Characterizations of the $G_2(4)$ and $L_3(4)$ near octagons

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Abstract

A triple $(\mathcal{S}, S, \mathcal{Q})$ consisting of a near polygon $\mathcal{S}$, a line spread S of $\mathcal{S}$ and a set $\mathcal{Q}$ of quads of $\mathcal{S}$ is called a *polygonal triple* if certain nice properties are satisfied, among which there is the requirement that the point-line geometry $\mathcal{S}'$ formed by the lines of S and the quads of $\mathcal{Q}$ is itself also a near polygon. This paper addresses the problem of classifying all near polygons $\mathcal{S}$ that admit a polygonal triple $(\mathcal{S}, S, \mathcal{Q})$ for which a given generalized polygon $\mathcal{S}'$ is the associated near polygon. We obtain several nonexistence results and show that the $G_2(4)$ and $L_3(4)$ near octagons are the unique near octagons that admit polygonal triples whose quads are isomorphic to the generalized quadrangle $W(2)$ and whose associated near polygons are respectively isomorphic to the dual split Cayley hexagon $H^D(4)$ and the unique generalized hexagon of order $(4, 1)$.

Mathematics Subject Classifications: 05B25, 51E12, 51E25

1 Introduction

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$. We denote by Γ the *collinearity graph* of $\mathcal{S}$, that is the graph whose vertices are the points of $\mathcal{S}$ with two distinct points being adjacent whenever they are collinear, i.e. incident with the same line. Then $\mathcal{S}$ is called a *near polygon* if Γ has finite diameter, if every two distinct points are incident with at most one line and if for every point x and every line L, there exists a unique point $\pi_L(x)$ on L nearest to x with respect to the distance function in Γ. If $d \in \mathbb{N}$ is the diameter of Γ, then the near polygon is moreover called a *near* $2d$*-gon*.

In [2] and [3], Anurag Bishnoi and the author constructed two new near octagons related to the respective simple groups $G_2(4)$ and $L_3(4)$. These so-called $G_2(4)$ and $L_3(4)$

near octagons have many geometrical and structural properties in common. Several of these properties can be summarized by saying that the two near octagons admit so-called polygonal triples and octagonal pairs (see Sections 2.2 and 2.4 for definitions). The fact that the near octagons admit polygonal triples and octagonal pairs imply that they are related to generalized polygons, a well studied class of point-line geometries. In fact, the connection with generalized polygons is two-fold. On the one hand, the near octagons have subgeometries that are generalized polygons and on the other hand a certain generalized polygon can be defined on those lines of the near octagon that are contained in at least two of these subgeometries.

Already in the early phases of their discovery it was clear that the $G_2(4)$ and $L_3(4)$ near octagons are related to various interesting objects in finite geometry, group theory and graph theory. Many of these relationships are explained in [2, 3]. The $G_2(4)$ near octagon for instance is related to the Suzuki chain of finite simple groups, one of the most fundamental objects in finite group theory, and to the Suzuki chain of (strongly regular) graphs. The $L_3(4)$ near octagon is moreover a subgeometry of the $G_2(4)$ near octagon.

As already mentioned, both near octagons have substructures that are generalized polygons. These form an important class of point-line geometries that include the (axiomatic) projective planes. They were introduced by Jacques Tits [16] in 1959 and have ever since been intensively studied [19]. Many of the known examples arise from classical groups or groups of Lie type (as $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$), and they naturally arise in extremal graph theory as those point-line geometries whose incidence graphs have diameter n and girth $2n$ for some integer $n \geqslant 2$.

As already mentioned, the $G_2(4)$ and $L_3(4)$ near octagons belong to the family of near polygons. This is an important class of point-line geometries which were introduced by Shult and Yanushka [13] in 1980 because of their connection with certain line systems in Euclidean spaces. All generalized n-gons with n even and all connected bipartite graphs with finite diameter are examples of near polygons, but many more examples exist. The standard examples seem to be the dual polar spaces [6] which are closely related to the polar spaces of Jaques Tits [17]. Near polygons also play a fundamental role in graph theory. The collinearity graphs of the so-called regular near polygons are regarded in Chapter 6 of [4] as one of the four main classes of distance-regular graphs.

