Sharp minimum degree conditions for the existence of disjoint theta graphs

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Submitted: Jun 23, 2020; Accepted: Aug 10, 2021; Published: Sep 10, 2021

Abstract

In 1963, Corrádi and Hajnal showed that if G is an n-vertex graph with $n \geqslant 3k$ and $\delta(G) \geqslant 2k$, then G will contain k disjoint cycles; furthermore, this result is best possible, both in terms of the number of vertices as well as the minimum degree. In this paper we focus on an analogue of this result for theta graphs. Results from Kawarabayashi and Chiba et al. showed that if $n = 4k$ and $\delta(G) \geqslant \lceil \frac{5}{2}k \rceil$, or if n is large with respect to k and $\delta(G) \geqslant 2k+1$, respectively, then G contains k disjoint theta graphs. While the minimum degree condition in both results are sharp for the number of vertices considered, this leaves a gap in which no sufficient minimum degree condition is known. Our main result in this paper resolves this by showing if $n \geqslant 4k$ and $\delta(G) \geqslant \lceil \frac{5}{2}k \rceil$, then G contains k disjoint theta graphs. Furthermore, we show this minimum degree condition is sharp for more than just $n = 4k$, and we discuss how and when the sharp minimum degree condition may transition from $\lceil \frac{5}{2}k \rceil$ to $2k+1$.

Mathematics Subject Classifications: 05C35

1 Introduction

All graphs in this paper are simple, unless otherwise noted. Additionally, "disjoint" is always taken to mean "vertex-disjoint." We say "G contains H" to mean that the graph G contains the graph H as a subgraph. Given a graph G, we use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of G, respectively, and for a vertex v, we use $v \in G$ to denote $v \in V(G)$. For a subgraph H of G, and for a vertex $v \in G$ (where v is not necessarily in H), the neighborhood of v in H is denoted by $N_H(v)$, and the number of neighbors of v in H will be written by $d_H(v)$. We use $|G|$ for $|V(G)|$, $\overline{G}$ for the complement of G, and $\delta(G)$ for its minimum degree. Furthermore, $\sigma_2(G)$ denotes the minimum Ore degree (sometimes called the minimum degree sum), which is given by the minimum of $d_G(x) + d_G(y)$ over all non-adjacent pairs of distinct vertices x and y in G (when G is complete, we say $\sigma_2(G) = \infty$).

The notation K_n is used to denote the complete graph on n vertices, and $K_{k_1,\dots,k_t}$ is the complete t-partite graph with parts of size $k_1, \dots, k_t$. For graphs G and H, $G \cup H$ represents the disjoint union of G and H, and $G \vee H$ is the join of G and H. Furthermore for positive integers $p \geqslant 2$, pH denotes the disjoint union of p copies of H.

A chorded cycle is a graph with a spanning cycle and at least one additional edge. A theta graph is a graph formed by connecting two distinct vertices with three independent paths. The notation $\theta_{i,j,k}$ is used to denote a theta graph formed by connecting two distinct vertices with three independent paths of length i, j, and k, respectively (length refers to the number of edges). For example K_4^-, which is obtained by removing an edge from K_4, can also be denoted as $\theta_{1,2,2}$. Lastly, the Paw is the 4-vertex graph formed by adding an edge to $K_{1,3}$.

In 1963, Corrádi and Hajnal verified a conjecture of Erdős by proving the following.

Theorem 1 (Corrádi and Hajnal [3])**.** *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph with $n \geqslant 3k$ and $\delta(G) \geqslant 2k$, then G contains k disjoint cycles.*

This result is sharp both in terms of the number of vertices as well as the minimum degree. Additionally, Theorem 1 has been extended in several ways by finding sharp minimum degree conditions guaranteeing the existence of a fixed number of objects such as chorded cycles [4], chorded cycles with at least two chords [5], and even combinations of cycles and chorded cycles [1, 10], to name a few. In this paper we consider an analogue to Theorem 1 regarding theta graphs, with the first result towards this type of analogue given in [6].

Theorem 2 (Kawarabayashi [6])**.** *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph with $n = 4k$ and $\delta(G) \geqslant \frac{5}{2}k$, then G contains a K_4^--factor.*

Note that a K_4^--factor is a spanning subgraph consisting of disjoint copies of K_4^-. Since the smallest theta graph possible is K_4^-, we immediately deduce the following corollary, which is actually equivalent to Theorem 2. We add the ceiling notation for the purposes of better discussing the sharpness of the statement.

Corollary 3. *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph with $n = 4k$ and $\delta(G) \geqslant \lceil \frac{5}{2}k \rceil$, then G contains k disjoint theta graphs.*

Both Theorem 2 and Corollary 3 are sharp in multiple senses. First, since no theta graph exists on fewer than four vertices, we cannot have k disjoint theta graphs in a graph with fewer than $4k$ vertices. Furthermore, $K_{k-1,\lfloor \frac{3k+1}{2} \rfloor,\lceil \frac{3k+1}{2} \rceil}$ is a graph on $4k$ vertices with minimum degree $\lceil \frac{5}{2}k \rceil - 1$ that does not have k disjoint theta graphs (this is discussed in Section 2). This shows the minimum degree condition in Theorem 2 and Corollary 3 is best possible.

However Corollary 3 is restricted to only when $n = 4k$. This leads to a natural question of what minimum degree conditions suffice to guarantee k disjoint theta graphs when $n > 4k$? This question was partially answered by Chiba et al. in the following theorem and subsequent corollary.

Theorem 4 (Chiba-Fujita-Kawarabayashi-Sakuma [2])**.** *Let $k \in \mathbb{Z}^+$, let $\alpha = 9600k^2 + 36k + 11$, and let $c_k = 4(3k+1)\alpha^{3(k-1)}(3(2k-1))^{2(k-1)} + (12+\alpha)(k-1)$. If G is an n-vertex graph with $n \geqslant c_k$ and $\delta(G) \geqslant 2k$, then one of the following holds:*

(i) G contains k vertex-disjoint theta graphs;

(ii) $k \geqslant 2$ and $\overline{K_{2k-1}} \vee pK_2 \subseteq G \subseteq K_{2k-1} \vee pK_2$ for some integer $p \geqslant k$;

(iii) $k = 1$ and each block of G is either a K_2 or a cycle.

Corollary 5. *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph with $n \geqslant c_k$ and $\delta(G) \geqslant 2k+1$, then G contains k disjoint theta graphs.*

The example given in (ii) of Theorem 4 shows that the minimum degree condition in Corollary 5 is sharp. Thus, Corollaries 3 and 5 provide sharp minimum degree conditions of $\lceil \frac{5}{2}k \rceil$ and $2k+1$, respectively, when $n = 4k$ and $n \geqslant c_k$, respectively. The purpose of this paper is to address the gap when $4k < n < c_k$ in which we have no minimum degree condition, let alone a sharp one, that guarantees the existence of k disjoint theta graphs.

That said, in [7], it is shown that every n-vertex graph with $n \geqslant 4k$ and $\delta(G) \geqslant \frac{1}{2}(n+k-1)$ has k disjoint copies of K_4^-, and furthermore the minimum degree condition in this statement is sharp. However in this paper we do not restrict ourselves to only finding disjoint copies of K_4^-. As a result, it is possible that a much lower minimum degree condition suffices to guarantee k disjoint theta graphs. Indeed, our following main result extends the minimum degree condition in Theorem 2 and Corollary 3 for all n-vertex graphs with $n \geqslant 4k$.

Theorem 6. *For all $k \in \mathbb{Z}^+$, if G is an n-vertex graph with $n \geqslant 4k$ and $\delta(G) \geqslant \lceil \frac{5k}{2} \rceil$, then G contains k disjoint theta graphs.*

As mentioned, $K_{k-1,\lfloor \frac{3k+1}{2} \rfloor, \lceil \frac{3k+1}{2} \rceil}$ is a $4k$-vertex graph with minimum degree $\lceil \frac{5}{2}k \rceil - 1$, that does not contain k disjoint theta graphs. Thus, Theorem 6 is best possible for general graphs, both in terms of the number of vertices as well as the minimum degree.

The breakdown of this paper is as follows. In Section 2, we construct graphs that show the minimum degree condition in Theorem 6 is sharp for all n-vertex graphs with $4k \leqslant n < 5k$. The proof of Theorem 6 is distributed across Sections 3 and 4. In particular, Section 3 contains notation and structural lemmas that will be used throughout the cases presented in Section 4. Lastly in Section 5, we use a result of Komlós in [8] to consider if perhaps the minimum degree condition in Theorem 6 can be reduced to $2k+1$ if we restrict n to be at least $5k$. In addition, we use a result of Kühn, Osthus, and Treglown in [9] to discuss minimum degree sum versions, and we pose questions regarding potential future work.

2 Sharpness

In this section we construct a family of graphs that show Theorem 6 is sharp when $4k \leqslant n < 5k$. In the following we will use the identities that for $x, y \in \mathbb{Z}$, $\lfloor x - \frac{y}{2} \rfloor = x - \lceil \frac{y}{2} \rceil$, $\lceil x - \frac{y}{2} \rceil = x - \lfloor \frac{y}{2} \rfloor$, and $x = \lfloor \frac{x}{2} \rfloor + \lceil \frac{x}{2} \rceil$.

Define $G(t, n) = K_{t, \lfloor \frac{n-t}{2} \rfloor, \lceil \frac{n-t}{2} \rceil}$. For $k, n \in \mathbb{Z}^+$ such that $4k+1 \leqslant n \leqslant 5k$, define $H_{k,n} = G(5k-n, n)$. Observe that the partite sets of $H_{k,n}$ are of size $5k-n$, $\lfloor n - \frac{5k}{2} \rfloor = n - \lceil \frac{5k}{2} \rceil$, and $\lceil n - \frac{5k}{2} \rceil = n - \lfloor \frac{5k}{2} \rfloor$. Note that since $n \leqslant 5k$, it is possible that $5k - n = 0$ making $H_{k,n}$ bipartite. Further as $n \geqslant 4k+1$, the partite set of size $5k-n$ is the smallest partite set, and it has size at most $k-1$. So $\delta(H_{k,n}) = (5k-n) + (n - \lceil \frac{5k}{2} \rceil) = \lfloor \frac{5k}{2} \rfloor$. However, the partite sets of size $n - \lfloor \frac{5k}{2} \rfloor$ and $n - \lceil \frac{5k}{2} \rceil$ have the same size exactly when k is even. Thus when k is even, $\lfloor \frac{5k}{2} \rfloor = \lceil \frac{5k}{2} \rceil$ so that $\delta(H_{k,n}) = \lceil \frac{5k}{2} \rceil$. When k is odd, $\lfloor \frac{5k}{2} \rfloor = \lceil \frac{5k}{2} \rceil - 1$ so that $\delta(H_{k,n}) = \lceil \frac{5k}{2} \rceil - 1$.

We claim that the only way to find k disjoint theta graphs in $H_{k,n}$ is to use every vertex. Observe that the maximum number of 4-vertex theta graphs in $H_{k,n}$ is $5k-n$, which leaves $n - 4(5k-n) = 5n - 20k$ vertices in $H_{k,n}$. The maximum number of theta graphs on at least five vertices that can be formed from $5n - 20k$ vertices is $\frac{5n-20k}{5} = n - 4k$. This yields a total of $(5k-n) + (n-4k) = k$ disjoint theta graphs. Thus, the maximum number of disjoint theta graphs that can be found in $H_{k,n}$ is k, and furthermore this requires all the vertices of $H_{k,n}$.

Define $H'_{k,n}$ to be the graph formed from $H_{k,n}$ by deleting a vertex from the largest partite set (i.e., the partite set of size $n - \lfloor \frac{5k}{2} \rfloor$). We claim that $H'_{k,n}$ is our sharpness example to Theorem 6, when $4k \leqslant n < 5k$. First observe that $4k \leqslant |H'_{k,n}| < 5k$, and by the work in the previous paragraph, $H'_{k,n}$ does not have k disjoint theta graphs. To find the minimum degree of $H'_{k,n}$, recall $\delta(H_{k,n}) = \lceil \frac{5k}{2} \rceil$ when k is even and $\delta(H_{k,n}) = \lceil \frac{5k}{2} \rceil - 1$ when k is odd. When k is even, the partite sets of size $n - \lfloor \frac{5k}{2} \rfloor$ and $n - \lceil \frac{5k}{2} \rceil$ are the same, so that forming $H'_{k,n}$ lowers the minimum degree by exactly one; hence $\delta(H'_{k,n}) = \lceil \frac{5k}{2} \rceil - 1$. When k is odd, the partite set of size $n - \lfloor \frac{5k}{2} \rfloor$ has exactly one more vertex than the partite set of size $n - \lceil \frac{5k}{2} \rceil$, so that forming $H'_{k,n}$ does not change the minimum degree; hence $\delta(H'_{k,n}) = \lceil \frac{5k}{2} \rceil - 1$. It is also worth noting that when k is odd, $\sigma_2(H'_{k,n}) = 2(\lceil \frac{5k}{2} \rceil - 1) = 5k - 1$, and when k is even, $\sigma_2(H'_{k,n}) = 2(\lceil \frac{5k}{2} \rceil - 1) = 5k - 2$.

