

Spectral extremal results for hypergraphs

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Abstract

Let F be a graph. A hypergraph is called Berge F if it can be obtained by replacing each edge in F by a hyperedge containing it. Given a family of graphs \mathcal{F} , we say that a hypergraph H is Berge \mathcal{F} -free if for every $F \in \mathcal{F}$, the hypergraph H does not contain a Berge F as a subhypergraph. In this paper we investigate on the connections between spectral radius of the adjacency tensor and structural properties of a linear hypergraph. In particular, we obtain a spectral version of Turán-type problems over linear k -uniform hypergraphs by using spectral methods.

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1 Introduction

A hypergraph $H = (V, E)$ consists of a vertex set V and an edge (hyperedge) set E , where each edge is a nonempty subset of V . A hypergraph is called k -uniform if each edge is a k -element subset of V . A 2-uniform hypergraph is simply called a graph. Two vertices x and y are said to be adjacent if there is an edge that contains both of these vertices. A hypergraph H is called linear if every two edges have at most one vertex in common.

For a fixed k -uniform family \mathcal{F} , the Turán number of \mathcal{F} , denoted by $ex_k(n, \mathcal{F})$, is the maximum number of edges of an \mathcal{F} -free hypergraph on n vertices. Similarly, given a family of k -uniform linear hypergraphs \mathcal{F} , the linear Turán number of \mathcal{F} , denoted $ex_k^{\text{lin}}(n, \mathcal{F})$, is the maximum number of edges in an \mathcal{F} -free k -uniform linear hypergraph on n vertices.

Turán type extremal problems in graphs and hypergraphs are the central topic of extremal combinatorics and have a vast literature. For a survey of recent results we refer the reader to [11, 18, 22]. Only a handful of results are known about the asymptotic behaviour of Turán numbers for hypergraphs. One of the most active subjects is Berge hypergraphs. The classical definition of a hypergraph cycle due to Berge is the following: a Berge cycle C_t of length $t \geq 2$ is an alternating sequence of distinct vertices (other than first and last) and distinct edges of the form $v_1 l_1 v_2 l_2 \cdots v_t l_t$ where $v_i, v_{i+1} \in l_i$ for each $i \in \{1, 2, \dots, t-1\}$ and $v_t, v_1 \in l_t$.

Gerbner and Palmer [14] gave the following natural generalization of the definitions of Berge graphs. Let $F = (V(F), E(F))$ be a graph and $\mathcal{B} = (V(\mathcal{B}), E(\mathcal{B}))$ be a hypergraph. We say \mathcal{B} is Berge F if there is a bijection $\phi : E(F) \rightarrow E(\mathcal{B})$ such that $e \subseteq \phi(e)$ for all $e \in E(F)$. In other words, given a graph F , we can obtain a Berge F by replacing each edge of F with a hyperedge that contains it. Given a family of graphs \mathcal{F} , we say that a hypergraph H is Berge \mathcal{F} -free if for every $F \in \mathcal{F}$, the hypergraph H does not contain a Berge F as a subhypergraph. The maximum possible number of edges in a Berge \mathcal{F} -free hypergraph on n vertices is the Turán number of Berge \mathcal{F} .

Linear Turán extremal problems have been studied mostly implicitly. For example, the famous upper bound of Ruzsa and Szemerédi [26] on triple systems not carrying three edges on six vertices is equivalent to $ex_3^{\text{lin}}(n, T) = o(n^2)$ where T is the linear triangle. For $l \geq 2$, Füredi and Özkahya [10] showed $ex_3^{\text{lin}}(n, C_{2l+1}) \leq 2ln^{1+1/l} + 9ln$. For the even case, it is easy to show $ex_3^{\text{lin}}(n, C_{2l}) \leq O(n^{1+1/l})$. Recently, linear Turán numbers were studied in [5, 8, 12, 15, 28].

Our aim is to consider a spectral version of hypergraph Turán problems, i.e., spectral extremal hypergraph theory, which is the subset of extremal problems where invariants are based on the eigenvalues or eigenvectors of a hypergraph. The most natural such invariant is the maximal absolute value of the eigenvalues of the adjacency tensor of a hypergraph H , called its spectral radius. Because the spectral radius is a close correlate of the number of edges in a hypergraph, the following problem is a natural spectral analog of hypergraph Turán problems: *What is the maximum spectral radius of hypergraphs of order n , not containing a given \mathcal{F} ?*

In fact, spectral extremal graph theory has a substantial history, with many important results. Examples include Stanley's bound [27], theorems of Wilf [29] relating eigenvalues

of graphs to their chromatic number, and many other examples. Much of recent work in spectral extremal graph theory is due to Nikiforov, who has considered maximizing the spectral radius over several families of graphs; see [23]. Although the area is still difficult and underdeveloped, we believe that ultimately a spectral approach to extremal hypergraph theory will turn out to be a fruitful and interesting accompaniment to “conventional” extremal theory, see e.g., [1, 17].