Besides several infinite families, also several examples of "sporadic" near polygons are known. These are near polygons which do not seem to belong in a natural way to a bigger family of near polygons, and some of them have an automorphism group that is closely related to a finite simple group. In fact, there seem to be six sporadic examples with such an automorphism group, namely the two near hexagons related to the Mathieu groups M_{12} and M_{24} constructed in [13], the near hexagon related to the simple group $U_4(3)$ constructed in [1], the near octagon related to the Hall-Janko group HJ constructed in [5], and the $G_2(4)$ and $L_3(4)$ near octagons constructed in [2, 3]. It is interesting to observe that the first four examples were constructed within a period of five years after the introduction of near polygons in [13]. It then took another 30 years before the following two examples were found in [2, 3].

One of our motivations for the present paper was to investigate whether the $G_2(4)$

and $L_3(4)$ near octagons are really sporadic, that is, to investigate whether they naturally belong to a larger family of near polygons. In order to achieve this goal, we need a certain framework in which we would like to search for new examples. The common properties that the $G_2(4)$ and $L_3(4)$ near octagons share can hereby serve as a starting point. These considerations already led in Section 4 of [3] to the family of near octagons having an octagonal pair and in [8] to the introduction of the family of near polygons having a polygonal triple. It turned out in [8] that not only the $G_2(4)$ and $L_3(4)$ near octagons admit such polygonal triples, but also other families of known near polygons (although sometimes in a more hidden form). None of these other examples can however play the role of an infinite family that naturally accommodates the $G_2(4)$ and $L_3(4)$ near octagons.

There exist relationships between near octagons admitting a polygonal pair and near octagons admitting an octagonal pair. These will be recalled in Section 2.4 and further investigated in Section 3. The intention of this paper is to classify near polygons that have certain polygonal triples. Our main results are summarized in Theorem 2. Although we do not find any new examples of near polygons and thus also no hint for infinite families to accommodate the $G_2(4)$ and $L_3(4)$ near octagons, we find instead new structural characterization results for these near octagons. To prove our main results, we need to develop some algorithms on the one hand (Section 5), and to make some computer computations on the other hand (Section 6). It is interesting to note that the characterization result for the $G_2(4)$ near octagon that we will obtain is basically computer free. We also note that the nonexistence of certain polygonal triples in Theorem 2 will be proved by algebraic combinatorial techniques in Section 4.

2 Technical definitions and main results

2.1 Basic notions and properties of near polygons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon as defined in Section 1. Very often, we regard the lines of $\mathcal{S}$ as sets of points. If this is the case, then incidence is just containment and the lines of $\mathcal{S}$ coincide with the maximal cliques of the collinearity graph Γ, showing that $\mathcal{S}$ is uniquely determined by Γ. If x and y are two points of $\mathcal{S}$, then we denote by $\mathrm{d}(x, y)$ the distance between x and y in Γ. If x is a point of $\mathcal{S}$, then $\Gamma_i(x)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from x. We denote the set $\{x\} \cup \Gamma_1(x)$ also by $x^\perp$.

If L_1 and L_2 are two lines of a near $2d$-gon $\mathcal{S}$, then $\mathrm{d}(L_1, L_2)$ denotes the minimal distance between a point of L_1 and a point of L_2. If $\mathrm{d}(L_1, L_2)$ attains its maximal possible value $d-1$, then the lines L_1 and L_2 are called *opposite*. The lines L_1 and L_2 are called *parallel* if for every $i \in \{1, 2\}$ and every point $x_i \in L_i$, there exists a unique point $x_{3-i} \in L_{3-i}$ at distance $\mathrm{d}(L_1, L_2)$ from x_i. Two opposite lines are always parallel. A set S of lines of $\mathcal{S}$ is called a *line spread* if every point is contained in a unique line of S.

A near polygon is said to have *order* (s, t) if every point is incident with precisely $t+1$ lines and if every line is incident with precisely $s+1$ points. A finite near polygon $\mathcal{S}$ is said to be *regular with parameters* s, t, t_i, $i \in \{0, 1, \ldots, d\}$, if $\mathcal{S}$ has order (s, t) and if for every two points x and y at distance i, there are precisely $t_i + 1$ lines through y

containing a point at distance $i-1$ from x. We then have $t_0 = -1$, $t_1 = 0$ and $t_d = t$. Many interesting examples of finite near polygons are regular. The regular near polygons are precisely the finite near polygons with a distance-regular collinearity graph.