Thus $H'_{k,n}$ shows that Theorem 6 is sharp when $4k \leqslant n < 5k$. Furthermore, $H'_{k,4k}$ is exactly the graph constructed in [6] used to show Theorem 2 and Corollary 3 are sharp.

3 Setup and Structural Lemmas

In this section we provide the setup behind our proof of Theorem 6. In addition, we present notation and structural lemmas that will be used throughout the proof of Theorem 6, which is primarily contained in Section 4.

3.1 Notation

Let G be a graph, $v \in V(G)$, and A and B be two subsets of $V(G)$, not necessarily disjoint. We let $N_B(v)$ denote $N_G(v) \cap B$, and let both $\|v, B\|$ and $d_B(v)$ denote $|N_B(v)|$. We also let $\|A, B\| = \sum_{v \in A} \|v, B\|$. For every collection of subgraphs $\mathcal{H}$ of G, we let $V(\mathcal{H}) = \bigcup_{H \in \mathcal{H}} V(H)$. If H is a subgraph of G, we often replace $V(H)$ with H in the above notation (e.g., $N_H(v) = N_{V(H)}(v)$, $\|v, H\| = \|v, V(H)\|$, and $\|A, H\| = \|A, V(H)\|$). Similarly, we often replace $V(\mathcal{H})$ with $\mathcal{H}$ when $\mathcal{H}$ is a collection of subsets of G (e.g., $\|A, \mathcal{H}\| = \|A, V(\mathcal{H})\|$). Furthermore, this notation is symmetric so that $\|A, B\| = \|B, A\|$.

If G is a graph and $A \subseteq V(G)$, we let $G[A]$ denote the subgraph of G induced by the vertices of A. If H is a subgraph of G, we let $H + A = G[V(H) \cup A]$ and $H - A = G[V(H) \setminus A]$. If $|A|$ is small, we often replace A with the vertices of A in the above notation (e.g., if $A = \{v\}$, we use $H + v = H + A$ and $H - v = H - A$). If F is a subgraph of G, we let $H + F = H + V(F)$ and $H - F = H - V(F)$.

A $(\mathcal{T}, R)$-partition of a graph G is a partition of G into a collection, denoted by $\mathcal{T}$, of $k-1$ disjoint theta graphs, and an induced subgraph R that contains no theta graph. We will use the notation $T \in \mathcal{T}$ to mean that T is one of the disjoint theta graphs in the collection $\mathcal{T}$. Here we emphasize that for every vertex $v \in G$, either $v \in R$, or $v \in T$ for some $T \in \mathcal{T}$, and furthermore, $d_G(v) = \|v, \mathcal{T}\| + \|v, R\|$.

3.2 Setup

We now begin the proof of Theorem 6. Suppose on the contrary that for some $k \in \mathbb{Z}^+$, there exist n-vertex graphs with $n \geqslant 4k$ and minimum degree at least $\lceil \frac{5}{2}k \rceil$ that do not contain k disjoint theta graphs. Among these graphs choose G to be one that is edge-maximal with respect to not having k disjoint theta graphs. That is, G does not contain k disjoint theta graphs, however for all $e \notin E(G)$, $G + e$ does contain k disjoint theta graphs. Since G cannot be complete (otherwise it would contain k disjoint theta graphs as $n \geqslant 4k$), such $e \notin E(G)$ exists.

Since $G + e$ contains k disjoint theta graphs, G must contain $k-1$ disjoint theta graphs, and furthermore these theta graphs cover all but at least four vertices of G. That is, G contains a $(\mathcal{T}, R)$-partition in which $|R| \geqslant 4$.

Among all $(\mathcal{T}, R)$-partitions of G, we define an optimal $(\mathcal{T}, R)$-parition to be one such that all of the following hold:

(O1) $|V(\mathcal{T})|$ is minimized;

(O2) subject to (O1), the number of chorded cycles in $\mathcal{T}$ is maximized;

(O3) subject to (O1) and (O2), the length of the longest path in R is maximized;

(O4) subject to (O1) - (O3), when $R \in \{P_4, C_4, \text{ Paw}\}$, $R \in \{C_4, \text{ Paw}\}$ is preferred over P_4;

(O5) subject to (O1) - (O4), when $R \in \{C_4, \text{ Paw}\}$, $G[T] \cong K_4$ is preferred over $G[T] \cong K_4^-$, for $T \in \mathcal{T}$;

(O6) if all previous have been satisfied, we prefer $R \cong$ Paw over $R \cong C_4$.

Since we have already shown G contains a $(\mathcal{T}, R)$-partition in which $|R| \geqslant 4$, (O1) implies that $|R| \geqslant 4$ even when the partition is optimal. In all the following, we now fix a $\mathcal{T}$ and R such that they form an optimal $(\mathcal{T}, R)$-partition, and let P denote a longest path in R.

3.3 Structural Lemmas

We now prove a series of structural lemmas that will be used throughout Sections 3 and 4.

Lemma 7. *Let $v \in R$ and $T \in \mathcal{T}$. Then $\|v, T\| \leqslant 4$, and furthermore:*

1. *if $\|v, T\| = 4$, then $|T| = 4$;*
2. *if $\|v, T\| = 3$, then either $|T| = 4$, or*
   (a) *$G[T] \cong \theta_{1,2,3}$, and the neighors of v are exactly the vertices not incident to the chord of T, or*
   (b) *$G[T] \cong K_{2,3}$, and $G[T + v] \cong K_{3,3}$.*

Proof. Fix $v \in R$ and $T \in \mathcal{T}$. Observe first that every theta graph has a Hamiltonian path; fix such a path and call it P_T. For any two vertices v_1, v_2 on P_T, we let $v_1 P_T v_2$ denote the subpath of P_T starting at v_1 and ending at v_2.

If $\|v, T\| \geqslant 4$, then let x_1, x_2, x_3, x_4 be neighbors of v appearing along P_T in this order (not necessarily consecutive). Furthermore, choose these so that v has no other neighbors in $x_1 P_T x_4$. Observe that we can replace T with a theta graph formed from v and $x_2 P_T x_4$, which is a chorded cycle. This implies x_1 and x_2 are consecutive along P_T, and furthermore, x_1 is an endpoint of P_T, otherwise we contradict (O1) in both instances. By symmetry, x_3 and x_4 are also consecutive along P_T, and x_4 is the other endpoint of P_T. This also implies T is a chorded cycle, otherwise we contradict (O2). So T has a Hamiltonian cycle, and up to relabelling the vertices and P_T, x_2 and x_3 are consecutive along P_T. Thus $|T| = 4$, and if $\|v, T\| \geqslant 4$, then $\|v, T\| = 4$.

Now suppose $\|v, T\| = 3$ with neighbors x_1, x_2, x_3 appearing along P_T in this order (not necessarily consecutive). In all the following, assume $|T| \geqslant 5$.

Claim 8. *We may assume x_1 and x_3 are the endpoints of P_T.*

Proof. If not then without loss of generality, let p_1 be the endpoint of P_T such that x_1 is an interior vertex on the path $p_1 P_T x_2$. By replacing T with the chorded cycle formed from v and $x_1 P_T x_3$, we see that p_1 and x_1 must be consecutive along P_T otherwise we contradict (O1), and additionally x_3 is an endpoint of P_T. This same replacement now implies that T is a chorded cycle, otherwise we contradict (O2).

Since T is a chorded cycle, it has a Hamiltonian cycle; denote the Hamilton cycle by $C_T = p_1 P_T x_3 p_1$ where we can rechoose P_T if necessary so that x_3 is an endpoint. We see that there is at most one vertex between x_i and x_{i+1} for $i \in \{1, 2\}$, otherwise v

together with the path from x_{i+1} along C_T to x_i that includes p_1, forms a theta graph with fewer vertices than T, contradicting (O1). Furthermore, if x_i and x_{i+1} are consecutive along C_T, then we can form our desired Hamiltonian path in T with, up to relabelling, x_1 and x_3 as its endpoints. So we may assume $|T| = 6$, and label the vertices so that $C_T = p_1x_1p_2x_2p_3x_3p_1$.

Recall that T must be a chorded cycle. If a chord of the form x_1x_2 exists, then $G[\{v, x_1, p_2, x_2\}]$ forms K_4^-. If a chord of the form p_1p_2 exists, then $G[\{v, p_1, x_1, p_2, x_2\}]$ forms a theta graph on five vertices. If a chord of the form x_1p_3 exists, then $G[\{v, x_1, p_3, x_3, p_1\}]$ forms a theta graph on five vertices. Each of these contradict (O1), and since every chord in T must look like one of these, this proves the claim. $\square$

If T is a chorded cycle, then by relabelling the vertices and P_T, we may assume P_T together with the edge x_1x_3 creates a Hamiltonian cycle of T. If there are at least two interior vertices in $x_1P_Tx_2$, then we can replace T with a chorded cycle formed from x_1, v, and $x_2P_Tx_3$, contradicting (O1). By symmetry, the same holds for $x_2P_Tx_3$, and since $|T| \geqslant 5$, there is exactly one interior vertex in each of $x_1P_Tx_2$ and $x_2P_Tx_3$; call them w_1 and w_2, respectively.

If x_2 is incident to a chord, say x_2x_3, then we replace T with the theta graph $G[\{v, x_2, w_2, x_3\}]$, contradicting (O1). A similar argument holds if x_2x_1 is a chord. If x_1w_2 is a chord, then we replace T with the theta graph $G[\{v, x_1, x_3, w_2\}]$, contradicting (O1). A similar argument holds if x_3w_1 is a chord. Thus, the only chord in T is w_1w_2 so that $G[T] \cong \theta_{1,2,3}$ and the neighbors of v are exactly the vertices not incident to the chord of T.

So we assume T is not a chorded cycle. Since T is a theta graph, x_1 and x_3 are adjacent to distinct vertices u and u', respectively. Furthermore, we may assume u is an interior vertex to $u'P_Tx_3$. If x_2 is a vertex in x_1P_Tu' (other than x_1), then we can replace T with the chorded cycle formed from v, $x_1P_Tx_2$, and uP_Tx_3, which contradicts (O2). Thus by symmetry x_2 is an interior vertex in uP_Tu'. If there are two interior vertices in uP_Tu', then we can replace T with the theta graph formed from v, x_1P_Tu', and uP_Tx_3, contradicting (O1). Thus, x_2 is the only interior vertex to uP_Tu'. If there is an interior vertex to x_1P_Tu', then we can replace T with a theta graph formed from v and $u'P_Tx_3$, contradicting (O1). So by symmetry, P_T is $x_1u'x_2ux_3$ so that $G[T] \cong K_{2,3}$ and $G[T + v] \cong K_{3,3}$. This proves the lemma. $\square$

As a result of Lemma 7, we adopt the following conventions to label the vertices of T based on $G[T]$. If $|T| = 4$ for some $T \in \mathcal{T}$, then we use the convention that $V(T) = \{x_1, x_2, y_1, y_2\}$ where x_1x_2 is a chord and y_1y_2 may or may not exist. If $G[T] \cong K_{2,3}$, then let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2\}$ be the bipartite sets of T. If $G[T] \cong \theta_{1,2,3}$, then let $V(T) = \{f_1, f_2, g_1, g_2, g_3\}$ so that f_1, f_2 are the vertices of degree 3, g_3 is adjacent to both f_1 and f_2, and f_ig_i forms an edge for $i \in \{1, 2\}$.

Lemma 9. *Let $v, v' \in R$ and $T \in \mathcal{T}$ such that $\|v, T\| \geqslant \|v', T\|$. Then each of the following hold:*

- *if $G[T] \cong \theta_{1,2,3}$, then $\|v', T\| \leqslant 2$.*

- *if $G[T] \cong K_{2,3}$ and $vv' \in E(G)$, then $\|v', T\| \leqslant 2$*
- *if $G[T] \cong K_{2,3}$, $vv' \in E(G)$, and $\|\{v, v'\}, T\| = 5$, then $N_T(v) = \{a_1, a_2, a_3\}$ and $N_T(v') = \{b_1, b_2\}$.*

Proof. Suppose $G[T] \cong \theta_{1,2,3}$ and $\|v, T\| \geqslant \|v', T\| \geqslant 3$. Then by Lemma 7, $N_T(v) = N_T(v') = \{g_1, g_2, g_3\}$. However, $G[\{v, v', g_1, g_2\}]$ is a theta graph with fewer vertices than T, contradicting (O1).

