[V_2])| \leqslant \binom{|V_2|}{2} + \left\lfloor \frac{|V_2|^2 + 3|V_2|}{6} \right\rfloor.$$

Then the number of edges in G is: for the first three cases,

$$\begin{aligned}
|E(G)| &= |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)| \\
&\leqslant 6 + \binom{n-3}{2} + \left\lfloor \frac{(n-3)^2 + 3(n-3)}{6} \right\rfloor + 4(n-3) \\
&= \binom{n+1}{2} + \left\lfloor \frac{(n-3)^2 + 3(n-3)}{6} \right\rfloor \\
&= \binom{n}{2} + \left\lfloor \frac{n^2 - 6n + 9 + 3(n-3) + 6n}{6} \right\rfloor \\
&= \binom{n}{2} + \left\lfloor \frac{n^2 + 3n}{6} \right\rfloor,
\end{aligned}$$

for Case 4,

$$\begin{aligned}
|E(G)| &= |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)| \\
&\leqslant 2 + \binom{n-2}{2} + \left\lfloor \frac{(n-2)^2 + 3(n-2)}{6} \right\rfloor + 2(n-3) \\
&= \binom{n}{2} - 1 + \left\lfloor \frac{(n-2)^2 + 3(n-2)}{6} \right\rfloor \\
&= \binom{n}{2} + \left\lfloor \frac{(n-2)^2 + 3(n-2) - 6n}{6} \right\rfloor \\
&= \binom{n}{2} + \left\lfloor \frac{n^2 - 7n - 2}{6} \right\rfloor \\
&< \binom{n}{2} + \left\lfloor \frac{n^2 + 3n}{6} \right\rfloor.
\end{aligned}$$

The induction step is finished. It follows that $\pi(\{T_1, T_2\}) \leqslant \frac{4}{3}$. □

Lemma 16. $\pi(H_8) = \frac{4}{3}$.

Proof. We first prove $\pi(H_8) \leqslant \frac{4}{3}$. To show this, we prove that H_8 is T_1 and T_2-colorable, i.e there are graph homomorphisms from H_8 to T_1 and from H_8 to T_2.

For T_1: We define a map f by $f(ACX) = f(BCX) = 4, f(ADY) = f(BDY) = 3$, $f(ACY) = f(BCY) = 2, f(ADX) = f(BDX) = 1$. One can check that f is a graph homomorphism from H_8 to T_1.

For T_2: We define a map g by $g(ACX) = g(ADX) = 1$, $g(ADY) = g(ACY) = 3$, $g(BDX) = g(BDY) = 2, g(BCY) = g(BCX) = 4$. It is easy to check that g is a graph homomorphism from H_8 to T_2.

For any positive integer n, let G_n be a 2-colored graph on n vertices such that $h_n(G_n) \geqslant \pi(T_1, T_2) + \epsilon = \pi(T_1(s), T_2(s)) + \epsilon$, for any $s \geqslant 2$ and $\epsilon > 0$. Then G_n contains $T_1(s)$ or $T_2(s)$ as subgraph, further G_n contains H_8 as subgraph. Then $\pi(H_8) \leqslant \pi(\{T_1, T_2\})$. By Lemma 15, $\pi(H_8) \leqslant \frac{4}{3}$. By Remark 14, if $\pi(H_8) < \frac{4}{3}$, then $\pi(H_8) = 1$, while H_8 is not T-colorable, a contradiction. Thus it must be the case $\pi(H_8) = \frac{4}{3}$. □

Remark 17. As we know, if $\pi(H) < \frac{3}{2}$, it must be colorable by G_c and its variations, then it must be be colorable by H_8 according to Lemma 8. Thus $\pi(H) \in \{1, \frac{4}{3}\}$.

For convenience, we use numbers to represent vertices: $ACX = 1, ADY = 2, ACY = 3, ADX = 4, BDX = 5, BCY = 6, BCX = 7, BDY = 8$. Then H_8 has edges:

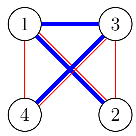
$$E_r(H_8) = \{12, 13, 24, 34, 16, 37, 48, 25, 35, 18, 46, 27\};$$

$$E_b(H_8) = \{56, 57, 68, 78, 26, 15, 47, 38, 35, 18, 46, 27\}.$$

Now we are ready to finish the proof of Theorem 4.