The rest of this paper is organized as follows. In the next section, the bulk of the necessary notation and the basic facts have been presented, including the definitions and properties of eigenvalues of tensors and hypergraphs. We develop a few new tools in order to provide spectral analogues of extremal hypergraph problems. In Section 3 we consider spectral analogues of Turán-type problems for hypergraphs and investigate bounds on the maximum spectral radius of linear k -uniform hypergraphs with girth at least five by using the spectral methods proved in Section 2.

2 Preliminaries

2.1 Spectra of tensors

In 2005, Qi [25] and Lim [20] independently introduced the concept of tensor eigenvalues and the spectra of tensors. An order k dimension n real tensor¹ $\mathcal{T} = (\mathcal{T}_{i_1 \dots i_k})$ consists of n^k real entries $\mathcal{T}_{i_1 \dots i_k}$ for $1 \leq i_1, i_2, \dots, i_k \leq n$. Evidently, a vector of dimension n is a tensor of order 1 and a matrix is a tensor of order 2. \mathcal{T} is called symmetric if the value of $\mathcal{T}_{i_1 \dots i_k}$ is invariant under any permutation of the indices i_1, i_2, \dots, i_k . Given a vector $x \in \mathbb{R}^n$, $\mathcal{T}x^k$ is a real number and $\mathcal{T}x^{k-1}$ is an n -dimensional vector. $\mathcal{T}x^k$ and the i th component of $\mathcal{T}x^{k-1}$ are given by:

$$\mathcal{T}x^k = \sum_{i_1, i_2, \dots, i_k \in [n]} \mathcal{T}_{i_1 i_2 \dots i_k} x_{i_1} x_{i_2} \dots x_{i_k}. \quad (1)$$

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2, \dots, i_k \in [n]} \mathcal{T}_{i i_2 \dots i_k} x_{i_2} \dots x_{i_k}. \quad (2)$$

The former is simply tensor contraction of \mathcal{T} with the k -th outer product of x with itself, and the latter is the i -th coordinate of the contraction of \mathcal{T} with the $(k-1)$ -st outer product of x with itself. Note that the symmetry of \mathcal{T} makes these contractions well-defined without specifying which indices are summed over. Let \mathcal{T} be an order k dimension n real tensor. For some $\lambda \in \mathbb{C}$, if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the eigenequation

$$\mathcal{T}x^{k-1} = \lambda x^{[k-1]}, \quad (3)$$

then λ is called an eigenvalue of \mathcal{T} and x is its corresponding eigenvector, where $x^{[k-1]} := (x_1^{k-1}, x_2^{k-1}, \dots, x_n^{k-1})^T \in \mathbb{C}^n \setminus \{0\}$.

If x is a real eigenvector of \mathcal{T} , then clearly the corresponding eigenvalue λ is real. In this case, λ is called an H -eigenvalue and x is called an H -eigenvector associated with

¹Sometimes known as a “hypermatrix” or simply, “matrix.”

λ . Furthermore, if x is nonnegative and real, we say λ is an H^+ -eigenvalue of \mathcal{T} . If x is positive and real, λ is said to be an H^{++} -eigenvalue of \mathcal{T} . The maximal absolute value of the eigenvalues of \mathcal{T} is called the spectral radius of \mathcal{T} , denoted by $\rho(\mathcal{T})$.

In 2012, Cooper and Dutle [6] defined the adjacency tensor of a k -uniform hypergraph H . The adjacency tensor $\mathcal{A} = \mathcal{A}(H)$ is an order k dimension n symmetric tensor defined by

$$\mathcal{A}_{i_1 \dots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1, \dots, i_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

This definition generalizes adjacency matrices, and its theory is a natural starting point for spectral hypergraph theory. For a vector x of dimension n and a subset $U \subseteq V$, we write

$$x^U = \prod_{v_i \in U} x_{v_i}.$$

The product $\mathcal{A}x^k$ therefore has an interpretation as follows:

$$\mathcal{A}x^k = k \sum_{e \in E(H)} x^e. \quad (4)$$

The right-hand side (without the factor of k) is sometimes known as the Lagrangian polynomial of H .

For nonnegative tensors, we have a generalization of the Perron-Frobenius theorem, see [4, 6, 13, 30]. Let $\mathcal{T} = (\mathcal{T}_{i_1 \dots i_k})$ be an order k dimension n nonnegative tensor. If for any nonempty proper index subset $\alpha \subset \{1, 2, \dots, n\}$, there is at least an entry $\mathcal{T}_{i_1 \dots i_k} > 0$, where $i_1 \in \alpha$ and at least an $i_j \notin \alpha$ for $j = 2, \dots, k$, then \mathcal{T} is called nonnegative weakly irreducible tensor. It was proved that a k -uniform hypergraph H is connected if and only if its adjacency tensor $\mathcal{A}(H)$ is weakly irreducible (see [13, 30]). Let $\rho(H)$ denote the spectral radius of a hypergraph H . By the Perron-Frobenius theorem, if H is connected, the eigenvector $x = (x_1, x_2, \dots, x_n)^T$ corresponding to $\rho(H)$, known as the principal eigenvector, can be chosen to be strictly positive. Throughout the paper, we only consider connected and simple hypergraphs.