Now suppose $G[T] \cong K_{2,3}$ and $vv' \in E(G)$. If $\|v, T\| \geqslant \|v', T\| \geqslant 3$, then by Lemma 7, $N_T(v) = N_T(v') = \{a_1, a_2, a_3\}$. However, $G[\{v, v', a_1, a_2\}]$ is a theta graph with fewer vertices than T, contradicting (O1). If $\|\{v, v'\}, T\| = 5$, then $\|v, T\| = 3$ and $\|v', T\| = 2$. By Lemma 7, $N_T(v) = \{a_1, a_2, a_3\}$. If say $v'a_1 \in E(G)$, then we can replace T with $G[\{v, v', a_1, b_1, a_2\}]$, which contains $\theta_{1,2,3}$, increasing the number of chorded cycles in $\mathcal{T}$, contradicting (O2). $\square$

Lemma 10. *Let p be an endpoint of P and $v \in R \setminus P$. Then for all $T \in \mathcal{T}$, both of the following hold:*

1. *if $\|v, T\| = 4$, then $\|p, T\| = 0$;*
2. *$\|\{p, v\}, T\| \leqslant 6$, and if equality holds, then $G[T] \cong K_4^-$, $\|p, T\| = 4$, and $N_T(v) = \{y_1, y_2\}$.*

Proof. Fix $T \in \mathcal{T}$ and $v \in R \setminus P$. If $\|v, T\| = 4$, then by Lemma 7, $|T| = 4$. Since we are assuming $v \in R \setminus P$ exists, we do not necessarily prefer K_4 over K_4^- in $\mathcal{T}$. Thus, for all $u \in T$, replacing T with the theta graph in $T - u + v$ results in a partition of G satisfying (O1) and (O2). So if p has any neighbor in T, say u, then this new optimal partition contradicts (O3). This proves 10.1.

Now suppose $\|\{p, v\}, T\| \geqslant 6$. By Lemma 7 and 10.1, $3 \leqslant \|p, T\| \leqslant 4$ and $2 \leqslant \|v, T\| \leqslant 3$. Further, by Lemmas 7 and 9, either $|T| = 4$ or $G[T] \cong K_{2,3}$. If $G[T] \cong K_{2,3}$, then $N_T(p) = N_T(v) = \{a_1, a_2, a_3\}$. However, we can replace T with $T - a_1 + v$ and P with $P + a_1$, contradicting (O3).

So $|T| = 4$. Since $3 \leqslant \|p, T\| \leqslant 4$ and $2 \leqslant \|v, T\| \leqslant 3$, there exists $u \in N_T(p)$ such that v is adjacent to at least two vertices in $T - u$. If $G[T] \cong K_4$, then we can replace T with $T - u + v$, and P with $P + u$, contradicting (O3). Thus, $G[T] \cong K_4^-$.

If $N_T(v) \neq \{y_1, y_2\}$, then without loss of generality, we may assume $vx_1 \in E(G)$ and furthermore, $py_1 \in E(G)$, as $\|p, T\| \geqslant 3$. If $T - y_1 + v$ forms a theta graph, then we can replace T with this theta graph, and replace P with $P + y_1$, contradicting (O3). Therefore, we must have $\|v, T\| = 2$, $\|p, T\| = 4$, with $vy_1 \in E(G)$. However, we can replace T with $T - y_2 + v$ and P with $P + y_2$, which again contradicts (O3).

Thus $N_T(v) = \{y_1, y_2\}$, and since we assumed $\|\{p, v\}, T\| \geqslant 6$, Lemma 7 implies we must have equality with $\|p, T\| = 4$. $\square$

Lemma 11. *Suppose $|P| \geqslant 2$. Let p and p' be the endpoints of P, let $v \in R \setminus P$, and let $T \in \mathcal{T}$. If $F = \{p, p', v\}$, then $\|F, T\| \leqslant 8$, and if equality holds, then $\|v, T\| = 0$.*

Proof. Let $F = \{p, p', v\}$ and let $T \in \mathcal{T}$ such that $\|F, T\| \geqslant 8$. Recall that since $v \in R \setminus P$ exists, we do not prefer K_4 over K_4^- in $\mathcal{T}$.

If $\|v, T\| = 4$, then by Lemma 10.1, $\|\{p, p'\}, T\| = 0$, which contradicts $\|F, T\| \geqslant 8$.

If $\|v, T\| = 3$, then by Lemma 10.2, $\|p, T\| \leqslant 2$ and $\|p', T\| \leqslant 2$. However this contradicts $\|F, T\| \geqslant 8$.

Suppose $\|v, T\| = 2$. If $\|p, T\| = 4$, then by Lemma 10.2, $G[T] \cong K_4^-$ and $N_T(v) = \{y_1, y_2\}$. If p' is adjacent to either y_i, then we can replace T with the theta graph $T - y_i + p$, and we can replace P with the path formed from $P - p + y_i + v$, which is longer than P contradicting (O3). Therefore since $\|F, T\| \geqslant 8$, we must have $N_T(p') = \{x_1, x_2\}$. However, we can replace T with the theta graph in $T - y_1 + p'$, and we can replace P with the longer path formed from $P - p' + y_1 + v$, again contradicting (O3).

So $\|p, T\| \leqslant 3$ and by symmetry, $\|p', T\| \leqslant 3$. Furthermore, since $\|F, T\| \geqslant 8$, we have equality and $\|p, T\| = \|p', T\| = 3$. By Lemmas 7 and 9, either $|T| = 4$ or $G[T] \cong K_{2,3}$. If $G[T] \cong K_{2,3}$, then by Lemma 7 $N_T(p) = N_T(p') = \{a_1, a_2, a_3\}$. If v is adjacent to any a_i, then we can replace T with $T - a_i + p$ and replace P with the path $P - p + a_i + v$, contradicting (O3). So $N_T(v) = \{b_1, b_2\}$. However we can replace T with $T - a_1 + v$ and replace P with the path $P + a_1$, contradicting (O3). Thus, $|T| = 4$.

Suppose first that $N_T(v) = \{y_1, y_2\}$. Since $\|p, T\| = 3$, we may assume without loss of generality that p is adjacent to y_1. However, since $\|p', T\| = 3$, we can replace T with $T - y_1 + p'$ and replace P with $P - p' + y_1 + v$ contradicting (O3). Thus $N_T(v) \neq \{y_1, y_2\}$ and without loss of generality, we assume v is adjacent to x_1. Since $\|v, T\| = 2$, v has exactly one more neighbor in T, and without loss of generality we may assume that it is not y_1. Note that $T - y_1 + v$ is a theta graph. Thus, if either p or p' is adjacent to y_1, we can replace T with $T - y_1 + v$ and replace P with $P + y_1$ contradicting (O3). So $N_T(p) = N_T(p') = \{x_1, x_2, y_2\}$. If the other neighbor of v is x_2, then we replace T and P with $T - y_2 + v$ and $P + y_2$ respectively. If the other neighbor of v is y_2, then we replace T and P with $T - x_2 - y_1 + p + v$ and $P - p + x_2 + y_1$, respectively. In either situation we contradict (O3), and this completes the case where $\|v, T\| = 2$.

If $\|v, T\| = 1$, then without loss of generality we may assume $\|p, T\| = 4$ and $\|p', T\| \geqslant 3$. By Lemma 7, $|T| = 4$. Let u be the neighbor of v in T. If u is also a neighbor of p', then we replace T and P with $T - u + p$ and $P - p + u + v$, respectively, which contradicts (O3). So u is not a neighbor of p'. However, we can now replace T and P with $T - u + p'$ and $P - p' + u + v$, respectively, which contradicts (O3).

Thus, $\|v, T\| = 0$, and by Lemma 7, $\|F, T\| = 8$. ☐

Lemma 12. *Let H be a subgraph of G such that $G[V(H)]$ contains no theta graph. Suppose Q_1 and Q_2 are disjoint paths in H. Then $\|Q_1, Q_2\| \leqslant 2$.*

Proof. Suppose on the contrary that $\|Q_1, Q_2\| \geqslant 3$. If there exists a vertex $v \in Q_1$ such that $\|v, Q_2\| \geqslant 3$, then $G[v + Q_2]$ contains a theta graph. Therefore, there exist two distinct vertices $v_1, v_2 \in Q_1$ such that $\|v_i, Q_2\| \geqslant 1$ for $i \in \{1, 2\}$. Let u_i be the neighbor of v_i in Q_2, where possibly $u_1 = u_2$. Observe that $v_1 Q_1 v_2$ together with $u_1 Q_2 u_2$ forms a cycle. So if any edge exists between Q_1 and Q_2 other than $v_1 u_1$ and $v_2 u_2$, we can find a theta graph in $G[V(Q_1) \cup V(Q_2)]$. ☐

4 Proof of Theorem 6

This section contains the majority of the proof of Theorem 6 broken into cases depending on R. We first show that $V(R) = V(P)$; that is, R contains a Hamiltonian path. We then show that $|R| \leqslant 4$. Recall that since we are considering an optimal $(\mathcal{T}, R)$-partition of G, we can conclude that $|R| = 4$. Since R has no theta graph and has a Hamiltonian path, this implies $R \in \{P_4, C_4, \text{ Paw}\}$. We then show $R \cong$ Paw, and ultimately derive a contradiction in this final case.

4.1 $\boldsymbol{V(R) \neq V(P)}$

Suppose $V(R) \neq V(P)$; that is, $R \setminus P \neq \emptyset$. We first show that R is not an independent set, so that $|P| \geqslant 2$ with distinct endpoints. We then find a vertex v in $R \setminus P$ such that $\|v, R\| \leqslant 2$, and use Lemma 11 to arrive at a contradiction.

Lemma 13. *R is not an independent set; that is, $|P| \geqslant 2$.*

Proof. Suppose on the contrary, that R is an independent set. Then we can view any vertex in R as the endpoint of P. So for any $v, v' \in R$, by Lemma 10, $\|\{v, v'\}, T\| \leqslant 6$ for all $T \in \mathcal{T}$, and if equality holds, then $\|v, T\| = 4$ and $N_T(v') = \{y_1, y_2\}$. However, since v and v' can both play the role of p, we cannot have equality so that $\|\{v, v'\}, T\| \leqslant 5$.

So

$$2(\frac{5}{2}k) \leqslant \|\{v, v'\}, \mathcal{T}\| + \|\{v, v'\}, R\| \leqslant 5(k-1) + 0,$$

which is a contradiction. □

Lemma 14. *There exists a vertex $v \in R \setminus P$ such that $\|v, R\| \leqslant 2$.*

Proof. Let Q be a longest path in $R \setminus P$. If Q is trivial, say just a vertex v, then the only possible neighbors of v in R are vertices on P. Thus, $\|v, P\| \leqslant 2$, else there exists a theta graph in R. So we may assume Q is nontrivial, and let q and q' be its endpoints.

Suppose that $\|q, R\| \geqslant \|q', R\| \geqslant 3$ so that $\|\{q, q'\}, R\| \geqslant 6$. Since these are endpoints of a longest path in $R \setminus P$, the only neighbors of q and q' are on Q and P. By Lemma 12, $\|Q, P\| \leqslant 2$, and in particular, $\|\{q, q'\}, P\| \leqslant 2$. If either $\|q, Q\| \geqslant 3$ or $\|q', Q\| \geqslant 3$, then we can find a theta graph in $G[Q]$. Therefore we must have $\|q, Q\| = \|q', Q\| = \|\{q, q'\}, P\| = 2$.

Since we are assuming $\|q, R\| \geqslant \|q', R\| \geqslant 3$, we must also have $\|q, P\| = \|q', P\| = 1$. However, we can easily find a cycle using all the vertices of Q and the vertices of P between the neighbors of q and q' on P, and we have extra edges in $G[Q]$ so that we find a theta graph in R. So either $\|q, R\| \leqslant 2$ or $\|q', R\| \leqslant 2$, which proves the lemma. □

By Lemma 14, there exists $v \in R \setminus P$ such that $\|v, R\| \leqslant 2$. For $i \in \{0, 1, 2, 3, 4\}$ define $\mathcal{T}_i = \{T \in \mathcal{T} : \|v, T\| = i\}$, and let $t_i = |\mathcal{T}_i|$. Thus, $t_0 + t_1 + t_2 + t_3 + t_4 = k - 1$. Note that $\|v, R\| \leqslant 2$ so that

$$\frac{5}{2}k \leqslant \|v, \mathcal{T}\| + \|v, R\| \leqslant 0t_0 + 1t_1 + 2t_2 + 3t_3 + 4t_4 + 2. \tag{1}$$

Let $F = \{p, p', v\}$ where p and p' are the endpoints of P. By Lemma 11, $\|F, \mathcal{T}_i\| \leqslant 8t_i$ for $i \in \{0, 1, 2\}$. By Lemma 10, $\|F, \mathcal{T}_3\| \leqslant 7t_3$ and $\|F, \mathcal{T}_4\| = 4t_4$. Because p and p' are the ends of a longest path P in R, $\|\{p, p'\}, R\| = \|\{p, p'\}, P\|$. To avoid a theta graph, $\|\{p, p'\}, P\| \leqslant 4$. Thus $\|F, R\| \leqslant 6$ so that

$$3(\frac{5}{2}k) \leqslant \|F, \mathcal{T}\| + \|F, R\| \leqslant 8t_0 + 8t_1 + 8t_2 + 7t_3 + 4t_4 + 6. \tag{2}$$

Adding (1) and (2), yields the following contradiction:

$$\begin{aligned} 10k &\leqslant 8t_0 + 9t_1 + 10t_2 + 10t_3 + 8t_4 + 8 \\ &\leqslant 10(t_0 + t_1 + t_2 + t_3 + t_4) + 8 \\ &= 10(k - 1) + 8 \\ &= 10k - 2. \end{aligned}$$

This completes the case in which $V(R) \neq V(P)$.