Proof of Items 2 and 3 in Theorem 4. By Remark 14, Remark 17 and Lemma 9, the Turán densities of all bipartite 2-colored graphs are in the set $\{1, \frac{4}{3}, \frac{3}{2}\}$. To show Item 2, let H be a 2-colored graph with $\pi(H) = \frac{4}{3}$, then H must be H_8-colorable. One can check if H does not contain T as a subgraph, then H must be T-colorable, implying $\pi(H) = 1$, a contradiction. By excluding the bipartite 2-colored graphs in Item 2, we obtain the result in Item 3. $\square$

Example 2. Let T_3 be the following 2-colored graph, T_3 is non-degenerate and $\pi(T_3) = \frac{4}{3}$.



T_3

4 The degenerate {2, 3}-graphs

In this section, we study degenerate $\{2,3\}$-graphs and show an application of the study of 2-edge-colored graphs on the Turán density of $\{2,3\}$-graphs. A $\{2,3\}$-graph is a non-uniform hypergraph where each edge consists of 2 or 3 vertices. Given a $\{2,3\}$-graph G, we call an edge of cardinality i as an i-edge, and use $E_i(G)$ to represent the set of i-edges. Thus G can be represented by $G = (V(G), E_2(G), E_3(G))$. A 2-edge e is called a double edge if $e \subset f$, for some 3-edge $f \in E_3(G)$. For convenience, we use the form of ac to denote the edge $\{a, b\}$ and use abc to denote the edge $\{a, b, c\}$. The notation $H_n^{\{2,3\}}$ represents a $\{2,3\}$-graph on n vertices, $K_n^{\{2,3\}}$ represents the complete hypergraph on n vertices with edge set $\binom{[n]}{2} \cup \binom{[n]}{3}$.

Given a family of $\{2,3\}$-graphs $\mathcal{H}$, the Turán density of $\mathcal{H}$ is defined to be:

$$\pi(\mathcal{H}) = \lim_{n\to\infty} \pi_n(\mathcal{H}) = \lim_{n\to\infty} \max \left\{ \frac{|E_2(G)|}{\binom{n}{2}} + \frac{|E_3(G)|}{\binom{n}{3}} \right\},$$

where the maximum is taken over all H-free hypergraphs G on n vertices satisfying $G \subseteq K_n^{\{2,3\}}$, and G is $\mathcal{H}$-free $\{2,3\}$-graph. Please refer to [3] for details on the Turán density of non-uniform hypergraphs.

Next let us see some definitions and results for $\{2,3\}$-graphs.

Definition 3. [10] Let H be a hypergraph containing some 2-edges. The 2-subdivision of H is a new hypergraph H' obtained from H by subdividing each 2-edge simultaneously. Namely, if H contains t 2-edges, add t new vertices $x_1, \ldots, x_t$ to H and for $i = 1, 2, \ldots, t$ and replace the 2-edge $\{u_i, v_i\}$ with $\{u_i, x_i\}$ and $\{x_i, v_i\}$.

Theorem 18. *[10] Let H' be the 2-subdivision of H. If H is degenerate, then so is H'.*

Definition 4. [10] The suspension of a hypergraph H, denoted by $S(H)$, is the hypergraph with $V = V(H) \cup \{v\}$ where $\{v\}$ is a new vertex not in $V(H)$, and the edge set $E = \{e \cup \{v\} : e \in E(H)\}$. We write $S^t(H)$ to denote the hypergraph obtained by iterating the suspension operation t-times, i.e. $S^2(H) = S(S(H))$ and $S^3(H) = S(S(S(H)))$, etc.

Proposition 1. [10] For any family of hypergraphs $\mathcal{H}$ we have that $\pi(S(\mathcal{H})) \leqslant \pi(\mathcal{H})$.