2.2 Useful tools

In this section we present some useful tools which help to recast into spectral theory some classical results on hypergraphs and their proofs. We also introduce some additional notation employed below.

Let $H = (V, E)$ be a connected simple hypergraph on n vertices and m edges. For a vertex v , let $L(v)$ be the set of edges containing v and N_v be the neighborhood of v , i.e., $N_v = \{x \in V \setminus \{v\} \mid v, x \in l \text{ for some } l \in E\}$. The degree of a vertex v , which is denoted by $d(v)$, is defined as the number of edges containing v , i.e., $d(v) = |L(v)|$. For any two vertices u and v , let N_{uv} be the set of common neighbors of u and v . The codegree of u and v , denoted by $d(u, v)$, is the number of edges containing both u and v in H . Let Δ ,

Δ_2 and \bar{d} denote the maximum degree, the maximum codegree and the average degree of H , respectively. It is easy to verify that $\bar{d} = km/n$.

For a set $X \subseteq V$, let $E_t(X) = \{l \mid l \in E \text{ and } |l \cap X| = t\}$ and $e_t(X) = |E_t(X)|$. Similarly, let $E_t^v(X) = \{l \mid l \in E, v \in l \text{ and } |l \cap X| = t\}$ and $e_t^v(X) = |E_t^v(X)|$. If X and Y are disjoint sets of vertices of H , we write $E(X, Y)$ as the set of edges containing some vertices in X and the others in Y and $e(X, Y)$ the number of edges in $E(X, Y)$, i.e., $E(X, Y) = \{e \in E(H) \mid H \cap X \neq \emptyset \text{ and } H \cap Y \neq \emptyset\}$ and $e(X, Y) = |E(X, Y)|$.

Lemma 1. *Let H be a connected simple k -uniform hypergraph and ρ be the spectral radius of the adjacency tensor of H . Then*

$$\rho^2 \leq \frac{1}{k-1} \sum_{t=1}^k \sum_{e \in E_t(N_u)} \sum_{v \in N_u \cap e} d(u, v) \quad (5)$$

where u is the vertex corresponding to a maximum entry of the principal eigenvector.

Proof. Let x be an eigenvector corresponding to ρ . For a vertex $v \in V(H)$, we will use x_v to denote the eigenvector entry of x corresponding to v . For any $v \in V(H)$, the eigenvector equation is

$$\rho x_v^{k-1} = \sum_{\{v, i_2, \dots, i_k\} \in E} x_{i_2} x_{i_3} \cdots x_{i_k}. \quad (6)$$

By the Perron-Frobenius Theorem, x has all positive entries, and it will be convenient for us to normalize so that the maximum entry of x is 1. Let u denote the vertex with maximum eigenvector entry equal to 1. Throughout the paper, we will use u to denote the vertex with maximum eigenvector entry equal to 1. If there are multiple such vertices, choose and fix u arbitrarily among them. Since $x_u = 1$, (6) becomes

$$\rho = \sum_{\{u, i_2, \dots, i_k\} \in E} x_{i_2} x_{i_3} \cdots x_{i_k}. \quad (7)$$

Apply AM-GM inequality to (7), we have

$$\rho \leq \frac{1}{k-1} \sum_{\{u, i_2, \dots, i_k\} \in E} (x_{i_2}^{k-1} + x_{i_3}^{k-1} + \cdots + x_{i_k}^{k-1}). \quad (8)$$

The next inequality is a simple consequence of our normalization and an easy double counting argument, but will be used extensively throughout the paper and therefore warrants special attention. Multiplying both sides of (8) by ρ and applying (6) gives

$$\begin{aligned} \rho^2 &\leq \frac{1}{k-1} \sum_{\{u, i_2, \dots, i_k\} \in E} \rho x_{i_2}^{k-1} + \rho x_{i_3}^{k-1} + \cdots + \rho x_{i_k}^{k-1} \\ &= \frac{1}{k-1} \sum_{\{u, i_2, \dots, i_k\} \in E} \left[\sum_{e \in L(i_2)} x^{e \setminus \{i_2\}} + \sum_{e \in L(i_3)} x^{e \setminus \{i_3\}} + \cdots + \sum_{e \in L(i_k)} x^{e \setminus \{i_k\}} \right]. \end{aligned}$$

Since the maximum entry of x is normalized to 1, it is obvious that $x^{e \setminus \{v\}} \leq 1$ for any vertex $v \in V(H)$. Thus

$$\rho^2 \leq \frac{1}{k-1} \sum_{\{u, i_2, \dots, i_k\} \in E} \left[\sum_{e \in L(i_2)} 1 + \sum_{e \in L(i_3)} 1 + \dots + \sum_{e \in L(i_k)} 1 \right]. \quad (9)$$

