On automorphisms of direct products
of Cayley graphs on abelian groups

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Abstract

Let X and Y be connected Cayley graphs on abelian groups, such that no two distinct vertices of X have exactly the same neighbours, and the same is true about Y. We show that if the number of vertices of X is relatively prime to the number of vertices of Y, then the direct product $X \times Y$ has only the obvious automorphisms (namely, the ones that come from automorphisms of its factors X and Y). This was not previously known even in the special case where $Y = K_2$ has only two vertices. The proof of this special case is short and elementary. The general case follows from the special case by standard arguments.

Mathematics Subject Classifications: 05C25, 05C76

1 Introduction

The *canonical bipartite double cover* [13] of a graph X is the bipartite graph BX with $V(BX) = V(X) \times \{0, 1\}$, where

$$(v, 0) \text{ is adjacent to } (w, 1) \text{ in } BX \iff v \text{ is adjacent to } w \text{ in } X.$$

Letting S_2 be the symmetric group on the 2-element set $\{0, 1\}$, it is clear that $\operatorname{Aut} X \times S_2$ is a subgroup of $\operatorname{Aut} BX$. If this subgroup happens to be all of $\operatorname{Aut} BX$, then it is easy to see (and well known) that X must be connected, and must also be "twin-free" (see Definition 4.1 below).

B. Fernandez and A. Hujdurović [5] recently established that the converse is true when X is a circulant graph of odd order. This had been conjectured by Y.-L. Qin, B. Xia, and S. Zhou [11, Conj. 1.3], who proved the special case where X has prime order. See

the introductions of [5] and [11] for additional history and motivation. All graphs in this paper are undirected, with no multiple edges, but loops are allowed

Circulant graphs are examples of "Cayley graphs" (see Definition 2.1 below), and both sets of authors asked whether the converse can be generalized to all Cayley graphs on abelian groups of odd order ([5, Problem 3.3] and [11, p. 157]). This note provides a short, elementary proof that the desired generalization is indeed true:

Theorem 1.1. *If X is a twin-free, connected Cayley graph on a finite abelian group of odd order, then* $\operatorname{Aut} BX = \operatorname{Aut} X \times S_2$.

Remark 1.2. A graph X is said to be *stable* if $\operatorname{Aut} BX = \operatorname{Aut} X \times S_2$ [11, p. 155], so the theorem can be rephrased as the statement that:

> *Every twin-free, connected Cayley graph on a finite abelian group of odd order is stable.*

(However, the term "stable graph" is ambiguous, because it also has other meanings in graph theory [1, 6].)

Example 1.3 (Hujdurović-Mitrović [personal communication]). The word "abelian" cannot be deleted from the statement of the theorem. For example, if

$$G = \langle\, a, x \mid a^3 = x^7 = 1,\ a^{-1}xa = x^2 \,\rangle$$

is the nonabelian group of order 21, then $X = \operatorname{Cay}\big(G; \{a^{\pm 1}, x^{\pm 1}, (ax)^{\pm 1}\}\big)$ is twin-free and connected, but it can easily be verified by computer that $|\operatorname{Aut} X| = 42$ and $|\operatorname{Aut} BX| = 252$. (This was discovered with MAGMA and confirmed with sagemath.)

Remark 1.4 ([5, Rem. 1.3]). For any Cayley graph X on an abelian group of odd order, the theorem makes it possible to obtain the automorphism group of BX from the automorphism group of X. For simplicity, let us assume that X is loopless. Then there exist integers $c, d \geqslant 1$, and a twin-free, connected, abelian Cayley graph Y of odd order, such that $X \cong \overline{K_c} \wr (Y \wr \overline{K_d})$. Then, by a well-known theorem of Sabidussi [12] on the automorphism group of a wreath product of graphs, the theorem implies that $\operatorname{Aut} BX = S_c \wr \big((\operatorname{Aut} Y \times \mathbb{Z}_2) \wr S_d\big)$.