4.2 $\boldsymbol{V(R) = V(P)}$

In the remainder of Section 4, we assume $V(R) = V(P)$. That is, P is a Hamiltonian path in R, and $\|v, R\| = \|v, P\|$ for all $v \in G$. Recall that $|R| \geqslant 4$. So we may define distinct vertices p, q, q', and p' so that p and p' are the endpoints of P, and q and q' are the neighbors of p and p', respectively, along P.

Lemma 15. *Suppose $|R| \geqslant 5$. Then for all $r \in V(P) \setminus \{p, q, q', p'\}$, if $F = \{p, q, r, q', p'\}$, then $\|F, T\| \leqslant 12$ for all $T \in \mathcal{T}$.*

Proof. Fix $T \in \mathcal{T}$ and suppose $\|F, T\| \geqslant 13$. Thus, there exists some $v \in F$ such that $\|v, T\| \geqslant 3$. So by Lemma 7, either $|T| = 4$, $G[T] \cong \theta_{1,2,3}$, or $G[T] \cong K_{2,3}$. If $G[T] \cong \theta_{1,2,3}$, then by Lemmas 7 and 9, there exists at most one vertex $v \in F$ such that $\|v, T\| = 3$, and as a result, $\|F - v, T\| \leqslant 8$. However, this contradicts $\|F, T\| \geqslant 13$.

Similarly if $G[T] \cong K_{2,3}$, Lemmas 7 and 9 imply $\|\{p, q\}, T\| \leqslant 5$, $\|\{p', q'\}, T\| \leqslant 5$, and $\|r, T\| \leqslant 3$. Therefore equality holds in each of these inequalities, and $N_T(r) = \{a_1, a_2, a_3\}$. Then $G[\{a_1, a_2, b_1, p, q\}]$ is a theta graph, and $G[\{a_3, b_2, q', p'\}]$ is a C_4. Now we can connect a_3 to q' via an independent path through r, creating another theta graph, a contradiction.

So we may assume $|T| = 4$. Suppose that for each $i \in \{1, 2\}$, $G[\{p, q, x_i, y_i\}]$ contains a theta graph. Let $Q_1 = x_1y_1$, $Q_1' = x_2y_2$, and Q_2 be the path from r to p' along P. By Lemma 12, if $\|Q_1, Q_2\| \geqslant 3$, then $G[V(Q_1) \cup V(Q_3)]$ contains a theta graph. However we are also assuming $G[\{p, q, x_2, y_2\}]$ does as well, a contradiction. So $\|Q_1, Q_2\| \leqslant 2$, and similarly, $\|Q_1', Q_2\| \leqslant 2$. Thus, $\|\{r, q', p'\}, T\| \leqslant 4$, which implies $\|\{p, q\}, T\| \geqslant 9$, which cannot happen by Lemma 7.

So without loss of generality, suppose $G[\{p, q, x_1, y_1\}]$ does not contain a theta graph, but suppose $G[\{p, q, x_2, y_2\}]$ does contain a theta graph. Let $Q_1 = x_1y_1$, $Q_2 = pq$, and Q_2' be the path from r to p' along P. By Lemma 12, $\|Q_1, Q_2\| \leqslant 2$ and $\|Q_1, Q_2'\| \leqslant 2$; the former is true because $G[\{p, q, x_1, y_1\}]$ does not contain a theta graph, and the latter is true because otherwise we can replace T with two disjoint theta graphs in a manner similar to the previous case. Since $\|F,T\| \geqslant 13$, we have $\|\{x_2,y_2\},F\| \geqslant 9$. So either $\|x_2,F\| = 5$ or $\|y_2,F\| = 5$. If $\|y_2,F\| = 5$, then for any vertex $u \in \{p,q,q',p'\}$, $P-u+y_2$ contains a theta graph. Therefore $T-y_2+u$ cannot form a theta graph, which implies $\|u,T\| \leqslant 2$ for each $u \in \{p,q,q',p'\}$. However this implies $\|r,T\| \geqslant 5$ contradicting Lemma 7. So we may assume $\|x_2,F\| = 5$ and $\|y_2,F\| = 4$. Observe that both $P-p-q+x_2$ and $P-q'-p'+x_2$ contain theta graphs. Let $Q_1 = y_1x_1y_2$, $Q_2 = pq$, and $Q_2' = q'p'$, so that by Lemma 12 $\|Q_1,Q_2\| \leqslant 2$ and $\|Q_1,Q_2'\| \leqslant 2$, otherwise we replace T with two disjoint theta graphs similar to the above. Thus $\|\{p,q\},T\| \leqslant 4$ and $\|\{p',q'\},T\| \leqslant 4$, which implies $\|r,T\| \geqslant 5$, contradicting Lemma 7.

Thus we may assume $G[\{p,q,x_1,y_1\}]$ and $G[\{p,q,x_2,y_2\}]$ do not contain theta graphs. So by Lemma 12, $\|\{p,q\},T\| \leqslant 4$, and by symmetry, $\|\{p',q'\},T\| \leqslant 4$. Thus $\|r,T\| \geqslant 5$, again contradicting Lemma 7. □

Lemma 16. $|R| = 4$.

Proof. If on the contrary $|R| \geqslant 5$, then there exists $r \in V(P) \setminus \{p,q,q',p'\}$, and we can let $F = \{p,q,r,q',p'\}$. Note that $\|r,P\| \leqslant 4$, otherwise R will contain a theta graph. Furthermore, since $|R| = |P| \geqslant 5$, $P-p-q$ has a spanning path on at least three vertices, and we can consider pq as a path on two vertices. Therefore by Lemma 12, $\|\{p,q\},P-p-q\| \leqslant 2$, which implies $\|\{p,q\},P\| \leqslant 4$, and by symmetry, $\|\{q',p'\},P\| \leqslant 4$. Thus $\|F,R\| \leqslant 12$. By Lemma 15,

$$5(\frac{5}{2}k) \leqslant \|F,\mathcal{T}\| + \|F,R\| \leqslant 12(k-1)+12,$$

which yields $12.5k \leqslant 12k$ and $k \leqslant 0$, a contradiction.

So $|R| \leqslant 4$, and since we are considering an optimal $(\mathcal{T},R)$-partition so that $|R| \geqslant 4$, we have $|R| = 4$ as desired. □

Since R has a Hamiltonian path and contains no theta graphs, $R \in \{P_4, C_4, \text{ Paw}\}$. We consider each case separately.

4.2.1 $R \cong P_4$

Observe that when $R \cong P_4$, we still do not prefer K_4 over K_4^- in $\mathcal{T}$. However in this case, we will seek to replace $\mathcal{T}$ and R with a new partition satisfying (O1) - (O3), in which R is replaced with either C_4 or the Paw, contradicting (O4).

Lemma 17. *If $R \cong P_4$, then $\|R,T\| \leqslant 10$ for all $T \in \mathcal{T}$.*

Proof. Suppose $\|R,T\| \geqslant 11$. Then either there exists $u \in R$ such that $\|u,T\| = 4$, or there exists adjacent vertices in R, v and w such that $\|v,T\| = \|w,T\| = 3$. In either case, by Lemmas 7 and 9, $|T| = 4$.

Suppose $\|p,T\| = 4$. Note that for all $v \in T$, $T-v+p$ is a theta graph. Thus, for all $v \in T$, $\|v,P\| \leqslant 2$ as otherwise we can replace T with $T-v+p$ and replace R with the

C_4 or Paw in $P - p + v$ which contradicts (O4). However this implies $\|R, T\| \leqslant 8$ in this case.

Now suppose $\|p, T\| = 3$. Without loss of generality, we may assume p is adjacent to both x_1 and y_1. Note that $\|y_2, P - p\| \leqslant 1$, otherwise we can replace T with the theta graph $T - y_2 + p$ and replace R with either C_4 or the Paw in $P - p + y_2$, contradicting (O4). Note also that $\|x_2, P - p\| \leqslant 2$, otherwise we can replace T with the theta graph $P - p + x_2$ and replace R with the Paw in $T - x_2 + p$, contradicting (O4). Similarly, $\|y_1, P - p\| \leqslant 2$. Lastly, $\|x_1, P - p\| \leqslant 2$, otherwise we can replace T with the theta graph $P - p + x_1$ and replace R with either C_4 or the Paw in $T - x_1 + p$, contradicting (O4) (recall that $\|p, T\| = 3$ so that p is adjacent to either x_2 or y_2). Therefore $\|T, P - p\| \leqslant 7$. However, since $\|p, T\| = 3$, we get $\|P, T\| \leqslant 10$ a contradiction.

So $\|p, T\| \leqslant 2$ and by symmetry, $\|p', T\| \leqslant 2$. By Lemma 7 and the assumption $\|R, T\| \geqslant 11$, we have $7 \leqslant \|\{q, q'\}, T\| \leqslant 8$ and $3 \leqslant \|\{p, p'\}, T\| \leqslant 4$. Without loss of generality, we may assume $qy_1, q'y_1 \in E(G)$. If either p or p' is adjacent to two vertices of $\{x_1, x_2, y_2\}$, say p, then we can replace T with the theta graph $T - y_1 + p$ and replace R with the Paw in $P - p + y_1$. Now because $\|\{p, p'\}, T\| \geqslant 3$, we can conclude without loss of generality that $py_1 \in E(G)$ and p' is adjacent to a vertex of $\{x_1, x_2, y_2\}$. Then we can replace T with the theta graph $P - p' + y_1$ and replace R with the Paw in $T - y_1 + p'$, a contradiction. □

Since $R \cong P_4$, $\|P, R\| = 6$. Thus by Lemma 17,

$$4(\frac{5}{2}k) \leqslant \|P, \mathcal{T}\| + \|P, R\| \leqslant 10(k-1) + 6,$$

which yields $10k \leqslant 10k - 4$, a contradiction.

4.2.2 $R \cong C_4$

We continue to label the vertices in R as p, q, q', p', only now we know $pp' \in E(G)$ and $\|P, R\| = 8$. Recall that now K_4 is preferred over K_4^-.

It is worth mentioning that when $R \in \{C_4,\ \text{Paw}\}$, the arguments used in some of the proofs are similar to those in [6], and at times, are exactly the same. However our proofs must additionally consider more possibilities for T, and so we include these arguments as well as the ones done in [6] for the sake of completeness. We also note that in some instances, our arguments are more concise.

Lemma 18. *For all $T \in \mathcal{T}$, $\|R, T\| \leqslant 10$, unless $G[R + T] \cong (K_4 \cup K_1) \vee \overline{K_3}$, in which case $\|R, T\| = 11$.*

Proof. Suppose $\|R, T\| \geqslant 11$. Then either there exists $u \in R$ such that $\|u, T\| = 4$, or there exists adjacent vertices in R, v and w such that $\|v, T\| = \|w, T\| = 3$. In either case, by Lemmas 7 and 9, $|T| = 4$.

Claim 19. $G[T] \cong K_4$.

Proof. Let $G[T] \cong K_4^-$ so that $y_1y_2 \notin E(G)$. Suppose $\|p, T\| = 4$. Note that $\|y_1, R\| \leqslant 2$, otherwise we replace T with the K_4 in $T - y_1 + p$ and replace the C_4 in R with either another C_4 or Paw in $R - p + y_1$. Similarly, $\|y_2, R\| \leqslant 2$. Note also that $\|x_1, R\| \leqslant 3$, otherwise we replace T with two disjoint theta graphs in $T - x_1 + p$ and $R - p + x_1$. Similarly, $\|x_2, R\| \leqslant 3$, however this contradicts $\|R, T\| \geqslant 11$.