We now estimate the right side of (9) in two different ways. On one hand, for a fixed edge $\{u, i_2, \dots, i_k\} \in E$, we consider the edge set $L(i_j)$ for $2 \leq j \leq k$. The edge set $L(i_j)$ can be considered the union of the k disjoint subsets of edges incident to i_j which have intersection with N_u of cardinality $t = 1$ through $t = k$. That is, $L(i_j) = \bigcup_{t=1}^k E_t^{i_j}(N_u)$. Then we have $\sum_{e \in L(i_j)} 1 = \sum_{t=1}^k (\sum_{e \in E_t^{i_j}(N_u)} 1)$. Combining with (9), this gives

$$\rho^2 \leq \frac{1}{k-1} \sum_{\{u, i_2, \dots, i_k\} \in E} \left[\sum_{t=1}^k \left(\sum_{e \in E_t^{i_2}(N_u)} 1 + \sum_{e \in E_t^{i_3}(N_u)} 1 + \dots + \sum_{e \in E_t^{i_k}(N_u)} 1 \right) \right] \quad (10)$$

On the other hand, for an arbitrary edge $l = \{i_1, i_2, \dots, i_k\}$ with $|l \cap N_u| = t$, without loss of generality, assume $l \cap N_u = \{i_1, i_2, \dots, i_t\}$. Then the contribution of the edge l on the right-hand side of (10) is just the sum of the number of edges containing both the vertices u and i_j over $1 \leq j \leq t$. In other words, the edge l appears in the right summation exactly $d(u, i_1) + d(u, i_2) + \dots + d(u, i_t)$ times.

Thus (10) becomes

$$\begin{aligned} \rho^2 &\leq \frac{1}{k-1} \left[\sum_{e \in E_1(N_u)} \sum_{v \in N_u \cap e} d(u, v) + \dots + \sum_{e \in E_k(N_u)} \sum_{v \in N_u \cap e} d(u, v) \right] \\ &= \frac{1}{k-1} \sum_{t=1}^k \sum_{e \in E_t(N_u)} \sum_{v \in N_u \cap e} d(u, v). \end{aligned}$$

This completes the proof. \square

Note that Lemma 1 illustrates a relationship between spectral radius of the adjacency tensor and structural properties of hypergraphs. Combined with estimates of the codegree, one can obtain results such as the following.

Lemma 2. *Let H be a connected simple k -uniform hypergraph with maximum codegree Δ_2 and ρ be the spectral radius of the adjacency tensor of H . Let u be the vertex with maximum eigenvector entry. Then*

- (1) $\rho^2 \leq \frac{\Delta_2}{k-1} [e_1(N_u) + 2e_2(N_u) + \dots + ke_k(N_u)];$
- (2) $\rho^2 \leq \frac{\Delta_2}{k-1} \sum_{v \in N_u} d(v).$

Proof. Since the maximum codegree of H is Δ_2 , by Lemma 1, we have

$$\rho^2 \leq \frac{\Delta_2}{k-1} [e_1(N_u) + 2e_2(N_u) + \cdots + ke_k(N_u)] = \frac{\Delta_2}{k-1} \sum_{v \in N_u} d(v) \quad (11)$$

where the last equality follows because both sums count the number of pairs (v, e) where $v \in e \in \bigcup_t E_t(N_u)$. \square

It is clear that the codegree of each pair of adjacent vertices in H is exactly 1 if H is a linear hypergraph. We get the following result.

Corollary 3. *Let H be a connected simple k -uniform linear hypergraph and ρ be the spectral radius of the adjacency tensor of H . Let u be the vertex with maximum eigenvector entry. Then*

$$(1) \quad \rho^2 \leq \frac{1}{k-1} [e_1(N_u) + 2e_2(N_u) + \cdots + ke_k(N_u)];$$

$$(2) \quad \rho^2 \leq \frac{1}{k-1} \sum_{v \in N_u} d(v).$$

For linear hypergraphs, the results in Corollary 3 are our main tools and we will use this technique in the sequel.

3 Main Results

The purpose of this section is to illustrate the use of the tools developed in Section 2, which translate nonspectral extremal problems into spectral results. We also determine bounds on the maximum spectral radius of uniform hypergraphs with girth at least five.

3.1 Spectral radius of linear hypergraphs without Fan^k

To illustrate this technique, we first give a spectral version result corresponding to the linear Turán number of Fan^k . We include this as a quick way for the reader to become acquainted with our notation.

For $k \geq 2$, the k -fan Fan^k is the k -uniform linear hypergraph with k edges f_1, \dots, f_k which pairwise intersect in the a single vertex v , and an additional edge g which intersects all f_i in a vertex different from v . Füredi and Gyárfás studied the linear Turán number of Fan^k in [12]. They proved that $ex_k^{\text{lin}}(n, \text{Fan}^k) \leq \frac{n^2}{k^2}$, asymptotically a factor of $1 + 1/(k-1)$ smaller than the maximum number of edges in a k -uniform linear hypergraph, $\binom{n}{2} / \binom{k}{2}$. The Turán number of Fan^k on k -uniform hypergraphs was determined by Mubayi and Pikhurko in [21].