The canonical bipartite double cover BX can be realized as the *direct product* $X \times K_2$ (see Definition 5.1), and the theory of direct products (see Section 5) implies that the theorem can be generalized by replacing $BX = X \times K_2$ with $X \times Y$, where Y is any graph in a much more general family:

Corollary 1.5. *Let X be a twin-free, connected Cayley graph on a finite abelian group of odd order, and let Y be any twin-free, connected graph, such that either:*

1. *Y is not bipartite, and $|V(Y)|$ is relatively prime to $|V(X)|$, or*

2. *Y is bipartite, with bipartition $V(Y) = Y_0 \cup Y_1$, such that*

   (a) $|Y_0|$ and $|Y_1|$ are relatively prime to $|V(X)|$, and

(b) either $|Y_0| \neq |Y_1|$, or Y has an automorphism that interchanges Y_0 and Y_1.

Also assume that neither X nor Y is the one-vertex trivial graph. Then

$$\mathrm{Aut}(X \times Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y.$$

The statement of the corollary becomes much simpler if we assume that Y is also an abelian Cayley graph:

Corollary 1.6. *Let X and Y be twin-free, connected Cayley graphs on abelian groups, such that $|V(X)|$ is relatively prime to $|V(Y)|$. Also assume that neither X nor Y is the one-vertex trivial graph. Then*

$$\mathrm{Aut}(X \times Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y.$$

Remark 1.7. In Theorem 1.1, the Cayley graph X can be allowed to have an edge-colouring that is invariant under translation by elements of G (see Remark 3.4(2)). However, the proofs of Corollaries 1.5 and 1.6 do not allow colours on the edges.

Here is an outline of the paper:

§1. Introduction

§2. A crucial lemma

§3. Comments on the lemma (optional)

§4. Proof of the main theorem

§5. Review of direct products (proof of Corollaries 1.5 and 1.6)

2 A crucial lemma

As was already mentioned in the introduction, all graphs in this paper are undirected, with no multiple edges. Loops are allowed, but they are not necessary for any of the arguments, so readers are welcome to assume that all graphs are simple. Readers at the other extreme, who want to discuss multiple edges (or edge-colourings), are referred to Remark 3.4, but these complications are forbidden in this section.

Definition 2.1 ([7, p. 34])**.** Let S be a *symmetric* subset of an abelian group G. (This means that $-s \in S$, for all $s \in S$.) The corresponding *Cayley graph* $\mathrm{Cay}(G; S)$ is the graph whose vertices are the elements of G, and with an edge joining the vertices g and h if and only if $g = s + h$ for some $s \in S$.

We now state a simple observation that is probably already in the literature somewhere. (Although the same proof also applies to Cayley *digraphs*, we state the result only for Cayley *graphs*, because they are the topic of this note.)

Lemma 2.2. *Let φ be an automorphism of a Cayley graph $\mathrm{Cay}(G; S)$, and let $k \in \mathbb{Z}^+$. If*

1. *G is abelian, and*

2. *$ks \neq kt$ for all $s, t \in S$, such that $s \neq t$,*

then φ is an automorphism of $\mathrm{Cay}(G; kS)$, where $kS = \{\, ks \mid s \in S \,\}$.

Proof. Write $k = p_1 p_2 \cdots p_r$, where each p_i is prime, and let $k_i = p_1 p_2 \cdots p_i$ for $0 \leqslant i \leqslant r$. We will prove by induction on i that φ is an automorphism of $\mathrm{Cay}(G, k_i S)$. The base case is true by assumption, since $k_0 S = 1S = S$.

For $v, w \in G$, let $\#(v, w)$ be the number of walks of length p_i from v to w in the graph $\mathrm{Cay}(G, k_{i-1}S)$. These walks are in one-to-one correspondence with the p_i-tuples $(s_1, s_2, \ldots, s_{p_i})$ of elements of $k_{i-1}S$, such that $s_1 + s_2 + \cdots + s_{p_i} = w - v$. Since G is abelian, any cyclic rotation of $(s_1, s_2, \ldots, s_{p_i})$ also corresponds to a walk from v to w. Therefore, the set of these walks can be partitioned into sets of cardinality p_i, unless $w = p_i s + v$, for some $s \in k_{i-1}S$, in which case there is a walk of the form $v, s + v, 2s + v, \ldots, p_i s + v = w$. (Also note that s is unique, if it exists, by assumption (2).) Hence, we see that

$$\#(v, w) \not\equiv 0 \pmod{p_i} \quad \Longleftrightarrow \quad v \text{ is adjacent to } w \text{ in } \mathrm{Cay}(G; p_i k_{i-1}S).$$

