Quickly proving Diestel's normal spanning tree criterion

Max Pitz

Department of Mathematics
Universität Hamburg
20146 Hamburg, Germany

max.pitz@uni-hamburg.de

Submitted: Jun 4, 2020; Accepted: Aug 24, 2021; Published: Sep 24, 2021

Abstract

We present two short proofs for Diestel's criterion that a connected graph has a normal spanning tree provided it contains no subdivision of a countable clique in which every edge has been replaced by uncountably many parallel edges.

Mathematics Subject Classifications: 05C05, 05C63

1 Overview

This paper continues a line of inquiry started in [8] with the aim to find efficient algorithms for constructing normal spanning trees in infinite graphs. A rooted spanning tree T of a graph G is called *normal* if the end vertices of any edge of G are comparable in the natural tree order of T. Intuitively, all the edges of G run 'parallel' to branches of T, but never 'across'.

Every countable connected graph has a normal spanning tree, but uncountable graphs might not, as demonstrated by complete graphs on uncountably many vertices. While exact characterisations of graphs with normal spanning trees exist, see e.g. [6, 7], these may be hard to verify in practice. The most applied sufficient condition for normal spanning trees is the following criterion due to Halin [4], and its strengthening due to Diestel [2], see also [7, §6] for an updated proof.

Theorem 1 (Halin)**.** *Every connected graph without a $TK^{\aleph_0}$ has a normal spanning tree.*

Theorem 2 (Diestel)**.** *Every connected graph without fat $TK^{\aleph_0}$ has a normal spanning tree.*

Here, a $TK^{\aleph_0}$ is a subdivision of the countable clique $K^{\aleph_0}$, and a *fat* $TK^{\aleph_0}$ is a subdivision of the multigraph obtained from a $K^{\aleph_0}$ by replacing every edge with $\aleph_1$ parallel edges.

Until recently, only fairly involved proofs of these results were available: Halin's original proof employing his theory of simplicial decompositions [4], and Diestel's proof strategy building on the forbidden minor characterisation for normal spanning trees [2, 7]. In [8], however, the present author found a simple greedy algorithm which constructs the desired normal spanning tree in Halin's Theorem 1 in just ω many steps. The purpose of this note is to provide two simple proofs also for Theorem 2, one of them again an ω-length algorithm.

Notably, this algorithm also yields a new, local version of Theorem 2: Given a set of vertices U of a connected graph G, there exists a normal tree of G containing U provided every fat $TK^{\aleph_0}$ in G can be separated from U by a finite set of vertices, see Theorem 5 below. Furthermore, both Halin's and Diestel's criterions are sufficient for a normal spanning tree, but by no means necessary, as a fat $TK^{\aleph_0}$ by itself does admit a normal spanning tree. Our algorithm allows us to strengthen these results in the following way: A connected graph has a normal spanning tree if and only if its vertex set is a countable union of sets each separated from any fat $TK^{\aleph_0}$ by a finite set of vertices, Theorem 6.

2 Tree orders and normal trees

We follow the notation in [1]. The *tree-order* $\leqslant_T$ of a tree T with root r is defined by setting $u \leqslant_T v$ if u lies on the unique path from r to v in T. Then $\leqslant_T$ is a partial order. For a vertex v of T, let $\lceil v \rceil := \{t \in T \colon t \leqslant_T v\}$ be the *down-closure* v in T, the nodes on the $r - v$ path in T.

For rooted trees that are not necessarily spanning, one generalises the notion of normality as follows: A rooted tree $T \subseteq G$ is *normal (in G)* if the end vertices of any T-path in G (a path in G with end vertices in T but all edges and inner vertices outside of T) are comparable in the tree order of T. If T is spanning, this clearly reduces to the definition given in the introduction. If $T \subseteq G$ is normal, then the set of neighbours $N(D)$ of any component D of $G - T$ forms a chain in T, i.e. all vertices of $N(D)$ are comparable in $\leqslant_T$. Moreover, incomparable nodes v, w of any normal tree $T \subseteq G$ are separated in G by $\lceil v \rceil \cap \lceil w \rceil$.