For the precise definition and basic properties of generalized n-gons with $n \geqslant 3$, we refer to the monograph [19]. In this paper, we only need generalized n-gons with $n = 2d$ even and for these geometries one can give the following alternative definition. A *generalized* $2d$*-gon* with $d \geqslant 2$ can be viewed as a near $2d$-gon having the properties that every point is incident with at least two lines and for every two points x and y at distance $i \in \{1, 2, \dots, d-1\}$, there is a unique neighbour of y at distance $i-1$ from x. A finite generalized $2d$-gon of order (s,t) is regular with parameters $t_1, t_2, \dots, t_{d-1}$ equal to 0.

Every generalized $2d$-gon with at least three points on each line and at least three lines through each point must have an order, see e.g. [19, Corollary 1.5.3]. This property is no longer valid for general near polygons.

A set Q of points of a near polygon $\mathcal{S}$ is called a *quad* if the following three properties hold:

- every line of $\mathcal{S}$ that has two points in Q has all its points in Q;
- if x and y are two points of Q at distance 2, then every common neighbour of x and y is also contained in Q;
- the point-line geometry $\widetilde{Q}$ defined on Q by those lines that have all their points in Q is a generalized quadrangle.

Although the notion of quad might look technical at first, quads naturally occur in the theory of near polygons. In fact, Shult and Yanushka proved in [13, Proposition 2.5] that two points of a near polygon at distance 2 are contained in a (necessarily unique) quad if certain mild conditions are satisfied. This fundamental result of Shult and Yanushka can be applied to many near polygons, including the $G_2(4)$ and $L_3(4)$ near octagons. The $G_2(4)$ and $L_3(4)$ near octagons indeed have many quads and for each quad Q of these near octagons, the generalized quadrangle $\widetilde{Q}$ is isomorphic to $W(2)$. Here, $W(2)$ is the smallest member of the family of the symplectic generalized quadrangles. The symplectic generalized quadrangle $W(q)$ with q a prime power is defined as the geometry of the points and lines of the projective space $\mathrm{PG}(3,q)$ that are totally isotropic with respect to a given symplectic polarity, see [12].

A point x of a near polygon is called *classical* with respect to one of its quads Q if there exists a unique point $\pi_Q(x) \in Q$ such that $\mathrm{d}(x,y) = \mathrm{d}(x, \pi_Q(x)) + \mathrm{d}(\pi_Q(x), y)$ for every point $y \in Q$. A point x of a near polygon is called *ovoidal* with respect to one of its quads Q if the points of Q nearest to x form an *ovoid* of $\widetilde{Q}$, that is, a set of points of Q having a unique point in common with each line of $\widetilde{Q}$. Shult and Yanushka [13, Proposition 2.6] proved that if $\mathcal{S}$ is a near polygon having the property that every line is incident with at least three points, then for every point-quad pair (x, Q), we have that x is either classical or ovoidal with respect to Q.

A quad Q of a near polygon is called *classical* if every point of $\mathcal{S}$ is classical with respect to Q. The fact that certain quads are classical is a situation that quite often occurs in the theory of near polygons. In the standard examples of the dual polar spaces, it is even the case that all quads are classical, and this property can be used to provide a characterization for this family of near polygons, see [6]. Also for other families of near polygons, as those with a polygonal triple or an octagonal pair (see Sections 2.2 and 2.4), it can be proved that certain quads need to be classical.

2.2 Polygonal triples

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near $(2d+2)$-gon, $d \geqslant 1$, having a line spread S and a family $\mathcal{Q}$ of quads for which the following hold:

(PT1) For every point x of $\mathcal{S}$, the quads of $\mathcal{Q}$ through x all contain the unique line L_x of S through x and partition the set of lines through x that are distinct from L_x.

(PT2) The point-line geometry $\mathcal{S}'$ with point set S, line set $\mathcal{Q}$ and natural incidence (i.e. containment) is a near $2d$-gon.

Then we call $(\mathcal{S}, S, \mathcal{Q})$ a *polygonal triple* with *associated near polygon* $\mathcal{S}'$. The elements of $\mathcal{Q}$ are called the *quads of the polygonal triple*. For polygonal triples, $\mathcal{Q}$ is uniquely determined by $\mathcal{S}$ and S as it consists of all quads of $\mathcal{S}$ containing a line of S. Polygonal triples were introduced and studied in [8].

