A generalization of the Bollobás set pairs inequality

Jason O'Neill*
Department of Mathematics
University of California San Diego
California, U.S.A.
jmoneill@ucsd.edu

Jacques Verstraete*
Department of Mathematics
University of California San Diego
California, U.S.A.
jverstra@math.ucsd.edu

Submitted: June 7, 2020; Accepted: June 7, 2021; Published: Jul 2, 2021

Abstract

The Bollobás set pairs inequality is a fundamental result in extremal set theory with many applications. In this paper, for $n \geqslant k \geqslant t \geqslant 2$, we consider a collection of k families $\mathcal{A}_i : 1 \leqslant i \leqslant k$ where $\mathcal{A}_i = \{A_{i,j} \subset [n] : j \in [n]\}$ so that $A_{1,i_1} \cap \cdots \cap A_{k,i_k} \neq \varnothing$ if and only if there are at least t distinct indices $i_1, i_2, \ldots, i_k$. Via a natural connection to a hypergraph covering problem, we give bounds on the maximum size $\beta_{k,t}(n)$ of the families with ground set $[n]$.

Mathematics Subject Classifications: 05D05, 05D40, 05C65

1 Introduction

A central topic of study in extremal set theory is the maximum size of a family of subsets of an n-element set subject to restrictions on their intersections. Classical theorems in the area are discussed in Bollobás [2]. In this paper, we generalize one such theorem, known as the Bollobás set pairs inequality or two families theorem [3]:

Theorem 1. (Bollobás) *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be families of finite sets, such that $A_i \cap B_j \neq \varnothing$ if and only if $i, j \in [m]$ are distinct. Then*

$$\sum_{i=1}^{m} \binom{|A_i \cup B_i|}{|A_i|}^{-1} \leqslant 1. \tag{1}$$

For convenience, we refer to a pair of families $\mathcal{A}$ and $\mathcal{B}$ satisfying the conditions of Theorem 1 as a *Bollobás set pair*. The inequality above is tight, as we may take the pairs (A_i, B_i) to be distinct partitions of a set of size $a + b$ with $|A_i| = a$ and $|B_i| = b$ for $1 \leqslant i \leqslant \binom{a+b}{a}$.

*Supported by NSF award DMS-1800332.

The latter inequality was proved for $a = 2$ by Erdős, Hajnal and Moon [5], and in general has a number of different proofs [11, 12, 14, 17, 18]. A geometric version was proved by Lovász [17, 18], who showed that if $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ are respectively a-dimensional and b-dimensional subspaces of a linear space and $\dim(A_i \cap B_j) = 0$ if and only if $i, j \in [m]$ are distinct, then $m \leqslant \binom{a+b}{a}$.

1.1 Main Theorem

Theorem 1 has been generalized in a number of different directions in the literature [6, 9, 13, 16, 21, 24]. In this paper, we give a generalization of Theorem 1 from the case of two families to $k \geqslant 3$ families of sets with conditions on the k-wise intersections. For $2 \leqslant t \leqslant k$, a *Bollobás* (k,t)*-tuple* is a sequence $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ of set families $\mathcal{A}_j = \{A_{j,i} : 1 \leqslant i \leqslant m\}$ where $\bigcap_{j=1}^k A_{j,i_j} \neq \varnothing$ if and only if at least t of the indices $i_1, i_2, \ldots, i_k$ are distinct. We refer to m as the *size* of the Bollobás (k,t)-tuple. Let $[m]_{(t)}$ denote the set of sequences of t distinct elements of $[m]$ and fix a surjection $\phi : [k] \to [t]$. For $\sigma \in [m]_{(t-1)}$, set $\sigma(t) = \sigma(1)$ and define $A_{1,\sigma}(\phi) = \bigcap_{j:\phi(j)=1} A_{j,\sigma(1)}$ and, for $2 \leqslant j \leqslant t$, we define

$$A_{j,\sigma}(\phi) = \bigcap_{h:\phi(h)=j} A_{h,\sigma(j)} \backslash \bigcup_{h=1}^{j-1} A_{h,\sigma}(\phi).$$

Using this notation, we generalize (1) as follows:

Theorem 2. *Let $k \geqslant t \geqslant 2$ and $m \geqslant t$, let $\phi : [k] \to [t]$ be a surjection, and let $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ be a Bollobás (k,t)-tuple of size m. Then*

$$\sum_{\sigma \in [m]_{(t-1)}} \binom{|A_{1,\sigma}(\phi) \cup A_{2,\sigma}(\phi) \cup \cdots \cup A_{t,\sigma}(\phi)|}{|A_{1,\sigma}(\phi)| \;\; |A_{2,\sigma}(\phi)| \;\; \cdots \;\; |A_{t,\sigma}(\phi)|}^{-1} \leqslant 1. \tag{2}$$

We show in Section 2.1 that this inequality is tight for all $k \geqslant t = 2$, but do not have an example to show that this inequality is tight for any $t > 2$.

For $n \geqslant k \geqslant t \geqslant 2$, let $\beta_{k,t}(n)$ denote the maximum m such that there exists a Bollobás (k,t)-tuple of size m consisting of subsets of $[n]$. Then (1) gives $\beta_{2,2}(n) \leqslant \binom{n}{\lfloor n/2 \rfloor}$ which is tight for all $n \geqslant 2$. Letting $H(q) = -q \log_2 q - (1-q)\log_2(1-q)$ denote the standard binary entropy function, we prove the following theorem:

Theorem 3. *For $k \geqslant 3$ and large enough n,*

$$\frac{1}{k} \leqslant \frac{\log_2 \beta_{k,2}(n)}{n} \leqslant H\left(\frac{1}{k}\right) \leqslant \frac{\log_2(ke)}{k}. \tag{3}$$

For $k \geqslant t \geqslant 3$ and large enough n,

$$\frac{\log_2 e}{\binom{k}{t-1}(t+1)t^{t-1}} \leqslant \frac{\log_2 \beta_{k,t}(n)}{n} \leqslant \frac{2}{\binom{k}{t-1}(t-1)^{t-3}}. \tag{4}$$

This determines $\log_2 \beta_{k,2}(n)$ up to a factor of order $\log_2 k$ and $\log_2 \beta_{k,t}(n)$ up to a factor of order t^3. We leave it as an open problem to determine the asymptotic value of $(\log_2 \beta_{k,t}(n))/n$ as $n \to \infty$ for any $k \geqslant 3$ and $t \geqslant 2$. A natural source for lower bounds on $\beta_{k,t}(n)$ comes from the probabilistic method – see the random constructions in Section 3.1 which establish the lower bounds in Theorem 3. To prove Theorem 3, we use a natural connection to hypergraph covering problems.

