A generalization of the Bollobás set pairs inequality

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Abstract

The Bollobás set pairs inequality is a fundamental result in extremal set theory with many applications. In this paper, for $n \geq k \geq t \geq 2$, we consider a collection of $k$ families $A_i : 1 \leq i \leq k$ where $A_i = \{ A_{i,j} \subset [n] : j \in [n] \}$ so that $A_{1,i_1} \cap \cdots \cap A_{k,i_k} \neq \emptyset$ if and only if there are at least $t$ distinct indices $i_1, i_2, \ldots, i_k$. Via a natural connection to a hypergraph covering problem, we give bounds on the maximum size $\beta_{k,t}(n)$ of the families with ground set $[n]$.

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1 Introduction

A central topic of study in extremal set theory is the maximum size of a family of subsets of an $n$-element set subject to restrictions on their intersections. Classical theorems in the area are discussed in Bollobás [2]. In this paper, we generalize one such theorem, known as the Bollobás set pairs inequality or two families theorem [3]:

**Theorem 1.** (Bollobás) Let $\mathcal{A} = \{ A_1, A_2, \ldots, A_m \}$ and $\mathcal{B} = \{ B_1, B_2, \ldots, B_m \}$ be families of finite sets, such that $A_i \cap B_j \neq \emptyset$ if and only if $i, j \in [m]$ are distinct. Then

$$\sum_{i=1}^{m} \left( \frac{|A_i \cup B_i|}{|A_i|} \right)^{-1} \leq 1. \quad (1)$$

For convenience, we refer to a pair of families $\mathcal{A}$ and $\mathcal{B}$ satisfying the conditions of Theorem 1 as a *Bollobás set pair*. The inequality above is tight, as we may take the pairs $(A_i, B_i)$ to be distinct partitions of a set of size $a + b$ with $|A_i| = a$ and $|B_i| = b$ for $1 \leq i \leq \binom{a+b}{a}$.

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The latter inequality was proved for \( a = 2 \) by Erdős, Hajnal and Moon [5], and in general has a number of different proofs [11, 12, 14, 17, 18]. A geometric version was proved by Lovász [17, 18], who showed that if \( A_1, A_2, \ldots, A_m \) and \( B_1, B_2, \ldots, B_m \) are respectively \( a \)-dimensional and \( b \)-dimensional subspaces of a linear space and \( \dim(A_i \cap B_j) = 0 \) if and only if \( i, j \in [m] \) are distinct, then \( m \leq \binom{n+a}{a} \).

1.1 Main Theorem

Theorem 1 has been generalized in a number of different directions in the literature [6, 9, 13, 16, 21, 24]. In this paper, we give a generalization of Theorem 1 from the case of two families to \( k \geq 3 \) families of sets with conditions on the \( k \)-wise intersections. For \( 2 \leq t \leq k \), a Bollobás \((k,t)\)-tuple is a sequence \((A_1, A_2, \ldots, A_k)\) of set families \( A_j = \{A_{j,i} : 1 \leq i \leq m\} \) where \( \bigcap_{j=1}^{k} A_{j,i,j} \neq \emptyset \) if and only if at least \( t \) of the indices \( i_1, i_2, \ldots, i_k \) are distinct. We refer to \( m \) as the size of the Bollobás \((k,t)\)-tuple. Let \([m]_t\) denote the set of sequences of \( t \) distinct elements of \([m]\) and fix a surjection \( \phi : [k] \to [t] \).

For \( \sigma \in [m]_{(t-1)} \), set \( \sigma(t) = \sigma(1) \) and define \( A_{1,\sigma}(\phi) = \bigcap_{j : \phi(j) = 1} A_{j,\sigma(1)} \) and, for \( 2 \leq j \leq t \), we define

\[
A_{j,\sigma}(\phi) = \bigcap_{h : \phi(h) = j} A_{h,\sigma(h)} \setminus \bigcup_{h=1}^{j-1} A_{h,\sigma(h)}. 
\]

Using this notation, we generalize (1) as follows:

**Theorem 2.** Let \( k \geq t \geq 2 \) and \( m \geq t \), let \( \phi : [k] \to [t] \) be a surjection, and let \((A_1, A_2, \ldots, A_k)\) be a Bollobás \((k,t)\)-tuple of size \( m \). Then

\[
\sum_{\sigma \in [m]_{(t-1)}} \left( \frac{|A_{1,\sigma}(\phi) \cup A_{2,\sigma}(\phi) \cup \cdots \cup A_{t,\sigma}(\phi)|}{|A_{1,\sigma}(\phi)| \cdot |A_{2,\sigma}(\phi)| \cdots |A_{t,\sigma}(\phi)|} \right)^{-1} \leq 1. 
\]

We show in Section 2.1 that this inequality is tight for all \( k \geq t = 2 \), but do not have an example to show that this inequality is tight for any \( t > 2 \).

