Cameron–Liebler k-sets in AG(n, q)

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Abstract

We study Cameron–Liebler k-sets in the affine geometry, so sets of k-spaces in AG(n, q). This generalizes research on Cameron–Liebler k-sets in the projective geometry PG(n, q). Note that in algebraic combinatorics, Cameron–Liebler k-sets of AG(n, q) correspond to certain equitable bipartitions of the association scheme of k-spaces in AG(n, q), while in the analysis of Boolean functions, they correspond to Boolean degree 1 functions of AG(n, q).

We define Cameron–Liebler k-sets in AG(n, q) in a similar way as Cameron–Liebler k-sets in PG(n, q), such that its characteristic vector is a linear combination of point-pencils. In particular, we investigate the relationship between Cameron–Liebler k-sets in AG(n, q) and PG(n, q). As a by-product, we calculate the character table of the association scheme of affine lines. Furthermore, we characterize the smallest examples of Cameron–Liebler k-sets.

This paper focuses on AG(n, q) for $n > 3$, while the case for Cameron–Liebler line classes in AG$(3, q)$ was already treated separately.

Mathematics Subject Classifications: 05B20, 05B25

1 Introduction

The investigation of Cameron–Liebler line classes in the projective geometry PG(n, q) goes back to Cameron and Liebler in 1982 [4]. Their motivation was the investigation of the subgroup structure of PGL$(n + 1, q)$. Particularly, a line orbit of a subgroup of

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PGL$(n+1,q)$ acting on PG(n,q) with the same number of point- and line-orbits is a Cameron–Liebler line class.

The concept of Cameron–Liebler line classes was rediscovered several times, see the introduction of [8] for a short overview. In particular, algebraic combinatorialists studied equitable bipartitions as a natural generalization of perfect codes under various names in several highly symmetric families of graphs such as hypercubes and Johnson graphs. Similarly, they also correspond to Boolean degree 1 functions in the analysis of Boolean functions.

In the special case of PG$(3,q)$, a Cameron–Liebler line class can be defined as a family of lines which intersects all spreads of PG$(3,q)$ in exactly x lines for some constant x [4]. We call x the *parameter* of the Cameron–Liebler line class. In PG$(3,q)$, there exists a list of examples which we refer to as *trivial*: (1) the empty set with parameter $x=0$, (2) all lines through a fixed point with parameter $x=1$, (3) all lines in a fixed plane with parameter $x=1$, (4) the union of (2) and (3), when disjoint, with parameter $x=2$, and (5)–(8) the complements of (1)–(4) with parameters $x=q^2+1, q^2, q^2, q^2-1$. Cameron and Liebler conjectured that these are the only examples. This was disproven by Drudge who found an example with parameter $x=5$ in PG$(3,3)$ [10]. Nowadays there are several infinite families of non-trivial examples known [1, 5, 12, 14]. In contrast to this, there are no non-trivial examples known for $n>3$. Hence, there is some difference in behaviour between $n=3$ and $n>3$. This carries over to AG(n,q), where this paper handles the case $n>3$, while we treat the case $n=3$ separately in [9].

Cameron–Liebler line classes were generalized to k-spaces of PG(n,q) in [2, 8]. These are families of k-spaces which lie in the span of the point-(k-space) incidence matrix. We call such families *Cameron–Liebler k-sets of* PG(n,q). Note that if $\mathcal{L}$ is a Cameron–Liebler k-set of PG(n,q), then its parameter x is defined by $|\mathcal{L}| = x\begin{bmatrix} n \\ k \end{bmatrix}_q$ where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the Gaussian binomial coefficient. Analogously, we call a family $\mathcal{L}$ of k-spaces which lies in the span of the (affine) point-(k-space) incidence matrix a *Cameron–Liebler k-set of* AG(n,q). Moreover, if $|\mathcal{L}| = x\begin{bmatrix} n \\ k \end{bmatrix}_q$, we say that x is the parameter of the Cameron–Liebler k-set $\mathcal{L}$.

After some preliminaries, we start our paper with some general properties of Cameron–Liebler k-sets in Section 3. In particular, we show the following.

Theorem 1. *Let $\mathcal{L}$ be a Cameron–Liebler k-set with parameter x in PG(n,q) which does not contain k-spaces in some hyperplane H. Then $\mathcal{L}$ is a Cameron–Liebler k-set with parameter x of* AG$(n,q) \cong$ PG$(n,q) \setminus H$.

We also obtained a result on the converse of Theorem 1. In particular, we show the following.

Theorem 2. *If $\mathcal{L}$ is a Cameron–Liebler k-set of* AG(n,q) *with parameter x, then $\mathcal{L}$ is a Cameron–Liebler k-set of* PG(n,q) *with parameter x in the projective closure* PG(n,q) *of* AG(n,q).

We prove this theorem in general, but for the case $k=1$ we give an alternative proof using the character table of the association scheme of affine lines. This association scheme has been investigated before as it is a well-known 3-class association scheme, see [25] for a more detailed study. We could not find the character table of the affine lines scheme in the literature, so we provide the latter in Section 4. While for $\mathrm{PG}(n,q)$ the character tables of the association scheme of k-spaces is explicitly known due to Delsarte [7] and Eisfeld [11], the determination of the character tables of the association scheme of k-spaces in $\mathrm{AG}(n,q)$ is still open.

A 3-class association scheme has four common eigenspaces V_0, V_1, V_2, V_3, where V_0 is spanned by the all-ones vector. In our ordering, we provide explicit bases for $V_0 + V_1$ and $V_0 + V_3$, and we give a spanning set for $V_0 + V_2 + V_3$.

An immediate consequence of Theorem 2 is that the following results for Cameron–Liebler k-sets of $\mathrm{PG}(n,q)$ are also valid for Cameron–Liebler k-sets of $\mathrm{AG}(n,q)$.

Theorem 3. *[2, Theorem 4.9] There are no Cameron–Liebler k-sets in $\mathrm{AG}(n,q)$, with $n \geqslant 3k+2$ and $q \geqslant 3$, of parameter*

$$2 \leqslant x \leqslant \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2} - \frac{k^2}{4} - \frac{3k}{4} - \frac{3}{2}} (q-1)^{\frac{k^2}{4} - \frac{k}{4} + \frac{1}{2}} \sqrt{q^2+q+1}.$$

The statement in [2, Theorem 4.9] is slightly different and its proof currently contains an error. The authors of [2] have submitted an erratum. The version in this article is based on the recent corrected version on arXiv.* For $n \geqslant \frac{5}{2}k + \frac{3}{2}$, a similar bound is given in [19, Theorem 7]. For $n = 2k+1$, there is a better bound.

Theorem 4. *[22, (Metsch, Theorem 1.4)] For $k \geqslant 3$, there are no non-trivial Cameron–Liebler k-sets with parameter x in $\mathrm{AG}(2k+1,q)$ for $2 \leqslant x \leqslant q/5$ and $q \geqslant q_0$ for some universal constant q_0.*

Note that [22, Theorem 1.4] requires that $k < q \log q - q - 1$. This condition can be removed, see [18, Theorem 1.8]. Complementary to the two previous results, Theorem 1 also implies that the situation is known for small q.

Theorem 5. *[13, Theorem 1.4] Let $n \geqslant 2k+1 > 3$. Then there are no non-trivial Cameron–Liebler k-sets in $\mathrm{AG}(n,q)$ for $q \leqslant 5$.*

We conclude with Section 6, where we obtain a classification of the smallest Cameron–Liebler k-sets of $\mathrm{AG}(n,q)$.

Theorem 6. *All Cameron–Liebler k-sets of $\mathrm{AG}(n,q)$ with parameter $x \leqslant 2$ are trivial.*

Note that this cannot be deduced from the literature on $\mathrm{PG}(n,q)$. We are also able to classify all Cameron–Liebler sets of hyperplanes in $\mathrm{AG}(n,q)$, this will be done in Section 7. We conclude with suggestions for future work in Section 8.

*See arXiv:1805.09539v3 [math.CO]

2 Preliminaries

Consider a prime p and let $q = p^h$, with $h \geqslant 1$. Then consider $\mathrm{PG}(n, q)$, and $\mathrm{AG}(n, q)$, for $n > 2$, as the n-dimensional projective, and affine, space over $\mathbb{F}_q$ respectively. Suppose that we consider a hyperplane π_∞ in $\mathrm{PG}(n, q)$, which from now on will be called the hyperplane at infinity. Then we can consider all the points, lines, planes, and other spaces that do not lie inside π_∞. In this way we obtain the affine space $\mathrm{AG}(n, q)$.

The following notation will be used throughout this article.

Definition 7. For $a, b \in \mathbb{N}$, we denote the *Gaussian binomial coefficient* by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q^a - 1) \cdots (q^{a-b+1} - 1)}{(q^b - 1) \cdots (q - 1)}.$$

The Gaussian binomial coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q$ equals the number of $(b-1)$-spaces in $\mathrm{PG}(a-1, q)$. Here we define that $\begin{bmatrix} a \\ b \end{bmatrix}_q = 0$ if $b > a$.

In general we take $k \geqslant 1$ with $n \geqslant k + 1$, unless otherwise stated. Note that if we also ask that $(k + 1) \mid (n + 1)$, then it automatically follows that $n \geqslant 2k + 1$.

Definition 8. Consider $\mathrm{PG}(n, q)$, or $\mathrm{AG}(n, q)$ respectively.

1. A *partial k-spread* is a set of pairwise disjoint k-spaces.

2. A *conjugated switching k-set* is a pair of disjoint partial k-spreads that cover the same set of points.

3. A *k-spread* is a partial k-spread that partitions the point set of $\mathrm{PG}(n, q)$, or $\mathrm{AG}(n, q)$ respectively.

Remark 9. Since there is a lot of interaction between $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$ we want to clarify some formulations.

- Suppose that we have a set of k-spaces $\mathcal{L}$ in $\mathrm{PG}(n, q)$. Then the *restriction* of $\mathcal{L}$ to $\mathrm{AG}(n, q)$ is the set of k-spaces of $\mathcal{L}$ that are not contained in π_∞.

- We say that two k-spaces are *projectively (or affinely) disjoint* if they intersect in the empty set in $\mathrm{PG}(n, q)$ (or $\mathrm{AG}(n, q)$ respectively).

We will give some examples of k-spreads in $\mathrm{AG}(n, q)$, which we will denote by respectively type I, II and III for future purposes.

Lemma 10. *Consider the affine space* $\mathrm{AG}(n, q)$ *and the corresponding projective space* $\mathrm{PG}(n, q)$*. Then the following k-sets* $\mathcal{S}$ *are k-spreads in* $\mathrm{AG}(n, q)$*.*

1. *(*Type I*) Every k-spread in* $\mathrm{PG}(n, q)$ *restricted to the affine space.*

2. *(*Type II*) Consider a $(k-1)$-space K in π_∞ and define the set $\mathcal{S}$ as the set of all affine k-spaces through K.*

3. *(*Type III*) Consider an $(n-2)$-space π_{n-2} in π_∞, then there are exactly q other hyperplanes through π_{n-2} not equal to π_∞. Call these hyperplanes π_i, for $i \in \{1, \dots, q\}$. If we select for every hyperplane π_i a $(k-1)$-space $\tau_i \subseteq \pi_{n-2}$ (not all equal), then we can define the k-spread*

$$\mathcal{S} := \{K \mid K \text{ a } k\text{-space in } \mathrm{AG}(n,q),\ \ \tau_i \subseteq K \subseteq \pi_i \text{ for some } i \in \{1, \dots, q\}\}.$$

Proof.