1.2 Covering hypergraphs

Theorem 1 has a wide variety of applications, from saturation problems [3, 19] to covering problems for graphs [11, 20], complexity of 0-1 matrices [23], geometric problems [1], counting cross-intersecting families [7], crosscuts and transversals of hypergraphs [24, 25, 26], hypergraph entropy [15, 22], and perfect hashing [8, 10]. In this section, we give an application of our main results to hypergraph covering problems. For a k-uniform hypergraph H, let $f(H)$ denote the minimum number of complete k-partite k-uniform hypergraphs whose union is H. In the case of graph covering, a simple connection to the Bollobás set pairs inequality (1) may be described as follows. Let $K_{n,n} \setminus M$ denote the complement of a perfect matching $M = \{x_i y_i : 1 \leqslant i \leqslant n\}$ in the complete bipartite graph $K_{n,n}$ with parts $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. If $H_1, H_2, \dots, H_m$ are complete bipartite graphs in a minimum covering of $K_{n,n} \setminus M$, then let $A_i = \{j : x_i \in V(H_j)\}$ and $B_i = \{j : y_i \in V(H_j)\}$. Setting $\mathcal{A} = \{A_i\}_{i \in [m]}$ and $\mathcal{B} = \{B_i\}_{i \in [m]}$, it is straightforward to check that $(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair, and Theorem 1 applies to give

$$f(K_{n,n} \backslash M) = \min\{m : \binom{m}{\lceil m/2 \rceil} \geqslant n\}. \tag{5}$$

In a similar way, Theorem 2 applies to covering complete k-partite k-uniform hypergraphs. Let $K_{n,n,\dots,n}$ denote the complete k-partite k-uniform hypergraph with parts $X_i = \{x_{ij} : j \in [n]\}$ for $i \in [k]$. Let $H_{k,t}(n)$ denote the subhypergraph consisting of hyperedges $\{x_{1,i_1}, x_{2,i_2}, \dots, x_{k,i_k}\}$ such that at least t of the indices $i_1, i_2, \dots, i_k$ are distinct, and set $f_{k,t}(n) = f(H_{k,t}(n))$. Then there is a one-to-one correspondence between Bollobás (k,t)-tuples of subsets of $[m]$ and coverings of $H_{k,t}(n)$ with m complete k-partite k-graphs. We let $\beta_{k,t}(m)$ be the maximum size of a Bollobás (k,t)-tuple of subsets of $[m]$, so that

$$f_{k,t}(n) = \min\{m : \beta_{k,t}(m) \geqslant n\}. \tag{6}$$

This correspondence together with Theorem 2 will be exploited to prove

$$f_{k,2}(n) \geqslant \min\{m : \binom{m}{\lceil m/k \rceil} \geqslant n\} \tag{7}$$

which is partly an analog of (5). More generally, we prove the following theorem:

Theorem 4. *For $k \geqslant 3$ and large enough n,*

$$\frac{k}{\log_2(ke)} \leqslant \frac{1}{H(\frac{1}{k})} \leqslant \frac{f_{k,2}(n)}{\log_2 n} \leqslant k. \tag{8}$$

For $k \geqslant t \geqslant 3$ and large enough n,

$$\binom{k}{t-1}\frac{(t-1)^{t-3}}{2} \leqslant \frac{f_{k,t}(n)}{\log_2 n} \leqslant \frac{(t+1)t^{t-1}}{\log_2 e}\binom{k}{t-1}. \tag{9}$$

The bounds on $\beta_{k,t}(n)$ in Theorem (3) follow immediately from this theorem and (6). Equation (9) gives the order of magnitude for each $t \geqslant 3$ as $k \to \infty$, but for $t = 2$, Equation (8) has a gap of order $\log_2 k$. From (7), we obtain $\beta_{k,2}(n) \leqslant \binom{n}{\lfloor n/k \rfloor}$. It is perhaps unsurprising that the asymptotic value of $f_{k,t}(n)/\log_2 n$ as $n \to \infty$ is not known for any $k > 2$, since a limiting value of $f(K_n^k)/\log_2 n$ is not known for any $k > 2$ – see Körner and Marston [15] and Guruswami and Riazanov [10].

1.3 Organization and notation

Given a subset $A \subset [n]$, let $A^c := [n] \setminus A$ be the complement of A in $[n]$. For positive integers $k \leqslant n$, let $(n)_{(k)} = (n)(n-1)\cdots(n-k+1)$ denote the falling factorial. This paper is organized as follows. In Section 2, we prove Theorem 2. In Section 2.1, we construct a Bollobás $(k,2)$-tuple which achieves equality in Theorem 2 and in Section 2.2, we construct a Bollobás $(k,2)$-tuple which gives the lower bound in Equation (3). The upper bound on $f_{k,t}(n)$ in Theorem 4 comes from a probabilistic construction in Section 3.1, and the proof of the lower bound on $f_{k,t}(n)$ is given in Section 3.3; we prove (7) in Section 3.2.