The following well-known consequence of Jung's criterion [6] about the existence of normal spanning trees has been pointed out in [5, Lemma 7.2] and will be used later.

Fact 3. *Let G be a graph with a normal spanning tree. Then for every connected subgraph $C \subseteq G$ and every $r \in C$ there is a normal spanning tree of C with root r.*

For distinct vertices v, w of G we denote by $\kappa(v, w) = \kappa_G(v, w)$ the (vertex-)connectivity between v and w in G, i.e. the largest size of a family of independent (i.e. pairwise internally-disjoint) $v - w$ paths. If v and w are non-adjacent, this is by Menger's theorem for infinite graphs [1, Proposition 8.4.1] equivalent to the minimal size of a $v - w$ separator in G.

Note that a fat $TK^{\aleph_0}$ has only $\aleph_1$ many edges in total. Hence, the following fact observed by Halin [3, (15)] follows readily by selecting the desired paths for a $TK^{\aleph_0}$ recursively in ω_1 steps, so at any point during the construction there are only countably many vertices to avoid.

Fact 4. *Let U be a countable set of vertices in G. There is a fat $TK^{\aleph_0}$ with branch vertices U if and only if $\kappa(u,v)$ is uncountable for all $u \neq v \in U$.*

3 The first proof

First proof of Theorem 2. By induction on $|G|$. We may assume that $|G|$ is uncountable. Suppose we have a continuous[1] increasing ordinal-indexed sequence $(G_i : i < \sigma)$ of induced subgraphs all of size less than $|G|$ with $G = \bigcup_{i<\sigma} G_i$ such that

(i) the end vertices of any G_i-path[2] in G have infinite connectivity in G_i, and

(ii) the end vertices of any G_i-path in G have uncountable connectivity in G.

Then we can construct normal spanning trees T_i of G_i extending each other all with the same root by (transfinite) recursion on i. If $\ell < \sigma$ is a limit, by continuity of our sequence we may simply define $T_\ell = \bigcup_{i<\ell} T_i$. For the successor case, suppose that T_i is already defined. By (ii), the neighbourhood $N(C)$ is finite for every component C of $G_{i+1} - G_i$ (otherwise we get a fat $TK^{\aleph_0}$ by (ii) and Fact 4), and by (i), $N(C)$ lies on a chain of T_i (as incomparable vertices in T_i are separated in G_i by the intersection of their finite down-closures). Let $t_C \in N(C)$ be maximal in the tree order of T_i, and let r_C be a neighbour of t_C in C. By the induction hypothesis and Fact 3, C has a normal spanning tree T_C with root r_C. Then T_i together with all T_C and edges $t_C r_C$ is a normal spanning tree T_{i+1} of G_{i+1}. Once the recursion is complete, $T = \bigcup_{i<\sigma} T_i$ is the desired normal spanning tree of G. It remains to construct a sequence $(G_i : i < \sigma)$ with (i) and (ii). The reader familiar with elementary submodel techniques such as in [9] may wish to take a continuous increasing chain $(M_i : i < \sigma)$ with $\sigma = cf(|G|)$ of $<|G|$-sized elementary submodels M_i of a large enough fragment of ZFC with $G \in M_i$, such that $G \subseteq \bigcup_{i<\sigma} M_i$. Then $G_i = G \cap M_i$ is as required.