1. Consider a projective k-spread $\mathcal{S}$, then it is clear that every two k-spaces of $\mathcal{S}$ are affinely disjoint. Secondly, $\mathcal{S}$ restricted to $\mathrm{AG}(n,q)$ partitions the point set of $\mathrm{AG}(n,q)$, since its extension reaches every (affine) point in $\mathrm{PG}(n,q)$.

2. Trivial.

3. It is clear that all these elements are disjoint. Thus we only need to prove that for every affine point p there exists an element of $\mathcal{S}$ that contains it. Consider for this point p the hyperplane $\langle p, \pi_{n-2}\rangle$, then this is a hyperplane through π_{n-2}. Without loss of generality we may assume that it is π_i. Such that $\langle p, \tau_i\rangle$ is a k-space in $\mathcal{S}$ which contains p. This proves that $\mathcal{S}$ is indeed a k-spread. □

Remark 11. The size of a k-spread in $\mathrm{AG}(n,q)$ is equal to $\frac{q^n}{q^k} = q^{n-k}$, where q^k is the number of points in an affine k-space and q^n is the total number of points in $\mathrm{AG}(n,q)$. An analogous result can be obtained in $\mathrm{PG}(n,q)$, where the size of a k-spread is known to be $\frac{q^{n+1}-1}{q^{k+1}-1}$. Note that this number is only an integer if $(k+1) \mid (n+1)$, so it follows that this is a necessary condition for the existence of k-spreads in $\mathrm{PG}(n,q)$. It is proven in [17, Corollary 4.17] that this is also a sufficient condition.

Definition 12. Let us denote the set of k-spaces in $\mathrm{PG}(n,q)$, and $\mathrm{AG}(n,q)$, by Π_k, and Φ_k, respectively. If we number the points and the k-spaces in these spaces, then we can define the point-(k-space) incidence matrix P_n and A_n respectively. These matrices are $0,1$-valued matrices with a 1 on position (i,j) if and only if point i lies on k-space j.

We now give a special construction for the matrix P_n.

Construction 13 (Incidence matrix)**.** Consider now the point-(k-space) incidence matrix P_n of $\mathrm{PG}(n,q)$, where the rows correspond to the points and the columns correspond to the elements of Π_k. We order the rows and columns in such a way that the first rows and columns correspond to the affine points and affine k-spaces respectively. Then P_n is of the following form:

$$P_n = \begin{bmatrix} A_n & \bar{0} \\ B_2 & P_{n-1} \end{bmatrix}, \tag{1}$$

where A_n is the incidence matrix of $\mathrm{AG}(n,q)$, where again the rows correspond to the points and the columns correspond to the elements of Φ_k. The matrix $\bar{0}$ is the zero-matrix and the part that remains unnamed, we call B_2.

We will use the notation of Construction 13 in the following results. These results give some information about the characteristic vector of Cameron–Liebler k-sets. This characteristic vector is a $0,1$ valued vector, which contains a 1 on position i if and only if the ith k-space belongs to the Cameron–Liebler k-set.

Lemma 14. *[9, Lemma 2.4] Consider a set $\mathcal{L}$ of k-spaces in $\mathrm{AG}(n,q)$ such that $\chi_{\mathcal{L}} \in (\ker(A_n))^\perp = Im(A_n^T)$. Then it follows for every affine k-spread $\mathcal{S}$ that*

$$|\mathcal{L} \cap \mathcal{S}| = x,$$

for a fixed integer x.

Remark 15. We should also note that in general it holds that $\mathrm{Im}(P_n^T) = (\ker(P_n))^\perp$ and $\mathrm{Im}(A_n^T) = (\ker(A_n))^\perp$.

Theorem 16. *[9, Theorem 2.3] Consider the projective space $\mathrm{PG}(n,q)$ and consider a set of k-spaces $\mathcal{L}$. If its characteristic vector $\chi_{\mathcal{L}} \in (\ker(P_n))^\perp$ and $\mathcal{L}$ also contains no k-spaces at infinity, then $\chi_{\mathcal{L}}$ restricted to the affine space belongs to $(\ker(A_n))^\perp$.*

2.1 Cameron–Liebler k-sets in $\mathrm{PG}(n,q)$

Our goal here is to state some important results that are known for Cameron–Liebler k-sets in $\mathrm{PG}(n,q)$. We start with the definition of Cameron–Liebler k-sets in $\mathrm{PG}(n,q)$.

Definition 17. A Cameron–Liebler k-set $\mathcal{L}$ in $\mathrm{PG}(n,q)$ is a set of k-spaces such that for its characteristic vector $\chi_{\mathcal{L}}$, it holds that $\chi_{\mathcal{L}} \in \mathrm{Im}(P_n^T)$. We say that $\mathcal{L}$ has parameter $x := |\mathcal{L}|/\begin{bmatrix} n \\ k \end{bmatrix}$.

Remark 18. The fact that $\chi_{\mathcal{L}} \in \mathrm{Im}(P_n^T)$ states that $\chi_{\mathcal{L}}$ is a linear combination of the rows of P_n^T. In some literature, for example [13], the characteristic vector $\chi_{\mathcal{L}}$ is called a Boolean degree 1 function in $\mathrm{PG}(n,q)$. Similarly we can consider $\chi_{\mathcal{L}} \in \mathrm{Im}(A_n^T)$ for the incidence matrix A_n in $\mathrm{AG}(n,q)$.

In this section we list some results on Cameron–Liebler k-spaces in $\mathrm{PG}(n,q)$. We refer to [2] for more information.

Theorem 19. *[2, Theorem 2.9] Let $\mathcal{L}$ be a non-empty set of k-spaces in $\mathrm{PG}(n,q)$, $n \geqslant 2k+1$, with characteristic vector χ, and x so that $|\mathcal{L}| = x\begin{bmatrix} n \\ k \end{bmatrix}_q$. Then the following properties are equivalent.*

1. *The set $\mathcal{L}$ is a Cameron–Liebler k-set.*
2. *For every k-space K, the number of elements of $\mathcal{L}$ disjoint from K is equal to $(x - \chi(K))\begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q q^{k^2+k}$.*
3. *For every pair of conjugated switching sets $\mathcal{R}$ and $\mathcal{R}'$, $|\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'|$.*

If $\mathrm{PG}(n,q)$ has a k-spread, then the following property is equivalent to the previous ones.

4 For every k-spread $\mathcal{S}$, $|\mathcal{L} \cap \mathcal{S}| = x$.

Example 20. [2, Example 3.2] The following k-sets are examples of Cameron–Liebler k-sets in $\mathrm{PG}(n, q)$.

1. The set of all the k-spaces through a fixed point is an example of a Cameron–Liebler k-set of parameter $x = 1$.
2. If we consider the set of k-spaces inside a fixed hyperplane, then this is a Cameron–Liebler k-set of parameter $x = \frac{q^{(n-k)}-1}{q^{(k+1)}-1}$. Note that x is only an integer if and only if $(k+1) \mid (n+1)$.

In order to give some context on the study of Cameron–Liebler k-sets in $\mathrm{PG}(n, q)$, we now give some classification results

Theorem 21. *[13, Theorem 4.1] Let $q \in \{2, 3, 4, 5\}$. Then all Cameron–Liebler k-sets in $\mathrm{PG}(n, q)$ are of the form of Example 20, if $k, n-k \geqslant 2$ and either (a) $n \geqslant 5$ or (b) $n = 4$ and $q = 2$.*

Theorem 22. *[2, Theorem 4.1] Let $\mathcal{L}$ be a Cameron–Liebler k-set with parameter $x = 1$ in $\mathrm{PG}(n, q)$, $n \geqslant 2k + 1$. Then $\mathcal{L}$ consists out of all the k-spaces through a fixed point or $n = 2k + 1$ and $\mathcal{L}$ is the set of all the k-spaces in a hyperplane of PG($2k + 1, q$).*

Theorem 23. *[2, Theorem 4.2] There are no Cameron–Liebler k-sets in PG(n, q) with parameter $x \in]0, 1[$ and if $n \geqslant 3k + 2$, then there are no Cameron–Liebler k-sets with parameter $x \in]1, 2[$.*

3 Cameron–Liebler k-sets in $\mathrm{AG}(n, q)$

Definition 24. A Cameron–Liebler k-set $\mathcal{L}$ in $\mathrm{AG}(n, q)$ is a set of k-spaces such that for its characteristic vector $\chi_{\mathcal{L}}$, it holds that $\chi_{\mathcal{L}} \in \mathrm{Im}(A_n^T)$. Here we say that $\mathcal{L}$ has parameter $x := |\mathcal{L}| / \left[{n \atop k}\right]$.

Due to Lemma 10, we know that for every value of n, the affine space $\mathrm{AG}(n, q)$ contains k-spreads. By Lemma 14, we find that Cameron–Liebler k-sets have a constant intersection number with k-spreads. This number will be equal to the parameter x of the Cameron–Liebler k-set. This is proven in the next lemma.

Lemma 25. *Suppose that $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n, q)$, then it holds for every k-spread $\mathcal{S}$ that*

$$|\mathcal{L} \cap \mathcal{S}| = x,$$

where x is the parameter of $\mathcal{L}$.

Proof. Due to Lemma 14, it follows that for every k-spread $\mathcal{S}$ it holds that $|\mathcal{L} \cap \mathcal{S}| = c$, for a certain fixed integer c. To prove that $c = x$, we double count the pairs $(K, \mathcal{S})$, where $\mathcal{S}$ is a k-spread of type II and $K \in \mathcal{L} \cap \mathcal{S}$. So we obtain that

$$x \begin{bmatrix} n \\ k \end{bmatrix}_q \cdot 1 = \begin{bmatrix} n \\ k \end{bmatrix}_q \cdot c.$$

This proves the lemma. □

Remark that this statement holds for every n and k, since in each case we have these k-spreads. We now give some basic properties.

Lemma 26. *Consider $\mathcal{L}$ and $\mathcal{L}'$ to be Cameron–Liebler k-sets with parameter x and x' both in $\mathrm{AG}(n, q)$ or both in $\mathrm{PG}(n, q)$ respectively, then the following properties hold.*

1. *If $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n, q)$ or $\mathrm{PG}(n, q)$, then we have that $0 \leqslant x \leqslant q^{n-k}$ or $0 \leqslant x \leqslant \frac{q^{n+1}-1}{q^{k+1}-1}$ respectively.*
2. *If $\mathcal{L} \cap \mathcal{L}' = \varnothing$, then $\mathcal{L} \cup \mathcal{L}'$ is a Cameron–Liebler k-set of parameter $x + x'$.*
3. *If $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{L} \setminus \mathcal{L}'$ is a Cameron–Liebler k-set of parameter $x - x'$.*
4. *If $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n, q)$ or $\mathrm{PG}(n, q)$, then the complement of $\mathcal{L}$ in $\mathrm{AG}(n, q)$ or $\mathrm{PG}(n, q)$ is a Cameron–Liebler k-set with parameter $q^{n-k} - x$ or $\frac{q^{n+1}-1}{q^{k+1}-1} - x$ respectively.*

Proof. This lemma follows due to Lemma 25 and [2, Lemma 3.1] for the projective case. □

We now give some general results that will give connections between Cameron–Liebler k-sets in $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$. These results will be similar to the results obtained in [9].

Here we prove Theorems 1 and 2 from the introduction.