We recall from [8] some properties of polygonal triples which will be useful later. Suppose $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple with associated near polygon $\mathcal{S}'$. Then any two lines L_1 and L_2 of S are parallel, and the distance in $\mathcal{S}'$ between them is equal to $\mathrm{d}(L_1, L_2)$. Every line of $\mathcal{S}$ not belonging to S is contained in a unique quad of $\mathcal{Q}$. Regarding the quads of $\mathcal{Q}$, the following additional properties hold. For every quad $Q \in \mathcal{Q}$, the set of lines of S contained in Q is a line spread of $\widetilde{Q}$. Every quad $Q \in \mathcal{Q}$ is classical in $\mathcal{S}$, and for every line $L \in S$, also $\pi_Q(L) := \{\pi_Q(x) \,|\, x \in L\}$ is a line belonging to S.

In [8], several examples and families of polygonal triples were described. Among the near polygons having a polygonal triple, we have the $G_2(4)$ and $L_3(4)$ near octagons, some dual polar spaces and some so-called product and glued near polygons. In the case of the $G_2(4)$ or the $L_3(4)$ near octagon, the quads of the polygonal triple are all the quads of this near octagon and the lines of the line spread are all the lines of the near octagon that are contained in at least two quads.

We conclude this subsection by discussing two instances where polygonal triples can be completely classified. In classification attempts (as the ones of the present paper), the cases that correspond to one of these two instances can therefore often be omitted from the treatment.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a near polygon. Let $\overline{\Gamma}$ be the bipartite graph defined on the set $\overline{\mathcal{P}} := \mathcal{P} \times \{+, -\}$ by calling two distinct vertices (x_1, ϵ_1) and (x_2, ϵ_2) adjacent whenever $\mathrm{d}(x_1, x_2) \leqslant 1$ and $\epsilon_2 = -\epsilon_1$. As $\overline{\Gamma}$ is bipartite, it is the collinearity graph of a near polygon $\overline{\mathcal{S}}$. If we put $L_x := \{(x, +), (x, -)\}$ for every $x \in \mathcal{P}$, then the set $S := \{L_x \,|\, x \in \mathcal{P}\}$ is a line spread of $\overline{\mathcal{S}}$ and the set $\mathcal{Q} := \{L \times \{+, -\} \,|\, L \in \mathcal{L}\}$ is a set of quads. By [9], $(\overline{\mathcal{S}}, S, \mathcal{Q})$ is a polygonal triple with associated near polygon isomorphic to $\mathcal{S}$, and every near polygon with two points on each line admitting a polygonal triple is obtained in the above described way. A near polygon having two points on each line is also called *thin*.

Suppose $\mathcal{S}$ is a near polygon and $s \in \mathbb{N} \setminus \{0\}$. By considering $s+1$ isomorphic copies of $\mathcal{S}$ and joining the corresponding points to form lines of size $s+1$, we obtain a new near polygon which we denote by $\mathcal{S} \times \mathbb{L}_{s+1}$. In the special case that $\mathcal{S}$ is a line $\mathbb{L}_{s'+1}$ containing $s'+1$ points, then $\mathcal{S} \times \mathbb{L}_{s+1} = \mathbb{L}_{s'+1} \times \mathbb{L}_{s+1}$ is called an $(s'+1) \times (s+1)$-*grid* (or a *grid* for short). The set S of all lines of $\mathcal{S} \times \mathbb{L}_{s+1}$ joining corresponding points is a line spread S of $\mathcal{S} \times \mathbb{L}_{s+1}$. If $\mathcal{Q}$ is the set of all quads containing a line of S, then by [8] $(\mathcal{S} \times \mathbb{L}_{s+1}, S, \mathcal{Q})$ is a polygonal triple whose associated near polygon is isomorphic to $\mathcal{S}$ and for which all quads of $\mathcal{Q}$ are grids. In fact, we are able to prove the following.

