Eckhoff's problem on convex sets in the plane

Adam S. Jobson
Department of Mathematics
University of Louisville
Louisville, KY 40292, U.S.A.
adam.jobson@louisville.edu

André E. Kézdy
Department of Mathematics
University of Louisville
Louisville, KY 40292, U.S.A.
kezdy@louisville.edu

Jenő Lehel
Alfréd Rényi Institue of Mathematics
Budapest, Hungary
and
Department of Mathematics
University of Louisville
Louisville, KY 40292, U.S.A.
lehelj@renyi.hu

Submitted: Oct 28, 2020; Accepted: Jun 11, 2021; Published: Aug 27, 2021

Abstract

Eckhoff proposed a combinatorial version of the classical Hadwiger–Debrunner (p, q)-problems as follows. Let $\mathcal{F}$ be a finite family of convex sets in the plane and let $m \geqslant 1$ be an integer. If among every $\binom{m+2}{2}$ members of $\mathcal{F}$ all but at most $m-1$ members have a common point, then there is a common point for all but at most $m-1$ members of $\mathcal{F}$. The claim is an extension of Helly's theorem ($m = 1$). The case $m = 2$ was verified by Nadler and by Perles. Here we show that Eckhoff 's conjecture follows from an old conjecture due to Szemerédi and Petruska concerning 3-uniform hypergraphs. This conjecture is still open in general; its solution for a few special cases answers Eckhoff's problem for $m = 3, 4$. A new proof for the case $m = 2$ is also presented.

Mathematics Subject Classifications: 52A10, 52A35, 05C62, 05D05, 05D15, 05C65

1 Introduction

The subject of this note is a combinatorial version of the classical Hadwiger–Debrunner (p, q)-problems proposed by Eckhoff [2] (see also [1]). A family $\mathcal{F}$ of convex sets in the plane has the $\Delta(m)$*-property* if $\mathcal{F}$ has at least $|\mathcal{F}|-m+1$ sets with non-empty intersection. We restate Eckhoff's conjecture using this notation.

Problem 1. (Eckhoff [2, Problem 6]) Let $m \geqslant 1$, $k = \binom{m+2}{2}$ be integers, and let $\mathcal{F}$ be a family of at least k convex sets in $\mathbb{R}^2$. If every k members of $\mathcal{F}$ has the $\Delta(m)$-property, then $\mathcal{F}$ also has the $\Delta(m)$-property.

Due to Helly's theorem [5], Problem 1 has a positive answer for $m = 1$. The claim was verified also for $m = 2$ by Nadler [8] and by Perles [9]. In this note we show that Eckhoff's conjecture follows from an old conjecture due to Szemerédi and Petruska [10] on 3-uniform hypergraphs.

In Section 2, Problem 1 is restated first (Problem 2) in terms of 2-representable 3-uniform hypergraphs. The Szemerédi-Petruska conjecture, as reformulated by Lehel and Tuza [11, Problem 18.(a)] states that $\binom{m+2}{2}$ is the maximum order of a 3-uniform τ-critical hypergraph with transversal number m. Thus Eckhoff's conjecture becomes equivalent to a particular instance of a general extremal hypergraph problem (Theorem 6). The Szemerédi-Petruska conjecture is verified for $m = 2, 3, 4$ (see [7]) using the concept of 3-uniform τ–critical hypergraphs, cross-intersecting set-pair systems, and τ-critical graphs; this solves Eckhoff's problem for $m = 3, 4$, with a new proof for $m = 2$ (Corollary 7).

Eckhoff made the remark that the value of k in Problem 1 is not expected to be tight. Examples in Section 5 show that $k = \binom{m+2}{2}$ cannot be lowered for $m = 2, 3$, but it is not optimal for $m = 4$.

2 Convex hypergraphs

Given a family $\mathcal{F}$ of convex sets in $\mathbb{R}^2$, let H be the 3-uniform *intersection hypergraph* defined by vertex set $V(H) = \{F : F \in \mathcal{F}\}$ and edge set $E(H) = \{\{A, B, C\} : A, B, C \in \mathcal{F} \text{ and } A \cap B \cap C \neq \varnothing\}$.