Proof of Theorem 1. Suppose that $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{PG}(n, q)$ that misses the set of k-spaces in π_∞. So the characteristic vector of $\mathcal{L}$ in $\mathrm{PG}(n, q)$ lies inside $\mathrm{Im}(P_n^T)= (\ker(P_n))^\perp$. Then, by Theorem 16, we obtain that for its characteristic vector $\chi_\mathcal{L}$ in $\mathrm{AG}(n, q)$ it holds that $\chi_\mathcal{L} \in (\ker(A_n))^\perp = \mathrm{Im}(A_n^T)$. Here A_n is the point-(k-space) incidence matrix of the affine space from Construction 13. Due to the size of $\mathcal{L}$, the parameter remains the same. This proves the assertion. □

Proof of Theorem 2. Consider the characteristic vector $\chi_\mathcal{L}$ of the k-set $\mathcal{L}$, then we know that $\chi_\mathcal{L} \in \mathrm{Im}(A_n^T)$. Here A_n is the point-(k-space) incidence matrix of $\mathrm{AG}(n, q)$. Due to Construction 13, we know that

$$\begin{pmatrix} \chi_\mathcal{L} \\ \bar{0} \end{pmatrix} \in \ \mathrm{Im}(P_n^T),$$

with $\bar{0}$ the vector of the correct dimension that only contains zeroes. Note that $\binom{\chi_{\mathcal{L}}}{\bar{0}}$ is in fact the characteristic vector of $\mathcal{L}$ in $\mathrm{PG}(n,q)$. So $\mathcal{L}$ is by definition a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$. Due to the size of $\mathcal{L}$, the parameter remains the same. □

Theorem 27. *Suppose that $\mathcal{L}$ is a Cameron–Liebler k-set of parameter x in $\mathrm{PG}(n,q)$. Then $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$ of the same parameter x if and only if $\mathcal{L}$ is skew to the set of k-spaces in the hyperplane at infinity π_∞ of the affine space.*

Proof. Suppose that $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$ that misses the set of k-spaces in π_∞. Then, by Theorem 1, we obtain that $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$ that has the same parameter x.

Let $\mathcal{L}$ be a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$ whose restriction is a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$ of the same parameter x. Then we can define the restriction of $\mathcal{L}$ to $\mathrm{AG}(n,q)$ by $\mathcal{L}'$. Using Theorem 2, we know that $\mathcal{L}'$ is a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$. So, by Lemma 26, it follows that $\mathcal{L}\setminus\mathcal{L}'$ is a Cameron–Liebler k-set of parameter $x=0$ in $\mathrm{PG}(n,q)$. This Cameron–Liebler k-set would only contain k-spaces in the hyperplane at infinity. So clearly $\mathcal{L}\setminus\mathcal{L}'=\varnothing$. Thus $\mathcal{L}$ does not contain k-spaces at infinity. □

Theorem 28. *If there exists an affine Cameron–Liebler k-set with parameter x in $\mathrm{AG}(n,q)$, then there exists a Cameron–Liebler k-set of parameter $x+\frac{q^{n-k}-1}{q^{k+1}-1}$ in the projective closure $\mathrm{PG}(n,q)$.*

Proof. Due to Theorem 2, we know that $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$. Using Lemma 26 and Example 20, we can extend every Cameron–Liebler k-set in $\mathrm{AG}(n,q)$ as follows: $\mathcal{L}' := \mathcal{L}\cup\{K\in\Pi_k \mid K\subseteq\pi_\infty\}$. This set is a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$ of parameter $x+\frac{q^{n-k}-1}{q^{k+1}-1}$. □

Remark 29. If we now combine Theorems 27 and 28, we find another interesting observation. Recall that every Cameron–Liebler k-set of parameter x in $\mathrm{AG}(n,q)$ can be extended by adding all k-spaces at infinity to a Cameron–Liebler k-set of parameter $x+\frac{q^{n-k}-1}{q^{k+1}-1}$ in $\mathrm{PG}(n,q)$. But the other way also holds. Suppose we have a Cameron–Liebler k-set of parameter $x+\frac{q^{n-k}-1}{q^{k+1}-1}$ in $\mathrm{PG}(n,q)$ that contains all the k-spaces at infinity. Then, by Lemma 26, we can remove all the k-spaces at infinity and obtain a Cameron–Liebler k-set of parameter x in $\mathrm{AG}(n,q)$.

3.1 Equivalent definitions

Our goal will be to prove the following theorem, which presents equivalent definitions for Cameron–Liebler k-sets in $\mathrm{AG}(n,q)$.

Theorem 30. *Consider the affine space $\mathrm{AG}(n,q)$, for $n\geqslant 2k+1$, and let $\mathcal{L}$ be a set of k-spaces such that $|\mathcal{L}|=x\begin{bmatrix}n\\k\end{bmatrix}_q$ for a positive integer x. Then the following properties are equivalent.*

1. *$\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$.*

2. For every k-spread $\mathcal{S}$, it holds that $|\mathcal{L} \cap \mathcal{S}| = x$.

3. For every pair of conjugated switching k-sets $\mathcal{R}$ and $\mathcal{R}'$, $|\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'|$.

If $k = 1$ and we thus consider Cameron–Liebler line classes, then the following property is equivalent to the previous ones.

4. For every line ℓ, the number of elements of $\mathcal{L}$ affinely disjoint to ℓ is equal to

$$\left(q^2 \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q + 1\right)(x - \chi_{\mathcal{L}}(\ell)) \tag{2}$$

and through every point at infinity there are exactly x lines of $\mathcal{L}$.

To do this, we will need the following statement. Note that for every subspace τ of $\mathrm{AG}(n,q)$, we will denote $[\tau]_k := \{K \mid K \in \Phi_k, K \subseteq \tau\}$.

Theorem 31. *Let $\mathcal{L}$ be a collection of k-spaces in $\mathrm{AG}(n,q)$ such that property (2) of Theorem 30 holds. Suppose that τ_A is an arbitrary i-dimensional subspace in $AG(n,q)$, with $i \geqslant \max\{k+1, 3\}$. Then property (2) also holds for $\mathcal{L} \cap [\tau_A]_k$ in the (affine) subspace τ_A with respect to the k-spreads in τ_A.*

Proof. Consider the projective closure $\mathrm{PG}(n,q)$ of $\mathrm{AG}(n,q)$ and let π_∞ be the hyperplane at infinity. Let τ_A be an i-dimensional space in $\mathrm{AG}(n,q)$ and let τ be its projective closure, hence $\dim(\tau \cap \pi_\infty) \geqslant k$.

Our goal is to prove that $\mathcal{L}$ restricted to τ_A satisfies property (2) with x. Pick a $(k-1)$-space I in $\tau \cap \pi_\infty$ and denote E as the set of affine k-spaces through I and not in τ_A. Then it is clear that every k-space $K \in E$ is affinely disjoint towards every k-space in τ_A. It is also true that no two k-spaces in E share an affine point, and yet as a set they cover all affine points not in τ_A. So if we would choose a k-spread $\mathcal{S}$ in τ_A, then we can always extend this k-spread to a k-spread in $\mathrm{AG}(n,q)$ in the following way

$$\mathcal{S}' := \mathcal{S} \cup E.$$

Note that for every k-spread $\mathcal{S}$ in τ_A, we can use the same E. So we have that

$$x = |\mathcal{L} \cap \mathcal{S}'| = |\mathcal{L} \cap \mathcal{S}| + |\mathcal{L} \cap E|.$$

Hence, we have for every k-spread $\mathcal{S}$ in τ_A that $|(\mathcal{L} \cap [\tau_A]_k) \cap \mathcal{S}| = x - |\mathcal{L} \cap E|$. But this last term is a constant, due to the fact that E was fixed for every k-spread $\mathcal{S}$. This proves the theorem. □

We now can prove the main theorem.

Proof of Theorem 30. We first prove the equivalence between the first 3 statements and then we prove the equivalence with statement 4.

- **From (1) to (3):** Suppose that $\mathcal{L}$ is a Cameron–Liebler k-set with characteristic vector $\chi_{\mathcal{L}}$. Let $\mathcal{R}$ and $\mathcal{R}'$ be a pair of conjugated switching sets with characteristic vectors $\chi_{\mathcal{R}}$ and $\chi_{\mathcal{R}'}$ respectively. Then it holds, due to the definition of a pair of conjugated switching sets, that

$$\chi_{\mathcal{R}} - \chi_{\mathcal{R}'} \in \ker(A_n),$$

where A_n is the point-line incidence matrix of $\mathrm{AG}(n,q)$. Since $\chi_{\mathcal{L}} \in \mathrm{Im}(A_n^T) = (\ker(A_n))^{\perp}$, we have that

$$\chi_{\mathcal{L}} \cdot (\chi_{\mathcal{R}} - \chi_{\mathcal{R}'}) = 0.$$

This concludes the statement.

- **From (3) to (2):** Since for every pair of k-spreads $\mathcal{S}_1$ and $\mathcal{S}_2$, it holds that $\mathcal{S}_1 \setminus \mathcal{S}_2$ and $\mathcal{S}_2 \setminus \mathcal{S}_1$ are a pair of conjugated switching sets, we know that $|\mathcal{L} \cap \mathcal{S}_1| = c = |\mathcal{L} \cap \mathcal{S}_2|$. So we only need to show that $c = x$. To obtain this we double count the pairs $(K, \mathcal{S})$, with $K \in \mathcal{S} \cap \mathcal{L}$ and $\mathcal{S}$ a k-spread of type II. Hence, due to the fact that $|\mathcal{L}| = x\begin{bmatrix} n \\ k \end{bmatrix}$ and the number of k-space of type II through a fixed k-spread equals 1, we get that

$$x\begin{bmatrix} n \\ k \end{bmatrix} \cdot 1 = \begin{bmatrix} n \\ k \end{bmatrix} \cdot c.$$

Thus $x = c$, which completes the assertion.

- **From (2) to (1):** If $n = 2k+1$, then we know, due to Lemma 10 (1) and Theorem 19, that $\mathcal{L}$ is a Cameron–Liebler k-set of parameter x in $\mathrm{PG}(n,q)$. Hence due to Theorem 1, the assertion follows.
Suppose now that $n > 2k+1$. Here we will use similar techniques as in the proof of Theorem 2.9 in [2], (more specifically in the step (7) to (1)). Let τ be an arbitrary $(2k+1)$-dimensional subspace, then we can consider $\mathcal{L} \cap [\tau]_k$ in the affine space τ. Due to Theorem 31, we obtain that $\mathcal{L} \cap [\tau]_k$ satisfies Property (2) in τ. Hence, using the previous observation, we obtain that $\mathcal{L} \cap [\tau]_k$ is a Cameron–Liebler k-set of a certain parameter in τ. Note that this space τ was chosen arbitrarily. Thus it follows for every $(2k+1)$-dimensional subspace of $\mathrm{AG}(n,q)$ that, for the characteristic vector of $\mathcal{L} \cap [\tau]_k$, it holds that

$$\chi_{\mathcal{L} \cap [\tau]_k} \in \mathrm{Im}(A_\tau^T),$$

with A_τ the incidence matrix of τ. So we have that $\chi_{\mathcal{L} \cap [\tau]_k}$ is a linear combination of the rows of A_τ. Note that due to the fact that A_τ has full row rank, it holds that this linear combination is unique. We only need to show that $\chi_{\mathcal{L}}$ is uniquely defined by the vectors $\chi_{\mathcal{L} \cap [\tau]_k}$, with τ varying over all $(2k+1)$-spaces in $\mathrm{AG}(n,q)$.
We first want to show that for every two $(2k+1)$-spaces τ and τ' the coefficients of the row corresponding to a point in $\tau \cap \tau'$ in the linear combination of $\chi_{\mathcal{L} \cap [\tau]_k}$ and $\chi_{\mathcal{L} \cap [\tau']_k}$ are equal.
Consider the subspace $\tau \cap \tau'$, and consider the corresponding columns of A_n. Then using the fact that $A_{\tau \cap \tau'}$ also has full row rank, we conclude that the linear combination of the rows that give $\chi_{\mathcal{L} \cap [\tau \cap \tau']_k}$ is unique. Note that this unique linear

combination has the same coefficients for the rows corresponding with points in $\tau \cap \tau'$ as $\chi_{\mathcal{L}\cap[\tau]_k}$ and $\chi_{\mathcal{L}\cap[\tau']_k}$ has respectively. Here we also used the fact that an entry of A_n corresponding with a point of $\tau \setminus \tau'$ or $\tau' \setminus \tau$ and a k-space in $\tau \cap \tau'$ is zero. Thus we may conclude that the common rows in $\chi_{\mathcal{L}\cap[\tau]_k}$ and $\chi_{\mathcal{L}\cap[\tau']_k}$ have the same coefficient.