Proposition 1. *Let $(\mathcal{S}, S, \mathcal{Q})$ be a polygonal triple for which every quad of $\mathcal{Q}$ is a grid. Then $\mathcal{S}$ is isomorphic to $\mathcal{S}' \times \mathbb{L}$, where $\mathcal{S}'$ is the near polygon associated with $(\mathcal{S}, S, \mathcal{Q})$ and $\mathbb{L}$ is a line.*

Proof. Consider a fixed line $L \in S$. For every point $x \in L$, let A_x denote the set of all points y for which x is the unique point of L nearest to y. The point set $\mathcal{P}$ of $\mathcal{S}$ is then the disjoint union $\bigcup_{x \in L} A_x$.

Suppose K is a line of $\mathcal{S}$ not belonging to S. There is a unique quad $Q \in \mathcal{Q}$ containing K. This quad is classical in $\mathcal{S}$ and $L' := \pi_Q(L)$ is a line of $\widetilde{Q}$ belonging to S. The intersection $L' \cap K$ is a singleton $\{y\}$, and as Q is classical we know that K is contained in A_x, where $x \in L$ is the unique point of L nearest to y.

Now, each line of S intersects each A_x in a unique point and each quad of $\mathcal{Q}$ intersects each A_x in a line, showing that the subgeometry induced on each A_x, $x \in L$, is isomorphic to $\mathcal{S}'$. It is now also clear that $\mathcal{S} \cong \mathcal{S}' \times \mathbb{L}$, where $\mathbb{L}$ is any line of the same size as L. $\square$

2.3 The main result

The polygonal triples $(\mathcal{S}, S, \mathcal{Q})$ for which all lines of $\mathcal{S}$ are thin or for which all quads of $\mathcal{Q}$ are grids are easy to describe, see Section 2.2. The smallest generalized quadrangle which is neither *thin* nor a *grid* is the symplectic generalized quadrangle $W(2)$ of order $(2, 2)$. In this paper, we study and classify polygonal triples for which all quads are isomorphic to $W(2)$. Note that this includes the case of the $G_2(4)$ and $L_3(4)$ near octagons as all quads in these near octagons are isomorphic to $W(2)$. The following is our main result.

Theorem 2. *Suppose $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple with associated near polygon $\mathcal{S}'$ such that all quads of $\mathcal{Q}$ are isomorphic to $W(2)$. Then the following hold.*

- *If $\mathcal{S}' \cong H^D(4)$, then $\mathcal{S}$ is isomorphic to the $G_2(4)$ near octagon.*
- *If $\mathcal{S}' \cong GH(4,1)$, then $\mathcal{S}$ is isomorphic to the $L_3(4)$ near octagon.*
- *$\mathcal{S}'$ cannot be isomorphic to $H(4)$, $T(4, 64)$, $GO(4, 1)$, $RT(4, 2)$, $\mathcal{F}(H(4))$ and $\mathcal{F}(H^D(4))$.*

In Theorem 2, the generalized polygons $H^D(4)$, $GH(4,1)$, $H(4)$, $T(4,64)$, $GO(4,1)$, $RT(4,2)$, $\mathcal{F}(H(4))$ and $\mathcal{F}(H^D(4))$ are respectively isomorphic to the dual split Cayley generalized hexagon of order $(4,4)$, the unique generalized hexagon of order $(4,1)$, the split Cayley generalized hexagon of order $(4,4)$, the dual twisted triality hexagon of order $(4,64)$, the unique generalized octagon of order $(4,1)$, the Ree-Tits octagon of order $(4,2)$, the flag geometry of $H(4)$ and the flag geometry of $H^D(4)$. We note here that $\mathcal{F}(H(4))$ and $\mathcal{F}(H^D(4))$ are two generalized dodecagons of order $(4,1)$. We refer to [19] for the precise definitions of the generalized polygons $H^D(4)$, $H(4)$, $T(4,64)$ and $RT(4,2)$. The *flag geometry* $\mathcal{F}(\mathcal{S})$ of a point-line geometry $\mathcal{S}$ is defined as the point-line geometry whose points are the *flags* of $\mathcal{S}$, i.e. the unordered point-line pairs of $\mathcal{S}$, and whose lines are the points and lines of $\mathcal{S}$, with incidence being reverse containment. The generalized hexagon $GH(4,1)$ is isomorphic to the flag geometry of the projective plane $\mathrm{PG}(2,4)$ of order 4 and the generalized octagon $GO(4,1)$ is isomorphic to the flag geometry of the symplectic generalized quadrangle $W(4)$.