A 3-uniform hypergraph H, that is the intersection hypergraph of some family $\mathcal{F}$ of planar convex sets is called a 2*-representable* or *convex* hypergraph. Observe that a k-clique $N \subset V$ of the intersection hypergraph indicates that the k convex sets of $\mathcal{F}$ corresponding to the vertices of N have a common point in the plane, due to Helly's theorem. Eckhoff's problem is stated next in terms of convex hypergraphs.

Problem 2. Let $m \geqslant 1$ and $n \geqslant \binom{m+2}{2}$ be integers, and let H be a 2-representable 3-uniform hypergraph of order n. If $\omega(H[X]) \geqslant \binom{m+2}{2} - m + 1$, for every $X \subseteq V$, $|X| = \binom{m+2}{2}$, then $\omega(H) \geqslant n - m + 1$.

Observe that by Helly's theorem, a family $\mathcal{F}$ of k convex sets in $\mathbb{R}^2$ has the $\Delta(m)$-property if and only if the 3-uniform intersection hypergraph H defined by $\mathcal{F}$ has clique number $\omega(H) \geqslant k - m + 1$. This implies the equivalence of Problem 1 and Problem 2.

3 τ-critical 3-uniform hypergraphs

Let $H = (V, E)$ be an r-uniform hypergraph. For $X \subset V$ define the *subhypergraph* $H[X]$ on vertex set X with all those edges in E that are contained by X. For $e \in E$, denote $H - e$ the *partial hypergraph* with vertex set V and edge set $E \setminus \{e\}$. Let $\widehat{H} = (V, \widehat{E})$ be the r-uniform hypergraph obtained as the complement of H with $\widehat{E}$ containing all r-element subsets of V not in E.

The *transversal number* of a hypergraph H is defined by

$$\tau(H) = min\{|T| : T \subset V,\ e \cap T \neq \varnothing,\ \text{for each } e \in E\}.$$

A hypergraph H is τ*-critical* if it has no isolated vertex $(\bigcup_{e\in E} e = V)$ and $\tau(H - e) = \tau(H) - 1$ for every $e \in E$. Let $v_{max}(r, t)$ be the maximum order of an r-uniform τ-critical hypergraph H with $\tau(H) = t$. This function was introduced and investigated by Gyárfás et al. [4] and by Tuza [11, Section 4.2].

Denote $\omega(H)$ the *clique number* of H defined as the maximum cardinality of a subset $N \subset V$ such that every r-element set of N belongs to E.

Lemma 3.

(a) If $\widehat{H}$ is a τ-critical r-uniform hypergaph, then the maximum cliques of H have no common vertex.

(b) If the maximum cliques of an r-uniform hypergaph H have no common vertex, then $|V| \leqslant v_{max}(r, t)$, where $t = \tau(\widehat{H})$.

Proof. Notice that $N \subset V$ is a minimum cardinality transversal of $\widehat{H}$ if and only if $T = V \setminus N$ is the vertex set of a maximum cardinality clique of H.

(a) By definition, $\widehat{H}$ has no isolated vertex. Furthermore, for every $x \in V$ and $e \in \widehat{E}$, $x \in e$, we have $\tau(\widehat{H}[V \setminus \{x\}]) \leqslant \tau(\widehat{H} - e) = \tau(\widehat{H}) - 1$. Then the union of $\{x\}$ with a $(\tau(\widehat{H}) - 1)$-element transversal of $\widehat{H}[V \setminus \{x\}]$ forms a minimum transversal of $\widehat{H}$. Therefore, every $x \in V$ belongs to some minimum transversal of $\widehat{H}$. Equivalently, the complements of the minimum transversals of $\widehat{H}$, the maximum cliques of H, have no common vertex.

(b) Because the maximum cliques in H have no common vertex, the union of their complement in V, that is, the union of the t-element transversals of $\widehat{H}$, is equal to V. Let H' be a τ-critical partial hypergraph of $\widehat{H}$ with vertex V' and $\tau(H') = t$. We claim that $|V'| = |V|$.