For \( n \geq k \geq t \geq 2 \), let \( \beta_{k,t}(n) \) denote the maximum \( m \) such that there exists a Bollobás \((k,t)\)-tuple of size \( m \) consisting of subsets of \([n]\). Then (1) gives \( \beta_{2,2}(n) \leq \binom{n}{|n/2|} \) which is tight for all \( n \geq 2 \). Letting \( H(q) = -q \log_2 q - (1-q) \log_2 (1-q) \) denote the standard binary entropy function, we prove the following theorem:

**Theorem 3.** For \( k \geq 3 \) and large enough \( n \),

\[
\frac{1}{k} \leq \frac{\log_2 \beta_{k,2}(n)}{n} \leq H\left(\frac{1}{k}\right) \leq \frac{\log_2 (ke)}{k}. 
\]

For \( k \geq t \geq 3 \) and large enough \( n \),

\[
\frac{\log_2 e}{\binom{k}{t-1} (t+1)^{t-1}} \leq \frac{\log_2 \beta_{k,t}(n)}{n} \leq \frac{2}{\binom{k}{t-1} (t-1)^{t-3}}. 
\]
This determines \( \log_2 \beta_{k,2}(n) \) up to a factor of order \( \log_2 k \) and \( \log_2 \beta_{k,t}(n) \) up to a factor of order \( t^4 \). We leave it as an open problem to determine the asymptotic value of \( (\log_2 \beta_{k,t}(n))/n \) as \( n \to \infty \) for any \( k \geq 3 \) and \( t \geq 2 \). A natural source for lower bounds on \( \beta_{k,t}(n) \) comes from the probabilistic method – see the random constructions in Section 3.1 which establish the lower bounds in Theorem 3. To prove Theorem 3, we use a natural connection to hypergraph covering problems.

1.2 Covering hypergraphs

Theorem 1 has a wide variety of applications, from saturation problems [3, 19] to covering problems for graphs [11, 20], complexity of 0-1 matrices [23], geometric problems [1], counting cross-intersecting families [7], crosscuts and transversals of hypergraphs [24, 25, 26], hypergraph entropy [15, 22], and perfect hashing [8, 10]. In this section, we give an application of our main results to hypergraph covering problems. For a \( k \)-uniform hypergraph \( H \), let \( f(H) \) denote the minimum number of complete \( k \)-partite \( k \)-uniform hypergraphs whose union is \( H \). In the case of graph covering, a simple connection to the Bollobás set pairs inequality (1) may be described as follows. Let \( K_{n,n} \setminus M \) denote the complement of a perfect matching \( M = \{x_iy_i : 1 \leq i \leq n\} \) in the complete bipartite graph \( K_{n,n} \) with parts \( X = \{x_1, x_2, \ldots, x_n\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \). If \( H_1, H_2, \ldots, H_m \) are complete bipartite graphs in a minimum covering of \( K_{n,n} \setminus M \), then let \( A_i = \{j : x_i \in V(H_j)\} \) and \( B_i = \{j : y_i \in V(H_j)\} \). Setting \( A = \{A_i\}_{i \in [m]} \) and \( B = \{B_i\}_{i \in [m]} \), it is straightforward to check that \( (A, B) \) is a Bollobás set pair, and Theorem 1 applies to give

\[
f(K_{n,n} \setminus M) = \min\{m : \left(\frac{m}{\lceil m/2 \rceil}\right) \geq n\}.
\]

In a similar way, Theorem 2 applies to covering complete \( k \)-partite \( k \)-uniform hypergraphs. Let \( K_{n,n,\ldots,n} \) denote the complete \( k \)-partite \( k \)-uniform hypergraph with parts \( X_i = \{x_{ij} : j \in [n]\} \) for \( i \in [k] \). Let \( H_{k,t}(n) \) denote the subhypergraph consisting of hyperedges \( \{x_{i_1, i_2, \ldots, i_k} : \text{at least } t \text{ of the indices } i_1, i_2, \ldots, i_k \text{ are distinct}\} \) such that at least \( t \) of the indices \( i_1, i_2, \ldots, i_k \) are distinct, and set \( f_{k,t}(n) = f(H_{k,t}(n)) \). Then there is a one-to-one correspondence between Bollobás \( (k, t) \)-tuples of subsets of \( [m] \) and coverings of \( H_{k,t}(n) \) with \( m \) complete \( k \)-partite \( k \)-graphs. We let \( \beta_{k,t}(m) \) be the maximum size of a Bollobás \( (k, t) \)-tuple of subsets of \( [m] \), so that

\[
f_{k,t}(n) = \min\{m : \beta_{k,t}(m) \geq n\}.
\]

This correspondence together with Theorem 2 will be exploited to prove

\[
f_{k,2}(n) \geq \min\{m : \left(\frac{m}{\lceil m/k \rceil}\right) \geq n\}
\]

which is partly an analog of (5). More generally, we prove the following theorem:

**Theorem 4.** For \( k \geq 3 \) and large enough \( n \),

\[
\frac{k}{\log_2(k \epsilon)} \leq \frac{1}{H(\frac{1}{k})} \leq \frac{f_{k,2}(n)}{\log_2 n} \leq k.
\]
For $k \geq t \geq 3$ and large enough $n$,

$$\left(\begin{array}{c} k \\ t - 1 \end{array}\right) \frac{(t-1)^{t-3}}{2} \leq \frac{f_{k,t}(n)}{\log_2 n} \leq \frac{(t+1)t^{-1}}{\log_2 e} \left(\begin{array}{c} k \\ t - 1 \end{array}\right).$$

The bounds on $\beta_{k,t}(n)$ in Theorem (3) follow immediately from this theorem and (6). Equation (9) gives the order of magnitude for each $t \geq 3$ as $k \to \infty$, but for $t = 2$, Equation (8) has a gap of order $\log_2 k$. From (7), we obtain $\beta_{k,2}(n) \leq \binom{n}{\lfloor n/k \rfloor}$. It is perhaps unsurprising that the asymptotic value of $f_{k,t}(n)/\log_2 n$ as $n \to \infty$ is not known for any $k > 2$, since a limiting value of $f(K^k_n)/\log_2 n$ is not known for any $k > 2$ – see Körner and Marston [15] and Guruswami and Riazanov [10].