Theorem 2 covers precisely those cases where the quads of $\mathcal{Q}$ are isomorphic to $W(2)$ and $\mathcal{S}'$ is a known finite generalized $2d$-gon with $d \geqslant 3$. Note that as a line spread of $W(2)$ contains five lines, the generalized polygon $\mathcal{S}'$ must have five points per line, and thus have order $(4,t)$ for some $t \in \mathbb{N} \setminus \{0\}$ by [19, Section 1.5]. A complete classification of all finite generalized polygons of order $(4,t)$ does not exist at present. That is why we restrict to the known examples. A worse situation even occurs for (general) near polygons of order $(4,t)$ for which only a limited number of classification results are known.

The computational methods that we originally developed to characterize the $G_2(4)$ and $L_3(4)$ near octagons in terms of polygonal triples also had the potential to treat cases where these polygonal triples have arbitrary generalized $2d$-gons as associated near polygons. It therefore seemed natural to present our results in such a more general setting. Polygonal triples for which the associated near polygons have quads can often be studied with the aid of theoretical methods. In particular, in [8, Theorem 7] a complete classification has been obtained of all polygonal triples for which the associated near polygons are generalized quadrangles. That is why we also have made the assumption here that d is at least three.

In Section 4, we show the nonexistence of certain polygonal triples $(\mathcal{S}, S, \mathcal{Q})$ by computing the multiplicities of the eigenvalues of $\mathcal{S}$ and expressing that these are all nonnegative integers. These nonexistence results already cover certain of the cases mentioned in Theorem 2, namely the cases when $\mathcal{S}'$ is isomorphic to $T(4,64)$ or $GO(4,1)$.

We wish to note here that our characterization of the $G_2(4)$ near octagon is basically computer free. Indeed, our algorithms already imply (without additional computer computations) that there is at most one polygonal triple $(\mathcal{S}, S, \mathcal{Q})$ (up to some obvious form of isomorphism) for which all quads are isomorphic to $W(2)$ and for which the corresponding generalized hexagon $\mathcal{S}'$ is isomorphic to $H^D(4)$. However, also in this case, as a verification of our methods, we have used our algorithms to reconstruct $\mathcal{S}$ from $\mathcal{S}' \cong H^D(4)$ and to check that $\mathcal{S}$ is indeed a near octagon with similar properties as the $G_2(4)$ near octagon.

2.4 Octagonal pairs

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a finite near octagon of order (s,t) and S is a line spread of $\mathcal{S}$. For every point $x \in \mathcal{P}$, let L_x denote the unique line of S containing x. We define the following sets of points of $\mathcal{S}$:

- For every point x of $\mathcal{S}$, we define $\Gamma_1'(x) := L_x \setminus \{x\}$ and $\Gamma_1''(x) := \Gamma_1(x) \setminus \Gamma_1'(x)$.
- For every point x of $\mathcal{S}$ and every $i \in \{2,3\}$, $\Gamma_i'(x)$ denotes the set of points of $\Gamma_i(x)$ that are collinear with a point of $\Gamma_{i-1}'(x)$, and we put $\Gamma_i''(x) := \Gamma_i(x) \setminus \Gamma_i'(x)$.

Suppose moreover that there exists a positive divisor $t_2 \neq t$ of t such that the following hold for every point x of $\mathcal{S}$:

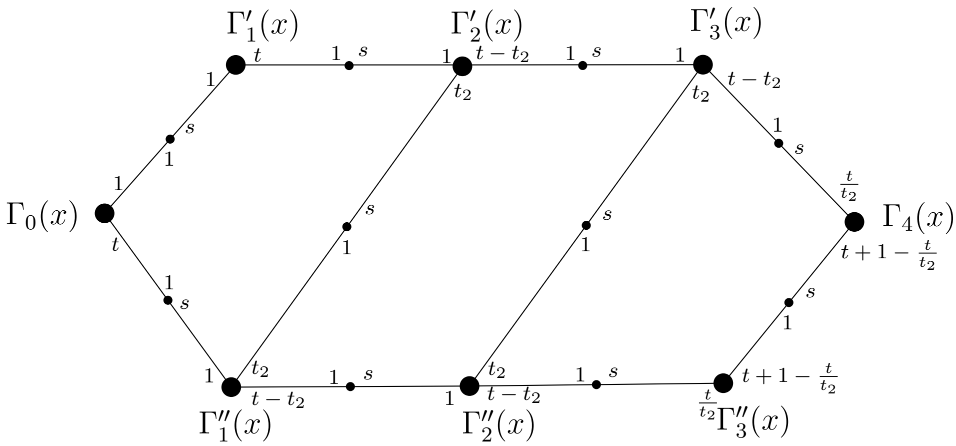
(P1) Every point of $\Gamma_2'(x)$ is incident with t_2 lines meeting $\Gamma_1''(x)$.