Theorem 19. *[3] Let R be a set of distinct positive integers with $|R| \geqslant 2$ and $R \neq \{1, 2\}$. Then a non-trivial degenerate R-graph always exists.*

A chain C^R is a special R-graph containing exactly one edge of each size such that any pair of these edges are comparable under inclusion relation. In [3], we say a degenerate R-graph is *trivial* if it is a subgraph of a blow-up of the chain C^R. By Theorem 19, there exist non-trivial degenerate $\{2,3\}$-graphs. The $\{2,3\}$-graph $H = \{12, 123\}$ is a chain, thus it is degenerate. By Theorem 18, the subdivision $H' = \{14, 24, 123\}$ is also degenerate, but it is non-trivial. As showed in [10], $H^0 = S(K_2^{1,2}) = \{13, 12, 123\}$ is not degenerate, and $\pi(H^0) = \frac{5}{4}$.

So what does the degenerate $\{2,3\}$-graph look like? To answer this question, we may need to construct a family of $\{2,3\}$-graphs G_n with $h_n(G_n) > (1+\epsilon)$ for some $\epsilon > 0$. Here are three $\{2,3\}$-graphs with edge density greater than 1.

Note that for any R-graph H (with possible loops), one can construct the family of H-colorable R-graph by blowing up H in certain way. The langrangian of H is the maximum edge density of the H-colorable R-graph that one can get this way. For more details of R-graphs with loops, blow-up, and Lagrangian, please refer to [3]. In this part, we will use an easy-understood way to calculate the edge densities.

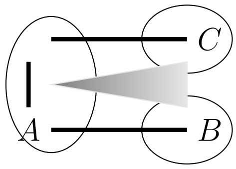
Example 3. A $\{2,3\}$-graph $G_1^{\{2,3\}}$ is a blowing-up of the general hypergraph H_1 with vertex set $\{a, b, c\}$ and edge set $\{aa, ab, ac, abc\}$, if there exists a partition of vertex set such that $V(G_1^{\{2,3\}}) = A \cup B \cup C$ and every 2-edge meets two vertices in A (or B, or C), every 3-edge meets A, B, C one vertex respectively. In other words,

$$E(G_1^{\{2,3\}}) = \binom{A}{2} \cup \binom{A}{1}\binom{B}{1} \cup \binom{A}{1}\binom{C}{1} \cup \binom{A}{1}\binom{B}{1}\binom{C}{1}.$$

Let $|A| = xn$ and $|B| = |C| = \frac{1-x}{2}n$ for some value $x \in (0,1)$. We have

$$\begin{aligned} h_n(G_1^{\{2,3\}}) &= \frac{\binom{xn}{2} + \binom{xn}{1}\binom{(1-x)n}{1}}{\binom{n}{2}} + \frac{xn(\frac{(1-x)n}{2})^2}{\binom{n}{3}} \\ &= x^2 + 2x(1-x) + \frac{3}{2}x(1-x)^2 + o_n(1) \\ &= \frac{7}{2}x - 4x^2 + \frac{3}{2}x^3 + o_n(1). \end{aligned}$$

The above value reaches the maximum value $\frac{245}{243} + o_n(1)$ at $x = \frac{7}{9}$.



$G_1^{\{2,3\}}$: $h_n(G_1^{\{2,3\}}) = \frac{245}{243}$ at $|A| = \frac{7}{9}n$.

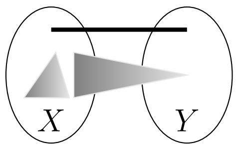
Example 4. A $\{2,3\}$-graph $G_2^{\{2,3\}}$ is a blowing-up of the general hypergraph H_2 with vertex set $\{x, y\}$ and edge set $\{xy, xxx, xxy\}$, if there exists a partition of vertex set such that $V(G_2^{\{2,3\}}) = X \cup Y$ and every 2-edge meets one vertex in X and one vertex in Y, every 3-edge either meet three vertices in X or two vertices in X plus one vertex in Y. Actually $G_2^{\{2,3\}}$ is H_2-colorable. In other words,

$$E(G_2^{\{2,3\}}) = \binom{X}{3} \cup \binom{X}{2}\binom{Y}{1} \cup \binom{X}{1}\binom{Y}{1}.$$