1.3 Organization and notation

Given a subset $A \subset [n]$, let $A^c := [n] \setminus A$ be the complement of $A$ in $[n]$. For positive integers $k \leq n$, let $(n)_{(k)} = (n)(n-1)\cdots(n-k+1)$ denote the falling factorial. This paper is organized as follows. In Section 2, we prove Theorem 2. In Section 2.1, we construct a Bollobás $(k,2)$-tuple which achieves equality in Theorem 2 and in Section 2.2, we construct a Bollobás $(k,2)$-tuple which gives the lower bound in Equation (3). The upper bound on $f_{k,t}(n)$ in Theorem 4 comes from a probabilistic construction in Section 3.1, and the proof of the lower bound on $f_{k,t}(n)$ is given in Section 3.3; we prove (7) in Section 3.2.

2 Proof of Theorem 2

Given a Bollobás set $(k,t)$-tuple $(A_1,\ldots,A_k)$ with $A_j = \{A_{j,i} : 1 \leq i \leq m\}$ and a surjection $\phi : [k] \to [t]$, consider $A_\ell(\phi) : 1 \leq \ell \leq t$ where $A_\ell(\phi) = \{A_{\ell,i}(\phi) : 1 \leq i \leq m\}$ and $A_{\ell,i}(\phi) = \bigcap_{h: \phi(h) = \ell} A_{h,i}$.

It follows that $(A_1(\phi),\ldots,A_t(\phi))$ is a Bollobás set $(t,t)$-tuple and hence it suffices to prove Theorem 2 in the case where $t = k$. In this setting, surjections $\phi : [k] \to [k]$ simply permute the $k$ families and as such we suppress the notation of $\phi$ for the remainder of this section. One of the proofs of Theorem 1, given a Bollobás set pair, defines a collection of chains $C_i$ for $i \in [m]$ and shows that these chains are necessarily disjoint. Similarly, given a Bollobás set $(k,k)$-tuple, we will define a collection of chains $C_\sigma$ for every ordered collection $\sigma$ of $(k-1)$ distinct indices of $[m]$ and show these chains are pairwise disjoint.

Let $(A_1,\ldots,A_k)$ with $A_j = \{A_{j,i} : 1 \leq i \leq m\}$ be a Bollobás set $(k,k)$-tuple, and set

$$X = \bigcup_{i=1}^m (A_{1,i} \cup A_{2,i} \cup \cdots \cup A_{k,i}).$$
with $|X| = n$. For $\sigma \in [m]_{(k - 1)}$, define a subset $C_\sigma$ of permutations $\pi : X \to [n]$ by

$$
C_\sigma := \left\{ \pi : X \to [n] : \max_{x \in A_1, \sigma} \pi(x) < \min_{y \in A_2, \sigma} \pi(y) \leq \max_{y \in A_3, \sigma} \pi(y) < \cdots < \min_{z \in A_k, \sigma} \pi(z) \right\}.
$$

Letting $U_\sigma := A_1, \sigma \cup \cdots \cup A_k, \sigma$, elementary counting methods give

$$
|C_\sigma| = \left( \frac{n}{|U_\sigma|} \right) |A_1, \sigma| \cdots |A_k, \sigma| (n - |U_\sigma|)! = n! \cdot \left( \frac{|U_\sigma|}{|A_1, \sigma| \cdots |A_k, \sigma|} \right)^{-1}. \quad (10)
$$

We will now prove a lemma which states that $\{C_\sigma\}_{\sigma \in [m]_{(k - 1)}}$ forms a disjoint collection of a permutations. The general proof only works for $k \geq 4$, so we first consider $k = 3$.

**Lemma 5.** If $\sigma_1, \sigma_2 \in [m]_{(2)}$ are distinct, then $C_{\sigma_1} \cap C_{\sigma_2} = \emptyset$.

**Proof.** Seeking a contradiction, suppose there exists $\pi \in C_{\sigma_1} \cap C_{\sigma_2}$. After relabeling, it suffices to consider the following five cases.

1. $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 4\}$
2. $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 3\}$
3. $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{1, 3\}$
4. $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{2, 3\}$
5. $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{3, 1\}$

In case (1), without loss of generality, max$\{\pi(x) : x \in A_{1,1}\} \leq$ max$\{\pi(x) : x \in A_{1,2}\}$ and thus $\pi \in C_{\sigma_2}$ yields

$$
\max_{x \in A_{1,1}} \pi(x) \leq \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y).
$$

Then as $A_{1,1} \cap A_{2,4} \cap A_{3,2} \neq \emptyset$, there exists $w \in A_{1,1} \cap A_{2,4} \cap A_{3,2}$. It follows that $w \notin A_{1,2}$ since if $w \in A_{1,2}$, then $w \in A_{1,2} \cap A_{2,4} \cap A_{3,2} \neq \emptyset$; a contradiction. But this yields a contradiction as

$$
\pi(w) \leq \max_{x \in A_{1,1}} \pi(x) \leq \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y) \leq \pi(w).
$$

In case (2), without loss of generality, max$\{\pi(x) : x \in A_{1,1}\} \leq$ max$\{\pi(x) : x \in A_{1,2}\}$ and we recover a similar contradiction as case (1) by noting that there exists $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ with $w \notin A_{1,2}$.

In case (3) we may assume max$\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} \leq$ max$\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\}$ and $\pi \in C_{1,3}$ yields max$\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\} <$ min$\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_2, \sigma)\}$. Thus

$$
\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} \leq \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_2, \sigma)\}
$$

and there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,3}$. It follows that $\pi(w) < \pi(w)$, a contradiction.