Since $p_i k_{i-1} = k_i$, the desired conclusion that $\varphi \in \mathrm{Aut}\,\mathrm{Cay}(G, k_i S)$ now follows from the induction hypothesis that $\varphi \in \mathrm{Aut}\,\mathrm{Cay}(G, k_{i-1}S)$ (and the observation that automorphisms preserve the value of the function $\#$). $\qquad\square$

3 Comments on the lemma

This section is optional.

Remarks 3.1. Two comments on assumption (2) of Lemma 2.2:

1. This assumption holds for all $S \subseteq G$ if and only if $\gcd\big(k, |G|\big) = 1$.

2. This assumption can be weakened. For example, if k is prime, then it suffices to assume, for each $s \in S$, that $|\{\, t \in S \mid ks = kt \,\}|$ is not divisible by k.

Remark 3.2. Lemma 2.2 may be of independent interest. For example, it provides a short, fairly elementary proof of the known classification of edge-transitive graphs of prime order. (See the following corollary.) Can it also simplify the proofs of other known results [10] on automorphism groups of circulant graphs?

Corollary 3.3 (Chao [2]). *Let $\mathbb{Z}_p$ be the cyclic group of order p, where p is prime. A connected Cayley graph $X = \mathrm{Cay}(\mathbb{Z}_p; S)$ on $\mathbb{Z}_p$ is edge-transitive if and only if S is a coset of a subgroup of the multiplicative group $\mathbb{Z}_p^\times$.*

Proof. ($\Rightarrow$) First, note that $0 \notin S$, because a connected, edge-transitive graph with p vertices cannot have loops. Since p is prime, this implies $S \subseteq \mathbb{Z}_p^\times$.

Now, let $A_0 = \{\, \varphi \in \operatorname{Aut} X \mid \varphi(0) = 0 \,\}$ be the stabilizer of the vertex 0 in $\operatorname{Aut} X$. For every $k \in \mathbb{Z}_p^\times$, Lemma 2.2 tells us that kS is A_0-invariant. Since $S = 1S$ is also A_0-invariant, this implies that $kS \cap S$ is A_0-invariant. However, A_0 is transitive on S (because $S = N_X(0)$ and X is edge-transitive), so this implies that either $kS = S$ or $S \cap kS = \emptyset$. Since this is true for all k (and $S \subseteq \mathbb{Z}_p^\times$), this means that S is a *block of imprimitivity* for the regular representation of $\mathbb{Z}_p^\times$ [3, p. 12]. So S is an orbit of some subgroup H of $\mathbb{Z}_p^\times$ [3, Thm. 1.5A, pp. 13–14]: $S = Hz$ for some $z \in \mathbb{Z}_p^\times$. Since we are dealing with the regular representation, this means that S is a coset of H.

($\Leftarrow$) This is the easy direction (and does not require the assumption that p is prime). Assume S is a coset of the subgroup H of $\mathbb{Z}_p^\times$. Note that H acts by automorphisms on the group $\mathbb{Z}_p$ (because multiplication by any $k \in \mathbb{Z}_p^\times$ is an automorphism of $\mathbb{Z}_p$). The set S is invariant under H; indeed, H is transitive on S (because S is a coset of H). The proof is now completed by a well-known, elementary argument [11, Lem. 2.8]: H is a group of automorphisms of the Cayley graph X (because it is a group of automorphisms of $\mathbb{Z}_p$ that fixes S). Since H fixes the vertex 0, and acts transitively on the set S of neighbours of 0, this implies that X is edge-transitive. $\square$

Remarks 3.4. Unlike in [5, 11], we do not need to assume that $\operatorname{Cay}(G; S)$ is a simple graph.

1. Graphs may have loops.

2. We can allow edge-colourings of $\operatorname{Cay}(G; S)$ that are invariant under translation by elements of G: the colour of an edge (v, w) must be the same as the colour of the edge $(v + g, w + g)$. Such colourings come from colourings of S: choose a colour for each element s of S (such that $\mathsf{colour}(s) = \mathsf{colour}(-s)$, for all $s \in S$), and then apply this colour to each edge of the form $(v, s + v)$. Automorphisms are required to preserve the edge-colouring.