Using all of these $(2k+1)$-spaces, we have that $\chi_{\mathcal{L}}$ is uniquely defined and $\chi_{\mathcal{L}} \in \mathrm{Im}(A_n^T)$. This proves the assertion.

- **Equivalence between (4) and the rest, when $k = 1$:** First if $\mathcal{L}$ is an affine Cameron–Liebler line class with parameter x, then, by Theorem 2, we get that $\mathcal{L}$ is a Cameron–Liebler line class in $\mathrm{PG}(n,q)$. Here we know that for every (affine) line ℓ, there are exactly

$$q^2 \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q (x - \chi_{\mathcal{L}}(\ell))$$

lines of $\mathcal{L}$ projectively disjoint to ℓ. So we only still need to consider the lines of $\mathcal{L}$ through the point $\ell \cap \pi_\infty$. But since this is a point at infinity, which gives a line spread of type II, we have a total of x lines of $\mathcal{L}$ through this point. Thus if we add those $x - \chi_{\mathcal{L}}(\ell)$ lines of $\mathcal{L}$ not equal to ℓ, we get a total of

$$\left(q^2 \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q + 1 \right) (x - \chi_{\mathcal{L}}(\ell))$$

lines of $\mathcal{L}$ disjoint to ℓ in $\mathrm{AG}(n,q)$.

Conversely, suppose that Property (2) holds, then we look at the corresponding projective space $\mathrm{PG}(n,q)$. We can see that of the

$$\left(q^2 \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q + 1 \right) (x - \chi_{\mathcal{L}}(\ell))$$

lines of $\mathcal{L}$ that are disjoint in $\mathrm{AG}(n,q)$ to an affine line ℓ, there are

$$q^2 \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q (x - \chi_{\mathcal{L}}(\ell))$$

elements of $\mathcal{L}$ projectively disjoint to ℓ.

If we now pick a line ℓ in π_∞, then there are

$$\frac{q^n - 1}{q - 1} - (q+1) = q^2 \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$$

points in π_∞ not in ℓ. Through every such point, there are exactly x lines of $\mathcal{L}$ that are disjoint to ℓ. If we combine these results we obtain that, by Theorem 19, it follows that $\mathcal{L}$ is a Cameron–Liebler line class in $\mathrm{PG}(n,q)$ with parameter x. Using Theorem 27, we see that $\mathcal{L}$ is also a Cameron–Liebler line class in $\mathrm{AG}(n,q)$ with the same parameter x. □

Remark 32. There is also another way to prove the equivalence of (1) and (2) for $k = 1$. For this, we will use association schemes. This will be done in Section 5.

Lemma 33. *Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are a pair of conjugated switching k-sets in* $\mathrm{PG}(n, q)$. *If we define $\mathcal{R}_A$ (and $\mathcal{R}'_A$) as the set of affine k-spaces of $\mathcal{R}$ (and $\mathcal{R}'$ respectively), then $\mathcal{R}_A$ and $\mathcal{R}'_A$ are conjugated switching k-sets in* $\mathrm{AG}(n, q)$.

Proof. Since $\mathcal{R} \cap \mathcal{R}' = \varnothing$, it is clear that

$$\mathcal{R}_A \cap \mathcal{R}'_A = \varnothing.$$

Since $\mathcal{R}_A$ and $\mathcal{R}'_A$ arose from $\mathcal{R}$ and $\mathcal{R}'$, we know that no two k-spaces in the same set intersect. Thus both are still partial k-spreads. So we only need to show that they still cover the same set of points. If an affine point p is covered by $\mathcal{R}_A$, then this point (which is also a projective point) is also covered by $\mathcal{R}$ and, hence, by $\mathcal{R}'$. Since this point was affine, the corresponding k-space of $\mathcal{R}'$ is contained in $\mathcal{R}'_A$ and, hence, the point is covered by $\mathcal{R}'_A$. □

This lemma also shows that there exist conjugated switching sets in $\mathrm{AG}(n, q)$, since they exist in $\mathrm{PG}(n, q)$. This fact implies that Theorem 30 does not have a trivial assumption.

4 The association scheme of affine lines

Our goal in this section is that we want to investigate the association scheme of lines in $\mathrm{AG}(n, q)$. We start with repeating some definitions of association schemes. If the reader is not familiar with association schemes, we refer to [3, 16].

Definition 34. [3, Section 2.1] Let X be a finite set. A *d-class association scheme* is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_d\}$ is a set of binary symmetrical relations with the following properties:

1. $\{\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_d\}$ is a partition of $X \times X$.

2. $\mathcal{R}_0$ is the identity relation.

3. There exist constants p^l_{ij} such that for $x, y \in X$, with $(x, y) \in \mathcal{R}_l$, there are exactly p^l_{ij} elements z with $(x, z) \in \mathcal{R}_i$ and $(z, y) \in \mathcal{R}_j$. These constants are called the *intersection numbers* of the association scheme.

In such a d-class association scheme we can define adjacency matrices as follows.

Definition 35. Consider a d-class association scheme $(X, \mathcal{R})$ where $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_d\}$ and $X = \{x_1, \ldots, x_n\}$. Then we can define $d + 1$ matrices $B_0, \ldots, B_d$ of dimension $n \times n$, such that

$$(B_k)_{ij} = \begin{cases} 1, \text{ if } (x_i, x_j) \in \mathcal{R}_k \\ 0, \text{ if } (x_i, x_j) \notin \mathcal{R}_k. \end{cases}$$

These matrices are called the *adjacency matrices* of the association scheme.

An important property of these adjacency matrices is that they can be diagonalized simultaneously, so we obtain maximal common (right) eigenspaces $V_0, \dots, V_d$. It is also known that these adjacency matrices span a $(d+1)$-dimensional commutative $\mathbb{C}$-algebra $\mathcal{A}$. This algebra is called the *Bose-Mesner algebra*, which has a basis of idempotents $\{E_i \mid 0 \leqslant i \leqslant d\}$. One can prove that every matrix E_i is the orthogonal projection to the eigenspace V_i. If we would consider the common eigenspaces, we can denote all the eigenvalues in a matrix. This matrix is called the eigenvalue matrix.

Definition 36. Consider a d-class association scheme $(X, \mathcal{R})$ where $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_d\}$ and $X = \{x_1, \dots, x_n\}$. Let $B_0, \dots, B_d$ be the adjacency matrices and $\{E_i \mid 0 \leqslant i \leqslant d\}$ be the idempotent basis of the Bose-Mesner algebra. Then the *eigenvalue matrix* $P = [P_{ij}]$ and the *dual eigenvalue matrix* $Q = [Q_{ij}]$ are the matrices for which it holds that

$$B_j = \sum_{i=0}^{d} P_{ij} E_i \text{ and } E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} B_i.$$

Here $0 \leqslant i, j \leqslant d$.

Since every E_i in the idempotent basis gives an orthogonal projection onto V_i, it is indeed true that the values P_{ij} are the eigenvalues. Another important fact is that $PQ = nI_{d+1} = QP$.

We now give a well-known example of such an association scheme.

Example 37. [16, Example 1.1.2] Consider the set of lines in $\mathrm{PG}(n, q)$, with $n \geqslant 3$. Then this is a finite set, which we will call Π_1. Consider now the following set of relations $\mathcal{R}' = \{\mathcal{R}'_0, \mathcal{R}'_1, \mathcal{R}'_2\}$. Then for ℓ and ℓ' in Π_1, we have that

- $(\ell, \ell') \in \mathcal{R}'_0$ if $\ell = \ell'$.
- $(\ell, \ell') \in \mathcal{R}'_1$ if they meet in a point.
- $(\ell, \ell') \in \mathcal{R}'_2$ when they do not meet at all.

It is well-known that $\Delta' = (\Pi_1, \mathcal{R}')$ gives an association scheme. This concept can be generalized to k-spaces in $\mathrm{PG}(n, q)$.

We try to define a similar association scheme for lines in $\mathrm{AG}(n, q)$. Note that due to the fact that there exists a concept of infinity in $\mathrm{AG}(n, q)$, this will lead to an increase of relations. Here we see that relation $\mathcal{R}'_1$ will split into two separate relations.

Construction 38. Consider the set Φ_1 of lines of $\mathrm{AG}(n, q)$, with $n \geqslant 3$. Then we can define a *3-class association scheme* $\Delta = (\Phi_1, \mathcal{R})$, where we denote the following relations $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\}$ as follows. Pick $\ell, \ell' \in \Phi_1$, then

- $(\ell, \ell') \in \mathcal{R}_0$ if $\ell = \ell'$.
- $(\ell, \ell') \in \mathcal{R}_1$ if they meet in an affine point.

- $(\ell, \ell') \in \mathcal{R}_2$ if they meet in a point at infinity.
- $(\ell, \ell') \in \mathcal{R}_3$ when they do not meet in the corresponding projective space.

In order to prove that this is an association scheme, we can refer to [20, Chapter 4], where the intersection numbers were explicitly calculated. Another way to view this, is as a semilattice and conclude, due to [7], that Δ is indeed an association scheme.

Let us consider Δ. If we number the lines of $\mathrm{AG}(n, q)$ in a fixed order

$$\left\{\ell_i \mid i \in \left\{0, \ldots, \frac{q^{n-1}(q^n-1)}{(q-1)} - 1\right\}\right\},$$

then we can define the adjacency matrices as B_0, B_1, B_2 and B_3. We know that these are $\frac{q^{n-1}(q^n-1)}{(q-1)} \times \frac{q^{n-1}(q^n-1)}{(q-1)}$ matrices over $\mathbb{C}$ that have common (right) eigenspaces. If we define these common (right) eigenspaces by V_0, V_1, V_2 and V_3, then we know that $\mathbb{C}^{\Phi_1} = V_0 \perp V_1 \perp V_2 \perp V_3$. Consider now the Bose-Mesner algebra $\mathcal{A}$ of the association scheme Δ, which will be a 4-dimensional $\mathbb{C}$-algebra. Then we know that $\mathcal{A}$ has a basis of idempotents $\{E_i \mid 0 \leqslant i \leqslant 3\}$, such that every E_i is the orthogonal projection onto V_i.

4.1 Calculating the eigenvalue matrix and dual eigenvalue matrix of Δ

In order to find the eigenvalue matrix P and the dual eigenvalue matrix Q, we need to define some other matrices known as the intersection matrices.