2 Proof of Theorem 2

Given a Bollobás set (k,t)-tuple $(\mathcal{A}_1,\ldots,\mathcal{A}_k)$ with $\mathcal{A}_j = \{A_{j,i} : 1 \leqslant i \leqslant m\}$ and a surjection $\phi : [k] \to [t]$, consider $\mathcal{A}_\ell(\phi) : 1 \leqslant \ell \leqslant t$ where $\mathcal{A}_\ell(\phi) = \{A_{\ell,i}(\phi) : 1 \leqslant i \leqslant m\}$ and

$$A_{\ell,i}(\phi) = \bigcap_{h:\phi(h)=\ell} A_{h,i}.$$

It follows that $(\mathcal{A}_1(\phi),\ldots,\mathcal{A}_t(\phi))$ is a Bollobás set (t,t)-tuple and hence it suffices to prove Theorem 2 in the case where $t = k$. In this setting, surjections $\phi : [k] \to [k]$ simply permute the k families and as such we suppress the notation of ϕ for the remainder of this section. One of the proofs of Theorem 1, given a Bollobás set pair, defines a collection of chains $\mathcal{C}_i$ for $i \in [m]$ and shows that these chains are necessarily disjoint. Similarly, given a Bollobás set (k,k)-tuple, we will define a collection of chains $\mathcal{C}_\sigma$ for every ordered collection σ of $(k-1)$ distinct indices of $[m]$ and show these chains are pairwise disjoint.

Let $(\mathcal{A}_1,\ldots,\mathcal{A}_k)$ with $\mathcal{A}_j = \{A_{j,i} : 1 \leqslant i \leqslant m\}$ be a Bollobás set (k,k)-tuple, and set

$$X = \bigcup_{i=1}^{m}(A_{1,i} \cup A_{2,i} \cup \cdots \cup A_{k,i})$$

with $|X| = n$. For $\sigma \in [m]_{(k-1)}$, define a subset $\mathscr{C}_\sigma$ of permutations $\pi : X \to [n]$ by

$$\mathscr{C}_\sigma := \left\{ \pi : X \to [n] : \max_{x \in A_{1,\sigma}} \pi(x) < \min_{y \in A_{2,\sigma}} \pi(y) \leqslant \max_{y \in A_{2,\sigma}} \pi(y) < \cdots < \min_{z \in A_{k,\sigma}} \pi(z) \right\}.$$

Letting $U_\sigma := A_{1,\sigma} \cup \cdots \cup A_{k,\sigma}$, elementary counting methods give

$$|\mathscr{C}_\sigma| = \binom{n}{|U_\sigma|} |A_{1,\sigma}|! \cdots |A_{k,\sigma}|!(n - |U_\sigma|)! = n! \cdot \binom{|U_\sigma|}{|A_{1,\sigma}| \cdots |A_{k,\sigma}|}^{-1}. \tag{10}$$

We will now prove a lemma which states that $\{\mathscr{C}_\sigma\}_{\sigma \in [m]_{(k-1)}}$ forms a disjoint collection of a permutations. The general proof only works for $k \geqslant 4$, so we first consider $k = 3$.

Lemma 5. *If $\sigma_1, \sigma_2 \in [m]_{(2)}$ are distinct, then $\mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2} = \varnothing$.*

Proof. Seeking a contradiction, suppose there exists $\pi \in \mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2}$. After relabeling, it suffices to consider the following five cases.

(1) $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 4\}$ (2) $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 3\}$
(3) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{1, 3\}$ (4) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{2, 3\}$
(5) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{3, 1\}$.

In case (1), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,2}\}$ and thus $\pi \in \mathscr{C}_{\sigma_2}$ yields

$$\max_{x \in A_{1,1}} \pi(x) \leqslant \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y).$$

Then as $A_{1,1} \cap A_{2,4} \cap A_{3,2} \neq \varnothing$, there exists $w \in A_{1,1} \cap A_{2,4} \cap A_{3,2}$. It follows that $w \notin A_{1,2}$ since if $w \in A_{1,2}$, then $w \in A_{1,2} \cap A_{2,4} \cap A_{3,2} \neq \varnothing$; a contradiction. But this yields a contradiction as

$$\pi(w) \leqslant \max_{x \in A_{1,1}} \pi(x) \leqslant \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y) \leqslant \pi(w).$$

In case (2), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,2}\}$ and we recover a similar contradiction as case (1) by noting that there exists $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ with $w \notin A_{1,2}$.

In case (3) we may assume $\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\}$ and $\pi \in \mathscr{C}_{1,3}$ yields $\max\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$. Thus

$$\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$$

and there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,3}$. It follows that $\pi(w) < \pi(w)$, a contradiction.

In case (4), if $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,2}\}$, then using $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ and noting $w \notin A_{1,2}$, we get a contradiction. Thus, we may assume otherwise and $\pi \in \mathcal{C}_{1,2}$ gives

$$\max_{x \in A_{1,2}} \pi(x) < \max_{x \in A_{1,1}} \pi(x) < \min_{z \in A_{3,1} \setminus (A_{1,1} \cup A_{2,2})} \pi(z).$$

This is a contradiction as there exists $w \in A_{1,2} \cap A_{2,3} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,2}$. In case (5), if $\max\{\pi(x) : x \in A_{1,1}\} \leqslant \max\{\pi(x) : x \in A_{1,3}\}$, then we may proceed as in the latter part of case (4) using $w \in A_{1,1} \cap A_{2,2} \cap A_{3,3}$ and $w \notin A_{2,1}$ and $w \notin A_{1,3}$ to get a contradiction. Otherwise, proceeding as in case (1) and noting there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$, but $w \notin A_{1,1}$ yields a contradiction. □

A similar argument yields the analog of Lemma 5 to the case where $k \geqslant 4$.