Because every vertex $x \in V \setminus V'$ belongs to some t-element transversal T of $\widehat{H}$, the set $T \setminus \{x\}$ is a $(t - 1)$-element transversal for all edges of $\widehat{H}$ not containing x; hence $\tau(H') < t$, a contradiction. Thus $|V| = |V'| \leqslant v_{max}(r, t)$ follows. □

Recall that $v_{max}(3, m)$ is the maximum order of a 3-uniform τ-critical hypergraph H with $\tau(H) = m$. The conjecture that $v_{max}(3, m) = \binom{m+2}{2}$ for every m [11, Problem 18.(a)] was verified only for a few small values of m:

Proposition 4 ([7]). *Let $m = 2, 3$, or 4, and $n > m$. If H is a 3-uniform hypergraph of order n with clique number $\omega(H) = n - m = k \geqslant 3$ and the intersection of the k-cliques of H is empty, then $n \leqslant \binom{m+2}{2}$.* □

Corollary 5. *$v_{max}(3, m) = \binom{m+2}{2}$, for $m = 2, 3$ and 4.*

Proof. For every $m \geqslant 1$, a 3-uniform τ-critical hypergraph of order $n = \binom{m+1}{2} + m + 1$ with transversal number m is obtained from the complete graph K_{m+1} by extending each edge with one vertex using additional distinct vertices. This construction implies $v_{max}(3, m) \geqslant \binom{m+2}{2}$.

Let $\widehat{H}$ be a τ-critical 3-uniform hypergraph with $\tau(\widehat{H}) = m$ and $|V| = v_{max}(3, m)$. By Lemma 3(a) and by applying Proposition 4, we obtain $|V| = v_{max}(3, m) \leqslant \binom{m+2}{2}$, $m = 2, 3, 4$. Thus $v_{max}(3, m) = \binom{m+2}{2}$ follows for $m = 2, 3$ and 4. □

4 Eckhoff's problem and τ-critical hypergraphs

Eckhoff's problem relates to the hypergraph extremal problem of determining $v_{max}(3, m)$ as is shown by the next theorem.

Theorem 6. *For $m \geqslant 1$ and $n \geqslant k \geqslant v_{max}(3, m)$, let $\mathcal{F}$ be a family of n convex sets in $\mathbb{R}^2$. If every k members of $\mathcal{F}$ have the $\Delta(m)$-property, then $\mathcal{F}$ has the $\Delta(m)$-property.*

Proof. Assume that the claim is not true. Let H_0 be a 3-uniform convex hypergraph of minimum order n_0 such that $\omega(H_0) \leqslant n_0 - m$, but $\omega(H_0[X]) \geqslant k - m + 1$, for every $X \subset V_0$, $|X| = k$. Notice that the definition of H_0 implies $n_0 > k$; furthermore, since n_0 is minimal, $\omega(H_0) = n_0 - m$.

We claim that the intersection of the maximum cliques of H_0 is empty. If $x \in V_0$ was a common vertex of all maximum cliques, then $H' = H_0[V_0 \setminus \{x\}]$ has order $n' = n_0 - 1$, and for its clique number we have $\omega(H') = \omega(H_0) - 1 = n_0 - m - 1 = n' - m$. At the same time, $\omega(H'[X]) \geqslant k - m + 1$, for every k-element subset $X \subset V_0 \setminus \{x\}$. Hence H' is a counterexample of order n', contradicting the minimality of n_0. Therefore, the maximum cliques of H_0 have no common vertex, and because $\tau(\widehat{H_0}) = n_0 - \omega(H_0) = m$, Lemma 3 implies $k < n_0 \leqslant v_{max}(3, m) \leqslant k$, a contradiction. □

As an immediate corollary of Theorem 6 and Proposition 5 we obtain an extensions of Helly's theorem together with a combinatorial proof for the case $m = 2$ (verified earlier by Nadler [8] and by Perles [9]).