Let $|X| = xn$ and $|Y| = (1-x)n$ for some value $x \in (0,1)$, we have

$$\begin{aligned} h_n(G_2^{\{2,3\}}) &= \frac{\binom{xn}{3} + \binom{xn}{2}\binom{(1-x)n}{1}}{\binom{n}{3}} + \frac{xn(1-x)n}{\binom{n}{2}} \\ &= x^3 + 3x^2(1-x) + 2x(1-x) + o_n(1) \\ &= 2x + x^2 - 2x^3 + o_n(1). \end{aligned}$$

The above value reaches the maximum value $\frac{19+13\sqrt{13}}{54} + o_n(1) \approx 1.21985\cdots + o_n(1)$ at $x = \frac{1+\sqrt{13}}{6}$.



$G_2^{\{2,3\}}$: $h_n(G_2^{\{2,3\}}) \approx 1.21985$ at $|X| = (\frac{1+\sqrt{13}}{6})n$.

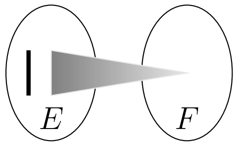
Example 5. A $\{2,3\}$-graph $G_3^{\{2,3\}}$ is a blowing-up of the general hypergraph H_3 with vertex set $\{e, f\}$ and edge set $\{ee, eef\}$, if there exists a partition of vertex set such that $V(G_2^{\{2,3\}}) = E \cup F$ and every 2-edge meets two vertices in E, every 3-edge meets two vertices in E plus one vertex in F. Actually $G_3^{\{2,3\}}$ is H_3-colorable. In other words,

$$E(G_3^{\{2,3\}}) = \binom{E}{2} \cup \binom{E}{2}\binom{Y}{1}.$$

Let $|E| = xn$ and $|F| = (1-x)n$ for some value $x \in (0,1)$, we have

$$\begin{aligned} h_n(G_3^{\{2,3\}}) &= \frac{\binom{xn}{2}}{\binom{n}{2}} + \frac{\binom{xn}{2}\binom{(1-x)n}{1}}{\binom{n}{3}} \\ &= x^2 + 3x^2(1-x) + o_n(1) \\ &= 4x^2 - 3x^3 + o_n(1). \end{aligned}$$

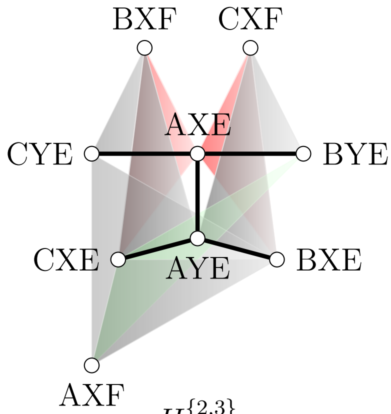
The above value reaches the maximum value $\frac{256}{243} + o_n(1)$ at $x = \frac{8}{9}$.



$G_3^{\{2,3\}}$: $h_n(G_3^{\{2,3\}}) = \frac{256}{243}$ at $|E| = \frac{8}{9}n$.

A degenerate $\{2,3\}$-graph must appear as subgraphs in all above $\{2,3\}$-graphs $G_1^{\{2,3\}}$, $G_2^{\{2,3\}}$ and $G_3^{\{2,3\}}$, thus it must appear as subgraph in the product of these hypergraphs. By taking this product, we get a 12-vertex $\{2,3\}$-graph which is $H_9^{\{2,3\}}$-colorable. Thus we have

Lemma 20. *The degenerate* $\{2,3\}$*-graphs must be* $H_9^{\{2,3\}}$*-colorable.*



$H_9^{\{2,3\}}$

The following theorem shows a relation between such $\{2,3\}$-graphs and the 2-colored graphs and can help us determine the upper bound for the Turán density of some $\{2,3\}$-graphs.