Lemma 6. *Let $k \geqslant 4$. If $\sigma_1, \sigma_2 \in [m]_{(k-1)}$ are distinct, then $\mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2} = \varnothing$.*

Proof. Since $\sigma_1, \sigma_2 \in [m]_{(k-1)}$ are distinct, there exists minimal $h \in [k-1]$ so that $\sigma_1(h) \neq \sigma_2(h)$. Seeking a contradiction, suppose there exists a $\pi \in \mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2}$. Without loss of generality,

$$\max\{\pi(x) : x \in A_{h,\sigma_1}\} \leqslant \max\{\pi(x) : x \in A_{h,\sigma_2}\} < \min\{\pi(z) : z \in A_{k,\sigma_2}\}.$$

Now, consider a bijection $\tau : [k-1] \setminus \{h\} \to [k-1] \setminus \{1\}$ which has no fixed points. As in Lemma 5, we want to show that there exists a $w \in A_{h,\sigma_1} \cap A_{k,\sigma_2}$ and consider two separate cases.

First, suppose that $\sigma_1(h) \notin \sigma_2([k-1])$. As $|\{\sigma_1(h), \sigma_2(1), \ldots, \sigma_2(k-1)\}| = k$, there exists

$$w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap \bigcap_{l \in [k-1] \setminus \{h\}} A_{l,\sigma_2(\tau(l))}. \tag{11}$$

Next, suppose that $\sigma_1(h) = \sigma_2(x)$ for some x. We now claim that $x \neq 1$. If $h = 1$, then this is trivial. If $h > 1$, then $\sigma_1(1) = \sigma_2(1)$, so $\sigma_1(h) \neq \sigma_2(1)$ since $\sigma_1(h) \neq \sigma_1(1)$. For τ as above, there exists $y \in [k-1] \setminus \{h\}$ so that $\tau(y) = x$. Taking γ distinct from $\{\sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}$, $|\{\sigma_1(h), \gamma, \sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}| = k$ and hence there exists

$$w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap A_{y,\gamma} \cap \bigcap_{l \in [k-1] \setminus \{y,h\}} A_{l,\sigma_2(\tau(l))}. \tag{12}$$

By construction, $w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)}$. Suppose there exists a $t \in [k-1] \setminus \{h\}$ so that $w \in A_{t,\sigma_2(t)}$. As τ has no fixed points, replacing the set in the k-wise intersection corresponding to $\mathcal{A}_t$ with $A_{t,\sigma_2(t)}$ in either (11) or (12), w is an element of this new k-wise intersection with $(k-1)$ distinct indices; a contradiction. If $w \in A_{h,\sigma_2(h)}$, then we may similarly replace $A_{h,\sigma_1(h)}$ with $A_{h,\sigma_2(h)}$ in the k-wise intersection in either (11) or (12) to get a contradiction. Thus, $w \notin A_{1,\sigma_2(1)} \cup \cdots \cup A_{k-1,\sigma_2(k-1)}$ and hence $w \in A_{h,\sigma_1} \cap A_{k,\sigma_2}$ so that $\pi(w) < \pi(w)$; a contradiction. □

Using Equation (10), Lemma 5, and Lemma 6, we are now able to prove Theorem 2 in the case where $t = k$. There are $n!$ total permutations, and Lemma 5 and Lemma 6 yield that each of which appears in at most one of the sets $\mathscr{C}_\sigma$ for $\sigma \in [m]_{(k-1)}$. Hence, using $|\mathscr{C}_\sigma|$ in Equation (10),

$$\sum_{\sigma \in [m]_{(k-1)}} |\mathscr{C}_\sigma| = \sum_{\sigma \in [m]_{(k-1)}} n! \cdot \binom{|A_{1,\sigma} \cup \cdots \cup A_{k,\sigma}|}{|A_{1,\sigma}| \cdots |A_{k,\sigma}|}^{-1} \leqslant n!$$

and thus the result follows by dividing through by $n!$.

2.1 Sharpness of Theorem 2

We give a simple construction establishing the sharpness of Theorem 2 for $k \geqslant t = 2$. Let $n \geqslant 4k$ and using addition modulo n, define $A_{1,i} = \{i\}^c$, $A_{j,i} = \{i-(j-1), i+(j-1)\}^c$ for $j \in [2, k-1]$, and $A_{k,i} = \{i-k+2, i-k+3, \ldots, i+k-2\}$. Letting $\mathcal{A}_j = \{A_{j,i}\}_{i\in[n]}$ for all $j \in [k]$, we will show $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ is a Bollobás $(k, 2)$-tuple. Since $|A_{1,i}| = n - 1$ and $|A_{2,i} \cap \cdots \cap A_{k,i}| = 1$, Theorem 2 with $t = 2$ and surjection $\phi : [k] \to [2]$ with $\phi(1) = 1$ and $\phi(i) = 2$ for $i \neq 1$ gives

$$1 \;\geqslant\; \sum_{i=1}^{n} \binom{|A_{1,i}| + |A_{2,i} \cap \cdots \cap A_{k,i}|}{|A_{1,i}|}^{-1} \;=\; \sum_{i=1}^{n} \frac{1}{n} \;=\; 1.$$

By construction, for all $i \in [n]$, $A_{1,i} \cap A_{2,i} \cap \cdots \cap A_{k,i} = \varnothing$. It thus suffices to show these are the only empty k-wise intersections. To this end, for $\boldsymbol{i} = (i_1, \ldots, i_{k-1})$, define

$$A(\boldsymbol{i}) := A_{1,i_1} \cap \cdots \cap A_{k-1,i_{k-1}}.$$

Lemma 7. *Let $\boldsymbol{i} = (i_1, \ldots, i_{k-1})$. If $A(\boldsymbol{i})^c = A_{k,i_k}$, then $i_1 = \cdots = i_k$.*

Proof. We proceed by induction on k where the result is trivial when $k = 2$. In the case where $k > 2$, $i_{k-1} - k + 2 = i_k + x$ for some x such that $-(k-2) \leqslant x \leqslant (k-2)$ and thus $i_{k-1} + (k-2) = i_{k-1} - (k-2) + (2k-4) = i_k + x + (2k-4)$.