Definition 39. Consider a d-class association scheme with intersection numbers p_{ij}^k. Then we can define the following $(d+1) \times (d+1)$ matrices for $i \in \{0, \ldots, d\}$

$$\mathcal{P}_i = [p_{ij}^k]_{k,j},$$

hence the (k, j)-entry is $\mathcal{P}_i(k, j) = p_{ij}^k$. These matrices are known as *intersection matrices.*

These intersection matrices for the association scheme of Construction 38 can be calculated:

$$\mathcal{P}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{P}_1 = \begin{pmatrix} 0 & q\left(\frac{q^n-1}{q-1} - 1\right) & 0 & 0 \\ 1 & (q-1)^2 + \left(\frac{q^n-1}{q-1} - 2\right) & q-1 & (q-1)\left(\frac{q^n-1}{q-1} - 1 - q\right) \\ 0 & q^2 & 0 & q\left(\frac{q^n-1}{q-1} - 1 - q\right) \\ 0 & q^2 & q & q\left(\frac{q^n-1}{q-1} - 1 - (q+1)\right) \end{pmatrix},$$

$$\mathcal{P}_2 = \begin{pmatrix} 0 & 0 & q^{n-1}-1 & 0 \\ 0 & q-1 & 0 & q^{n-1}-q \\ 1 & 0 & q^{n-1}-2 & 0 \\ 0 & q & 0 & q^{n-1}-1-q \end{pmatrix}$$

and

$$\mathcal{P}_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{q^2-(q+1)q^n+q^{2n-1}}{q-1} \\ 0 & -q^2+q^n & -q+q^{n-1} & \frac{q^3+q^2-\left(2\,q^2+q-1\right)q^{n-1}-q+q^{2n-1}}{q-1} \\ 0 & -\frac{q^3-q^{n+1}}{q-1} & 0 & \frac{q^3+q^2-(2\,q+1)q^n+q^{2n-1}}{q-1} \\ 1 & -\frac{q^3+q^2-q-q^{n+1}}{q-1} & -q-1+q^{n-1} & \frac{q^3+3\,q^2-\left(2\,q^2+2\,q-1\right)q^{n-1}-2\,q+q^{2n-1}}{q-1} \end{pmatrix}.$$

For these calculations we refer to [20, Chapter 4].

Lemma 40. *[3, page 45, Lemma 2.2.1] Consider a d-class association scheme together with the eigenvalue matrix P and the intersection matrices $\mathcal{P}_i$, for $i \in \{0, \dots, d\}$. Then*

$$P \cdot \mathcal{P}_i \cdot P^{-1} = \begin{pmatrix} P_{0i} & 0 & 0 & \dots & 0 \\ 0 & P_{1i} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P_{di} \end{pmatrix}.$$

Consequently, $\mathcal{P}_i$ and the adjacency matrix B_i have the same eigenvalues.

This lemma implies that the intersection matrices can be diagonalized simultaneously. In order to find P, we use the following theorem.

Theorem 41. *[3, Proposition 2.2.2] Consider a d-class association scheme and let u_i, for $i \in \{0, \dots, d\}$, be the set of common left normalized (column) eigenvectors of the intersection matrices. Here we mean with normalized, that $(u_i)_0 = 1$ for every i. Then the rows of the eigenvalue matrix P are the elements $(u_i)^T$.*

This lemma together with the following left (normalized) eigenvectors of the intersection matrices above, will give us the eigenvalue matrix.

$$u_0 = \left(1, -q + \frac{q^{n+1}-1}{q-1} - 1, q^{n-1}-1, q + q^n + \frac{q^{2n-1}-1}{q-1} - \frac{2\left(q^{n+1}-1\right)}{q-1} + 1\right)^T,$$

$$u_1 = \left(1, -q + \frac{q^n-1}{q-1} - 1, -1, q - \frac{q^n-1}{q-1} + 1\right)^T,$$

$$u_2 = (1, -q, -1, q)^T,$$

$$u_3 = \left(1, -q, q^{n-1}-1, q - q^{n-1}\right)^T.$$

These left eigenvectors were calculated by using Sage [24]. From this together with the lemma above, we can obtain the eigenvalue matrix P of the association scheme Δ, see Construction 38. So we conclude that

$$P = \begin{pmatrix} 1 & -\frac{q^2-q^{n+1}}{q-1} & q^{n-1}-1 & \frac{q^2-(q+1)q^n+q^{2n-1}}{q-1} \\ 1 & -\frac{q^2-q^n}{q-1} & -1 & \frac{q^2-q^n}{q-1} \\ 1 & -q & -1 & q \\ 1 & -q & q^{n-1}-1 & q-q^{n-1} \end{pmatrix} \tag{3}$$

and due to $PQ = q^{n-1}\left(\frac{q^n-1}{q-1}\right) I_4 = |\Phi_1| I_4 = QP$, we obtain that

$$Q = \begin{pmatrix} 1 & q^n-1 & -\frac{\left(q^2+1\right)q^n-q^2-q^{2n}}{q^2-q} & \frac{q^n-q}{q-1} \\ 1 & \frac{\left(q^2+1\right)q^n-q^2-q^{2n}}{q^2-q^{n+1}} & -\frac{\left(q^2+1\right)q^n-q^2-q^{2n}}{q^2-q^{n+1}} & -1 \\ 1 & \frac{q-q^{n+1}}{q^n-q} & \frac{\left(q^2+1\right)q^n-q^2-q^{2n}}{(q-1)q^n-q^2+q} & \frac{q^n-q}{q-1} \\ 1 & \frac{q-q^{n+1}}{q^n-q} & \frac{q^{n+1}-q}{q^n-q} & -1 \end{pmatrix}. \tag{4}$$

5 Cameron–Liebler line classes in $\mathrm{AG}(n, q)$

In this section we will give an alternative proof for the following statement, which is a special case of Theorem 30.

Theorem 42. *Suppose that $\mathcal{L}$ is a set of lines in* $\mathrm{AG}(n, q)$*, $n \geqslant 3$, such that for every line spread $\mathcal{S}$ it holds that*

$$|\mathcal{L} \cap \mathcal{S}| = x.$$

Then $\mathcal{L}$ is a Cameron–Liebler line class of parameter x.

To prove this theorem, we make use of the association scheme of Section 4. Let us recall Δ from Construction 38, then we first start with the concept of inner distributions.

5.1 Inner distribution

Definition 43. ([3, Section 2.5] and [21, Definition: Section 5, (10)]) Consider a d-class association scheme $(X, \mathcal{R})$ and let $\mathcal{L}$ be a subset of X, then we can consider its characteristic vector $\chi_{\mathcal{L}}$. For this vector we can define its *inner distribution* as the row vector $u = (u_0, u_1, u_2, \ldots, u_d)$ with elements in $\mathbb{R}$, for which it holds that

$$u_i = \frac{1}{|\mathcal{L}|}|\mathcal{R}_i \cap (\mathcal{L} \times \mathcal{L})|.$$

Remark 44. Note that for the inner distribution $u = (u_0, u_1, u_2, \ldots, u_d)$ of a certain characteristic vector $\chi_{\mathcal{L}}$, it holds that

$$u_i = \frac{1}{|\mathcal{L}|}\chi_{\mathcal{L}}^T \cdot B_i \cdot \chi_{\mathcal{L}},$$

for $0 \leqslant i \leqslant d$.

The following theorem will give us a way to observe in which eigenspaces of Δ a characteristic vector lies in.

Theorem 45. *([3, Lemma 2.5.1 and Proposition 2.5.2]) Consider a d-class association scheme $\Gamma = (X, \mathcal{R})$ and let $\mathcal{A}$ be its Bose-Mesner algebra. Denote the idempotent basis of $\mathcal{A}$ by $\{E_i \mid 0 \leqslant i \leqslant d\}$, with common eigenspaces $V_0, \ldots, V_d$. Then it follows for every subset $\mathcal{L}$ of X, that its characteristic vector $\chi_{\mathcal{L}} \in \mathbb{R}^d$ can be written as follows*

$$\chi_{\mathcal{L}} = a_0 v_0 + a_1 v_1 + \cdots + a_d v_d,$$

with $v_i \in V_i$ and $a_i \in \mathbb{R}$ for each $0 \leqslant i \leqslant d$. If u is the inner distribution of $\chi_{\mathcal{L}}$, then the following properties are equivalent for fixed $0 \leqslant i \leqslant d$

1. *$(u \cdot Q)_i = 0$, with Q the dual eigenvalue matrix of Γ.*

2. *$E_i \cdot \chi_{\mathcal{L}} = 0$.*

This last property implies that the projection of $\chi_{\mathcal{L}}$ onto the eigenspace V_i is zero, thus $a_i = 0$.

Now we mention the next very useful theorem stated in [7]. Our formulation is based on unpublished notes by Klaus Metsch.

Theorem 46. *[7, Theorem 6.8]Let $\Gamma = (X, \mathcal{R})$ be a d-class association scheme, with $\{E_i \mid 0 \leqslant i \leqslant d\}$ the idempotent basis of the Bose-Mesner algebra. Suppose G is a subgroup of $Aut(\Gamma)$ that acts transitively on X and whose orbits on $X \times X$ are the relations $\mathcal{R}_0, \ldots, \mathcal{R}_d$. Let χ and ψ be vectors of $\mathbb{R}^{|X|}$. Then the following two statements are equivalent.*

1. *For all $k \geqslant 1$, we have $E_k \cdot \chi = 0$ or $E_k \cdot \psi = 0$.*

2. *$\chi \cdot \psi^g$ is constant for all $g \in G$.*

Remark 47. We know that property (1) is equivalent with the fact that both vectors lie in opposite (common) eigenspaces besides V_0.

Remark 48. A second observation is that in the 3-class association scheme Δ, the group $\mathrm{AGL}(n, q)$ acts indeed transitively on pairs of lines of the same type in $\mathrm{AG}(n, q)$. It is also clear that elements of $\mathrm{AGL}(n, q)$ send line spreads of type I and type II to line spreads of type I and type II respectively.

The same happens for line spreads of type III, we explicitly proved this fact. But first we give a definition.

Definition 49. Let $\mathcal{S}$ be a line spread of type III, with the property that all the chosen points p_i in π_{n-2} are chosen differently. Then we call $\mathcal{S}$ a line spread of type III^+.

Lemma 50. *The affine collineation group $AGL(n, q)$ sends spreads of type III to spreads of type III. In particular, it sends spreads of type III^+ to spreads of type III^+.*

Proof. Consider $\mathcal{S}$ to be a line spread of type III, defined by an $(n-2)$-space $\pi_{n-2} \subseteq \pi_\infty$, the set of hyperplanes $H = \{\pi_i \mid i \in \{1, \ldots, q\}\}$ and the q points $p_i \in \pi_{n-2}$. If we now consider $\theta \in \mathrm{AGL}(n, q)$, then $\pi_{n-2}^\theta \subseteq \pi_\infty$ and all the hyperplanes of $H = \{\pi_i \mid i \in \{1, \ldots, q\}\}$ are sent to different hyperplanes through π_{n-2}^θ. Also all the points p_i are sent to points $p_i^\theta \in \pi_{n-2}^\theta$, which if they all are different points they shall remain so. We conclude that

$$\mathcal{S}^\theta = \{K^\theta \in \Phi_1 \mid p_i \in K \subseteq \pi_i \text{ for some } i\} = \{K' \in \Phi_1 \mid p_i^\theta \in K' \subseteq \pi_i^\theta \text{ for some } i\},$$

which is of the required form. ☐

5.2 About the common eigenspaces

In this section we give a basis for $V_0 \perp V_1$ and $V_0 \perp V_3$, and give a spanning set for $V_0 \perp V_2 \perp V_3$ in the association scheme Δ from Construction 38.

Definition 51. A point-pencil in $\mathrm{PG}(n, q)$ or $\mathrm{AG}(n, q)$ is the set of lines through a fixed point in $\mathrm{PG}(n, q)$ or $\mathrm{AG}(n, q)$ respectively.