(P2) Every point of $\Gamma_2''(x)$ is incident with a unique line meeting $\Gamma_1''(x)$.

(P3) Every point of $\Gamma_3'(x)$ is incident with t_2 lines meeting $\Gamma_2''(x)$.

(P4) Every point of $\Gamma_3''(x)$ is incident with $\frac{t}{t_2}$ lines meeting $\Gamma_2''(x)$.

If these properties hold, then we say that $(\mathcal{S}, S)$ is an *octagonal pair with parameters* (s,t,t_2). The family $\mathcal{F}$ of near octagons introduced and studied in Section 4 of [3] are precisely the near octagons that admit an octagonal pair. From [3], we know that the structure of $\mathcal{S}$ with respect to any of its points x can be described by the following diagram:



In this diagram, the edges denote lines and the big nodes denote points (belonging to the mentioned point sets). The number of lines through a given point meeting the various sets are mentioned around the big nodes, while the number of points on a given line contained in the various sets are mentioned around the small nodes. The following was proved in Section 4 of [3].

Proposition 3 ([3])**.** *Suppose $(\mathcal{S}, S)$ is an octagonal pair with parameters (s, t, t_2), $s \geqslant 2$, and denote by $\mathcal{Q}$ the set of quads of $\mathcal{S}$. Then $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple for which all quads have order (s, t_2) and whose associated near polygon is a generalized hexagon of order $(st_2, \frac{t}{t_2} - 1)$.*

By [2] and [3], we know that the $G_2(4)$ and the $L_3(4)$ near octagons admit octagonal pairs. In the case of the $G_2(4)$ near octagon, we have $(s, t_2, t) = (2, 2, 10)$, all quads are isomorphic to $W(2)$ and the associated generalized hexagon $\mathcal{S}'$ is isomorphic to the dual split Cayley generalized hexagon $H^D(4)$. In the case of the $L_3(4)$ near octagon, we have $(s, t_2, t) = (2, 2, 4)$, all quads are isomorphic to $W(2)$ and the associated generalized hexagon $\mathcal{S}'$ is the unique generalized hexagon $GH(4, 1)$ of order $(4, 1)$.

In Section 3, we prove the following converse of Proposition 3.

Theorem 4. *Suppose $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple for which the associated near polygon is a finite generalized hexagon having an order and for which all quads in $\mathcal{Q}$ are finite and have the same order. Then $(\mathcal{S}, S, \mathcal{Q})$ arises from an octagonal pair as described in Proposition 3.*

As the symplectic generalized quadrangle $W(2)$ is the unique generalized quadrangle of order $(2, 2)$ [12], we thus see by Proposition 3 and Theorem 4 that classifying octagonal pairs with parameters $(s, t, t_2) = (2, t, 2)$ is equivalent with classifying all polygonal triples whose quads are isomorphic to $W(2)$ and whose associated near polygons are generalized hexagons of order $(4, \frac{t}{2} - 1)$. Theorem 2 thus also provides a classification of all octogonal pairs with parameters $(s, t, t_2) = (2, t, 2)$ for which the associated generalized hexagon is either $GH(4, 1)$, $H(4)$, $H^D(4)$ or $T(4, 64)$.

3 On polygonal triples for which the associated near polygon is regular

In this section, we prove Theorem 4. The arguments that are necessary to achieve this goal allow to prove a more general result. We will therefore derive our results in a more general setting.

Let $(\mathcal{S}, S, \mathcal{Q})$ be a polygonal triple for which the corresponding near polygon $\mathcal{S}'$ has diameter d and is regular with parameters s, t, t_i, $i \in \{0, 1, \ldots, d\}$. Recall then that $t_0 = -1$, $t_1 = 0$, $t_d = t$ and that $\mathcal{S}$ has diameter $d + 1$. Many of the interesting finite near polygons are regular and so the treatment in this section basically covers all these examples.