3. Lemma 2.2 remains valid in this setting. To see this, let

$$S_c = \{\, s \in S \mid \mathsf{colour}(s) = c \,\}, \quad \text{for each colour } c.$$

Lemma 2.2 (as stated, without considering any edge-colourings) implies that φ is an automorphism of $\operatorname{Cay}(G; kS_c)$. Saying that this is true for every c is exactly the same as saying that φ is a (colour-preserving) automorphism of $\operatorname{Cay}(G; kS)$, if we let

$$\mathsf{colour}(t) = \{\, \mathsf{colour}(s) \mid t = ks, \ s \in S \,\} \quad \text{for each } t \in kS.$$

Also note that this proof only applies the original version of Lemma 2.2 to S_c, not all of S, so hypothesis (2) can be replaced with the weaker assumption that:

(2′) *for every colour* c, *we have* $ks \neq kt$ *for all* $s, t \in S_c$, *such that* $s \neq t$.

(If s and t have different colours, then it is not necessary to assume $ks \neq kt$.)

4. Technically, we do not allow graphs to have multiple edges. However, since the statements of the results only consider automorphism groups, not other graphical properties, the multiplicity of an edge can be encoded as part of its colour (or "label"). For example, an edge coloured "2B, 3R, W" could be thought of as representing 2 blue edges, 3 red edges, and a white edge, all with the same endpoints.

4 Proof of the main theorem

Definition 4.1 (Kotlov-Lovász [9])**.** A graph X is *twin-free* if there do not exist two distinct vertices v and w, such that $N_X(v) = N_X(w)$, where $N_X(v)$ denotes the set of neighbours of v in X.

Remark 4.2. Synonyms for "twin-free" include "irreducible" [5], "R-thin" [8, p. 91], and "vertex-determining" [11].

Let S be a symmetric subset of a finite abelian group G of odd order, such that the Cayley graph $X = \mathrm{Cay}(G; S)$ is twin-free and connected. Given $\varphi \in \mathrm{Aut}\, BX$, we wish to show that $\varphi \in \mathrm{Aut}\, X \times S_2$.

Note that
$$BX = \mathrm{Cay}\big(G \times \mathbb{Z}_2; S \times \{1\}\big),$$
and that BX is connected and bipartite, with bipartition sets $G \times \{0\}$ and $G \times \{1\}$. Since $\mathrm{Aut}\, X \times S_2$ contains an element that interchanges these two sets, we may assume
$$\varphi\big(G \times \{i\}\big) = G \times \{i\} \text{ for } i = 0, 1.$$

Remark 4.3. If the edges of X have been coloured (as allowed by Remark 3.4(2)):

1. Each element x of $N_X(v)$ is labelled with the colour of the edge from v to x. Therefore, saying that $N_X(v) = N_X(w)$ means that, for each $x \in N_X(v)$, the colour of the edge joining x to v is same as the colour of the edge joining x to w. Hence, an edge-coloured graph may be twin-free, even though its underlying uncoloured graph is not twin-free.

2. The edges of BX are coloured by colouring the edge from $(v, 0)$ to $(w, 1)$ with whatever colour appears on the edge from v to w in X.

Let $k = |G| + 1$, so $k \equiv 1 \pmod{|G|}$ and $k \equiv 0 \pmod 2$. Then, for every $(s, 1) \in S \times \{1\}$, we have $k(s, 1) = (s, 0)$, so Lemma 2.2 tells us that φ is an automorphism of the graph $\mathrm{Cay}\big(G \times \mathbb{Z}_2; S \times \{0\}\big)$ (which is the disjoint union of two copies of X). Hence, after multiplying by an element of $\mathrm{Aut}\, X \times \{\iota\}$, where ι is the identity element of S_2, we may assume that $\varphi(v) = v$ for all $v \in G \times \{0\}$.