Theorem 52. *([6, Theorem 9.5]) The point-line incidence matrix of* $\mathrm{AG}(n, q)$ *and* $\mathrm{PG}(n, q)$ *has full rank, which equals the number of points in* $\mathrm{AG}(n, q)$ *and* $\mathrm{PG}(n, q)$ *respectively. Hence the rows of these incidence matrices, which correspond to points and give point-pencils are linearly independent.*

Lemma 53. *[3, Lemma 2.2.1 (ii)] Consider the dual eigenvalue matrix Q in an association scheme, then $Q_{0i} = \dim(V_i)$.*

We now prove the following theorem that characterizes the space $V_0 \perp V_1$.

Theorem 54. *Consider the affine space $AG(n, q)$ and the 3-class association scheme Δ (see Construction 38). Then the point-pencils form a basis of the space $V_0 \perp V_1$.*

Proof. Let us first find the inner distribution of a point-pencil. It can be seen that this is equal to

$$u = \left(1, \frac{q^n - q}{q - 1}, 0, 0\right).$$

Thus we obtain that

$$u \cdot Q = \left(\frac{q^n - 1}{q - 1}, -\frac{(q + 1)q^{n-1} - 1 - q^{2n-1}}{q - 1}, 0, 0\right).$$

Hence these first two entries will never be zero for $n > 1$ and q a prime power. So Theorem 45 shows that all the point-pencils lie inside $V_0 \perp V_1$.

From Lemma 53 and the description of Q in Equation (4), we obtain that $\dim(V_0 \perp V_1) = 1 + (q^n - 1) = q^n$. This number is equal to the number of point-pencils in $\mathrm{AG}(n, q)$. Together with Lemma 52, we have that the point-pencils form a basis for the space $V_0 \perp V_1$. ☐

We now give a second result on these eigenspaces.

Lemma 55. *In the affine space $AG(n,q)$ with association scheme Δ (see Construction 38), we have the following:*

1. *The line spreads of type II form a basis for the space $V_0 \perp V_3$.*
2. *The space $V_0 \perp V_2 \perp V_3$ is spanned by line spreads of type III$^+$ and for the characteristic vector $\chi_{\mathcal{S}}$ of such a line spread $\mathcal{S}$, it holds that $E_2 \cdot \chi_{\mathcal{S}} \neq 0 \neq E_3 \cdot \chi_{\mathcal{S}}$.*

Proof.

1. This is done in a similar way as the previous lemma. The inner distribution of a line spread $\mathcal{S}_1$ of type II is equal to
$$s_1 = \left(1,\, 0,\, q^{n-1}-1,\, 0\right).$$
From this we obtain that
$$s_1 \cdot Q = \left(q^{n-1},\, 0,\, 0,\, \frac{q^{2n-1}-q^n}{q-1}\right).$$
The first and last entry will never be zero for $n>1$ and q a prime power. So from Theorem 45, we obtain that $\chi_{\mathcal{S}_1} \in V_0 \perp V_3$. Note that these line spreads are in fact subsets of point-pencils in the hyperplane at infinity in $\mathrm{PG}(n,q)$. But due to the fact that no two subsets contain the same line, we know that these line spreads are also linearly independent. From Lemma 53 and the description of Q in Equation (4), we obtain that
$$\dim(V_0 \perp V_3) = 1 + \frac{q^n-q}{q-1} = \frac{q^n-1}{q-1}.$$
This dimension is equal to the number of spreads of type II, which proves that these line spreads form a basis.
2. Analogously for a line spread $\mathcal{S}_2$ of type III$^+$. The inner distribution is equal to
$$s_2 = \left(1,\, 0,\, q^{n-2}-1,\, q^{n-1}-q^{n-2}\right),$$
such that
$$s_2 \cdot Q = \left(q^{n-1},\, 0,\, q^{2n-2}-q^{n-2},\, -\frac{(q^2-q+1)q^{n-2}-q^{2n-2}}{q-1}\right).$$
The first and third entry will never be zero for $n>1$ and q a prime power. The last entry needs some arguments. If $(q^2-q+1)q^{n-2}-q^{2n-2}=0$, then $q=0$ or $q(q-1)=q^n-1$ and thus $q=0$ or $q^{n-1}+\cdots+q^2+1=0$. This statement is never true if $n>1$ and q a prime power. Hence, using Theorem 45, we obtain that $\chi_{\mathcal{S}_2} \in V_0 \perp V_2 \perp V_3$ and especially we have that $E_2 \cdot \chi_{\mathcal{S}_2} \neq 0 \neq E_3 \cdot \chi_{\mathcal{S}_2}$.

To show that $V_0 \perp V_2 \perp V_3$ is spanned by line spreads of type III$^+$, we use Theorem 46. Suppose that these line spreads would span $V_0 \perp W_1$, with $V_2 \perp V_3 = W_1 \perp U_1$, then we want to show that $W_1 = V_2 \perp V_3$. If there exists a $\psi \in U_1 \setminus \{0\}$, then we know that $E_2 \cdot \psi \neq 0$ or $E_3 \cdot \psi \neq 0$. Let us now consider a line spread $\mathcal{S}$ of type III$^+$, then we know that its characteristic vector $\chi_{\mathcal{S}} \in V_0 \perp W_1 \subseteq V_0 \perp V_2 \perp V_3$. Hence $\chi_{\mathcal{S}}$ lies in the complementary space $V_0 \perp W_1$ of U_1, thus $\chi_{\mathcal{S}} \cdot \psi = 0$. Due to Lemma 50, we have that for every $\theta \in \mathrm{AGL}(n, q)$ it holds that $\chi_{\mathcal{S}^\theta} \cdot \psi = 0$. So from Theorem 46, we obtain that $E_2 \cdot \chi_{\mathcal{S}} = 0$ or $E_3 \cdot \chi_{\mathcal{S}} = 0$. This is a contradiction with the end of the preceding paragraph. □

5.3 The proof of Theorem 42

Proof. Consider the association scheme Δ from Construction 38 and let $\mathcal{L}$ be a line set in $\mathrm{AG}(n, q)$ such that for every line spread $\mathcal{S}$ it holds that $|\mathcal{L} \cap \mathcal{S}| = x$. Then our goal is to prove that $\chi_{\mathcal{L}} \in V_0 \perp V_1$, since, from Theorem 54, it then follows that $\chi_{\mathcal{L}} \in \mathrm{Im}(A_n^T)$ and hence $\mathcal{L}$ is a Cameron–Liebler line class of parameter x.

Consider $\mathcal{S}$ to be a line spread of type III$^+$. Such a line spread exists if we can choose q different points in π_{n-2}. This is clearly the case if $n \geqslant 3$. If we denote the characteristic vector of $\mathcal{S}$ by $\chi_{\mathcal{S}}$, we know by the definition of $\mathcal{L}$ that

$$\chi_{\mathcal{L}} \cdot \chi_{\mathcal{S}} = x.$$

In combination with Lemma 50, we know that

$$\chi_{\mathcal{L}} \cdot \chi_{\mathcal{S}^\theta} = x,$$

for all $\theta \in \mathrm{AGL}(n, q)$. Hence, from Theorem 46 and Lemma 55 (Property (2)) which states that $E_2 \cdot \chi_{\mathcal{S}} \neq 0$ and that $E_3 \cdot \chi_{\mathcal{S}} \neq 0$, we may conclude that $E_2 \cdot \chi_{\mathcal{L}} = 0 = E_3 \cdot \chi_{\mathcal{L}}$. Thus using Theorem 45, we obtain that

$$\chi_{\mathcal{L}} \in V_0 \perp V_1.$$

This proves the theorem. □

6 Classification results

In this section we will focus on some classification results of Cameron–Liebler k-sets in $\mathrm{AG}(n, q)$ with certain parameters. From now on we will use all the equivalent definitions of Theorem 30 to describe Cameron–Liebler k-sets. In order to obtain a classification result, we will need the following result.

Theorem 56. *[23, Theorem 3]*

Let $0 \leqslant t \leqslant k$ be positive integers. Let $\mathcal{S}$ be a set of k-spaces in $\mathrm{PG}(n, q)$, pairwise intersecting in at least a t-space. If $n \geqslant 2k + 1$, then

$$|\mathcal{S}| \leqslant \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q .$$

Equality holds if and only if $\mathcal{S}$ is the set of all k-spaces through a fixed t-space, or $n = 2k+1$ and $\mathcal{S}$ is the set of all k-spaces inside a fixed $(2k-t)$-space.

Before we give the classification results, the reader should keep Example 20 in mind, where we gave some examples of Cameron–Liebler k-sets in $\mathrm{PG}(n,q)$. Note that by restriction to $\mathrm{AG}(n,q)$ we actually obtain fewer examples or stronger conditions on Cameron–Liebler k-sets. We first give the following lemma.

Lemma 57. *A non-empty set of k-spaces contained in a hyperplane of* $\mathrm{AG}(n,q)$*, is not a Cameron–Liebler k-set in AG(n,q).*

Proof. Let $\mathcal{L}$ be a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$, that consists out of a set of k-spaces inside a hyperplane π. Pick a k-space $K \in \mathcal{L}$, which we can consider in the projective closure $\mathrm{PG}(n,q)$. Then we can define a type II k-spread $\mathcal{S}_1$ as the set of affine k-spaces through $K \cap \pi_\infty$. Analogously we can define $\mathcal{S}_2$ as the set of affine k-spaces through another $(k-1)$-space at infinity that does not lie in π. It is clear that

$$|\mathcal{L} \cap \mathcal{S}_1| \neq |\mathcal{L} \cap \mathcal{S}_2| = 0.$$

This is a contradiction with Lemma 14. □

This lemma gives the following classification result.

Theorem 58. *[13, Theorem 4.1] Let $q \in \{2,3,4,5\}$. Then all Cameron–Liebler k-sets in* $\mathrm{AG}(n,q)$ *consist out of all the k-spaces through a fixed point, if $k+1, n-k \geqslant 2$ and either (a) $n \geqslant 5$ or (b) $n = 4$ and $q = 2$.*

Proof. Here we use the combination of Theorem 2, Theorem 21 and Lemma 57. □

6.1 Cameron–Liebler k-sets with parameter $x = 1$ in $\mathrm{AG}(n,q)$

Example 59. Consider $\mathcal{L}$ as the set of k-spaces through a fixed affine point in $\mathrm{AG}(n,q)$. Then $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$ of parameter $x = 1$. This can be seen from Corollary 1 together with the fact that $\mathcal{L}$ also is a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$ of parameter $x = 1$.

Using Theorem 22, we know that for $n > 2k+1$ this example is the only example of a Cameron–Liebler k-set of parameter $x = 1$ in $\mathrm{PG}(n,q)$. If $n = 2k+1$, the set of all k-spaces in a hyperplane also gives an example of a Cameron–Liebler k-set with parameter $x = 1$. This fact gives the following theorem. The following theorem also proves the first part of Theorem 6 from the introduction.

Theorem 60. *Consider the affine space* $\mathrm{AG}(n,q)$ *and let $\mathcal{L}$ be a Cameron–Liebler k-set with parameter $x = 1$ in this affine space. If also $n \geqslant 2k+1$, then $\mathcal{L}$ consists of all the k-spaces through an affine point.*

Proof. Using Theorem 2, we obtain that every Cameron–Liebler k-set in $\mathrm{AG}(n,q)$ is a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$. The latter will have the same parameter $x = 1$. From Theorem 22 and Theorem 57, the assertion follows. □

We also will be able to improve this result for $n \geqslant k+2$, see Corollary 73.

6.2 Cameron–Liebler line classes of parameter $x = 2$ in $\mathrm{AG}(n, q)$

In this section, our goal will be to exclude the parameter $x = 2$ for Cameron–Liebler line classes in $\mathrm{AG}(n, q)$, with $n \geqslant 3$. To do this, we will need the following lemma.