Next, there is a y such that $-(k-2) \leqslant y \leqslant (k-2)$ with $i_{k-1} + (k-2) = i_k + y$, and since $n \geqslant 4k$, $x + 2k - 4 = y$ with equality over $\mathbb{Z}$ and moreover $i_{k-1} + (k-2) = i_k + (k-2)$ over $\mathbb{Z}$ and hence $i_k = i_{k-1}$. Removing these elements from each set, the result then follows by induction. □

If $A_{1,i_1} \cap \cdots \cap A_{k,i_k} = \varnothing$, then as $A(\boldsymbol{i}) = A_{1,i_1} \cap A_{2,i_2} \cap \cdots A_{k-1,i_{k-1}}$,

$$\varnothing = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}} \cap A_{k,i_k} = A(\boldsymbol{i}) \cap A_{k,i_k}.$$

The result follows by noting $|A(\boldsymbol{i})| \geqslant n - (2k-3)$, $|A_{k,i_k}| = 2k - 3$, and using Lemma 7.

2.2 An Explicit Construction

Let $k \geqslant 3$. An explicit construction of a Bollobás $(k, 2)$-tuple $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ where $|\mathcal{A}_i| = 2^n$ and each $\mathcal{A}_i$ consists of subsets of X for $|X| = kn$ may be described as follows. Let $I_j := \{x_{j,1}, x_{j,2}, \ldots, x_{j,k}\}$ and consider $X = I_1 \sqcup \cdots \sqcup I_n$. Now, for each $f : [n] \to [2]$ and $j \in [k]$, define

$$A_{j,f} := \{x_{1,f(1)+j-1}, \ \ldots \ , x_{n,f(n)+j-1}\}^c$$

where we work modulo k within the subscripts of I_j. It is straightforward to check that $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ is a Bollobás $(k, 2)$-tuple. This establishes the lower bound on $\beta_{k,2}(n)$ in Equation (3) and hence the upper bound on $f_{k,2}(n)$ in Equation (8).

3 Proof of Theorem 4

3.1 Upper bound on $f_{k,t}(n)$

We wish to find a covering of $H_{k,t}(n)$ with complete k-partite k-graphs and assume the parts of $H_{k,t}(n)$ are $X_1, X_2, \dots, X_k$. For each subset T of $[k]$ of size t, consider the uniformly random coloring $\chi_T : [n] \to T$. Given such a χ_T, let $Y_i \subset X_i$ be the vertices of color i for $i \in T$; that is $Y_i := \{x_{ij} : \chi(j) = i\}$ and $Y_i = X_i$ for $i \notin T$. Denote by $H(T, \chi)$ the (random) complete k-partite hypergraph with parts $Y_1, Y_2, \dots, Y_k$, and note that $H(T, \chi) \subset H_{k,t}(n)$. We place each $H(T, \chi)$ a total of N times independently and randomly where

$$N = \left\lfloor \frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} \right\rfloor$$

and produce $\binom{k}{t}N$ random subgraphs $H(T, \chi)$. For a set partition π of $[k]$, let $|\pi|$ denote the number of parts in the partition and index the parts by $[|\pi|]$. Given a set partition $\pi = (P_1, P_2, \dots, P_s)$, let

$$f(\pi, t) = \sum_{T \in [s]^{(t)}} \prod_{i \in T} |P_i|.$$

If U is the number of edges of $H_{k,t}(n)$ not in any of these subgraphs, then

$$\mathbb{E}(U) \leqslant \sum_{|\pi| \geqslant t} n^{|\pi|}(1 - t^{-t})^{Nf(\pi,t)} = \sum_{t \leqslant s \leqslant k} n^s \sum_{|\pi| = s} (1 - t^{-t})^{Nf(\pi,t)}. \tag{13}$$

For sufficiently large n, we claim that $\mathbb{E}(U) < 1$, which implies there exists a covering of $H_{k,t}(n)$ with at most $\binom{k}{t}N$ complete k-partite k-graphs, as required. The following technical lemma states that f is a decreasing function in the set partition lattice, and that $f(\pi, t)$ increases when we merge all but one element of a smaller part of π with a larger part of π:

Lemma 8. *Let $k \geqslant s \geqslant t \geqslant 2$, and let $\pi = (P_1, P_2, \dots, P_s)$ be a partition of $[k]$.*

(i) *If π' is a refinement of π with $|\pi'| = s + 1$, then $f(\pi, t) \leqslant f(\pi', t)$.*

(ii) *If $|P_1| \geqslant |P_2| \geqslant 2$ and $a \in P_2$, and π' is the partition $(P_1', P_2', \dots, P_s')$ of $[k]$ with $P_1' = P_1 \cup P_2 \setminus \{a\}$ and $P_2' = \{a\}$ and with $P_i' = P_i$ for $3 \leqslant i \leqslant s$, then $f(\pi', t) \leqslant f(\pi, t)$.*

The proof of Lemma 8 part (i) is in Appendix A and the proof of (ii) is similar to the proof of (i). By Lemma 8, a set partition of $[k]$ into s parts which minimizes $f(\pi, t)$ consists of one part of size $k - s + 1$ and $s - 1$ singleton parts and hence

$$\min\{f(\pi, t) : |\pi| = s\} = (k - s + 1)\binom{s-1}{t-1} + \binom{s-1}{t}. \tag{14}$$

In what follows, we denote a set partition of $[k]$ into s parts which minimizes $f(\pi, t)$ by π_s.