Theorem 4 ([12]). *One has $ex_k^{\text{lin}}(n, \text{Fan}^k) \leq \frac{n^2}{k^2}$ for all $k \geq 2$. The only extremal hypergraphs are the transversal designs on n vertices with k groups.*

A transversal design $T(n, k)$ on n vertices with k groups is a k -partite hypergraph with groups of equal size (thus n is a multiple of k) and each pair of vertices from different groups is contained in exactly one hyperedge. Such designs have long been known to exist for all sufficiently large n (as a function of k), due to their connection with mutually orthogonal latin squares (MOLS). After adopting some notations and results of Füredi and Gyárfás, we prove the spectral analog of Füredi and Gyárfás's result and obtain the following extremal spectral result. Theorem 5 also implies that the maximum spectral radius of the adjacency tensor of hypergraphs is closely related to the Turán numbers of hypergraphs.

Theorem 5. *Let \mathcal{H} denote the set of linear k -uniform hypergraphs of order n , $n \equiv 0 \pmod{k}$, with forbidden Fan^k and ρ be the maximum spectral radius of hypergraphs in \mathcal{H} . For n sufficiently large, we have $\rho = \frac{n}{k}$.*

Proof. Let $H = (V, E)$ be a k -uniform Fan^k -free linear hypergraph on n vertices. Let u be the vertex with maximum eigenvector entry. Set $B_u = V \setminus N_u$, where N_u is the neighborhood of the vertex u . Suppose $f \in E_k(N_u)$, the vertices $\{v_i\}_{i=1}^k$ of f must belong to k distinct edges $\{e_i\}_{i=1}^k$ containing u since H is linear. But then, $\{f, e_1, \dots, e_k\}$ is the edge set of a Fan^k , a contradiction. In other words, $e_k(N_u) = 0$. On one hand, by Corollary 3, we have

$$\begin{aligned} \rho^2 &\leq \frac{1}{k-1} [e_1(N_u) + 2e_2(N_u) + \dots + (k-1)e_{k-1}(N_u)] \\ &\leq \frac{1}{k-1} [(k-1)e_1(N_u) + (k-1)e_2(N_u) + \dots + (k-1)e_{k-1}(N_u)] \\ &\leq e_1(N_u) + e_2(N_u) + \dots + e_{k-1}(N_u) \\ &= e(N_u, B_u) \end{aligned} \tag{12}$$

By Theorem 4, we know that $|E(H)| \leq \frac{n^2}{k^2}$. Thus $e(N_u, B_u) \leq \frac{n^2}{k^2}$. After using (12), we have $\rho^2 \leq \frac{n^2}{k^2}$, i.e., $\rho \leq \frac{n}{k}$.

On the other hand, because a transversal design $T(n, k)$ is an $\frac{n}{k}$ -regular linear hypergraph without Fan^k , then $\rho(T(n, k)) = \frac{n}{k}$. Since ρ is the maximum spectral radius of hypergraphs in \mathcal{H} , we have $\rho \geq \frac{n}{k}$.

This completes the proof. \square

3.2 Spectral radius of linear hypergraphs without Berge C_4

One of the first results concerning Turán numbers of Berge cycles is due to Lazebnik and Verstraëte [19]. Very recently this was strengthened by Ergemlidze, Győri and Methuku [8] who showed that $ex_3^{\text{lin}}(n, \{C_4\}) \leq \frac{1}{6}n^{\frac{3}{2}} + O(n)$ (which is tight, due to a construction from [19]). Gerbner, Methuku and Vizer [15] proved that $ex_k^{\text{lin}}(n, \{K_{2,t}\}) \leq \frac{\sqrt{t-1}}{k(k-1)}n^{\frac{3}{2}} + O(n)$ which indicates $ex_k^{\text{lin}}(n, \{C_4\}) \leq \frac{1}{k(k-1)}n^{\frac{3}{2}} + O(n)$. In this subsection, we give a spectral version of Ergemlidze, Győri and Methuku's results on k -uniform linear hypergraphs. Throughout this subsection, suppose $k \geq 3$.

As before, let H be a k -uniform linear hypergraph of order n and size m containing no Berge C_4 . Our plan is to first upper bound $\sum_{v \in N_u} d(v)$ for the vertex u in H . After that, by Corollary 3, we get an upper bound of the spectral radius of a hypergraph containing no Berge C_4 .