In case (4), if max$\{\pi(x) : x \in A_{1,1}\} \leq$ max$\{\pi(x) : x \in A_{1,2}\}$, then using $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ and noting $w \notin A_{1,2}$, we get a contradiction. Thus, we may assume otherwise and $\pi \in C_{1,2}$ gives

$$
\max_{x \in A_{1,2}} \pi(x) < \max_{x \in A_{1,1}} \pi(x) < \min_{z \in A_{3,1} \setminus (A_{1,1} \cup A_{2,2})} \pi(z).
$$
This is a contradiction as there exists \( w \in A_{1,2} \cap A_{2,3} \cap A_{3,1} \) with \( w \notin A_{1,1} \) and \( w \notin A_{2,2} \). In case (5), if \( \max \{ \pi(x) : x \in A_{1,1} \} \leq \max \{ \pi(x) : x \in A_{1,3} \} \), then we may proceed as in the latter part of case (4) using \( w \in A_{1,1} \cap A_{2,2} \cap A_{3,3} \) and \( w \notin A_{2,1} \) and \( w \notin A_{1,3} \) to get a contradiction. Otherwise, proceeding as in case (1) and noting there exists \( w \in A_{1,3} \cap A_{2,2} \cap A_{3,1} \), but \( w \notin A_{1,1} \) yields a contradiction.

A similar argument yields the analog of Lemma 5 to the case where \( k \geq 4 \).

**Lemma 6.** Let \( k \geq 4 \). If \( \sigma_1, \sigma_2 \in [m]_{(k-1)} \) are distinct, then \( \mathcal{C}_{\sigma_1} \cap \mathcal{C}_{\sigma_2} = \emptyset \).

**Proof.** Since \( \sigma_1, \sigma_2 \in [m]_{(k-1)} \) are distinct, there exists minimal \( h \in [k-1] \) so that \( \sigma_1(h) \neq \sigma_2(h) \). Seeking a contradiction, suppose there exists a \( \pi \in \mathcal{C}_{\sigma_1} \cap \mathcal{C}_{\sigma_2} \). Without loss of generality,

\[
\max \{ \pi(x) : x \in A_{h,\sigma_1} \} \leq \max \{ \pi(x) : x \in A_{h,\sigma_2} \} < \min \{ \pi(z) : z \in A_{k,\sigma_2} \}.
\]

Now, consider a bijection \( \tau : [k-1] \setminus \{ h \} \to [k-1] \setminus \{ h \} \) which has no fixed points. As in Lemma 5, we want to show that there exists a \( w \in A_{h,\sigma_1} \cap A_{k,\sigma_2} \) and consider two separate cases.

First, suppose that \( \sigma_1(h) \notin \sigma_2([k-1]) \). As \( |\{\sigma_1(h), \sigma_2(1), \ldots, \sigma_2(k-1)\}| = k \), there exists

\[
w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap \bigcap_{l \in [k-1] \setminus \{ h \}} A_{l,\sigma_2(\tau(l))}.
\]

Next, suppose that \( \sigma_1(h) = \sigma_2(x) \) for some \( x \). We now claim that \( x \neq 1 \). If \( h = 1 \), then this is trivial. If \( h > 1 \), then \( \sigma_1(1) = \sigma_2(1) \), so \( \sigma_1(h) \neq \sigma_2(1) \) since \( \sigma_1(h) \neq \sigma_1(1) \). For \( \tau \) as above, there exists \( y \in [k-1] \setminus \{ h \} \) so that \( \tau(y) = x \). Taking \( \gamma \) distinct from \( \{\sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\} \), \( |\{\sigma_1(h), \gamma, \sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}| = k \) and hence there exists

\[
w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap A_{y,\gamma} \cap \bigcap_{l \in [k-1] \setminus \{ h \}} A_{l,\sigma_2(\tau(l))}.
\]

By construction, \( w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \). Suppose there exists a \( t \in [k-1] \setminus \{ h \} \) so that \( w \in A_{t,\sigma_1(1)} \). As \( \tau \) has no fixed points, replacing the set in the \( k \)-wise intersection corresponding to \( A_t \) with \( A_{t,\sigma_2(1)} \) in either (11) or (12), \( w \) is an element of this new \( k \)-wise intersection with \( (k-1) \) distinct indices; a contradiction. If \( w \in A_{h,\sigma_1(h)} \), then we may similarly replace \( A_{h,\sigma_1(h)} \) with \( A_{h,\sigma_2(h)} \) in the \( k \)-wise intersection in either (11) or (12) to get a contradiction. Thus, \( w \notin A_{1,\sigma_2(1)} \cup \cdots \cup A_{k-1,\sigma_2(k-1)} \) and hence \( w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \) so that \( \pi(w) < \pi(w) \); a contradiction.

Using Equation (10), Lemma 5, and Lemma 6, we are now able to prove Theorem 2 in the case where \( t = k \). There are \( n! \) total permutations, and Lemma 5 and Lemma 6 yield that each of which appears in at most one of the sets \( \mathcal{C}_{\sigma} \) for \( \sigma \in [m]_{(k-1)} \). Hence, using \( |\mathcal{C}_{\sigma}| \) in Equation (10),

\[
\sum_{\sigma \in [m]_{(k-1)}} |\mathcal{C}_{\sigma}| = \sum_{\sigma \in [m]_{(k-1)}} n! \cdot \left( \frac{|A_{1,\sigma} \cup \cdots \cup A_{k,\sigma}|}{|A_{1,\sigma}| \cdots |A_{k,\sigma}|} \right)^{-1} \leq n!
\]

and thus the result follows by dividing through by \( n! \).
2.1 Sharpness of Theorem 2