For n large enough, and all s where $t \leqslant s \leqslant k$, we will show

$$\frac{\sum_{|\pi|=t}(1-t^{-t})^{Nf(\pi,t)}}{\sum_{|\pi|=s}(1-t^{-t})^{Nf(\pi,t)}} \geqslant n^{s-t}.$$

Replacing the numerator with its largest term and each term in denominator with its largest term,

$$\frac{\sum_{|\pi|=t}(1-t^{-t})^{Nf(\pi,t)}}{\sum_{|\pi|=s}(1-t^{-t})^{Nf(\pi,t)}} \geqslant \frac{(1-t^{-t})^{Nf(\pi_t,t)}}{S(k,s)(1-t^{-t})^{Nf(\pi_s,t)}} = \frac{1}{S(k,s)}(1-t^{-t})^{N(f(\pi_t,t)-f(\pi_s,t))}$$

where $S(k,s)$ is the Stirling number of the second kind. Taking $n \geqslant S(k,s)$, we will show in Appendix B that

$$\frac{1}{S(k,s)}(1-t^{-t})^{N(f(\pi_t,t)-f(\pi_s,t))} \geqslant n^{s-t}. \tag{15}$$

Therefore, the index $s = t$ maximizes the right hand side of Equation (13), and hence

$$\mathbb{E}[U] \leqslant (k-t+1)(n^t)\sum_{|\pi|=t}(1-t^{-t})^{Nf(\pi,t)} < (k-t+1)n^t S(k,t)(1-t^{-t})^{N(k-t+1)} < 1$$

for our choice of N provided $n \geqslant kS(k,t)$. Thus,

$$f_{k,t}(n) \leqslant \binom{k}{t}\frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} = \frac{(t+1)t^{t-1}}{\log_2 e}\binom{k}{t-1}\log_2 n.$$

3.2 Lower bound on $f_{k,2}(n)$

In this section, we show

$$f_{k,2}(n) \geqslant \min\{m : \binom{m}{\lceil m/k \rceil} \geqslant n\}. \tag{16}$$

Let $\{H_1, H_2, \ldots, H_m\}$ be a covering of $H_{k,2}(n)$ with $m = f_{k,2}(n)$ complete k-partite k-graphs. We recall $H_{k,2}(n) = K_{n,n,\ldots,n} \backslash M$, where M is a perfect matching of $K_{n,n,\ldots,n}$. For $i \in [k]$ and $j \in [n]$, define $A_{i,j} = \{H_r : x_{ij} \in V(H_r)\}$ and $\mathcal{A}_i = \{A_{i,j} : 1 \leqslant j \leqslant n\}$. As in (6), $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ is a Bollobás $(k,2)$-tuple of size n. For convenience, for each $i \in [k]$, let $\phi_i : [k] \to [2]$ be so that $\phi_i^{-1}(1) = \{i\}$. Taking the sum of inequality from Theorem 2 with $t = 2$ over all $i \in [k]$,

$$\sum_{i=1}^{k}\sum_{j=1}^{n}\binom{|A_{1,j}(\phi_i) \cup A_{2,j}(\phi_i)|}{|A_{1,j}(\phi_i)|}^{-1} \leqslant k. \tag{17}$$

We use this inequality to give a lower bound on $f_{k,2}(n) = m$. First we observe

$$\sum_{r=1}^{m}|V(H_r)| = \sum_{j=1}^{n}\sum_{i=1}^{k}|A_{i,j}| = \sum_{j=1}^{n}\sum_{i=1}^{k}|A_{1,j}(\phi_i)|. \tag{18}$$

Let ∂H denote the set of $(k-1)$-tuples of vertices contained in some edge of a hypergraph H. Then

$$\sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} |A_{2,j}(\phi_i)|. \tag{19}$$

Putting the above identities together,

$$\sum_{r=1}^{m} |V(H_r)| + \sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|). \tag{20}$$

We note $|\partial H_r \cap \partial M| \leqslant |V(H_r)|/(k-1)$, and therefore

$$\sum_{r=1}^{m} |\partial H_r \cap \partial M| \leqslant \frac{1}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{21}$$

It follows that

$$\sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) \leqslant \frac{k}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{22}$$

Subject to the linear inequalities (18) and (22), the left side of (17) is minimized when $kn|A_{1,j}(\phi_i)| = \sum_{r=1}^{m} |V(H_r)|$ and $kn(|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) = (k-1)|A_{1,j}(\phi_i)|$. Since $|V(H_r)| \leqslant (k-1)n$ for all $r \in [m]$, (17) implies $\binom{m}{\lceil m/k \rceil} \geqslant n$, which gives (16). □

3.3 Lower bound on $f_{k,k}(n)$

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,k}(n)$ with complete k-partite k-graphs, so $m = f(H_{k,k}(n))$. Given a k-partite k-graph H, consider its 2-shadow $\partial_2(H) = \{R \subset V(H) : |R| = k-2, R \subset e \text{ for some } e \in H\}$. Let $\partial_2(\mathcal{H}) = \bigcup_{i=1}^{m} \partial_2(H_i)$.

Given $R \in \partial_2(\mathcal{H})$ and $H_i \in \mathcal{H}$, let $H_i(R) := \{e \in \binom{V(H_i)}{2} : e \cup R \in H_i\}$ be the possibly empty link graph of the edge R in the hypergraph H_i and let $V(H_i(R))$ be the set of vertices in the link graph. Observe that double counting yields

$$\sum_{R \in \partial_2(\mathcal{H})} \left(\sum_{i=1}^{m} |V(H_i(R))| \right) = \sum_{i=1}^{m} \left(\sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right). \tag{23}$$

An optimization argument yields $|\partial_2(H_i)|$ is maximized when the parts of H_i are of equal or nearly equal maximal size. Since $|V(H_i(R))| \leqslant 2(n-k+2)$, the right hand side of Equation (23) is bounded above by

$$\sum_{i=1}^{m} \left(\sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right) \leqslant m \cdot \binom{k}{2} \cdot \left(\frac{n}{k}\right)^{k-2} \cdot 2(n-k+2). \tag{24}$$

For a lower bound on the left hand side of Equation (23), fix $R \in \partial_2(\mathcal{H})$ and without loss of generality suppose that $R = \{x_{1,1}, \ldots, x_{k-2,k-2}\}$. Let $Y = [k-1, n]$. Let $K_{Y,Y}$ be the complete bipartite graph with two distinct copies of Y and $\mathcal{M} = \{(x_{k-1,i}, x_{k,i} : i \in Y\}$ be a perfect matching in $K_{Y,Y}$. Then, $\{H_1(R), \ldots, H_m(R)\}$ forms a biclique cover of $K_{Y,Y} \setminus \mathcal{M}$. Applying the convexity result of Tarjan [23, Lemma 5],

$$\sum_{i=1}^{m} |V(H_i(R))| \geqslant (n-k+2)\log_2(n-k+2).$$

Noting that $|\partial_2(\mathcal{H})| = \binom{k}{2}(n)_{(k-2)}$, the left hand side of Equation (23) is bounded below by

$$\sum_{R \in \partial_2(\mathcal{H})} \left(\sum_{i=1}^{m} |V(H_i(R))| \right) \geqslant \binom{k}{2} (n)_{(k-2)} (n-k+2)\log_2(n-k+2). \tag{25}$$

Comparing the bounds from Equation (24) and Equation (25),

$$m \geqslant \frac{(n)_{(k-2)} \log_2(n-k+2)}{2\left(\frac{n}{k}\right)^{k-2}} \geqslant \frac{k^{k-2}}{2} \log_2 n$$

provided that n is large enough.