Now suppose $\|p, T\| = 3$. Without loss of generality, $N_T(p)$ is either $\{x_1, x_2, y_1\}$ or $\{x_1, y_1, y_2\}$. Suppose first $N_T(p) = \{x_1, x_2, y_1\}$. Note that $\|y_2, R\| \leqslant 1$, otherwise we can replace T with the K_4 in $T - y_2 + p$ and replace R with either another C_4 or Paw in $R - p + y_2$. Note also that $\|y_1, R\| \leqslant 3$, as otherwise we can replace T with two disjoint theta graphs in $T - y_1 + p$ and $R - p + y_1$. Lastly, $\|x_i, R\| \leqslant 3$ for $i \in \{1, 2\}$, otherwise we can replace T with the theta graph in $R - p + x_i$ and replace R with the Paw in $T - x_i + p$, contradicting (O6). However, this implies $\|R, T\| \leqslant 10$, a contradiction.

So we assume $\|p, T\| = 3$ and $N_T(p) = \{x_1, y_1, y_2\}$. Note that $\|x_2, R\| \leqslant 2$, otherwise we can replace T with two disjoint theta graphs in $T - x_2 + p$ and $R - p + x_2$. Furthermore, if $\|x_2, R\| = 2$, then $x_2q' \notin E(G)$, otherwise we replace T with the theta graph in $T - x_2 + p$ and replace the C_4 in R with the Paw in $R - p + x_2$. Similarly, $\|y_i, R\| \leqslant 3$ for $i \in \{1, 2\}$, and if equality holds, then $y_iq' \notin E(G)$. Now, if $\|y_i, R\| \leqslant 2$ for each $i \in \{1, 2\}$, then we obtain $\|R, T\| \leqslant 10$, which is a contradiction. So without loss of generality, suppose $\|y_1, R\| = 3$ so that y_1 is adjacent to p, q, and p'. Observe that $py_1p'p$ forms a triangle. Since $y_1q' \notin E(G)$, $\|q', T\| \leqslant 1$, otherwise we could replace T with the theta graph in $T - y_1 + q'$ and replace the C_4 in R with the theta graph in $R - q' + y_1$. Similarly, $\|q, T\| \leqslant 2$. However, this implies $\|R, T\| \leqslant 10$, a contradiction.

So $\|p, T\| \leqslant 2$. However since $G[R] \cong C_4$, by symmetry, $\|v, T\| \leqslant 2$ for all $v \in R$. But now $\|R, T\| \leqslant 8$, a contradiction. This proves the claim. □

Here we label the vertices of T as x_1, x_2, x_3, x_4. Suppose $\|p, T\| = 4$. So for $x \in T$, $\|x, R\| \leqslant 3$, otherwise we replace T with two disjoint theta graphs in $T - x + p$ and $R - p + x$. Furthermore, $\|x, R\| = 3$ only if $xq' \notin E(G)$, otherwise we replace T with the K_4 in $T - x + p$ and replace the C_4 in R with the Paw in $R - p + x$. Since $\|R, T\| \geqslant 11$, we may assume $\|x_i, R\| = 3$ for $i \in \{1, 2, 3\}$ and $\|x_4, R\| \geqslant 2$. Thus for $i \in \{1, 2, 3\}$, $x_iq, x_ip' \in E(G)$ and $x_iq' \notin E(G)$. If $x_4p' \in E(G)$, then we replace T with the K_4 in $G[\{x_1, x_2, p, q\}]$ and replace the C_4 in R with the Paw in $G[\{x_3, x_4, q', p'\}]$. A similar contradiction arises if $x_4q \in E(G)$. So as $\|R, T\| \geqslant 11$, we must have $x_4q' \in E(G)$, which yields $G[R + T] \cong (K_4 \cup K_1) \vee \overline{K_3}$.

So suppose $\|p, T\| \leqslant 3$, and since $G[R] \cong C_4$, by symmetry $\|v, T\| \leqslant 3$ for all $v \in R$. Again by the symmetry of R, we may assume $\|\{p, q, q'\}, T\| = 9$ and $\|p', T\| \geqslant 2$. Without loss of generality, we may assume $N_T(p) = \{x_1, x_2, x_3\}$, $\{x_1, x_2\} \subset N_T(q)$, and $x_1 \in N_T(q')$. This implies p' cannot be adjacent to two vertices from $\{x_2, x_3, x_4\}$, otherwise we replace T with two disjoint theta graphs in $T - x_1 + p'$ and $R - p' + x_1$. As a result $\|p', T\| \leqslant 2$, however since $\|R, T\| \geqslant 11$, we have equality and in particular, $x_1p' \in E(G)$. However, we can replace T with two disjoint theta graphs in $T - x_1 + p$ and $R - p + x_1$. □

Lemma 20. *There exists $\tilde{T} \in \mathcal{T}$ such that $\|R, \tilde{T}\| \geqslant 11$; that is, $G[R + \tilde{T}] \cong (K_4 \cup K_1) \vee \overline{K_3}$.*

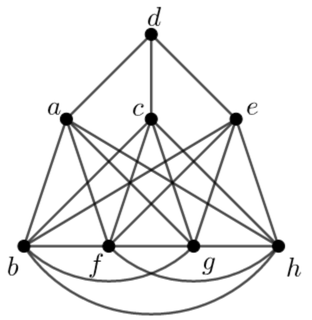

Figure 1: Labeling for $G[R + \tilde{T}]$

Proof. Suppose on the contrary, $\|R, T\| \leqslant 10$ for all $T \in \mathcal{T}$. Since $R \cong C_4$, we have $\|R, R\| = 8$. This yields

$$4(\frac{5}{2}k) \leqslant \|R, \mathcal{T}\| + \|R, R\| \leqslant 10(k-1) + 8 = 10k - 2,$$

which implies $10k \leqslant 10k - 2$, a contradiction.

Thus, there exists some $\tilde{T} \in \mathcal{T}$ such that $\|R, \tilde{T}\| \geqslant 11$. By Lemma 18, equality holds and $G[R + \tilde{T}] \cong (K_4 \cup K_1) \vee \overline{K_3}$. □

Let $\tilde{T} \in \mathcal{T}$ be as given by Lemma 20, and consider $G[R+\tilde{T}]$. We now introduce notation similar to that in [6]. Label the vertices $V(R) = \{a, b, c, d\}$ and $V(\tilde{T}) = \{e, f, g, h\}$ as follows. Label $d \in R$ such that $\|d, \tilde{T}\| = 1$, and let e be its neighbor in $\tilde{T}$. Let $a, b, c \in R$ such that $ad, cd \in E(G)$, and let $\{f, g, h\} = N_{\tilde{T}}(a)$. See Figure 1.

Let $F = \{a, b, c, d, e, f\}$, $X = \{a, c, e\}$, $Y = \{b, f\}$, and $A = X \cup Y$. Note that all vertices in X are symmetric and both vertices in Y are symmetric. Furthemore, we can replace $\tilde{T}$ and R with $G[\{a, b, g, h\}]$ and $G[\{c, d, e, f\}]$, respectively to obtain a new optimal partition, which we will denote by $(\mathcal{T}', R')$. Therefore, all previous lemmas that apply to $(\mathcal{T}, R)$, also apply to $(\mathcal{T}', R')$; for example, Lemma 7 implies $\|v, T\| \leqslant 4$ for all $v \in F$ and $T \in \mathcal{T}$.

In the following we seek bounds on $\|F, T\|$ for all $T \in \mathcal{T} \setminus \tilde{T}$, depending on $\|d, T\|$.

Lemma 21. *If $\|d, T\| = 4$ for some $T \in \mathcal{T} \setminus \tilde{T}$, then $\|F, T\| \leqslant 12$.*

Proof. Let $T \in \mathcal{T} \setminus \tilde{T}$ and assume $\|d, T\| = 4$. By Lemma 7, we know either $G[T] \cong K_4$ or $G[T] \cong K_4^-$. First we claim

$$N_T(X) \cap N_T(Y) = \emptyset. \tag{3}$$

Suppose, to the contrary, there is a vertex $v \in N_T(X) \cap N_T(Y)$. Without loss of generality, let $v \in N_T(e) \cap N_T(f)$. Then $G[d \cup V(T) - v]$, $G[v, e, f, g]$, and $G[a, b, c, h]$ all contain K_4^-, a contradiction. Note that if instead we have $v \in N_T(b) \cap N_T(f)$ for some $v \in T$, we can still find three copies of K_4^- by swapping b and g in the three previously listed. We thus also have

$$N_T(b) \cap N_T(f) = \emptyset. \tag{4}$$

By (3), the following is true for all $v \in V(T)$:

$$\text{Either } \|v,A\| = \|v,X\| \text{ and } \|v,Y\| = 0 \text{ or}$$
$$\|v,A\| = \|v,Y\| \text{ and } \|v,X\| = 0; \text{ hence } \|v,A\| \leqslant 3. \quad (5)$$

If $\|v,A\| \leqslant 2$ for all $v \in T$, then $\|F,T\| \leqslant 12$. So we must have some $v \in T$ with $\|v,A\| = 3$. We consider two cases: $\|y_1,A\| = 3$ and $G[T] \cong K_4^-$, or $\|x_1,A\| = 3$.

First assume $\|y_1,A\| = 3$ and $G[T] \cong K_4^-$. We claim $\|x_1,X\| = \|x_2,X\| = \|y_2,X\| = 0$. Suppose not and assume without loss of generality that e is adjacent to one of x_1, x_2, y_2. Then $G[e, x_1, x_2, y_2]$ contains Paw, $G[b, f, g, h]$ contains K_4, and $G[a, c, d, y_1]$ contains K_4^-, contradicting (O6). Thus $\|x_1,X\| = \|x_2,X\| = \|y_2,X\| = 0$ and by (4), $\|x_1,A\|, \|x_2,A\|, \|y_2,A\| \leqslant 1$. Now $\|A,T\| \leqslant 6$ so $\|F,T\| \leqslant 12$ as desired. Because y_1 and y_2 are symmetric in K_4^-, we may assume $\|y_1,A\| \leqslant 2$ and $\|y_2,A\| \leqslant 2$ for the rest of the proof when $G[T] \cong K_4^-$.

In the second case, assume $\|x_1,A\| = 3$. By (3), since $N_T(X) \cap N_T(Y) = \emptyset$, we may further assume $\|x_1,X\| = 3$.

We claim $\|x_2,X\| \leqslant 2$. Suppose not and assume $\|x_2,X\| = 3$. Then if $ey_1 \in E(G)$, then $G[a, c, d, x_1]$ contains K_4^-, $G[b, f, g, h]$ contains K_4, and when $G[T] \cong K_4^-$, $G[e, y_1, y_2, x_2]$ contains Paw which contradicts (O6) and when $G[T] \cong K_4$, $G[e, y_1, y_2, x_2]$ contains K_4^- which gives k disjoint theta graphs, a contradiction. So $ey_1 \notin E(G)$ and by symmetry we conclude, $\|y_1,X\| = 0$ and also $\|y_2,X\| = 0$. By (4), $\|y_1,A\| \leqslant 1$ and $\|y_2,A\| \leqslant 1$. Thus $\|A,T\| = \|x_1,A\| + \|x_2,A\| + \|y_1,A\| + \|y_2,A\| \leqslant 3+3+1+1 = 8$ so $\|F,T\| \leqslant 12$ as desired. Hence we may assume that $\|x_2,X\| \leqslant 2$.

If $G[T] \cong K_4$, then x_2, y_1, y_2 are all symmetric (replace x_2 with y_1 or y_2 in the argument above) and if $G[T] \cong K_4^-$, we have already argued that $\|y_1,A\| \leqslant 2$ and $\|y_2,A\| \leqslant 2$. Hence $\|v,A\| \leqslant 2$ for all $v \in \{x_2, y_1, y_2\}$. If $\|v,A\| \leqslant 1$ for any one $v \in \{x_2, y_1, y_2\}$, then $\|A,T\| \leqslant 8$ so $\|F,T\| \leqslant 12$ as desired. Hence we may assume $\|x_2,A\| = \|y_1,A\| = \|y_2,A\| = 2$. By (4) and (5), we have $\|x_2,X\| = \|y_1,X\| = \|y_2,X\| = 2$. In particular, since $\|\{x_2, y_1\}, X\| = 4$, we may assume without loss of generality, $ey_1, ex_2 \in E(G)$. Then $G[a, c, d, x_1]$ contains K_4^-, $G[b, f, g, h]$ contains K_4, and when $G[T] \cong K_4^-$, $G[e, y_1, y_2, x_2]$ contains Paw which contradicts (O6) and when $G[T] \cong K_4$, $G[e, y_1, y_2, x_2]$ contains K_4^- which gives k disjoint theta graphs, a contradiction. $\square$

Lemma 22. *If $\|d,T\| = 3$ for some $T \in \mathcal{T} \setminus \tilde{T}$, then $\|F,T\| \leqslant 13$.*

Proof. Let $T \in \mathcal{T} \setminus \tilde{T}$, suppose $\|d,T\| = 3$ and assume to start $G[T] \cong \theta_{1,2,3}$. Recall that $(\mathcal{T}, R)$ and $(\mathcal{T}', R')$ are both optimal partitions of G such that $d \in V(R) \cap V(R')$ and $F = V(R) \cup V(R')$. Therefore by Lemma 9, for all $v \in F - d$, $\|v,T\| \leqslant 2$, which implies $\|F,T\| \leqslant 13$ as desired.