We give a simple construction establishing the sharpness of Theorem 2 for $k \geq t = 2$. Let $n \geq 4k$ and using addition modulo $n$, define $A_{1,i} = \{i\}^c$, $A_{j,i} = \{i - (j - 1), i + (j - 1)\}^c$ for $j \in [2, k - 1]$, and $A_{k,i} = \{i - k + 2, i - k + 3, \ldots, i + k - 2\}$. Letting $A_j = \{A_{j,i}\}_{i \in [n]}$ for all $j \in [k]$, we will show $(A_1, \ldots, A_k)$ is a Bollobás $(k, 2)$-tuple. Since $|A_{1,i}| = n - 1$ and $|A_{2,i} \cap \cdots \cap A_{k,i}| = 1$, Theorem 2 with $t = 2$ and surjection $\phi : [k] \to [2]$ with $\phi(1) = 1$ and $\phi(i) = 2$ for $i \neq 1$ gives

$$1 \geq \sum_{i=1}^{n} \left( \frac{|A_{1,i}| + |A_{2,i} \cap \cdots \cap A_{k,i}|}{|A_{1,i}|} \right)^{-1} = \sum_{i=1}^{n} \frac{1}{n} = 1.$$ 

By construction, for all $i \in [n]$, $A_{1,i} \cap A_{2,i} \cap \cdots \cap A_{k,i} = \emptyset$. It thus suffices to show these are the only empty $k$-wise intersections. To this end, for $i = (i_1, \ldots, i_{k-1})$, define

$$A(i) := A_{1,i_1} \cap \cdots \cap A_{k-1,i_{k-1}}.$$ 

**Lemma 7.** Let $i = (i_1, \ldots, i_{k-1})$. If $A(i)^c = A_{k,i_k}$, then $i_1 = \cdots = i_k$.

**Proof.** We proceed by induction on $k$ where the result is trivial when $k = 2$. In the case where $k > 2$, $i_{k-1} - k + 2 = i_k + x$ for some $x$ such that $-(k - 2) \leq x \leq (k - 2)$ and thus $i_{k-1} + (k - 2) = i_k - 1 - (k - 2) + (2k - 4) = i_k + x + (2k - 4)$. Next, there is a $y$ such that $-(k - 2) \leq y \leq (k - 2)$ with $i_{k-1} + (k - 2) = i_k + y$, and since $n \geq 4k$, $x + 2k - 4 = y$ with equality over $\mathbb{Z}$ and moreover $i_{k-1} + (k - 2) = i_k + (k - 2)$ over $\mathbb{Z}$ and hence $i_k = i_{k-1}$. Removing these elements from each set, the result then follows by induction. \[\square\]

If $A_{1,i_1} \cap \cdots \cap A_{k,i_k} = \emptyset$, then as $A(i) = A_{1,i_1} \cap A_{2,i_2} \cap \cdots A_{k-1,i_{k-1}}$,

$$\emptyset = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}} \cap A_{k,i_k} = A(i) \cap A_{k,i_k}.$$ 

The result follows by noting $|A(i)| \geq n - (2k - 3)$, $|A_{k,i_k}| = 2k - 3$, and using Lemma 7.

2.2 An Explicit Construction

Let $k \geq 3$. An explicit construction of a Bollobás $(k, 2)$-tuple $(A_1, A_2, \ldots, A_k)$ where $|A_i| = 2^n$ and each $A_i$ consists of subsets of $X$ for $|X| = kn$ may be described as follows. Let $I_j := \{x_{j,1}, x_{j,2}, \ldots, x_{j,k}\}$ and consider $X = I_1 \sqcup \cdots \sqcup I_n$. Now, for each $f : [n] \to [2]$ and $j \in [k]$, define

$$A_{j,f} := \{x_{1,f(1)+j-1}, \ldots, x_{n,f(n)+j-1}\}^c$$ 

where we work modulo $k$ within the subscripts of $I_j$. It is straightforward to check that $(A_1, A_2, \ldots, A_k)$ is a Bollobás $(k, 2)$-tuple. This establishes the lower bound on $\beta_{k,2}(n)$ in Equation (3) and hence the upper bound on $f_{k,2}(n)$ in Equation (8).
3 Proof of Theorem 4

3.1 Upper bound on $f_{k,t}(n)$

We wish to find a covering of $H_{k,t}(n)$ with complete $k$-partite $k$-graphs and assume the parts of $H_{k,t}(n)$ are $X_1, X_2, \ldots, X_k$. For each subset $T$ of $[k]$ of size $t$, consider the uniformly random coloring $\chi_T : [n] \to T$. Given such a $\chi_T$, let $Y_i \subset X_i$ be the vertices of color $i$ for $i \in T$; that is $Y_i := \{x_{ij} : \chi(j) = i\}$ and $Y_i = X_i$ for $i \notin T$. Denote by $H(T, \chi)$ the (random) complete $k$-partite hypergraph with parts $Y_1, Y_2, \ldots, Y_k$, and note that $H(T, \chi) \subset H_{k,t}(n)$. We place each $H(T, \chi)$ a total of $N$ times independently and randomly where

$$N = \left\lfloor \frac{(t + 1)t^l \log_2 n}{(k - t + 1) \log_2 e} \right\rfloor$$

and produce $\binom{k}{t}N$ random subgraphs $H(T, \chi)$. For a set partition $\pi$ of $[k]$, let $|\pi|$ denote the number of parts in the partition and index the parts by $[|\pi|]$. Given a set partition $\pi = (P_1, P_2, \ldots, P_s)$, let

$$f(\pi, t) = \sum_{T \in [s]^{(t)}} \prod_{i \in T} |P_i|.$$ 

If $U$ is the number of edges of $H_{k,t}(n)$ not in any of these subgraphs, then

$$\mathbb{E}(U) \leq \sum_{|\pi| \geq t} n^{|\pi|} (1 - t^{-t})^{Nf(\pi, t)} = \sum_{s \leq t \leq k} n^s \sum_{|\pi| = s} (1 - t^{-t})^{Nf(\pi, t)}. \tag{13}$$