It is now easy to complete the proof, by using the assumption that X is twin-free. For all $g \in G$, we have

$$\begin{aligned}
N_X(g) \times \{0\} &= \varphi\big(N_X(g) \times \{0\}\big) && (\varphi(v) = v \text{ for all } v \in G \times \{0\}) \\
&= \varphi\big(N_{BX}(g,1)\big) && (\text{definition of } BX) \\
&= N_{BX}\big(\varphi(g,1)\big) && (\varphi \text{ is an automorphism}) \\
&= N_{BX}(g',1) && (\text{where } \varphi(g,1) = (g',1)) \\
&= N_X(g') \times \{0\} && (\text{definition of } BX).
\end{aligned}$$

Since X is twin-free, this implies $g = g'$, so $\varphi(g,1) = (g',1) = (g,1)$, which means $\varphi(v) = v$ for all $v \in G \times \{1\}$. Since this equality also holds for all $v \in G \times \{0\}$, we conclude that φ is the identity element of $\operatorname{Aut} BX$, and is therefore in the subgroup $\operatorname{Aut} X \times S_2$. $\qquad\square$

5 Review of direct products

Definition 5.1 ([8, p. 36]). The *direct product* of two graphs X and Y is the graph $X \times Y$ with $V(X \times Y) = V(X) \times V(Y)$, and

$$(x_1, y_1) \text{ is adjacent to } (x_2, y_2) \text{ in } X \times Y \iff \begin{array}{l} x_1 \text{ is adjacent to } x_2 \text{ in } X, \text{ and} \\ y_1 \text{ is adjacent to } y_2 \text{ in } Y. \end{array}$$

Remark 5.2 ([8, p. 36]). The literature has numerous other names for the direct product, including "tensor product," "Kronecker product," "cardinal product," and "conjunction."

The graph X in Theorem 1.1 is allowed to have edge-colours and multiple edges, but the theory of automorphisms of direct products does not seem to have been developed in this generality, so:

Assumptions 5.3. In this section (and, therefore, in Corollaries 1.5 and 1.6), graphs do *not* have edge-colours or multiple edges (but they may have loops).

As was mentioned in Section 1, we have $BX = X \times K_2$. Generalizing the comments there about $\operatorname{Aut} BX$, it is clear that $\operatorname{Aut} X \times \operatorname{Aut} Y$ is a subgroup of $\operatorname{Aut}(X \times Y)$, and that if this subgroup happens to be all of $\operatorname{Aut}(X \times Y)$, then X and Y must be connected, and must also be twin-free. (We ignore the situation where one of the graphs is the one-vertex trivial graph.)

For direct products of non-bipartite graphs, the converse holds if and only if a certain "coprimality" condition holds. However, instead of stating the full strength of this classical theorem of W. Dörfler, we present only a simpler, weakened version of the result:

Theorem 5.4 (Dörfler, cf. [4] or [8, Thm. 8.18, p. 103]). *Let X and Y be twin-free, connected, non-bipartite graphs of relatively prime orders. Then $\operatorname{Aut}(X \times Y) = \operatorname{Aut} X \times \operatorname{Aut} Y$.*

The situation is more complicated (and not yet understood) when one of the factors of a direct product is bipartite. However, the following facts shed some light, as will be seen in the proposition that follows.

Facts 5.5. Let X and Y be connected graphs.

1. The *Cartesian skeleton* of X is a certain graph $\mathcal{S}X$ that is defined from X, such that $V(\mathcal{S}X) = V(X)$ [8, Defn. 8.2, p. 95].

2. Every automorphism of $X \times Y$ is also an automorphism of $\mathcal{S}(X \times Y)$ [8, Prop. 8.11, p. 97].

3. If X and Y are twin-free (and have more than one vertex), then $\mathcal{S}(X \times Y) = \mathcal{S}X \square \mathcal{S}Y$ [8, Prop. 8.10, p. 96], where $\square$ denotes the *Cartesian product* [8, p. 35].

4. If $|V(X)|$ is relatively prime to $|V(Y)|$, then $\operatorname{Aut}(X \square Y) = \operatorname{Aut}X \times \operatorname{Aut}Y$ [8, Cor. 6.12, p. 70].