Next assume $G[T] \cong K_{2,3}$. By Lemma 7, $da_1, da_2, da_3 \in E(G)$. Furthermore, by Lemmas 7 and 9, $\|u,T\| \leqslant 3$ for all $u \in Y$ and $\|v,T\| \leqslant 2$ for all $v \in X$, respectively. We may assume there exists $v \in X$ such that $\|v,T\| = 2$, otherwise the lemma holds; without loss of generality say $\|c,T\| = 2$. If say $ca_1 \in E(G)$, then we can replace T with $G[c, d, a_1, a_2, b_1] \cong \theta_{1,2,3}$, which contradicts (O2). Thus, $N_T(c) = \{b_1, b_2\}$.

If $bb_i \in E(G)$ (or symmetrically fb_i), then T can be replaced in $\mathcal{T}$ with $G[b, c, b_1, b_2, a_1] \cong \theta_{1,2,3}$ which contradicts (O2). If $ba_i \in E(G)$ (or symmetrically fa_i), then T and $\tilde{T}$ can be replaced in $\mathcal{T}$ by $G[c, e, g, h] \cong K_4^-$ and a $\theta_{1,2,3}$ in $G[a, b, f, d, a_i]$ which contradicts (O2). Thus $\|b, T\| = \|f, T\| = 0$, and $\|F, T\| \leqslant 13$ as desired.

Now $|T| = 4$. We may assume that either $N_T(d) = \{x_1, x_2, y_1\}$, or $N_T(d) = \{x_1, y_1, y_2\}$ and $G[T] \cong K_4^-$.

Case 22.1. $N_T(d) = \{x_1, x_2, y_1\}$.

Using arguments similar to the ones used in the proofs of (3), (4), and (5), we can observe:

$$N_{T\setminus\{x_1,x_2\}}(X) \cap N_{T\setminus\{x_1,x_2\}}(Y) = \emptyset \text{ and } N_{T\setminus\{x_1,x_2\}}(b) \cap N_{T\setminus\{x_1,x_2\}}(f) = \emptyset. \tag{6}$$

Now (6) implies $\|y_i, A\| \leqslant 3$ for each $i \in \{1, 2\}$, and furthermore, equality implies $\|y_i, X\| = 3$, and if $\|y_i, A\| = 2$, then $N_A(y_i) \subseteq X$.

Next we claim $\|x_1, A\| \leqslant 3$. If not, then $\|x_1, X\| \geqslant 2$ and $\|x_1, Y\| \geqslant 1$. Because a, c, e are symmetric and b, f are symmetric, without loss of generality we may assume $ax_1, cx_1, fx_1 \in E(G)$. Now $G[b, e, g, h]$ is K_4, $G[a, c, f, x_1]$ contains K_4^-, and when $G[T] \cong K_4^-$, $G[d, x_2, y_1, y_2]$ contains Paw which contradicts (O6) and when $G[T] \cong K_4$, $G[d, x_2, y_1, y_2]$ contains K_4^- which gives k disjoint theta graphs, a contradiction. Hence $\|x_1, A\| \leqslant 3$ and by symmetry, $\|x_2, A\| \leqslant 3$.

We further claim that if $\|x_1, A\| = 3$, then $\|x_1, X\| = 3$. If this were not the case, by the argument in the previous paragraph, we must have $\|x_1, X\| = 1$ and $\|x_1, Y\| = 2$. Without loss of generality, say $x_1a \in E(G)$. Then $G[b, f, a, x_1]$ is K_4, $G[c, e, g, h]$ is K_4^-, and when $G[T] \cong K_4^-$, $G[d, x_2, y_1, y_2]$ contains Paw which contradicts (O6) and when $G[T] \cong K_4$, $G[d, x_2, y_1, y_2]$ contains K_4^- which gives k disjoint theta graphs, a contradiction. Thus if $\|x_1, A\| = 3$, then $\|x_1, X\| = 3$. By symmetry the same is true for x_2.

Now we may assume $\|T, A\| \geqslant 11$, else the lemma holds. So $\|v, A\| = 2$ for at most one $v \in T$, and $\|u, A\| = 3$ for all $u \in T - v$. Without loss of generality, we may assume $\|x_1, A\| = 3$, and by the previous paragraph, $\|x_1, X\| = 3$. Recall that since $\|y_i, A\| \geqslant 2$ for each $i \in \{1, 2\}$, $N_A(y_i) \subseteq X$. We also know that either $\|y_1, X\| = 3$ or $\|y_2, X\| = 3$.

Suppose $\|y_1, X\| = 3$, and without loss of generality $cy_2, ey_2 \in E(G)$. Now $G[b, f, g, h]$ is K_4 and $G[c, e, x_1, y_2]$ and $G[a, d, y_1, x_2]$ both contain K_4^- resulting in k disjoint theta graphs, a contradiction. Next suppose instead that $\|y_2, X\| = 3$, and without loss of generality, $cy_1, ey_1 \in E(G)$. Now $G[b, f, g, h]$ is K_4 and $G[c, d, y_1, x_2]$ and $G[a, e, x_1, y_2]$ both contain K_4^- resulting in k disjoint theta graphs, a contradiction. This completes the case.

Case 22.2. $N_T(d) = \{x_1, y_1, y_2\}$ and $G[T] \cong K_4^-$.

Using arguments similar to the ones used in the proofs of (3), (4), and (5), we can observe:

$$N_{T\setminus\{x_1\}}(X) \cap N_{T\setminus\{x_1\}}(Y) = \emptyset \text{ and } N_{T\setminus\{x_1\}}(b) \cap N_{T\setminus\{x_1\}}(f) = \emptyset. \tag{7}$$

This implies that for any $v \in T - x_1$, $\|v, A\| \leqslant 3$, and if $\|v, A\| \geqslant 2$, then $\|v, A\| = \|v, X\|$. Note that if we can show $\|T, A\| \leqslant 10$, then the lemma holds.

Suppose to start that $\|y_1, A\| = 3$ so that $\|y_1, X\| = 3$. If one of y_2, x_1, x_2 is adjacent to say $e \in X$, then $G[b, f, g, h]$ is K_4, $G[a, c, d, y_1]$ is K_4^-, and $G[e, x_1, x_2, y_2]$ contains Paw which contradicts (O6). Thus $\|T - y_1, X\| = 0$, which implies $\|x_1, A\| = \|x_1, Y\| \leqslant 2$, and by (7), $\|x_2, A\| \leqslant 1$ and $\|y_2, A\| \leqslant 1$. This implies $\|T, A\| \leqslant 7$ so that the lemma holds. So we may assume $\|y_1, A\| \leqslant 2$ and by symmetry, $\|y_2, A\| \leqslant 2$.

By (7), $\|x_2, A\| \leqslant 3$, and since $\|y_i, A\| \leqslant 2$ for $i \in \{1, 2\}$, we may assume $\|x_1, A\| \geqslant 4$, otherwise the lemma holds. Since $\|x_1, A\| \leqslant 5$ as $|A| = 5$, we may assume without loss of generality that $\|y_1, A\| = 2$ and $\|x_2, A\| \geqslant 2$, or again the lemma holds. This implies $\|y_1, A\| = \|y_1, X\|$ and $\|x_2, A\| = \|x_2, X\|$ so that y_1 and x_2 have a common neighbor in X, say e. Thus $G[d, e, y_1, x_2]$ contains K_4^-. Since $\|x_1, A\| \geqslant 4$, we may assume $x_1c, x_1b \in E(G)$, so that $G[b, c, x_1, y_2]$ contains a Paw. However, this leaves $G[a, f, g, h] \cong K_4$, and using these to replace T, R, and $\tilde{T}$, respectively, contradicts (O6).

This completes both cases and proves the lemma. □

Our next goal is to bound $\|F, T\|$ in the remaining cases when $\|d, T\| \leqslant 2$ for some $T \in \mathcal{T} \setminus \tilde{T}$. To do this, it will be helpful to look at $\|\{c, d\}, T\|$, which we will denote by $x = \|\{c, d\}, T\|$ in the following. By Lemma 7, $\|v, T\| \leqslant 4$ for all $v \in F$, so that:

$$\text{if } x \in \{0, 1, 2\}, \text{ then } \|F, T\| = \|\{a, b, e, f\}, T\| + x \leqslant 16 + x. \tag{8}$$

Note (8) holds for $x \geqslant 3$ as well but we can develop stronger inequalities in these cases which we do now. By Lemma 18, the only way for $\|R, T\| = 11$, is either $x = 4$ and $\|d, T\| \in \{1, 3\}$, or $x = 7$. The same is true for $\|R', T\| = 11$ (recall $R' = G[\{c, d, e, f\}]$). So if $x = 4$ and $\|d, T\| \in \{1, 3\}$, or if $x = 7$, then $\|F, T\| = \|R, T\| + \|R', T\| - x \leqslant 22 - x \leqslant 18$. For all other possibilities, we get $\|F, T\| \leqslant 20 - x$ as both $\|R, T\|, \|R', T\| \leqslant 10$ by Lemma 18.

Now using these last two inequalities for $\|F, T\|$, we have the following:

$$\begin{aligned} &\text{if } x = 4 \text{ and } \|d, T\| \in \{1, 3\}, \text{ then } \|F, T\| \leqslant 18, \\ &\qquad\qquad \text{otherwise if } x \in \{3, 4, 5, 6\} \text{ then } \|F, T\| \leqslant 20 - x. \end{aligned} \tag{9}$$

This leads immediately to the following lemma.

Lemma 23. *If $\|d, T\| \leqslant 1$ for some $T \in \mathcal{T} \setminus \tilde{T}$, then $\|F, T\| \leqslant 18$.*

Proof. Since $x = \|\{c, d\}, T\| \leqslant 5$, (8) and (9) immediately imply $\|F, T\| \leqslant 18$ as desired. □

Lemma 24. *If $\|d, T\| = 2$ for some $T \in \mathcal{T} \setminus \tilde{T}$, then $\|F, T\| \leqslant 16$.*

Proof. Here we still let $x = \|\{c, d\}, T\|$. If $\|d, T\| = 2$, then $2 \leqslant x \leqslant 6$. Since $\|d, T\| = 2$, if $x \in \{4, 5, 6\}$, then by (9), the lemma holds.

Suppose $x = 2$ so that $\|c, T\| = 0$. We may assume $\|\{a, b, e, f\}, T\| \geqslant 15$, otherwise the lemma holds. Without loss of generality, assume $\|a, T\| = \|b, T\| = 4$ and $\|e, T\|, \|f, T\| \geqslant 3$. By Lemma 7, $|T| = 4$. Let $\{u, v\} = N_T(d)$. Then $G[c, f, g, h]$ contains K_4, $G[a, b, V(T) \setminus$

$\{u, v\}]$ contains K_4^-, and when $G[T] \cong K_4$, $G[d, e, u, v]$ contains K_4^- resulting in k disjoint theta graphs and when $G[T] \cong K_4^-$, $G[d, e, u, v]$ contains Paw which contradicts (O6).

Now suppose $x = 3$ so that $\|c, T\| = 1$. Since $\|R, T\| \leqslant 10$, $\|\{a, b\}, T\| \leqslant 7$, and similarly $\|\{e, f\}, T\| \leqslant 7$. If in either of these two inequalities seven can be replaced with six, then $\|F, T\| \leqslant 16$ as desired. So we may assume $\|\{a, b\}, T\| = \|\{c, d\}, T\| = 7$, and furthermore by Lemma 7, $|T| = 4$. Because $\|\{e, f\}, T\| = 7$ and $\|d, T\| = 2$, there is some $v \in V(T)$ such that $vd, ve, vf \in E(G)$. Now $G[v, d, e, f]$ contains K_4^-. Because $\|\{a, b\}, T\| = 7$, one of a or b is adjacent to all vertices of $V(T) \setminus \{v\}$. Then either $G[a, V(T) \setminus \{v\}]$ and $G[b, c, g, h]$ both contain K_4^- or $G[b, V(T) \setminus \{v\}]$ and $G[a, c, g, h]$ both contain K_4^-; both situations yield k disjoint theta graphs, a contradiction. □

For $i \in \{0, 1, 2, 3, 4\}$ define $\mathcal{T}_i = \{T \in \mathcal{T} \setminus \tilde{T} : \|d, T\| = i\}$, and let $t_i = |\mathcal{T}_i|$. Thus, $t_0 + t_1 + t_2 + t_3 + t_4 = k - 2$. By Lemmas 21, 22, 23, and 24, we have:

$$6\left(\frac{5}{2}k\right) = 15k \leqslant \|F, \mathcal{T} \setminus \tilde{T}\| + \|F, R + \tilde{T}\| \leqslant 18t_0 + 18t_1 + 16t_2 + 13t_3 + 12t_4 + 30. \quad (10)$$

We also have the following equation based on the degree of d:

$$\frac{5}{2}k \leqslant d_G(d) = 0t_0 + t_1 + 2t_2 + 3t_3 + 4t_4 + 3. \quad (11)$$

Now taking $2\times$ (10) $+4\times$ (11) gives:

$$40k \leqslant 36t_0 + 40t_1 + 40t_2 + 38t_3 + 40t_4 + 72. \quad (12)$$

By the definition of the t_i, we have that $\sum_{i=0}^{4} t_i = k - 2$. From this equation we get:

$$(36t_0 + 40t_1 + 40t_2 + 38t_3 + 40t_4) + 72 \leqslant 40(k - 2) + 72 = 40k - 8. \quad (13)$$

Now (13) and (12) contradict each other. This concludes the case when $R \cong C_4$.