For sufficiently large $n$, we claim that $\mathbb{E}(U) < 1$, which implies there exists a covering of $H_{k,t}(n)$ with at most $\binom{k}{t}N$ complete $k$-partite $k$-graphs, as required. The following technical lemma states that $f$ is a decreasing function in the set partition lattice, and that $f(\pi, t)$ increases when we merge all but one element of a smaller part of $\pi$ with a larger part of $\pi$:

Lemma 8. Let $k \geq s \geq t \geq 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$.

(i) If $\pi'$ is a refinement of $\pi$ with $|\pi'| = s + 1$, then $f(\pi, t) \leq f(\pi', t)$.

(ii) If $|P_1| \geq |P_2| \geq 2$ and $\alpha \in P_2$, and $\pi'$ is the partition $(P_1', P_2', \ldots, P_s')$ of $[k]$ with $P_1' = P_1 \cup P_2 \setminus \{\alpha\}$ and $P_2' = \{\alpha\}$ and with $P_i' = P_i$ for $3 \leq i \leq s$, then $f(\pi', t) \leq f(\pi, t)$.

The proof of Lemma 8 part (i) is in Appendix A and the proof of (ii) is similar to the proof of (i). By Lemma 8, a set partition of $[k]$ into $s$ parts which minimizes $f(\pi, t)$ consists of one part of size $k - s + 1$ and $s - 1$ singleton parts and hence

$$\min\{f(\pi, t) : |\pi| = s\} = (k - s + 1) \binom{s - 1}{t - 1} + \binom{s - 1}{t}. \tag{14}$$

In what follows, we denote a set partition of $[k]$ into $s$ parts which minimizes $f(\pi, t)$ by $\pi_s$. 

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For \( n \) large enough, and all \( s \) where \( t \leq s \leq k \), we will show
\[
\sum_{|\pi|=t} (1 - t^{-t})^{Nf(\pi, t)} \geq n^{s-t}.
\]
Replacing the numerator with its largest term and each term in denominator with its largest term,
\[
\frac{\sum_{|\pi|=t}(1 - t^{-t})^{Nf(\pi, t)}}{\sum_{|\pi|=s}(1 - t^{-t})^{Nf(\pi, t)}} \geq \frac{(1 - t^{-t})^{Nf(\pi_1, t)}}{S(k, s)(1 - t^{-t})^{Nf(\pi_1, t)}} = \frac{1}{S(k, s)} (1 - t^{-t})^{Nf(\pi_1, t)-f(\pi_s, t)}
\]
where \( S(k, s) \) is the Stirling number of the second kind. Taking \( n \geq S(k, s) \), we will show in Appendix B that
\[
\frac{1}{S(k, s)} (1 - t^{-t})^{Nf(\pi_1, t)-f(\pi_s, t)} \geq n^{s-t}.
\]
Therefore, the index \( s = t \) maximizes the right hand side of Equation (13), and hence
\[
\mathbb{E}[U] \leq (k - t + 1)(n^t) \sum_{m=n} (1 - t^{-t})^{Nf(\pi, t)} < (k - t + 1)n^t S(k, t) (1 - t^{-t})^{N(k-t+1)} < 1
\]
for our choice of \( N \) provided \( n \geq kS(k, t) \). Thus,
\[
f_{k,t}(n) \leq \left( \frac{k}{t} \right) \frac{(t + 1)t^t \log_2 n}{(k - t + 1) \log_2 e} = \frac{(t + 1)t^{t-1}}{\log_2 e} \left( \frac{k}{t - 1} \right) \log_2 n.
\]

### 3.2 Lower bound on \( f_{k,2}(n) \)

In this section, we show
\[
f_{k,2}(n) \geq \min \{ m : \left( \frac{m}{[m/k]} \right) \geq n \}.
\]

Let \( \{ H_1, H_2, \ldots, H_m \} \) be a covering of \( H_{k,2}(n) \) with \( m = f_{k,2}(n) \) complete \( k \)-partite \( k \)-graphs. We recall \( H_{k,2}(n) = K_{n,n,\ldots,n} \setminus M \), where \( M \) is a perfect matching of \( K_{n,n,\ldots,n} \). For \( i \in [k] \) and \( j \in [n] \), define \( A_{i,j} = \{ H_r : x_{ij} \in V(H_r) \} \) and \( A_i = \{ A_{i,j} : 1 \leq j \leq n \} \). As in (6), \( (A_1, A_2, \ldots, A_k) \) is a Bollobás \( (k,2) \)-tuple of size \( n \). For convenience, for each \( i \in [k] \), let \( \phi_i : [k] \rightarrow [2] \) be so that \( \phi_i^{-1}(1) = \{ i \} \). Taking the sum of inequality from Theorem 2 with \( t = 2 \) over all \( i \in [k] \),
\[
\sum_{i=1}^k \sum_{j=1}^n \left( \frac{|A_{1,j}(\phi_i) \cup A_{2,j}(\phi_i)|}{|A_{1,j}(\phi_i)|} \right)^{-1} \leq k.
\]
We use this inequality to give a lower bound on \( f_{k,2}(n) = m \). First we observe
\[
\sum_{r=1}^m |V(H_r)| = \sum_{j=1}^n \sum_{i=1}^k |A_{i,j}| = \sum_{j=1}^n \sum_{i=1}^k |A_{1,j}(\phi_i)|.
\]
Let $\partial H$ denote the set of $(k-1)$-tuples of vertices contained in some edge of a hypergraph $H$. Then
\[
\sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} |A_{2,j}(\phi_i)|. \tag{19}
\]