Theorem 6. *Let \mathcal{H} denote the set of linear k -uniform hypergraphs of order n with forbidden Berge C_4 and ρ be the maximum spectral radius among hypergraphs in \mathcal{H} . For $k \geq 3$, $\rho \leq \sqrt{\frac{3}{2} + \frac{1}{2(k-1)}}(n-1)^{\frac{1}{2}} + O(\frac{1}{\sqrt{n}})$.*

To prove Theorem 6, we only need to upper bound $\sum_{v \in N_u} d(v)$ by Corollary 3. Notice that $d(v) = \sum_{t=1}^k e_t^v(N_u)$ for any $v \in N_u$. Our plan is to estimate via an upper bound on $e_t^v(N_u)$ for $1 \leq t \leq k$.

Set $B_u = V \setminus (N_u \cup \{u\})$. Define the set $S_x = \{\omega \in B_u \mid \exists l \in E_1^x(N_u) \text{ with } \omega \in l\}$ for any $x \in N_u$, i.e., the set of vertices contained in some edge which intersects N_u only at x . Note that $S_x \subset N_x \setminus (N_u \cup \{u\})$. The following results are necessary to our proof.

Lemma 7. *Let x and y be two adjacent vertices in H . Then $|N_{xy}| \leq 2k - 3$.*

Proof. Since x and y are adjacent, let l_{xy} be the edge containing both the vertices x and y . Suppose for the sake of a contradiction that $|N_{xy}| \geq 2k - 2$. Then there are at least k vertices in N_{xy} other than the common neighbors in l_{xy} . Since H is k -uniform, there must exist two distinct vertices v_1 and v_2 in N_{xy} such that the pairs xv_1 and xv_2 are contained in two distinct edges which are not the edge l_{xy} .

If the pairs yv_1 and yv_2 are contained in one edge incident to y , there must exist a vertex v_3 in N_{xy} such that yv_1 and yv_3 are contained in two distinct edges incident to y . Then either the edges containing xv_1 , yv_1 , yv_3 and xv_3 or the edges containing xv_2 , yv_2 , yv_3 and xv_3 form a Berge C_4 in H , a contradiction. Otherwise the pairs yv_1 and yv_2 are contained in two different edges incident to y , then the four edges containing the pairs xv_1 , xv_2 , yv_1 and yv_2 form a Berge C_4 in H , a contradiction. \square

Lemma 8. *Let x and y be two distinct vertices in N_u of H . If the pairs xu and yu are in distinct edges, then $S_x \cap S_y = \emptyset$.*

Proof. Let l_x, l_y be two edges incident to u such that $x \in l_x$ and $y \in l_y$. Since H is linear, it is obvious that $l_x \neq l_y$. If there were a vertex v in $S_x \cap S_y$, then each of the pairs xv and yv would be contained in edges l'_x and l'_y so that $l'_x \cap N_u = \{x\}$ and $l'_y \cap N_u = \{y\}$, whence $l'_x \neq l'_y$ and neither edge contains u . But then the four edges l_x, l_y, l'_x , and l'_y are a Berge C_4 , a contradiction. \square

Proof of Theorem 6. First we show that $\sum_{t=2}^k e_t^v(N_u)$ is no more than k for any $v \in N_u$ while $k \geq 3$. Suppose for a contradiction that there is a vertex $v \in N_u$ such that $\sum_{t=2}^k e_t^v(N_u) \geq k + 1$. Since H is linear, there is exactly one of these at least $k + 1$ edges containing u . Let l_i , $1 \leq i \leq k$, denote k other edges containing v . Since $|l_i \cap N_u| \geq 2$, we

select k distinct vertices $x_i \in l_i \cap N_u$ for $1 \leq i \leq k$. Then for the adjacent vertices u and v we have $|N_{uv}| \geq 2k - 2$, contradicting with Lemma 7. Thus $d(v) \leq e_1^v(N_u) + k$ for any $v \in N_u$.

Since H is linear and $e_1^v(N_u) \geq d(v) - k$ for any $v \in N_u$, we have $|N_u| = (k - 1)d(u)$ and $|S_v| \geq (k - 1)(d(v) - k)$. Thus,

$$\sum_{v \in N_u} |S_v| \geq \sum_{v \in N_u} (k - 1)(d(v) - k) = \sum_{v \in N_u} (k - 1)d(v) - k(k - 1)^2 d(u). \quad (13)$$

For each $1 \leq i \leq d(u)$, let l_i denote the i -th edge incident to u , and write $l_i = \{u, i_2, i_3, \dots, i_k\}$. Set $V_i = \bigcup_{j=2}^k S_{i_j}$ for each $1 \leq i \leq d(u)$. Obviously $\bigcup_{i=1}^{d(u)} V_i \subseteq B_u$.