For $t \geqslant 3$ and $t < k$, the lower bound on $f_{k,t}(n)$ in Theorem 4 is obtained from the lower bounds on $f_{t-1,t-1}(n-1)$ as follows: Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,t}(n)$ with complete k-partite k-graphs, so $m = f(H_{k,t}(n))$. Given $T \in \binom{[k]}{k-t+1}$, define $H_T \subset H_{k,t}(n)$ by

$$H_T := \{\{x_{1,i_1}, \ldots, x_{k,i_k}\} \in H_{k,t}(n) : i_j = 1 \;\forall\, j \in T\}.$$

It follows that at least $f_{t-1,t-1}(n-1)$ of the complete k-partite k-graphs in $\mathcal{H}$ are needed to cover H_T. Moreover, for distinct $T, T' \in \binom{[k]}{k-t+1}$, the corresponding complete k-partite k-graphs from $\mathcal{H}$ are necessarily pairwise disjoint and hence

$$f_{k,t}(n) \geqslant \binom{k}{k-t+1} f_{t-1,t-1}(n-1) \geqslant \binom{k}{t-1} \frac{(t-1)^{t-3}}{2} \log_2 n$$

provided that n is large enough.

4 Concluding remarks

• Our main theorem, Theorem 2 is tight for $t = 2$ and $k \geqslant 2$, as shown in Section 2.1. It would be interesting to generalize this example to $2 < t \leqslant k$ to determine whether Theorem 2 is tight in general. The first open case is $t = k = 3$.

• A particular case of the Bollobás set pairs inequality occurs when every set in $\mathcal{A}$ has size a and every set in $\mathcal{B}$ has size b, and one obtains the tight bound $|\mathcal{A}| \leqslant \binom{a+b}{b}$. The generalization to Bollobás (k,t)-tuples for $k \geqslant 3$ is equally interesting but wide open, as are potential generalizations to vector spaces – see Lovász [17, 18].

• Orlin [20] proved that the clique cover number $cc(K_n \backslash M)$ of a complete graph K_n minus a perfect matching M is precisely $\min\{m : 2\binom{m-1}{\lfloor m/2 \rfloor} \geqslant n\}$. Theorem 4 yields lower bounds on the clique cover number of the complement of a perfect matching M in the complete k-uniform hypergraph K_n^k:

Corollary 9. *Let $K_n^k \setminus M$ be the complement of a perfect matching in K_n^k. Then*

$$cc(K_n^k \setminus M) \geqslant \frac{\log_2 \frac{n}{k}}{H(\frac{1}{k})} \geqslant \frac{k \log_2 \frac{n}{k}}{\log_2(ke)}.$$

• It would be interesting to prove an analog of Equation (16) for $t \geqslant 3$. That is,

$$f_{k,t}(n) \geqslant \min\{m : \binom{m}{\alpha_1, \ldots, \alpha_t} \geqslant n_{(t-1)}\}$$

for some optimal $\alpha_1, \ldots, \alpha_t$. The difficulty here lies in determining effective bounds on $|A_{i,\sigma}(\phi)|$.

Acknowledgements

We would like to thank the anonymous referees for their helpful comments.

References

[1] N. Alon, G. Kalai, A simple proof of the upper bound theorem. European J. Combin. 6, no. 3, 211–214, 1985.

[2] B. Bollobás, Combinatorics. Set systems, hypergraphs, families of vectors and combinatorial probability. Cambridge University Press, Cambridge, 1986.

[3] B. Bollobás, On generalized graph, Acta Math. Acad. Sci. Hungar. 16, 447-452, 1965.

[4] P. Erdős, A. W. Goodman, L. Pósa, The representation of a graph by set intersections. Canad. J. Math. 18, 106–112, 1966.

[5] P. Erdős, A. Hajnal, W. Moon, A problem in graph theory, Amer. Math. Monthly 71, 1107–1110, 1964.

[6] P. Frankl, An extremal problem for two families of sets, European J. Combin. 3, 125–127, 1982.

[7] P. Frankl, A. Kupavskii, Counting intersecting and pairs of cross-intersecting families, Combin. Probab. Comput. 27, no. 1, 60–68, 2018.

[8] M. L. Fredman, J. Komlós, On the size of separating systems and families of perfect hash functions, SIAM Journal on Algebraic Discrete Methods, 5(1), 61–68, 1984.

[9] Z. Füredi, Geometrical solution of a problem for two hypergraphs, European J. Combin. 5, 133–136, 1984.

[10] V. Guruswami, A. Riazanov, Beating Fredman-Komlós for perfect k-hashing, ICALP, 2019.

[11] G. Hansel, Nombre minimal de contacts de fermature nécessaires pour réaliser une fonction booléenne symétrique de n variables, C. R. Acad. Sci. Paris, pp. 6037–6040, 1964.

[12] F. Jaeger, C. Payan, Nombre maximal d'arêtes d'un hypergraphe critique de rang h, C. R. Acad. Sci. Paris 273, 221–223, 1971.

[13] D. Kang, J. Kim, Y. Kim, On the Erdős-Ko-Rado theorem and the Bollobás theorem for t-intersecting families. European J. Combin. 47, 68–74, 2015.