In what follows, however, we assume no such familiarity and outline a direct construction: Enumerate $V(G) = \{v_i : i < \sigma = |G|\}$ and put $G_0 := \{v_0\}$. If $\ell < \sigma$ is a limit, set $G_\ell := \bigcup_{i<\ell} G_i$ and note that properties (i) and (ii) are preserved under increasing unions. To define G_{i+1} from G_i, we use a countable closure argument. Set $G_i^0 := G[G_i \cup v_{i+1}]$ and construct G_i^{n+1} from G_i^n by adding, for every pair $v, w \in V(G_i^n)$ with $\kappa_G(v,w)$ at most countable, an inclusion-wise maximal family of independent $v-w$ paths in G to G_i^n, and for all remaining pairs some $\aleph_0$ many independent $v-w$ paths in G to G_i^n. Then $G_{i+1} := G\big[\bigcup_{n\in\mathbb{N}} G_i^n\big]$ is as desired: any G_{i+1}-path from v to w witnesses that $\kappa_G(v,w)$ was uncountable, giving (ii), and so we have added infinitely many independent $v-w$ paths to G_{i+1} in the process, giving (i). □

[1] The sequence $(G_i : i < \sigma)$ is continuous if for every limit ordinal $\ell < \sigma$ we have $G_\ell = \bigcup_{i<\ell} G_i$.

[2] A path with end vertices in G_i but all edges and inner vertices outside of G_i.

4 The second proof

Our second proof extracts the closure properties (i) and (ii) in the previous construction, and combines them into a single algorithm constructing the normal spanning tree in ω many steps, avoiding ordinals and transfinite constructions altogether.

Second proof of Theorem 2. For every pair of distinct vertices v and w of G with $\kappa(v,w)$ at most countable, fix a maximal collection $\mathcal{P}_{v,w} = \{P^1_{v,w}, P^2_{v,w}, \dots\}$ of independent v–w paths in G. Construct a countable chain $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ of rayless normal trees in G with the same root $r \in V(G)$ as follows: Put $T_0 = \{r\}$, and suppose T_n has already been defined. Since T_n is a rayless normal tree, any component D of $G - T_n$ has a finite neighbourhood $N(D)$ in T_n, because $N(D)$ is linearly ordered in the tree ordering. Let $t_D \in N(D)$ be maximal in the tree order of T_n, and let r_D be a neighbour of t_D in D. For each pair $v \neq w \in N(D)$ with countable connectivity select the path $P^D_{v,w}$ with least index in $\mathcal{P}_{v,w}$ intersecting D. The argument in [1, Theorem 8.2.4] shows that any finite set of vertices in a connected graph is contained in a finite normal tree with arbitrarily prescribed root. Hence, in each component D there is a finite tree T_D with root r_D that is normal in D and contains all vertices of $P^D_{v,w} \cap D$ for all $v \neq w \in N(D)$ with $\kappa(v,w)$ at most countable. Then T_n together with all T_D and edges $t_D r_D$ is a rayless normal tree in G with root r extending T_n. This completes the construction.

The union $T = \bigcup_{n\in\mathbb{N}} T_n$ with root r is a normal tree in G. We claim that T is spanning unless G contains a fat $TK^{\aleph_0}$. If T is not spanning, consider a component C of $G - T$. Then $N(C) \subseteq T$ is infinite: otherwise, $N(C) \subseteq T_n$ for some $n \in \mathbb{N}$, but then we would have extended T_n into C because C was a component of $G - T_n$, a contradiction. For every n, let D_n be the unique component of $G - T_n$ containing C.

By Fact 4, it suffices to show that $\kappa(v,w)$ is uncountable for every $v \neq w \in N(C)$. Consider a T-path P from v to w with its interior $\mathring{P}$ completely contained in C. If $\kappa(v,w)$ was countable, then by maximality of $\mathcal{P}_{v,w}$ there is $P^k_{v,w} \in \mathcal{P}_{v,w}$ containing an interior vertex x of P. Let m be minimal with $v, w \in T_m$. Since the $P^{D_n}_{v,w}$ are pairwise distinct, the path $P^k_{v,w}$ was selected as $P^{D_n}_{v,w}$ for some n with $m \leqslant n \leqslant m+k$. But then $x \in P^k_{v,w} \cap \mathring{P} \subseteq P^{D_n}_{v,w} \cap D_n \subseteq T_{n+1} \subseteq T$ contradicts that P is a T-path. □

5 Local versions of Diestel's criterion

By a slight modification of this ω-length algorithm, one readily obtains a proof of the following results, which answer [7, Problem 3].