Putting the above identities together,
\[
\sum_{r=1}^{m} |V(H_r)| + \sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|). \tag{20}
\]

We note $|\partial H_r \cap \partial M| \leq |V(H_r)|/(k-1)$, and therefore
\[
\sum_{r=1}^{m} |\partial H_r \cap \partial M| \leq \frac{1}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{21}
\]

It follows that
\[
\sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) \leq \frac{k}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{22}
\]

Subject to the linear inequalities (18) and (22), the left side of (17) is minimized when $\text{kn}|A_{1,j}(\phi)| = \sum_{r=1}^{m} |V(H_r)|$ and $\text{kn}(|A_{1,j}(\phi)| + |A_{2,j}(\phi)|) = (k-1)|A_{1,j}(\phi)|$. Since $|V(H_r)| \leq (k-1)n$ for all $r \in [m]$, (17) implies $(m_{j/k}) \geq n$, which gives (16). \hfill \Box

### 3.3 Lower bound on $f_{k,k}(n)$

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,k}(n)$ with complete $k$-partite $k$-graphs, so $m = f(H_{k,k}(n))$. Given a $k$-partite $k$-graph $H$, consider its 2-shadow $\partial_2(H) = \{R \subset V(H) : |R| = k-2, R \subset e \text{ for some } e \in H\}$. Let $\partial_2(\mathcal{H}) = \bigcup_{i=1}^{m} \partial_2(H_i)$. Given $R \in \partial_2(\mathcal{H})$ and $H_i \in \mathcal{H}$, let $H_i(R) := \{e \in \binom{V(H_i)}{2} : e \cup R \in H_i\}$ be the possibly empty link graph of the edge $R$ in the hypergraph $H_i$ and let $V(H_i(R))$ be the set of vertices in the link graph. Observe that double counting yields
\[
\sum_{R \in \partial_2(\mathcal{H})} \left( \sum_{i=1}^{m} |V(H_i(R))| \right) = \sum_{i=1}^{m} \left( \sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right). \tag{23}
\]

An optimization argument yields $|\partial_2(H_i)|$ is maximized when the parts of $H_i$ are of equal or nearly equal maximal size. Since $|V(H_i(R))| \leq 2(n-k+2)$, the right hand side of Equation (23) is bounded above by
\[
\sum_{i=1}^{m} \left( \sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right) \leq m \cdot \left( \frac{k}{2} \right) \cdot \left( \frac{n}{k} \right)^{k-2} \cdot 2(n-k+2). \tag{24}
\]
For a lower bound on the left hand side of Equation (23), fix \( R \in \partial_2(H) \) and without loss of generality suppose that \( R = \{x_{1,1}, \ldots, x_{k-2,k-2}\} \). Let \( Y = [k-1,n] \). Let \( K_{Y,Y} \) be the complete bipartite graph with two distinct copies of \( Y \) and \( M = \{(x_{k-1,i}, x_{k,i} : i \in Y\} \) be a perfect matching in \( K_{Y,Y} \). Then, \( \{H_1(R), \ldots, H_m(R)\} \) forms a biclique cover of \( K_{Y,Y} \setminus M \). Applying the convexity result of Tarjan [23, Lemma 5],

\[
\sum_{i=1}^{m} |V(H_i(R))| \geq (n-k+2) \log_2(n-k+2).
\]

Noting that \(|\partial_2(H)| = \binom{k}{2}(n-k-2)\), the left hand side of Equation (23) is bounded below by

\[
\sum_{R \in \partial_2(H)} \left( \sum_{i=1}^{m} |V(H_i(R))| \right) \geq \binom{k}{2}(n-k-2)(n-k+2) \log_2(n-k+2). \tag{25}
\]

Comparing the bounds from Equation (24) and Equation (25),

\[
m \geq \frac{(n-k-2) \log_2(n-k+2)}{2 \left( \frac{n}{k} \right)^{k-2}} \geq \frac{k^{k-2}}{2} \log_2 n
\]

provided that \( n \) is large enough.

For \( t \geq 3 \) and \( t < k \), the lower bound on \( f_{k,t}(n) \) in Theorem 4 is obtained from the lower bounds on \( f_{k-t+1}(n-1) \) as follows: Let \( \mathcal{H} = \{H_1, H_2, \ldots, H_m\} \) be a minimal covering of \( H_{k,t}(n) \) with complete \( k \)-partite \( k \)-graphs, so \( m = f(H_{k,t}(n)) \). Given \( T \in \binom{[k]}{k-t+1} \), define \( H_T \subset H_{k,t}(n) \) by

\[
H_T := \{\{x_{1,i_1}, \ldots, x_{k,i_k}\} \in H_{k,t}(n) : i_j = 1 \ \forall \ j \in T\}.
\]

It follows that at least \( f_{k-t+1}(n-1) \) of the complete \( k \)-partite \( k \)-graphs in \( \mathcal{H} \) are needed to cover \( H_T \). Moreover, for distinct \( T, T' \in \binom{[k]}{k-t+1} \), the corresponding complete \( k \)-partite \( k \)-graphs from \( \mathcal{H} \) are necessarily pairwise disjoint and hence

\[
f_{k,t}(n) \geq \binom{k}{k-t+1} f_{k-t+1}(n-1) \geq \binom{k}{t-1} (t-1)^{t-3} \log_2 n
\]

provided that \( n \) is large enough.