[14] G. O. H. Katona, Solution of a problem of Ehrenfeucht and Mycielski, J. Combin. Theory, Ser. A 17, 265–266, 1974.

[15] J. Körner, K. Marton, New bounds for perfect hashing via information theory, European J. Combin. 9, 523–530, 1988.

[16] L. Lovász, Combinatorial Problems and Exercises, North-Holland, Amsterdam, New York, Oxford, 1979.

[17] L. Lovász, Flats in matroids and geometric graphs, in Combinatorial Surveys (P. J. Cameron, ed.), Academic Press, New York, pp. 45-86, 1977.

[18] L. Lovász, Topological and algebraic methods in graph theory, in Graph Theory and Related Topics (J. A. Bondy and U. S. R. Murty, eds.) Academic Press, New York, 1979, pp. 1–15. (Proc. of Tutte Conference, Waterloo, 1977).

[19] G. Moshkovitz, A. Shapira, Exact bounds for some hypergraph saturation problems. J. Combin. Theory Ser. B 111, 242–248, 2015.

[20] J. Orlin. Contentment in graph theory: Covering graphs with cliques. Indagationes Mathematicae (Proceedings), 80 (5) 406–424, 1977.

[21] J. Talbot. A new Bollobás-type inequality and applications to t-intersecting families of sets, Discrete Mathematics 285, 349–353, 2004.

[22] G. Simonyi, Graph entropy: A survey. In Combinatorial Optimization, W. Cook, L. Lovász, P. Seymour, Eds., DIMACS Series in Discrete Mathematics and Theoretical Computer Science; American Mathematical Society: Providence, RI, USA; Volume 20, pp. 399–441, 1995.

[23] T.G Tarjan, Complexity of lattice-configurations. Studia Sci. Math. Hungar. 10, no. 1–2, 203–211, 1975.

[24] Zs. Tuza, Critical hypergraphs and intersecting set-pair systems, J. Combin. Theory Ser. B 39, 134–145, 1985.

[25] Zs. Tuza, Applications of the set-pair method in extremal hypergraph theory, in: P. Frankl, et al. (Eds.), Extremal Problems for Finite Sets, Bolyai Society Mathematical Studies, Vol. 3, János Bolyai Mathematical Society, Budapest, pp. 479–514, 1994.

[26] Zs. Tuza, Applications of the set-pair method in extremal problems, II, in: D. Miklos, et al. (Eds.), Combinatorics, Paul Erdős is Eighty, Bolyai Society Mathematical Studies, Vol. 2, János Bolyai Mathematical Society, Budapest, pp. 459–490, 1996.

A Proof of Lemma 8(i)

Let $k \geqslant s \geqslant t \geqslant 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$. In this section, we will show that if π' is a refinement of π with $|\pi'| = s+1$, then $f(\pi, t) \leqslant f(\pi', t)$.

Proof. Let $\pi = P_1|P_2|\cdots|P_s$ and without loss of generality, $\pi' = P_x|P_y|P_2|\cdots|P_s$. Setting $\mathcal{T}(\overline{1}) = \{T \in [s]^{(t)} : 1 \notin T\}$ and $\mathcal{T}'(\overline{x}, \overline{y}) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x, y \notin T\}$, it follows that

$$\sum_{T \in \mathcal{T}(\overline{1})} \prod_{i \in T} |P_i| = \sum_{T \in \mathcal{T}'(\overline{x}, \overline{y})} \prod_{i \in T} |P_i|.$$

Now, letting $\mathcal{T}(1) = \{T \in [s]^{(t)} : 1 \in T\}$ and $\mathcal{T}'(x, \overline{y}) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \in T, y \notin T\}$ and $\mathcal{T}'(\overline{x}, y) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \notin T, y \in T\}$, we see that

$$\sum_{T \in \mathcal{T}(1)} \prod_{i \in T} |P_i| = \sum_{T \in \mathcal{T}'(\overline{x}, y)} \prod_{i \in T} |P_i| + \sum_{T \in \mathcal{T}'(x, \overline{y})} \prod_{i \in T} |P_i|$$

since $|P_1| = |P_x| + |P_y|$. Thus letting $\mathcal{T}'(x, y) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \in T, y \in T\}$,

$$f(\pi', t) - f(\pi, t) = \sum_{T \in \mathcal{T}'(x, y)} \prod_{i \in T} |P'_i|$$

and in particular $f(\pi, t) \leqslant f(\pi', t)$. □

B Proof of Equation (15)

Let $S(k, s)$ be the Stirling number of the second kind and $f(\pi)$ be as in Section 3. In this section we will show

$$\frac{1}{S(k, s)} (1 - t^{-t})^{N(f(\pi_t, t) - f(\pi_s, t))} \geqslant n^{s-t}.$$

Proof. First, we recall that

$$N = \left\lfloor \frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} \right\rfloor \quad \text{and} \quad f(\pi_s, t) = (k-s+1)\binom{s-1}{t-1} + \binom{s-1}{t}.$$

As a result, when $t \leqslant s < k$, a calculation yields that

$$f(\pi_{s+1}, t) - f(\pi_s, t) = (k-s)\binom{s-1}{t-2}. \tag{26}$$

Letting $n \geqslant S(k, t)$, after taking $\log_2(\cdot)$ on both sides of (15), it suffices to prove that

$$N \cdot \frac{f(\pi_s, t) - f(\pi_t, t)}{t^t} \bigg(-t^t \log_2(1 - t^{-t}) \bigg) \geqslant (s-t+1)\log_2(n). \tag{27}$$

Using the fact that $(1 - t^{-t})^{t^t} \leqslant e^{-1}$ and our choice of N, it suffices to show that

$$f(\pi_s, t) - f(\pi_t, t) \geqslant \frac{(s-t+1)(k-t+1)}{t+1}. \tag{28}$$

The inequality in (28) holds for all $k \geqslant s > t \geqslant 3$ by using (26). □