Theorem 21. *Let* $H = (V,\ E_r,\ E_b)$ *be a 2-colored graph, and* $H' = (V', E_2,\ E_3)$ *be a* $\{2,3\}$*-graph obtained from* H *by adding a new vertex* $v \notin (V)$ *such that* $V' = V \cup \{v\}$ *and* $E_2 = E_r$*, and* $E_3 = \{e' | e' = e \cup v, e \in E_b\}$*. Then* $\pi(H') \leqslant \pi(H)$*.*

Proof. Let n be positive integer, let $G = (V,\ E_2(G),\ E_3(G))$ be an arbitrary H'-free $\{2,3\}$-graph on n vertices. For any vertex $v \in V(G)$, let $G_v = (V(G) \setminus \{v\}, E_{v,2}, E_{v,3})$ be a 2-colored graph obtained form G, such that the red edges are $E_{v,2} = E_2(G)$, the blue edges are $E_{v,3} = \{u, w | \{vuw\} \in E_3\}$. Observe that G_v is H-free since G is H'-free. Thus $h_{n-1}(G_v) \leqslant \pi_n(H)$.

Since

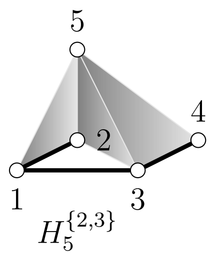
$$|E_2(G)| = \frac{1}{n-2} \sum_{v \in V(G)} |E_{v,2}| \text{ and } |E_3(G)| = \frac{1}{3} \sum_{v \in V(G)} |E_{v,3}|,$$

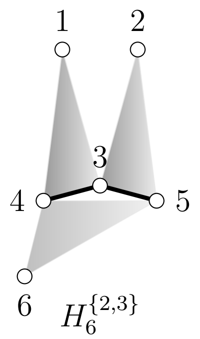
Then

$$\begin{aligned} h_n(G) &= \frac{|E_2(G)|}{\binom{n}{2}} + \frac{|E_3(G)|}{\binom{n}{3}} \\ &= \sum_{v \in V(G)} \frac{|E_{v,2}|}{(n-2)\binom{n}{2}} + \sum_{v \in V(G)} \frac{|E_{v,3}|}{3\binom{n}{3}} \\ &= \frac{1}{n} \sum_{v \in V(G)} \frac{|E_{v,2}|}{\binom{n-1}{2}} + \frac{1}{n} \sum_{v \in V(G)} \frac{|E_{v,3}|}{\binom{n-1}{2}} \\ &= \frac{1}{n} \sum_{v \in V(G)} \left(\frac{|E_{v,2}|}{\binom{n-1}{2}} + \frac{|E_{v,3}|}{\binom{n-1}{2}} \right) \\ &\leqslant \frac{1}{n} \sum_{v \in V(G)} h_{n-1}(G_v) \\ &\leqslant \pi(H). \end{aligned}$$

Therefore $\pi(H') \leqslant \pi(H)$. □

So far we couldn't give an upper bound of $\pi(H_9^{\{2,3\}})$, but we can show a subgraph of $\pi(H_9^{\{2,3\}})$ are degenerate using above theorem. Let us observe that if we remove a single vertex AXF and edges connecting to it, the resulting sub-hypergraph is $H_5^{\{2,3\}}$-colorable, where $H_5^{\{2,3\}} = ([5], \{12, 13, 34, 125, 135, 345\})$.





Observe that we can also obtain $H_5^{\{2,3\}}$ from T by adding vertex 5, and connect it with blue edges. Thus we have $\pi(H_5^{\{2,3\}}) = 1$.

In $H_9^{\{2,3\}}$, removing a single 2-edge connecting vertices AXE and AYE, the resulting subgraph is $H_6^{\{2,3\}}$-colorable, where $H_6^{\{2,3\}} = ([6], \{34, 35, 134, 235, 456\})$. However, we don't know the Turán density of $H_6^{\{2,3\}}$. We remark that determining the degenerate $\{2,3\}$-hypergraph is still unknown.

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