Theorem 5. *A set of vertices U in a connected graph G is contained in a normal tree of G provided every fat $TK^{\aleph_0}$ in G can be separated from U by a finite set of vertices.*

Proof. Let U be a set of vertices such that every fat $TK^{\aleph_0}$ in G can be separated from U by a finite set of vertices. Use the algorithm from Section 4, but only extend T_n into components D of $G - T_n$ with $U \cap D \neq \varnothing$. Additionally, make sure to include at least one vertex from $U \cap D$.

Then $T = \bigcup T_n$ is normal and it remains to argue that U is contained in T. Otherwise, there is a component C of $G - T$ containing a vertex from U. As in Section 4, this gives us a fat $TK^{\aleph_0}$ in G, and it is readily verified that this fat $TK^{\aleph_0}$ cannot be separated from U by a finite set of vertices, cf. [8, Proof of Theorem 3, item (2)]. □

Theorem 6. *A connected graph has a normal spanning tree if and only if its vertex set is a countable union of sets each separated from any fat $TK^{\aleph_0}$ by a finite set of vertices.*

Proof. For the forward implication, recall that the levels of any normal spanning tree can be separated by a finite set of vertices from any ray, and hence in particular from any fat $TK^{\aleph_0}$. Conversely, let $\{V_n \colon n \in \mathbb{N}\}$ be sets of vertices in G with $V(G) = \bigcup_{n\in\mathbb{N}} V_n$ such that each V_n can be separated from any fat $TK^{\aleph_0}$ by a finite set of vertices. Adapt the algorithm from Section 4, so that when extending T_n into a component D of $G - T_n$, we additionally include a vertex $v_D \in D \cap V_{n_D}$ where n_D minimal such that $V_{n_D} \cap D \neq \emptyset$.

Then $T = \bigcup T_n$ is normal and it remains to argue that it is spanning. Otherwise, there is a component C of $G - T$ and we may choose $n_C \in \mathbb{N}$ to be minimal such that C contains a vertex from V_{n_C}. As in Section 4, this component C gives us a fat $TK^{\aleph_0}$ in G, and it is readily verified that this fat $TK^{\aleph_0}$ cannot be separated from $\bigcup_{n \leqslant n_C} V_n$ by a finite set of vertices, cf. [8, Proof of Theorem 3, item (2)]. This, however, means that our fat $TK^{\aleph_0}$ cannot be separated from one of the V_n for some $n \leqslant n_C$, a contradiction. □

References

[1] Reinhard Diestel. *Graph Theory.* Springer, 5th edition, 2015.

[2] Reinhard Diestel. A simple existence criterion for normal spanning trees. *The Electronic Journal of Combinatorics*, 2016. #P2.33.

[3] Rudolf Halin. Unterteilungen vollständiger Graphen in Graphen mit unendlicher chromatischer Zahl. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 31, pages 156–165. Springer, 1967.

[4] Rudolf Halin. Simplicial decompositions of infinite graphs. In *Annals of Discrete Mathematics*, volume 3, pages 93–109. Elsevier, 1978.

[5] Rudolf Halin. Miscellaneous problems on infinite graphs. *Journal of Graph Theory*, 35(2):128–151, 2000.

[6] Heinz A. Jung. Wurzelbäume und unendliche Wege in Graphen. *Mathematische Nachrichten*, 41(1-3):1–22, 1969.

[7] Max Pitz. Proof of Halin's normal spanning tree conjecture. To appear in *Israel Journal of Mathematics.*

[8] Max Pitz. A unified existence theorem for normal spanning trees. *Journal of Combinatorial Theory, Series B*, 145:466–469, 2020.

[9] Lajos Soukup. Elementary submodels in infinite combinatorics. *Discrete Mathematics*, 311(15):1585–1598, 2011.