4.2.3 $R \cong$ Paw

Suppose there exists $T \in \mathcal{T}$ such that $G[T] \cong K_4^-$. When considering $G[R + T]$, suppose we replace T with a K_4 and replace R with a C_4. This results in a partition that satisfies (O1) - (O4), yet contradicts (O5), even though we have replaced $R \cong$ Paw with a C_4. We make note of this as we will use this in the arguments below.

We now label the vertices of R so that $V(R) = \{a, b, c, d\}$ where $\|a, R\| = 1$ and $\|b, R\| = 3$. Let $B = G[b, c, d]$. We consider $\|R, T\|$ based on the value of $\|a, T\|$ for $T \in \mathcal{T}$.

Lemma 25. *If $\|a, T\| \leqslant 1$ for some $T \in \mathcal{T}$, then $\|R, T\| \leqslant 12$*

Proof. Suppose $\|R, T\| \geqslant 13$. Lemma 7 implies $\|B, T\| \leqslant 12$ so that we must have $\|a, T\| = 1$, $\|B, T\| = 12$, and furthermore $|T| = 4$. Without loss of generality, ax_1 or ay_1 is an edge of G. Then $G[a, b, x_1, y_1]$ and $G[c, d, x_2, y_2]$ both contain K_4^-, a contradiction. □

Lemma 26. *If $\|a,T\| = 2$ for some $T \in \mathcal{T}$, then $\|R,T\| \leqslant 10$.*

Proof. Suppose $\|R,T\| \geqslant 11$. Then there must exist a vertex $v \in R$ such that $\|v,T\| \geqslant 3$. By Lemmas 7 and 9, if $G[T] \cong \theta_{1,2,3}$ or $G[T] \cong K_{2,3}$, we get at most two vertices of R each with three neighbors in T so that $\|R,T\| \leqslant 10$. Thus $|T| = 4$, and we consider three cases based on the neighbors of a: $ax_1, ax_2 \in E(G)$, $ax_1, ay_1 \in E(G)$ and $G[T] \cong K_4^-$, and $ay_1, ay_2 \in E(G)$ and $G[T] \cong K_4^-$.

Suppose to start $ax_1, ax_2 \in E(G)$ (note in this case $G[T]$ may be either K_4 or K_4^-). If $\|y_1, B\| \geqslant 2$, then $G[a, x_1, x_2, y_2]$ and $G[b, c, d, y_1]$ both contain K_4^-, a contradiction. Thus $\|y_1, B\| \leqslant 1$ and by symmetry, $\|y_2, B\| \leqslant 1$. Now $\|B,T\| \leqslant 8$ so $\|R,T\| \leqslant 10$ as desired.

Next assume $ax_1, ay_1 \in E(G)$ and $G[T] \cong K_4^-$. If $\|y_2, B\| \geqslant 2$, then $G[a, x_1, y_1, x_2]$ and $G[b, c, d, y_2]$ both contain K_4^-, a contradiction. If $\|x_2, B\| = 3$, then $G[b, c, d, x_2]$ is K_4 and $G[a, x_1, y_1, y_2]$ contains Paw contradicting (O5). Similarly, if $\|y_1, B\| = 3$, then $G[b, c, d, y_1]$ contains K_4 and $G[a, x_1, x_2, y_2]$ contains Paw contradicting (O5). Thus $\|B,T\| \leqslant 8$ so $\|R,T\| \leqslant 10$ as desired.

Finally assume $ay_1, ay_2 \in E(G)$ and $G[T] \cong K_4^-$. If $\|x_1, B\| = 3$, then $G[b, c, d, x_1]$ contains K_4 and $G[a, y_1, x_2, y_2]$ is C_4 contradicting (O5). Thus $\|x_1, B\| \leqslant 2$ and by symmetry $\|x_2, B\| \leqslant 2$. If $\|y_1, B\| = 3$, then $G[b, c, d, y_1]$ contains K_4 and $G[a, x_1, x_2, y_2]$ contains Paw, contradicting (O5). Thus $\|y_1, B\| \leqslant 2$ and by symmetry $\|y_2, B\| \leqslant 2$. Now $\|B,T\| \leqslant 8$ so $\|R,T\| \leqslant 10$ as desired. □

Lemma 27. *If $\|a,T\| = 3$ for some $T \in \mathcal{T}$, then $\|R,T\| \leqslant 8$.*

Proof. Suppose $\|a,T\| = 3$ and $\|R,T\| \geqslant 9$, and thus $\|B,T\| \geqslant 6$. If $G[T] \in \{\theta_{1,2,3}, K_{2,3}\}$, then there exists $v \in T$ such that $\|v, B\| \geqslant 2$. However, we can replace T with the theta graph in $B + v$, contradicting (O1).

So $|T| = 4$. We consider two cases based on the neighbors of a: $ax_1, ax_2, ay_1 \in E(G)$, and $ax_1, ay_1, ay_2 \in E(G)$ with $G[T] \cong K_4^-$.

First assume $ax_1, ax_2, ay_1 \in E(G)$. If for $i \in \{1, 2\}$, $\|y_i, B\| \geqslant 2$, then $T - y_i + a$ and $B + y_i$ each contain a K_4^-, a contradiction. When $G[T] \cong K_4^-$, we claim $\|y_2, B\| = 0$ and $\|x_1, B\|, \|x_2, B\| \leqslant 2$. For otherwise, if $\|y_2, B\| = 1$, then $G[a, x_1, x_2, y_1]$ is K_4 and $G[b, c, d, y_2]$ contains Paw which contradicts (O5). Also if $\|x_1, B\| = 3$, then $G[b, c, d, x_1]$ contains K_4 and $G[a, y_1, y_2, x_2]$ contains Paw which again contradicts (O5). By symmetry, it follows that $\|x_2, B\| \leqslant 2$. Thus when $G[T] \cong K_4^-$, we have $\|B,T\| \leqslant 5$, a contradiction to $\|R,T\| \geqslant 9$. When $G[T] \cong K_4$, x_1, x_2, y_1 are all symmetric so because we have already argued that $\|y_1, B\| \leqslant 1$, we also know by symmetry that $\|x_1, B\|, \|x_2, B\| \leqslant 1$. If $\|y_2, B\| \geqslant 2$, then $G[b, c, d, y_2]$ and $G[a, x_1, x_2, y_1]$ both contain K_4^-, a contradiction. Thus when $G[T] \cong K_4$, $\|B,T\| \leqslant 4$, again a contradiction.

Finally assume $ax_1, ay_1, ay_2 \in E(G)$ and $G[T] \cong K_4^-$. If $u \in \{x_2, y_1, y_2\}$, satisfies $\|u, B\| \geqslant 2$, then $T - u + a$ and $B + u$ each contain K_4^-, a contradiction. Since $\|B,T\| \geqslant 6$, we must have $\|x_1, B\| = 3$. However, $B + x_1$ is a K_4 and $T - x_1 + a$ is a C_4, contradicting (O5). □

Lemma 28. *If $\|a,T\| = 4$ for some $T \in \mathcal{T}$, then $\|R,T\| \leqslant 8$.*

Proof. By Lemma 7, $|T| = 4$. If $\|v, B\| \geqslant 2$ for some $v \in T$, then $B + v$ and $T - v + a$ both contain K_4^-, a contradiction. Thus $\|v, B\| \leqslant 1$ for all $v \in T$. Now $\|B, T\| \leqslant 4$ and $\|R, T\| \leqslant 8$ as desired. $\square$

For $i \in \{0, 1, 2, 3, 4\}$ define $\mathcal{T}_i = \{T \in \mathcal{T} : \|a, T\| = i\}$, and let $t_i = |\mathcal{T}_i|$. Thus, $t_0 + t_1 + t_2 + t_3 + t_4 = k - 1$. Looking at the degree of a, we have:

$$\frac{5}{2}k \leqslant d_G(a) = 0t_0 + 1t_1 + 2t_2 + 3t_3 + 4t_4 + 1. \tag{14}$$

By Lemmas 25, 26, 27, and 28, we get:

$$4\left(\frac{5}{2}k\right) \leqslant \|R, \mathcal{T}\| + \|R, R\| \leqslant 12t_0 + 12t_1 + 10t_2 + 8t_3 + 8t_4 + 8. \tag{15}$$

Now (14) $\times 4$ + (15) $\times 3$ gives:

$$40k \leqslant 36t_0 + 40t_1 + 38t_2 + 36t_3 + 40t_4 + 28. \tag{16}$$

By the definition of the t_i, we have that $\displaystyle\sum_{i=0}^{4} t_i = k - 1$. From this equation we get:

$$(36t_0 + 40t_1 + 38t_2 + 36t_3 + 40t_4) + 28 \leqslant 40(k - 1) + 28 = 40k - 12. \tag{17}$$

Now (16) and (17) contradict each other. This concludes the case when $R \cong$ Paw. The proof of Theorem 6 is now complete.

5 Minimum Degree Transition and Ore Versions

Theorem 6 shows that when $n \geqslant 4k$, every n-vertex graph with minimum degree at least $\lceil \frac{5}{2}k \rceil$ contains k disjoint theta graphs, and in Section 2, we show this condition is sharp when $4k \leqslant n < 5k$. At the same time Theorem 4 shows that when $n \geqslant c_k$ (as defined in Theorem 4), every n-vertex graph with minimum degree at least $2k + 1$ contains k disjoint theta graphs, and furthermore this condition is sharp for $n \geqslant c_k$. In this section, we discuss when and how the sharp minimum degree condition may transition from $\lceil \frac{5}{2}k \rceil$ to $2k+1$ as n goes from $5k$ to c_k. In addition, we consider minimum Ore degree versions, and we conclude with some questions for further research.

First, we present some terminology and notation that will be used in the theorems below. Given a graph H, an optimal coloring of H is a proper vertex-coloring using exactly $\chi(H)$ colors. Over all optimal colorings of H, the size of the smallest color class is denoted by $\sigma(H)$. The critical chromatic number of H, denoted $\chi_{cr}(H)$, is given by $\dfrac{(\chi(H) - 1)|H|}{|H| - \sigma(H)}$. As an example, $\sigma(K_{2,3}) = 2$ and $\chi_{cr}(K_{2,3}) = \frac{5}{3}$. This notation allows us to understand the following results of Komlós in [8], and Kühn, Osthus, and Treglown in [9], regarding near packings of graphs.

Theorem 29 (Komlós [8]). *For every graph H and $\epsilon > 0$, there exists $n_0 = n_0(H, \epsilon)$ such that for every $n \geqslant n_0$ the following holds: if G is an n-vertex graph with $\delta(G) \geqslant \left(1 - \frac{1}{\chi_{cr}(H)}\right) n$, then G contains a collection of disjoint copies of H that covers all but at most ϵn vertices of G.*

Theorem 30 (Kühn-Osthus-Treglown [9]). *For every graph H and for every $\epsilon > 0$, there exists $n_0 = n_0(H, \epsilon)$ such that for every $n \geqslant n_0$ the following holds: if G is an n-vertex graph with $\sigma_2(G) \geqslant 2\left(1 - \frac{1}{\chi_{cr}(H)}\right) n$, then G contains a collection of disjoint copies of H that covers all but at most ϵn vertices of G.*

Using these theorems, we can prove the following result regarding disjoint theta graphs.

Proposition 31. *For every rational number $\frac{a}{b} \geqslant 5$ and every real $\epsilon > 0$, there exists $k_0 = k_0(\epsilon, a, b)$ such that for every $k \geqslant k_0$ the following holds:*

(1) if G is an n-vertex graph where $n = \frac{ak}{b(1-\epsilon)}$ and $\delta(G) \geqslant \frac{2k}{1-\epsilon}$, then G contains k disjoint theta graphs, and

(2) if G is an n-vertex graph where $n = \frac{ak}{b(1-\epsilon)}$ and $\sigma_2(G) \geqslant \frac{4k}{1-\epsilon}$, then G contains k disjoint theta graphs.