### 4 Concluding remarks

- Our main theorem, Theorem 2 is tight for \( t = 2 \) and \( k \geq 2 \), as shown in Section 2.1. It would be interesting to generalize this example to \( 2 < t \leq k \) to determine whether Theorem 2 is tight in general. The first open case is \( t = k = 3 \).

- A particular case of the Bollobás set pairs inequality occurs when every set in \( \mathcal{A} \) has size \( a \) and every set in \( \mathcal{B} \) has size \( b \), and one obtains the tight bound \(|\mathcal{A}| \leq \binom{a+b}{a+b} \). The

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The generalization to Bollobás $(k,t)$-tuples for $k \geq 3$ is equally interesting but wide open, as are potential generalizations to vector spaces – see Lovász [17, 18].

• Orlin [20] proved that the clique cover number $cc(K_n \setminus M)$ of a complete graph $K_n$ minus a perfect matching $M$ is precisely $\min\{ m : 2^{(m-1)} \geq n \}$. Theorem 4 yields lower bounds on the clique cover number of the complement of a perfect matching $M$ in the complete $k$-uniform hypergraph $K_n^k$:

**Corollary 9.** Let $K_n^k \setminus M$ be the complement of a perfect matching in $K_n^k$. Then

$$cc(K_n^k \setminus M) \geq \frac{\log_2 \frac{n}{k}}{H(\frac{k}{k})} \geq \frac{k \log_2 \frac{n}{k}}{\log_2(ke)}.$$  

• It would be interesting to prove an analog of Equation (16) for $t \geq 3$. That is,

$$f_{k,t}(n) \geq \min\{ m : \left( \frac{m}{\alpha_1, \ldots, \alpha_t} \right) \geq n_{(t-1)} \}$$  

for some optimal $\alpha_1, \ldots, \alpha_t$. The difficulty here lies in determining effective bounds on $|A_{i,\sigma}(\phi)|$.

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**References**


A Proof of Lemma 8(i)

Let $k \geq s \geq t \geq 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$. In this section, we will show that if $\pi'$ is a refinement of $\pi$ with $|\pi'| = s + 1$, then $f(\pi, t) \leq f(\pi', t)$.

**Proof.** Let $\pi = P_1|P_2|\cdots|P_s$ and without loss of generality, $\pi' = P_s|P_y|P_2|\cdots|P_s$. Setting $\mathcal{T}(\overline{1}) = \{T \in [s]^{(t)} : 1 \notin T\}$ and $\mathcal{T}'(\overline{x}, \overline{y}) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \in T, y \notin T\}$, and in particular $f(\pi, t) = \sum_{T \in \mathcal{T}(\overline{1})} \prod_{i \in T} |P_i|$, it follows that

$$\sum_{T \in \mathcal{T}(\overline{1})} \prod_{i \in T} |P_i| = \sum_{T \in \mathcal{T}(\overline{x}, \overline{y})} \prod_{i \in T} |P_i| + \sum_{T \in \mathcal{T}'(\overline{x}, \overline{y})} \prod_{i \in T} |P_i|.$$

Now, letting $\mathcal{T}(1) = \{T \in [s]^{(t)} : 1 \in T\}$ and $\mathcal{T}'(\overline{x}, \overline{y}) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \in T, y \notin T\}$, we see that

$$\sum_{T \in \mathcal{T}(1)} \prod_{i \in T} |P_i| = \sum_{T \in \mathcal{T}'(\overline{x}, \overline{y})} \prod_{i \in T} |P_i|$$

since $|P_1| = |P_x| + |P_y|$. Thus letting $\mathcal{T}'(x, y) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \in T, y \in T\}$, and our choice of $N$, a calculation yields that $f(\pi', t) - f(\pi, t) = \sum_{T \in \mathcal{T}'(x, y)} \prod_{i \in T} |P_i'|$ and in particular $f(\pi, t) \leq f(\pi', t)$. \hfill \Box

B Proof of Equation (15)

Let $S(k, s)$ be the Stirling number of the second kind and $f(\pi)$ be as in Section 3. In this section we will show

$$\frac{1}{S(k, s)}(1 - t^{-t})^{N(f(\pi_{s+1}, t) - f(\pi_s, t))} \geq n^{s-t}.$$

**Proof.** First, we recall that

$$N = \left\lfloor \frac{(t+1)t^t \log_2 n}{(k-t+1) \log_2 e}\right\rfloor \quad \text{and} \quad f(\pi_s, t) = (k-s+1)\left(s-1\right)\left(t-1\right) + \left(s-1\right)\left(t\right).$$

As a result, when $t \leq s < k$, a calculation yields that

$$f(\pi_{s+1}, t) - f(\pi_s, t) = (k-s)\left(s-1\right)\left(t-2\right). \quad (26)$$

Letting $n \geq S(k, t)$, after taking $\log_2(\cdot)$ on both sides of (15), it suffices to prove that

$$N \cdot \frac{f(\pi_s, t) - f(\pi_{t, t})}{t^t} \left(-t^t \log_2(1 - t^{-t})\right) \geq (s-t+1) \log_2 n. \quad (27)$$

Using the fact that $(1 - t^{-t})^t \leq e^{-1}$ and our choice of $N$, it suffices to show that

$$f(\pi_s, t) - f(\pi_{t, t}) \geq \frac{(s-t+1)\left(k-t+1\right)}{t+1}. \quad (28)$$

The inequality in (28) holds for all $k \geq s > t \geq 3$ by using (26). \hfill \Box