5. If X is not bipartite, then $\mathcal{S}X$ is connected [8, Prop. 8.13(i), p. 98].

6. If Y is bipartite, then $\mathcal{S}Y$ has precisely two connected components, and their vertex sets are the bipartition sets of Y [8, Prop. 8.13(ii), p. 98].

The following straightforward consequence of these facts is presumably known to experts, but we do not have a reference. The gist is that, in order to understand the automorphism group of $X \times Y$, where Y is bipartite, it often suffices to understand the special case where $Y = K_2$.

Proposition 5.6. *Let X and Y be twin-free, connected graphs that have at least one edge, such that:*

1. *X is not bipartite,*

2. *Y is bipartite, with bipartition $V(Y) = Y_0 \cup Y_1$, such that*

    (a) $|Y_0|$ and $|Y_1|$ are relatively prime to $|V(X)|$, and

    (b) either Y has an automorphism that interchanges Y_0 and Y_1, or $|Y_0| \neq |Y_1|$, and

3. *$\operatorname{Aut}BX = \operatorname{Aut}X \times S_2$.*

Then $\operatorname{Aut}(X \times Y) = \operatorname{Aut}X \times \operatorname{Aut}Y$.

Proof. Let $\varphi \in \operatorname{Aut}(X \times Y)$. From Fact 6, we know that $\mathcal{S}Y$ has two connected components C_0 and C_1, where $V(C_0) = Y_0$ and $V(C_1) = Y_1$. By Facts 2 and 3, we have

$$\varphi \in \operatorname{Aut}\big(\mathcal{S}(X \times Y)\big) = \operatorname{Aut}(\mathcal{S}X \square \mathcal{S}Y) = \operatorname{Aut}\big(\mathcal{S}X \square (C_0 \cup C_1)\big).$$

If $|V(C_0)| = |V(C_1)|$, then (by hypothesis (2b)) Y has an automorphism that interchanges C_1 and C_2; therefore, we may assume that φ fixes each connected component of $\mathcal{S}X \square \mathcal{S}Y$. This means that φ restricts to an automorphism φ_i of $\mathcal{S}X \square C_i$ (for $i = 0, 1$). Since $\mathcal{S}X$ is connected (by Fact 5), and the connected component C_i is obviously also connected,

Fact 4 (and hypothesis (2a)) tells us that there exist permutations χ_i of $V(X)$ and η_i of Y_i, such that

$$\varphi(x,y) = \big(\chi_i(x), \eta_i(y)\big) \text{ for all } x \in V(X) \text{ and } y \in Y_i.$$

Choose adjacent vertices y_0 and y_1 of Y, with $y_i \in Y_i$, and let $y_i' = \eta_i(y_i)$. Let B and B' be the subgraphs of $X \times Y$ induced by $V(X) \times \{y_0, y_1\}$ and $V(X) \times \{y_0', y_1'\}$, respectively, so $B' = \varphi(B)$. By definition of the direct product, the maps $(x,i) \mapsto (x, y_i)$ and $(x, i) \mapsto (x, y_i')$ are isomorphisms from BX to B and B'. Therefore, the map $(x,i) \mapsto \big(\chi_i(x), i\big)$ is an automorphism of BX. So Assumption (3) tells us that $\chi_0 = \chi_1$. This means that the $V(X)$-component of $\varphi(x,y)$ depends only on x. (And we already knew that the $V(Y)$-component of $\varphi(x,y)$ depends only on y.) This easily implies $\varphi \in \operatorname{Aut} X \times \operatorname{Aut} Y$. $\square$

Proof of Corollary 1.5. Combine Theorem 5.4 and Proposition 5.6 (and note that Theorem 1.1 verifies hypothesis 5.6(3)). $\square$

Proof of Corollary 1.6. Since $|V(X)|$ is relatively prime to $|V(Y)|$, we may assume $|V(X)|$ is odd (by interchanging X and Y, if necessary). We may also assume Y is bipartite, for otherwise Corollary 1.5(1) applies. Now, we verify the hypotheses of Corollary 1.5(2). Since $|V(Y_0)| = |V(Y_1)| = |V(Y)|/2$, we know that hypothesis (2a) holds. Since Y is vertex-transitive (because it is a Cayley graph), hypothesis (2b) also holds. $\square$

Acknowledgments

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