Lemma 61. *Consider an affine Cameron–Liebler line class $\mathcal{L}$ with parameter $x = 2$ in $\mathrm{AG}(n, q)$, with $n \geqslant 4$. Then for every two points p_1 and p_2 in π_∞, there are two lines of $\mathcal{L}$ through each of them. These 4 lines generate at most a 3-space.*

Proof. Denote the lines of $\mathcal{L}$ through p_1 by ℓ_1 and ℓ_2, and denote the lines of $\mathcal{L}$ through p_2 by r_1 and r_2. We start by considering $\langle \ell_1, \ell_2, r_1 \rangle$, then we know that $\dim(\langle \ell_1, \ell_2, r_1 \rangle) \in$

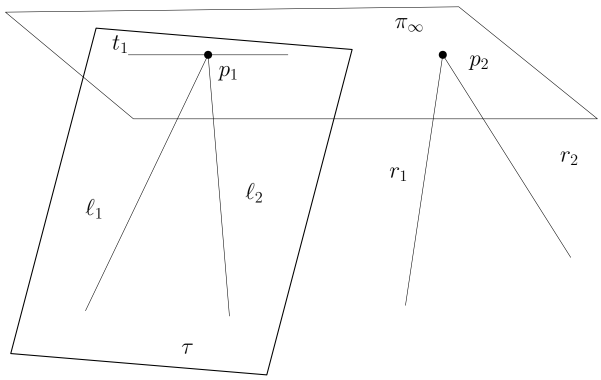

Figure 1: Sketch for the proof of Lemma 61

$\{2, 3, 4\}$. Suppose now that $\dim(\langle \ell_1, \ell_2, r_1 \rangle) = 4$. If we now call the plane $\langle \ell_1, \ell_2 \rangle = \tau$, then we know that $\dim(\tau \cap \pi_\infty) = 1$. This intersection line we call t_1, see Figure 1. Now we can find a $(n-2)$-space $\widetilde{\pi} \subseteq \pi_\infty$ such that

$$\langle t_1, p_2 \rangle \subseteq \widetilde{\pi}$$

and

$$\langle \ell_1, \ell_2, r_1 \rangle \cap \pi_\infty \nsubseteq \widetilde{\pi}.$$

This is possible, since first $\langle \ell_1, \ell_2, r_1 \rangle \cap \pi_\infty$ is a 3-dimensional space that contains $\langle t_1, p_2 \rangle$ as a 2-dimensional subspace. And, secondly, since $n \geqslant 4$, we know that $n - 2 \geqslant 2$.

Thus with the identity of Grassmann, we obtain that

$$\dim(\langle \widetilde{\pi}, \tau \rangle) = n - 2 + 2 - \dim(\widetilde{\pi} \cap \tau) = n - 1,$$

$$\dim(\langle \widetilde{\pi}, r_1 \rangle) = n - 2 + 1 - \dim(\widetilde{\pi} \cap r_1) = n - 1$$

and

$$\begin{aligned} \dim(\langle \ell_1, \ell_2, r_1, \widetilde{\pi} \rangle) &= \dim(\langle \ell_1, \ell_2, r_1 \rangle) + \dim(\widetilde{\pi}) - \dim(\langle \ell_1, \ell_2, r_1 \rangle \cap \widetilde{\pi}) \\ &= \dim(\langle \ell_1, \ell_2, r_1 \rangle) + \dim(\widetilde{\pi}) - \dim((\langle \ell_1, \ell_2, r_1 \rangle \cap \pi_\infty) \cap \widetilde{\pi}) \\ &= 4 + (n-2) - 2 = n. \end{aligned}$$

Thus we can conclude that $\langle \widetilde{\pi}, \ell_1, \ell_2 \rangle \neq \langle \widetilde{\pi}, r_1 \rangle$. So we can define a line spread $\mathcal{S}$ of $\mathrm{AG}(n, q)$ (of type† III) that contains ℓ_1, ℓ_2 and r_1, such that $|\mathcal{L} \cap \mathcal{S}| \geqslant 3$, which is a contradiction. Thus from this we can conclude that $\dim(\langle \ell_1, \ell_2, r_1 \rangle) \leqslant 3$. Analogously, we can obtain that $\langle \ell_1, \ell_2, r_2 \rangle$ and in general every space generated by three of these four lines is at most a 3-dimensional space. To show that these four lines span at most a 3-space, we need to consider some cases.

1. First if $p_2 \notin t_1$, then $\langle \ell_1, \ell_2, r_2 \rangle$ intersects $\langle \ell_1, \ell_2, r_1 \rangle$ in at least the point p_2 and the plane τ. So, since $p_2 \notin \tau$, both 3-spaces are the same. Hence, from now on, we assume that $p_2 \in t_1$.

2. If r_1 and/or r_2 are contained in τ, we are done, since these four lines span a plane or a 3-space.

3. If r_1 and r_2 are not contained in τ and $\langle r_1, r_2 \rangle \cap \tau = t_1$, then again we can conclude that all four lines lie in a 3-space.

4. If r_1 and r_2 are not contained in τ and $\langle r_1, r_2 \rangle \cap \tau \neq t_1$. Then $\langle r_1, r_2 \rangle \cap \pi_\infty \cap t_1 = p_2$ and we analogously obtain from previous cases that $\langle r_1, r_2, \ell_2 \rangle$ and $\langle r_1, r_2, \ell_1 \rangle$ are two 3-spaces that now contain $p_1 \notin \langle r_1, r_2 \rangle$ and $\langle r_1, r_2 \rangle$.

This proves the lemma. □

Let us now state the following known theorem.

Theorem 62 (Folklore)**.** *Consider a set of k-spaces $\mathcal{E}$ in $\mathrm{PG}(n, q)$, $1 \leqslant k \leqslant n-1$, such that every two k-spaces intersect in a $(k-1)$-space. Then $\mathcal{E}$ consists out of a subset of all the k-spaces through a fixed $(k-1)$-space or all the k-spaces inside a $(k+1)$-space.*

We are ready to state the main theorem. This theorem proves a second part of Theorem 6 in the introduction.

Theorem 63. *There does not exist a Cameron–Liebler line class $\mathcal{L}$ of parameter $x = 2$ in $\mathrm{AG}(n, q)$, $n \geqslant 3$.*

Proof. The case for $n = 3$ is proven in [9, Corollary 4.5], so we may suppose that $n \geqslant 4$. Suppose there exists a Cameron–Liebler line class $\mathcal{L}$ of parameter $x = 2$. Then we can define $\mathcal{E}$ as the set of planes, such that each plane is defined by a point at infinity and the two corresponding lines of $\mathcal{L}$ through this point. Due to Lemma 61 we know that these planes pairwise intersect in a line or coincide. Using Theorem 62, we can conclude that $\mathcal{E}$ consists out of a subset of all the planes through a fixed line or all the planes in a 3-space σ. If $\mathcal{E}$ would consist out of all the planes in a 3-space σ, then $\mathcal{L}$ is a set of lines inside σ and thus inside a certain hyperplane. This is a contradiction with Lemma 57. So we conclude that $\mathcal{E}$ consists out of planes through a fixed line ℓ.

If ℓ would be a line at infinity then $|\mathcal{L}| = 2(q+1)$, since every point at infinity belongs to two lines of $\mathcal{L}$. Note that for $n \geqslant 3$ this number is strictly smaller than the size of a

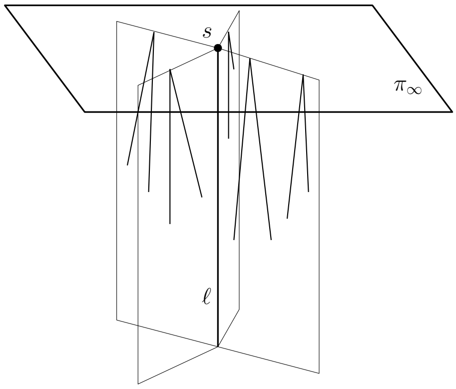

Figure 2: The last possible option in $\mathrm{AG}(n, q)$.

Cameron–Liebler line class in $\mathrm{AG}(n, q)$ of parameter $x = 2$, which has size $2\frac{q^n-1}{q-1}$. So the line ℓ should be affine. See Figure 2.

Note that $|\mathcal{E}| = \frac{q^{n-1}-1}{q-1}$, since every point at infinity will define exactly one plane in $\mathcal{E}$ by definition. Hence this is also the number of all planes through a line, such that we know that $\mathcal{E}$ consists out of all the planes through ℓ. Let us denote $s = \ell \cap \pi_\infty$. Now we can use Theorem 30, where we have shown that being an affine Cameron–Liebler line class in $\mathrm{AG}(n, q)$ is equivalent with the following statement. For every affine line ℓ_1, the number of affine lines in $\mathcal{L}$ disjoint to ℓ_1 in $\mathrm{AG}(n, q)$ is equal to

$$\left(q^2 \frac{q^{n-2}-1}{q-1} + 1\right)(x - \chi_{\mathcal{L}}(\ell_1)). \tag{5}$$

Consider now an affine line ℓ' through s that is contained in $\mathcal{L}$ and not equal to the intersection line ℓ. Note that this is always possible, since s belongs to exactly two lines of $\mathcal{L}$. Then all lines of $\mathcal{L}$ except those in the plane $\langle \ell, \ell' \rangle$, are disjoint to ℓ'. Since every other plane through ℓ has $2q$ lines of $\mathcal{L}$ skew to ℓ' and we also need to count the other line of $\mathcal{L}$ through s, this number is equal to

$$(|\mathcal{E}| - 1) \cdot 2q + 1.$$

With some calculations, we find that this equals

$$2q^2 \left(\frac{q^{n-2}-1}{q-1}\right) + 1.$$

This number should be equal to Equation (5). In this equation we fill in $\chi_{\mathcal{L}}(\ell') = 1$, and we obtain that there should be $\left(q^2 \frac{q^{n-2}-1}{q-1} + 1\right)(2-1)$ lines disjoint to $\ell' \in \mathcal{L}$. These two numbers are not equal. So there does not exist Cameron–Liebler line classes with parameter $x = 2$ in $\mathrm{AG}(n, q)$, $n \geqslant 4$ either. ☐

[†]This exists due to $n \geqslant 4$.

Remark 64. In contrast with the case for $x = 1$, we have proven the preceding theorem without using Theorem 2. The reason for this choice was that we are not able to deduce the preceding theorem from results in $\mathrm{PG}(n, q)$. Reducing to the projective case and using Theorem 21, we observe that there do not exist Cameron–Liebler k-sets of parameter $x = 2$ in $\mathrm{AG}(n, q)$ if $q \in \{2, 3, 4, 5\}$ together with $n \geqslant 5$ or $n = 4$, but $q = 2$.

While using Theorem 3, we obtain a non-existence result for $n \geqslant 5$ and $q \geqslant 3$. Both statements combined are weaker than the preceding theorem. This seems a small difference, but for general k the results are even stronger. This will be proven in Corollary 72.

6.3 Characterisation of parameter x of Cameron–Liebler k-sets in $\mathrm{AG}(n, q)$

Our goal here is to prove that there do not exist Cameron–Liebler k-sets in $\mathrm{AG}(n, q)$ of parameter $x = 2$, with $n \geqslant k + 2$. As we already briefly discussed, we can not deduce this result so far by using Theorem 2 and thus reducing to the projective case. If we would reduce to $\mathrm{PG}(n, q)$ and use Theorem 21 and Theorem 3, we would obtain a non-existence result for (1) $n \geqslant 3k+2$ and $q \geqslant 3$, (2) $n \geqslant 5$ with $q \in \{2, 3, 4, 5\}$ and (3) $n = 4$ with $q = 2$. Note that this statement is significantly weaker than the previous claim. In achieving this goal we will also obtain a minor non-existence condition on the parameters of certain Cameron–Liebler k-sets in $\mathrm{AG}(n, q)$. We will do so by connecting every Cameron–Liebler k-set to a Cameron–Liebler line class of the same parameter. For this we will need the following observation.