By Lemma 8, we have $V_i \cap V_j = \emptyset$ since $l_i \cap l_j = \{u\}$ by linearity. Thus $\sum_{i=1}^{d(u)} |V_i| \leq |B_u|$. For a fixed edge l_i , we only need to estimate the upper bound of $|S_{i_p} \cap S_{i_q}|$ where $i_p, i_q \in l_i$. Note that

$$|V_i| = \left| \bigcup_{j=2}^k S_{i_j} \right| \geq \sum_{j=2}^k |S_{i_j}| - \sum_{2 \leq p < q \leq k} |S_{i_p} \cap S_{i_q}|.$$

By Lemma 7, we have

$$|S_{i_p} \cap S_{i_q}| \leq |N_{i_p i_q} \setminus (N_u \cup \{u\})| \leq |N_{i_p i_q} \setminus (l_i \setminus \{x, y\})| \leq (2k - 3) - (k - 2) = k - 1$$

since the vertices p and q are adjacent. Then

$$|V_i| \geq \sum_{j=2}^k |S_{i_j}| - \binom{k-1}{2} (k-1) = \sum_{j=2}^k |S_{i_j}| - \frac{(k-1)^2(k-2)}{2}.$$

Thus we have

$$\sum_{j=2}^k |S_{i_j}| \leq |V_i| + \frac{(k-1)^2(k-2)}{2} \quad (14)$$

and

$$\sum_{i=1}^{d(u)} \sum_{j=2}^k |S_{i_j}| \leq \sum_{i=1}^{d(u)} |V_i| + \sum_{i=1}^{d(u)} \frac{(k-1)^2(k-2)}{2}. \quad (15)$$

Since $\sum_{i=1}^{d(u)} |V_i| \leq |B_u| \leq n$, applying (15), we obtain

$$\begin{aligned} \sum_{i=1}^{d(u)} \sum_{j=2}^k |S_{i_j}| &\leq \sum_{i=1}^{d(u)} |V_i| + \sum_{i=1}^{d(u)} \frac{(k-1)^2(k-2)}{2} \\ &\leq |B_u| + \frac{(k-1)^2(k-2)}{2} d(u) \end{aligned}$$

$$\leq n - |N_u| + \frac{(k-1)^2(k-2)}{2}d(u). \quad (16)$$

Combining this with (13), then

$$\sum_{v \in N_u} (k-1)(d(v) - k) \leq \sum_{i=1}^{d(u)} \sum_{j=2}^k |S_{ij}| \leq n + \frac{(k-1)^2(k-2)}{2}d(u) - (k-1)d(u). \quad (17)$$

The first inequality follows $|S_v| \geq (k-1)(d(v) - k)$ for any $v \in N_u$ and the second because $|N_u| = (k-1)d(u)$. Simplifying (17), we have

$$\sum_{v \in N_u} (k-1)d(v) \leq n + \frac{k(k-1)(3k-5)}{2}d(u). \quad (18)$$

Since H is linear, it is obvious that $d(u) \leq \frac{n-1}{k-1}$. Combining (18) with Corollary 3, we get

$$\rho^2 \leq \frac{n}{(k-1)^2} + \frac{k(3k-5)}{2(k-1)}d(u) \leq \left(\frac{3}{2} + \frac{1}{2(k-1)}\right)(n-1) + \frac{1}{(k-1)^2}. \quad (19)$$

That is, $\rho \leq \sqrt{\frac{3}{2} + \frac{1}{2(k-1)}}(n-1)^{\frac{1}{2}} + O(\frac{1}{\sqrt{n}})$. This completes the proof. \square

Note that $\sum_{t=2}^k e_t^v(N_u) \leq k$ holds if $k \geq 3$ in the proof of Theorem 6. For a graph G of order n containing no C_4 , we have $e_2^v(N_u) \leq 1$ and $\rho(G) \leq \sqrt{n}$ after applying $e_2^v(N_u) \leq 1$ to the proof of Theorem 6. Recall that Nikiforov [24] proved that $\rho^2 - \rho \leq n - 1$ where ρ is the spectral radius of a graph G of order n containing no C_4 . Obviously, $\rho(G) \leq \sqrt{n}$ is asymptotically equivalent to the Nikiforov's result when n is sufficiently large.

3.3 Spectral radius of linear hypergraphs with at least girth five

The question we consider in this subsection is to determine the maximum spectral radius of k -uniform linear hypergraphs on n vertices of girth at least five, while the maximum number of edges in graphs of girth five is an old problem of Erdős [7]. Lazebnik and Verstraëte [19] showed that $ex_3^{\text{lin}}(n, \{C_3, C_4\}) = n^{3/2}/6 + O(n)$. Very recently this was strengthened by Ergemlidze, Györi, and Methuku, who showed in [8] that $ex_3^{\text{lin}}(n, \{C_3, C_4\}) \sim ex_3^{\text{lin}}(n, \{C_4\})$. Timmons [28] proved that $ex_k^{\text{lin}}(n, \{C_3, K_{2,t+1}\}) \leq \frac{\sqrt{t}}{k(k-1)}n^{3/2} + O(n)$ which indicated $ex_k^{\text{lin}}(n, \{C_3, C_4\}) \leq \frac{1}{k(k-1)}n^{3/2} + O(n)$. Now we investigated on the spectral version of k -uniform linear hypergraphs on n vertices of girth at least five while the proof of the result below is substantially simpler and shorter than the above proof.