Corollary 7. *Let $1 \leqslant m \leqslant 4$, $k = \binom{m+2}{2}$, and let $\mathcal{F}$ be a family of at least k convex sets in $\mathbb{R}^2$. If every k members of $\mathcal{F}$ has the $\Delta(m)$-property, then $\mathcal{F}$ also has the $\Delta(m)$-property.* □

5 Concluding remarks

5.1 The best known general bound $v_{max}(3,m) \leqslant \frac{3}{4}m^2+m+1$ is obtained by Tuza[1] using the machinery of τ-critical hypergraphs. This bound combined with Theorem 6 yields the following finiteness result on Eckhoff's problem, for every m.

Corollary 8. *Let $\mathcal{F}$ be a family of at least $k \geqslant \frac{3}{4}m^2+m+1$ convex sets in $\mathbb{R}^2$. If every k members of $\mathcal{F}$ has the $\Delta(m)$-property, then $\mathcal{F}$ also has the $\Delta(m)$-property.* □

5.2 In Corollary 7 the value of k is optimal (the smallest possible) if there is a family of $n \geqslant k$ convex sets in $\mathbb{R}^2$ such that every $k-1$ members of $\mathcal{F}$ satisfy the $\Delta(m)$-property, but $\mathcal{F}$ fails it. It was proved by Nadler [8] that $k = \binom{m+2}{2}$ is optimal for $m=2$, but as noted by Eckhoff [1], it is 'somewhat unlikely' that it is optimal for every m. We address optimality for $m = 2,3,4$ by defining a family $\mathcal{F}_m$ of convex sets as follows.

$\mathcal{F}_2$: $m=2$, $k=6$. Let $\mathcal{F}_2$ be the family of $n=6$ line segments, taken each side of the triangle $T=(p,q,r)$ twice. Then any vertex of T covers only $4=n-m$ members of $\mathcal{F}_2$; meanwhile, when removing a copy of one side, say $\overline{qr}$, vertex p covers $(k-1)-(m-1)=4$ members.

$\mathcal{F}_3$: $m=3$, $k=10$. Let $p_0,p_1,p_2,p_3,p_4 \in \mathbb{R}^2$ be the vertices of a regular pentagon P, and let $\mathcal{F}_3$ be the family of $n=10$ convex sets: the five triangles $T_i=(p_i,p_{i+1},p_{i+2})$ plus the five quadrangles $Q_i=(p_i,p_{i+1},p_{i+2},p_{i+3})$, $0 \leqslant i \leqslant 4$, with (mod 5) index arithmetic. Notice that among eight members there are at least three triangles, and among three triangles the intersection of some two is a vertex of P, which covers only $7=n-m$ members of $\mathcal{F}_3$. On the other hand when removing some member C from $\mathcal{F}_3$, any vertex of P not in C covers $(k-1)-(m-1)=7$ members.

$\mathcal{F}_4$: $m=4$, $k=\binom{m+2}{2}-1=14$. Let $S=\{p_0,p_1,\ldots,p_7\}$ be the set of vertices of a regular octagon, and let $\mathcal{F}_4$ be the family of $n=14$ convex sets defined as follows. Take the eight hexagons determined by the vertex sets $S \setminus \{p_i,p_{i+1}\}$, $0 \leqslant i \leqslant 7$, and take the six quadrangles $Q_i=(p_i,p_{i+1},p_{i+2},p_{i+3})$, for $i \in \{1,2,3,5,6,7\}$, with (mod 8) index arithmetic. Notice that the undefined Q_0, Q_4 do not belong to $\mathcal{F}_4$, furthermore, the six quadrangles defined in $\mathcal{F}_4$ form three disjoint pairs. Taking one quadrangle from each pair plus the eight hexagons form a subfamily of 11 convex sets with no common point, thus at most $10=n-m$ members of $\mathcal{F}_4$ can be covered by one point. On the other hand, three intersecting quadrangles plus seven more hexagons contained in every subfamily $\mathcal{F}_4 \setminus \{C\}$, that is, $(k-1)-(m-1)=10$ members have a common point q of the plane as it is seen in Fig.1.