Proof. Fix a rational number $\frac{a}{b} \geqslant 5$ and a real number $\epsilon > 0$. Since $\frac{a}{b} \geqslant 5$, there exists $t \geqslant 3$ and $0 \leqslant r < b$ such that $a = (t+2)b + r$. Let $H = K_{2b, bt+r}$. Observe that $|H| = a$, and since $t \geqslant 3$, $\sigma(H) = 2b$. So $\chi_{cr}(H) = \frac{a}{tb+r}$. Furthermore, H contains b disjoint copies of $K_{2,3}$; that is, H contains b disjoint theta graphs.

Using this H and ϵ, let n_0 be given from Theorem 29. Let k_0 be such that $\frac{a}{b}k_0 \geqslant n_0$, let $k \geqslant k_0$, and let $n = \frac{ak}{b(1-\epsilon)}$. Observe

$$\left(1 - \frac{1}{\chi_{cr}(H)}\right) n = \left(1 - \frac{tb+r}{a}\right) \frac{ak}{b(1-\epsilon)} = \left(\frac{((t+2)b+r) - tb - r}{a}\right) \frac{ak}{b(1-\epsilon)} =$$

$$\left(\frac{2b}{a}\right) \frac{ak}{b(1-\epsilon)} = \frac{2k}{1-\epsilon}.$$

Therefore by Theorem 29, if G is an n-vertex graph with $n = \frac{ak}{b(1-\epsilon)}$ and $\delta(G) \geqslant \frac{2k}{1-\epsilon}$, then G contains a collection of disjoint copies of H that covers all but at most ϵn vertices. Since each copy of H contains b disjoint theta graphs, the number of disjoint theta graphs in G is at least $b(\frac{n-\epsilon n}{|H|}) = b(\frac{ak}{b(1-\epsilon)} \frac{(1-\epsilon)}{a}) = k$, which proves (1).

By Theorem 30, if G is an n-vertex graph with $n = \frac{ak}{b(1-\epsilon)}$ and $\sigma_2(G) \geqslant \frac{4k}{1-\epsilon}$, then G contains a collection of disjoint copies of H that covers all but at most ϵn vertices. The same calculation as above then proves (2). $\square$

Again using Theorem 30, we can obtain the following minimum Ore degree result for some n-vertex graphs with $4k \leqslant n < 5k$.

Proposition 32. *For every $\epsilon, 0 < \epsilon < \frac{1}{8}$, there exists $n_0 = n_0(\epsilon)$ such that for all $n \geqslant n_0$, if G is an n-vertex graph with $4k+8\epsilon n \leqslant n < 5k$ and $\sigma_2(G) \geqslant 5k+10\epsilon n$, then G contains k disjoint theta graphs.*

Proof. Fix ϵ where $0 < \epsilon < \frac{1}{8}$, let $C = \left\lfloor \frac{1}{\epsilon} \right\rfloor$, and let $h = 2\left\lfloor \frac{C(1-\epsilon)}{2} \right\rfloor$. For each graph on at most h vertices together with our choice of ϵ, Theorem 30 provides some n_0. In the following, let n_0 be the largest value over all possible n_0 for graphs on at most h vertices together with our choice of ϵ.

Choose $n \geqslant n_0$ and $k \geqslant 1$ such that $4k + 8\epsilon n \leqslant n < 5k$. Let G be an n-vertex graph with $\sigma_2(G) \geqslant 5k + 10\epsilon n$. Furthermore, let $k' = \left\lceil \frac{Ck}{n} \right\rceil$.

Let $G(t,n) = K_{t,\lfloor \frac{n-t}{2} \rfloor, \lceil \frac{n-t}{2} \rceil}$ where $1 \leqslant t \leqslant \frac{n}{4}$. Observe that $\chi_{cr}(G(t,n)) = \frac{2n}{n-t}$. Let $H = G(5k'-h, h)$ so that $|H| = h$. Our goal is to show that H has at least k' disjoint theta graphs, then show that Theorem 30 implies that all but at most ϵn vertices of G are covered by disjoint copies of H, and finally show that the number of copies of H in G is at least $\frac{k}{k'}$ so that G contains k disjoint theta graphs.

Claim 33. $4k' \leqslant h < 5k'$.

Proof. Observe:

$$h = 2\left\lfloor \frac{C(1-\epsilon)}{2} \right\rfloor \leqslant C(1-\epsilon) \text{ and } k' = \left\lceil \frac{Ck}{n} \right\rceil \geqslant \frac{Ck}{n}. \tag{18}$$

Thus, $\frac{h}{k'} \leqslant \frac{C(1-\epsilon)n}{Ck} < \frac{n}{k}$. Since $n < 5k$, we have $\frac{h}{k'} < \frac{n}{k} < 5$. Thus, $h < 5k'$.

To show $h \geqslant 4k'$, note $k' = \left\lceil \frac{Ck}{n} \right\rceil \leqslant \frac{Ck}{n}+1$. Further, $h = 2\left\lfloor \frac{C(1-\epsilon)}{2} \right\rfloor \geqslant 2\left(\frac{C(1-\epsilon)}{2} - 1\right) = C(1-\epsilon) - 2$. Since $C = \left\lfloor \frac{1}{\epsilon} \right\rfloor \leqslant \frac{1}{\epsilon}$, we can show $C(1-\epsilon) - 2 \geqslant C - 3$. Thus, $h \geqslant C - 3$.

Since $4k + 8\epsilon n \leqslant n$, we deduce $\frac{k}{n} \leqslant \frac{1-8\epsilon}{4}$. Observe:

$$\begin{aligned} \frac{k'}{h} &\leqslant \frac{\frac{Ck}{n}+1}{C-3} \\ &= \frac{k}{n} + \frac{1}{C-3} + \frac{k}{n}\frac{3}{C-3} \\ &\leqslant \frac{k}{n} + \frac{1}{C-3} + \frac{3}{C-3}\left(\frac{1-8\epsilon}{4}\right) \\ &= \frac{k}{n} + \frac{7-24\epsilon}{4(C-3)} \end{aligned}$$

We wish to show $\frac{7-24\epsilon}{4(C-3)} \leqslant 2\epsilon$, which is equivalent to showing $\frac{7}{8}\frac{1}{\epsilon} \leqslant C$. Indeed, since $C = \left\lfloor \frac{1}{\epsilon} \right\rfloor$, we have $\frac{1}{\epsilon} - 1 \leqslant C \leqslant \frac{1}{\epsilon}$, and since $\epsilon < \frac{1}{8}$, we get $\frac{7}{8}\frac{1}{\epsilon} < \frac{1}{\epsilon} - 1 \leqslant C$. Therefore $\frac{7-24\epsilon}{4(C-3)} \leqslant 2\epsilon$, and we have

$$\frac{k'}{h} \leqslant \frac{k}{n} + 2\epsilon. \tag{19}$$

So $h \geqslant \dfrac{4k'}{\frac{4k}{n} + 8\epsilon}$, and since $4k + 8\epsilon n \leqslant n$, we deduce that $\frac{4k}{n} + 8\epsilon \leqslant 1$. Thus, $h \geqslant 4k'$. □

Claim 34. *H has k' disjoint theta graphs.*

Proof. Since $h < 5k'$, we have $5k' - h > 0$. Thus, by the definition of H as $H = G(5k' - h, h)$, H is a tripartite graph. As a result, we can construct, $5k' - h$ disjoint copies of K_4^- that uses all $5k' - h$ vertices from the partite set of size $5k' - h$, and then uses $x_1 = \lfloor \frac{3}{2}(5k' - h) \rfloor$ and $x_2 = \lceil \frac{3}{2}(5k' - h) \rceil$ vertices from each of the remaining two partite sets (we will decide which later). This leaves us to construct $h - 4k'$ disjoint copies of $K_{2,3}$, which uses $y_1 = \lfloor \frac{5}{2}(h - 4k') \rfloor$ and $y_2 = \lceil \frac{5}{2}(h - 4k') \rceil$ vertices from each of the remaining two partite sets (we will decide which later).

Observe that $x_1 + x_2 + y_1 + y_2 = 2h - 5k' = h - (5k' - h)$, which is exactly the number of vertices in the remaining two partite sets. As a result, there are enough vertices in the two remaining partite sets to form all of our desired copies of K_4^- and $K_{2,3}$. By the construction of H, these remaining partite sets are as equal as possible, except that one may be larger than the other by exactly one. Observe that similarly, $x_1 + y_2$ and $x_2 + y_1$ are either the same or they differ by exactly one. So if $x_1 + y_2 = x_2 + y_1$, then because $x_1 + x_2 + y_1 + y_2 = h - (5k' - h)$, the two remaining partite sets also have the same size and we can find our desired theta graphs. If say $x_1 + y_2 = x_2 + y_1 + 1$, then similarly one of the partite sets has one more vertex than the other. So we take the $x_1 + y_2$ vertices from that larger set and $x_2 + y_1$ vertices from the smaller, to find our desired theta graphs. A similar argument holds if $x_1 + y_2 + 1 = x_2 + y_1$. In any case, we obtain k' disjoint theta graphs in H. □

To show Theorem 30 implies that all but at most ϵn vertices of G are covered by disjoint copies of H, note that by the construction of H, $\chi_{cr}(H) = \frac{2h}{2h-5k'}$. So $2\left(1 - \frac{1}{\chi_{cr}(H)}\right) n = \frac{5k'n}{h}$. By (19), $\frac{k'}{h} \leqslant \frac{k}{n} + 2\epsilon$, so that $\frac{5k'n}{h} \leqslant 5n(\frac{k}{n} + 2\epsilon) = 5k + 10\epsilon n$. Therefore, $\sigma_2(G) \geqslant 5k + 10\epsilon n \geqslant 2\left(1 - \frac{1}{\chi_{cr}(H)}\right) n$, and by Theorem 30, all but at most ϵn vertices of G are covered by disjoint copies of H.

In particular, G contains at least $\frac{n-\epsilon n}{h}$ disjoint copies of H, each of which contains k' disjoint theta graphs by Claim 34. Thus, G has at least $\frac{n-\epsilon n}{h} k'$ disjoint theta graphs. Recall that by (18), $h \leqslant C(1 - \epsilon)$ and $k' \geqslant \frac{Ck}{n}$, so we have $\frac{n-\epsilon n}{h} k' \geqslant \frac{n(1-\epsilon)Ck}{C(1-\epsilon)n} = k$ disjoint theta graphs in G, as desired. □

To the authors' knowledge, the only known n-vertex graphs with minimum degree $\lceil \frac{5}{2}k \rceil - 1$ that do not have k disjoint theta graphs, each satisfies $n < 5k$. So in some sense, (1) from Proposition 31 provides an indication that perhaps when $n \geqslant 5k$, a minimum degree near $2k$ (perhaps $2k + 1$) is sufficient enough to guarantee k disjoint theta graphs. That is to say, perhaps the sharp minimum degree in Theorem 6 transitions to the sharp minimum degree in Theorem 4 exactly at $n = 5k$ and then that minimum degree holds for all $n \geqslant 5k$. Proposition 32 and (2) from Proposition 31 seem to indicate a similar idea when considering minimum Ore degree versions. These results lead the authors to pose the following questions.

Question 35. Let $k \in \mathbb{Z}^+$, and let G be an n-vertex graph. When $n \geqslant 5k$ and $\delta(G) \geqslant 2k + 1$, will G always contain k disjoint theta graphs?

Question 36. Let $k \in \mathbb{Z}^+$, and let G be an n-vertex graph. When $n \geqslant 4k$ and $\sigma_2(G) \geqslant 5k$, will G always contain k disjoint theta graphs?

Question 37. Let $k \in \mathbb{Z}^+$, and let G be an n-vertex graph. When $n \geqslant 5k$ and $\sigma_2(G) \geqslant 4k+1$, will G always contain k disjoint theta graphs?

Recall that the graphs described in (ii) of Theorem 4, have at least $4k$ vertices, minimum degree $2k$, minimum Ore degree $4k$, and do not have k disjoint theta graphs. Therefore, if either Question 35 or Question 37 can be answered in the affirmative, then the results would be best possible. Similarly, $H'_{k,n}$ as described in Section 2, is an n-vertex graph with $4k \leqslant n < 5k$, such that when k is odd, $\sigma_2(H'_{k,n}) = 5k-1$, and does not contain k disjoint theta graphs. Hence if Question 36 can be answered in the affirmative, then the result would also be best possible.

Lastly, the authors would like to thank the anonymous referees for their careful reading of this manuscript. Their suggestions and comments have improved the quality of this paper.

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