Theorem 9. *Let \mathcal{H} denote the set of linear k -uniform hypergraphs of order n with girth at least five and ρ be the maximum spectral radius of hypergraphs in \mathcal{H} . For n sufficiently large, we have $\rho \leq \sqrt{\frac{n-1}{k-1}} + O(\frac{1}{\sqrt{n}})$.*

Proof. Let H be a linear hypergraphs of order n and size m with girth at least five. For an arbitrary and fixed vertex u , it is easy to verify that $E_{k-1}^v(N_u)$ is exactly the set of edges incident to u and $E_t^v(N_u) = \emptyset$ if $t \notin \{1, k-1\}$, where v is an arbitrary vertex in N_u .

Otherwise H contains a Berge C_3 , a contradiction. That is to say $\sum_{t=2}^k te_t(N_u) = (k-1)d(u)$.

Now we only need to provide an upper bound on $e_1(N_u)$.

As before, set $B_u = V \setminus \{N_u \cup \{u\}\}$ and $S_v = \{\omega \in B_u \mid \exists l \in E_1^v(N_u) \text{ with } \omega \in l\}$ for any $v \in N_u$. By Lemma 8, it is easy to verify that $S_x \cap S_y = \emptyset$ for an arbitrary pair x, y in N_u if x and y are nonadjacent. If x and y are adjacent, suppose for a contradiction that there is a vertex $z \in S_x \cap S_y$. Let l_{xy} be the edge containing both the vertices x and y . Similarly for l_{xz} and l_{yz} . Since H is linear, it is clear that l_{xy}, l_{xz} and l_{yz} form a Berge C_3 , a contradiction. Thus $e_1(N_u) \leq (n - (k-1)d(u))/(k-1)$.

By Corollary 3, we get

$$\rho^2 \leq \frac{1}{k-1} [e_1(N_u) + 2e_2(N_u) + \cdots + ke_k(N_u)] \leq \frac{n}{(k-1)^2} + \frac{(k-2)d(u)}{k-1}. \quad (20)$$

Since H is linear, it is obvious that $d(u) \leq \frac{n-1}{k-1}$, we get

$$\rho^2 \leq \frac{n-1}{k-1} + \frac{1}{(k-1)^2}. \quad (21)$$

Therefore $\rho \leq \sqrt{\frac{n-1}{k-1}} + O(\frac{1}{\sqrt{n}})$. □

If G be a graph of order n containing neither C_3 nor C_4 , it is easy to see that for n sufficiently large, Theorem 9 is asymptotically equivalent to the result of Favaron, Mahéo and Saclé [9] who showed that $\rho(G) \leq \sqrt{n-1}$ where $\rho(G)$ is the spectral radius of G .

4 Conclusion and open problems

Of course, the above results are just a small sample of spectral analogues for Turán-type hypergraph problems. Spectral extremal problems can be a rich font of interesting questions, since so many classical extremal problems have been investigated. See [17] for much more along these lines. Here we outline a few related to the above questions that we find particularly appealing.

1. What do the extremal spectral results for other hereditary *linear* hypergraph classes look like? For example, one might consider forbidden Berge cycles with lengths greater than 4, Berge $K_{2,t}$, the Fano plane, etc.
2. To get the upper bounds of Theorem 6 and Theorem 9, we used the inequality $d(u) \leq \frac{n-1}{k-1}$ which indicates that equality holds if there exists a vertex adjacent to other vertices in H . So how to construct the hypergraph with spectral radius attaining $O(\sqrt{n})$? As we know, the hyperstar containing either C_3 nor C_4 with $\frac{n-1}{k-1}$ edges sharing one vertex in common, can't reach the upper bound in Theorem 9, which is different from the case for graphs.

3. How would one generalize Lemma 1 to other tensors, such as the (signless) Laplacian tensor? How large can the maximum eigenvalue of the (signless) Laplacian tensor of a uniform hypergraph be if various subhypergraphs are forbidden?
4. Kalai asked the following beautiful question: Let \mathcal{F}_n denote the family of linear 3-uniform hypergraphs $H = (V, E)$ for which there exists an injective function $\phi : V \rightarrow \mathbb{R}^3$ so that the convex hulls of $\phi(e)$ and $\phi(f)$ intersect exactly in $\phi(e \cap f)$ for every $e, f \in E$. In other words, \mathcal{F}_n consists of those linear 3-uniform hypergraphs whose edges can be faithfully embedded as planar triangles in \mathbb{R}^3 . Then, if $f(n)$ is the maximum number of edges of any hypergraph in \mathcal{F}_n , is it true that $f(n) = o(n^2)$? Károlyi and Solymosi showed in [16] that $f(n) = \Omega(n^{3/2})$, but not much more is known about this question. We ask the spectral analogue: for $H \in \mathcal{F}_n$, is it true that $\rho(H) = o(n)$?

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