Lemma 65. *Let $\mathcal{L}$ be a Cameron–Liebler k-set with parameter x in* $\mathrm{AG}(n, q)$*. Then the number of elements of $\mathcal{L}$ through a fixed i-space at infinity, for $-1 \leqslant i \leqslant k - 2$, is equal to*

$$\begin{bmatrix} n-i-1 \\ k-i-1 \end{bmatrix}_q x.$$

Proof. Consider an i-space I at infinity. Then we can count all the elements of $\mathcal{L}$ through I, by counting the number of $(k-1)$-spaces through I inside π_∞ and multiplying this by the number of elements of $\mathcal{L}$ through each $(k-1)$-space. Both numbers are known, since through every $(k-1)$-space at infinity there are, by Lemma 14, in total x elements of $\mathcal{L}$. The assertion follows □

A remarkable observation is that $\begin{bmatrix} n-i-1 \\ k-i-1 \end{bmatrix}_q x$ equals the size of a Cameron–Liebler $(k-i-1)$-set in $\mathrm{AG}(n-i-1, q)$. This observation will lead to the following construction.

Construction 66. Consider $\mathcal{L}$ to be a Cameron–Liebler k-set in $\mathrm{AG}(n, q)$ of parameter x, and pick an i-space I at infinity, for $0 \leqslant i \leqslant k - 2$ and $n \geqslant k + 2$. Then, by Lemma 65, there are

$$\begin{bmatrix} n-i-1 \\ k-i-1 \end{bmatrix}_q x$$

k-spaces of $\mathcal{L}$ that contain I. Pick now an $(n-i-1)$-space π in $\mathrm{PG}(n, q)$ skew to I. Then every k-space of $\mathcal{L}$ through I will intersect π in a $(k-i-1)$-space in $\mathrm{PG}(n, q)$, see

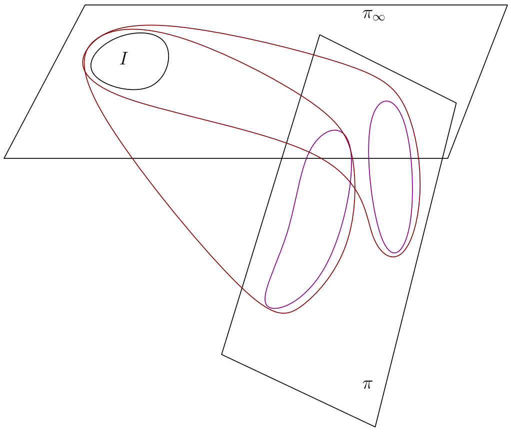

Figure 3: The red elements are k-spaces in $\mathcal{L}$ through I and we get the purple $(k-i-1)$-spaces after the projection onto the $(n-i-1)$-space π.

Figure 5. We shall denote this set of $(k-i-1)$-spaces by $\mathcal{J}$. Remark that π is in fact a projective space of dimension $n-i-1$, which is an affine space π_A ($\simeq \mathrm{AG}(n-i-1,q)$) of the same dimension with hyperplane at infinity equal to $\pi \cap \pi_\infty$.

We first give a result on k-spreads.

Lemma 67. *Consider Construction 66. Then every $(k-i-1)$-spread in π_A can be extended to an affine k-spread in $\mathrm{AG}(n,q)$, such that all the k-spaces in this k-spread contain I.*

Proof. Let $\mathcal{S}'$ be a $(k-i-1)$-spread in π_A, then

$$\mathcal{S} := \{\langle I, N\rangle \mid N \in \mathcal{S}'\}$$

is a k-spread in $\mathrm{AG}(n,q)$. □

Theorem 68. *Suppose that $n \geqslant 2k-i$. Consider Construction 66. Then the set $\mathcal{J}$ is a Cameron–Liebler $(k-i-1)$-spaces in π_A ($\simeq \mathrm{AG}(n-i-1,q)$), which has the same parameter x.*

Proof. Consider $\mathcal{L}$ and $\mathcal{J}$ as in Construction 66, then we need to prove that $\mathcal{J}$ is a Cameron–Liebler $(k-i-1)$-set with the same parameter x in π_A. We know that, due to Lemma 67, every $(k-i-1)$-spread $\mathcal{S}'$ in π_A can be extended to a k-spread $\mathcal{S}$ in $\mathrm{AG}(n,q)$ such that every element of $\mathcal{S}$ contains I. Since $\mathcal{L}$ is a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$, it holds that $|\mathcal{L} \cap \mathcal{S}| = x$. But since every k-space of $\mathcal{L} \cap \mathcal{S}$ contains I, it projects to an element of $\mathcal{J} \cap \mathcal{S}'$ and vice versa. So we have that $|\mathcal{J} \cap \mathcal{S}'| = x$. Using Theorem 30, which gives the condition on n, we have proven the theorem. □

Remark 69. Note that this construction cannot be done in a similar way for $\mathrm{PG}(n,q)$.

We now have found a way to reduce Cameron–Liebler k-sets to Cameron–Liebler line classes of the same parameter x. Hence, this will lead to a transfer of non-existence results.

Theorem 70. *Let $\mathcal{L}$ be a Cameron–Liebler k-set in $AG(n, q)$, with $n \geqslant k+2$. Suppose now that $\mathcal{L}$ has parameter x, then x satisfies every condition which holds for Cameron–Liebler line classes in $AG(n-k+1, q)$.*

Proof. We can use Theorem 68 for $i = k-2$, thus $n \geqslant k+2$, and obtain that there exists a Cameron–Liebler line class in $\mathrm{AG}(n-(k-2)-1, q)$ with the same parameter x. This proves the theorem. □

This theorem has the following consequences.

Theorem 71. *Suppose that $\mathcal{L}$ is a Cameron–Liebler $(n-2)$-set with parameter x of $\mathrm{AG}(n, q)$, then it holds that*

$$\frac{x(x-1)}{2} \equiv 0 \mod (q+1).$$

Proof. Use Theorem 70 for $n-2=k$ and the modular equality from [9, Corollary 4.3] or [15, Theorem 1.1]. □

The following corollary completes the proof of $x = 2$ in Theorem 6 in the introduction.

Corollary 72. *There do not exist Cameron–Liebler k-sets in $\mathrm{AG}(n, q)$ with parameter $x = 2$, with $n \geqslant k+2$.*

Proof. If there would exist a Cameron–Liebler k-set $\mathcal{L}$ of parameter $x = 2$ in $\mathrm{AG}(n, q)$, we can use Theorem 70 and obtain that there exists a Cameron–Liebler line class in $\mathrm{AG}(n-k+1, q)$ with parameter $x = 2$. Since $n-k+1 \geqslant 3$, we may use Theorem 63 to obtain a contradiction. □

We also have the following improvement of Theorem 60 To conclude Theorem 6.

Corollary 73. *The only Cameron–Liebler k-set $\mathcal{L}$ of parameter $x = 1$ in $\mathrm{AG}(n, q)$, with $n \geqslant k+2$, consists of the set of k-spaces through a fixed point.*

Proof. Again we use Theorem 70 to obtain a Cameron–Liebler line class $\mathcal{L}'$ of parameter $x = 1$ in $\mathrm{AG}(n-k+1, q)$. But combined this with Theorem 60 for $k = 1$ in $\mathrm{AG}(n-k+1, q)$, we obtain that $\mathcal{L}'$ consists out of all lines through a fixed point. Using Construction 66, it is easy to see that $\mathcal{L}$ is the set of all k-spaces through a point. □

7 Cameron–Liebler sets of hyperplanes in AG(n, q)

In this section we study Cameron–Liebler sets of hyperplanes in $\mathrm{AG}(n, q)$. We will be able to give a complete classification. This will be done by giving a classification of affine $(n-1)$-spreads.

Lemma 74. *The only $(n-1)$-spreads in $\mathrm{AG}(n, q)$ are spreads of type II.*

Proof. Let $\mathcal{S}$ be an $(n-1)$-spread, then we need to prove that $\mathcal{L}$ is of type II. Consider now the projective closure $\mathrm{PG}(n,q)$, then we know that every two hyperplanes of $\mathcal{S}$ will intersect in an $(n-2)$-space. Due to the fact that $\mathcal{S}$ is an affine $(n-1)$-spread, these intersections must lie at infinity. But since every affine hyperplane only has an $(n-2)$-dimensional intersection with infinity, all these $(n-2)$-spaces need to be the same. Hence, we have that $\mathcal{S}$ is of type II. □

The fact that we are able to classify all $(n-1)$-spreads in $\mathrm{AG}(n,q)$, gives us information how we can construct these Cameron–Liebler sets of hyperplanes in $\mathrm{AG}(n,q)$.

Theorem 75. *Let $\mathcal{L}$ be a set of affine hyperplanes in $\mathrm{AG}(n,q)$ and consider the projective closure $\mathrm{PG}(n,q)$. Then $\mathcal{L}$ is a Cameron–Liebler set of hyperplanes of parameter x if and only if $\mathcal{L}$ is a set of hyperplanes such that through every $(n-2)$-space at infinity we have chosen x arbitrary hyperplanes.*

Proof. The proof is similar as we have done for lines in Section 5. Hence, we find a 2-class association scheme $\Delta = (\Phi_{n-1}, \mathcal{R})$, with $\mathcal{R} := \{\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2\}$. Here $\mathcal{R}_0$ is the identical relation, while $\mathcal{R}_1$ is the relation that denotes that the two hyperplanes are disjoint and thus intersect in an $(n-2)$-space at infinity and $\mathcal{R}_2$ is the relation that denotes intersection. Using similar techniques as in Section 5, we obtain that

$$P = \begin{pmatrix} 1 & q-1 & \frac{q^{n+1}-1}{q-1} \\ 1 & q-1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & \frac{q^n-q}{q-1} & q^n-1 \\ 1 & \frac{q^n-q}{q-1} & -\frac{q^n-1}{q-1} \\ 1 & -1 & 0 \end{pmatrix}.$$

Due to Lemma 74, we have that the inner distribution of an $(n-2)$-spread is equal to the vector $v = (1, q-1, 0)$. Also point-pencils have inner distribution $w = (1, 0, \frac{q^n-1}{q-1} - 1)$. Using Theorem 45, we know that the $(n-2)$-spreads lie inside $V_0 \perp V_1$ and the point-pencils lie inside $V_0 \perp V_2$, in fact both sets will span these spaces. Combining this with Theorem 46, we find that a Cameron–Liebler $(n-2)$-set can be characterized by the constant intersection of $(n-2)$-spreads. Hence, due to Lemma 74, the assertion follows. □

8 Future research

We want to end this paper with some suggestions for further research. One could also attempt to classify more parameters of a Cameron–Liebler k-set in $\mathrm{AG}(n,q)$, since intuitively this will be less difficult than $\mathrm{PG}(n,q)$. We remind the reader of Theorem 2, which states that Cameron–Liebler k-sets in $\mathrm{AG}(n,q)$ are special cases of Cameron–Liebler k-sets in $\mathrm{PG}(n,q)$. Another interesting problem is to look for examples of Cameron–Liebler k-sets in $\mathrm{AG}(n,q)$. Note that it would be enough to find a Cameron–Liebler k-set in $\mathrm{PG}(n,q)$ that does not contain any k-spaces inside a hyperplane, see Theorem 1.

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