Family $\mathcal{F}_m$ shows that $k=\binom{m+2}{2}$ is optimal in Corollary 7 for $m=2,3$. Each of $\mathcal{F}_2$ and $\mathcal{F}_3$ is derived from a 3-uniform hypergraph witnessing $v_{max}(3,m)=\binom{m+2}{2}$. For $m=4$ the 3-uniform witness hypergraphs are not 2-representable. This fact was observed by Jobson et al. [6] when a similar method using convex hypergraphs was applied to

[1]Personal communication

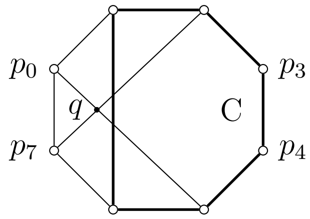

Figure 1: q covers ten members of $\mathcal{F}_4 \setminus \{C\}$

another geometry problem on convex sets in the plane [6]. Thus the optimum for $m = 4$ is less than $\binom{4+2}{2} = 15$; and $\mathcal{F}_4$ shows that for $m = 4$ the optimum value in Corollary 7 is actually $k = \binom{4+2}{2} - 1 = 14$.

5.3 In the light of the discussions above, Eckhoff's problem takes the form of an extremal problem asking for the smallest integer $k(m) \leqslant \binom{m+2}{2}$ such that Theorem 6 remains true when $v_{max}(3, m)$ is replaced with $k(m)$. The exact values, which we know are $k(1) = 3$, $k(2) = 6$, $k(3) = 10$, $k(4) = 14$, and we ask the question whether $k(m) = \Omega(m^2)$.

Acknowledgements

We would like to thank András Gyárfás for calling our attention to Eckhoff's problem, and for his valuable comments to an earlier version of the manuscript.

References

[1] J. Eckhoff. A survey of the Hadwiger–Debrunner (p, q)-problem. *in: Discrete and computational geometry, 347–377, Algorithms Combin.*, 25, Springer, Berlin, 2003. https://doi.org/10.1007/978-3-642-55566-4_16.

[2] J. Eckhoff. Problems in discrete geometry. *Convexity and discrete geometry including graph theory, 269–273, Springer Proc. Math. Stat.*, 148, Springer, 2016. https://doi.org/10.1007/978-3-319-28186-5_26.

[3] P. Erdős, and T. Gallai. On the maximal number of vertices representing the edges of a graph. *Közl. Magyar Tud. Akad. Mat. Kutató Int. Közl*, 6 (1961), 181–203.

[4] A. Gyárfás, J. Lehel, and Zs. Tuza. Upper bound on the order of τ-criti-cal hypergraphs. *J. Combin. Theory Ser.*, B 33 (1982), no. 2, 161–165. https://doi.org/10.1016/0095-8956(82)90065-X.

[5] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkte. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 32 (1923), 175–176.

[6] A. S. Jobson, A. E. Kézdy, J. Lehel, T. Pervenecki, and G. Tóth. Petruska's question on planar convex sets. *Discrete Math.*, 343 (2020), no. 9, 13pp.

[7] A. S. Jobson, A. E. Kézdy, and J. Lehel. The Szemerédi-Petruska conjecture for a few small values. *Eur. J. Math.*, May (2021) 8pp.
https://doi.org/10.1007/s40879-021-00466-9

[8] D. Nadler. Minimal 2-fold coverings of $\mathbb{E}^d$. *Geom. Dedicata*, 65 (1997), no 3. 305–312.

[9] M. A. Perles. A Helly type theorem for almost intersecting families. *Talk at the Convex Geometry meeting, Oberwolfach.* June 1993.

[10] E. Szemerédi, and G. Petruska. On a combinatorial problem I. *Studia Sci. Math. Hungar.*, 7 (1972), 363–374.

[11] Zs.Tuza. Critical hypergraphs and intersecting set-pair systems *J. Combin. Theory Ser. B*, 39 (1985), no. 2, 134–145.
https://doi.org/10.1016/0095-8956(85)90043-7