Suppose every quad of $\mathcal{Q}$ has order $(\widetilde{s}, \widetilde{t_2})$. Then $\mathcal{S}$ has order $(\widetilde{s}, \widetilde{t_2}(t + 1))$. As a line spread of a quad of $\mathcal{Q}$ contains $1 + \widetilde{s} \cdot \widetilde{t_2}$ lines, we have $s = \widetilde{s} \cdot \widetilde{t_2}$. For every point u of $\mathcal{S}$, we denote by L_u the unique line of S containing u. For every point x and every $i \in \{0, 1, \ldots, d + 1\}$, we denote by $\Gamma''_i(x)$ the set of all points $y \in \Gamma_i(x)$ for which y is the unique point of L_y nearest to $x \in L_x$, and we define $\Gamma'_i(x) := \Gamma_i(x) \setminus \Gamma''_i(x)$. Then $\Gamma'_0(x) = \Gamma''_{d+1}(x) = \emptyset$ and $\Gamma'_1(x) = L_x \setminus \{x\}$. If L is a line of S at distance i from L_x, then L contains a unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.

Lemma 5. *Let x be a point of $\mathcal{S}$ and $i \in \{1, 2, \dots, d\}$. Then no point of $\Gamma''_i(x)$ is collinear with a point of $\Gamma'_{i-1}(x)$.*

Proof. Suppose $y \in \Gamma''_i(x)$ is collinear with a point $z \in \Gamma'_{i-1}(x)$. As $y \in \Gamma''_i(x)$, we have $\mathrm{d}(L_x, L_y) = i$ and hence $\mathrm{d}(L_x, L_z) \geqslant i - 1$. As $z \in \Gamma_{i-1}(x)$, the point z must be the unique point of L_z nearest to x, i.e. $z \in \Gamma''_{i-1}(x)$, in contradiction with the fact that $z \in \Gamma'_{i-1}(x)$. $\square$

Lemma 6. *Let x be a point of $\mathcal{S}$ and $i \in \{1, 2, \dots, d\}$. Then no point of $\Gamma''_i(x)$ is collinear with a point of $\Gamma'_i(x)$.*

Proof. Suppose $y \in \Gamma''_i(x)$ is collinear with a point $z \in \Gamma'_i(x)$. Then $yz \notin S$ and so there exists a unique quad $Q \in \mathcal{Q}$ containing yz. The line $L_z \subseteq Q$ contains a point $u \in \Gamma''_{i-1}(x)$. The line $L_y \subseteq Q$ contains points of $\Gamma'_{i+1}(x)$. As x is classical with respect to Q, u is the unique point of Q nearest to x and $\mathrm{d}(u, y) = 1$, a contradiction, since z is the only point of yz collinear with u. $\square$

Lemma 7. *Let Q be a quad of $\mathcal{S}$. Put $i := d(x, Q)$. Let u denote the unique point of Q nearest to x. Then $u \in \Gamma''_i(x)$, $L_u \setminus \{u\} \subseteq \Gamma'_{i+1}(x)$, $u^\perp \setminus L_u \subseteq \Gamma''_{i+1}(x)$ and $\Gamma_2(u) \cap Q \subseteq \Gamma'_{i+2}(x)$.*

Proof. As u is the unique point of $L_u \subseteq Q$ nearest to $x \in L_x$, we have $u \in \Gamma''_i(x)$, $L_u \setminus \{u\} \subseteq \Gamma'_{i+1}(x)$ and $\mathrm{d}(L_x, L_u) = i$. Every line $L \in S \setminus \{L_u\}$ contained in Q lies at distance $i + 1$ from L_x and its unique point nearest to x lies at distance 1 from u as x is classical with respect to Q. Every other point of L lies at distance $i + 2$ from x and must belong to $\Gamma'_{i+2}(x)$, proving the remaining claims of the lemma. $\square$

We now use Lemma 7 to prove the following two lemmas.

Lemma 8. *Let $y \in \Gamma''_i(x)$ with $i \in \{0, 1, \dots, d\}$. Then:*

- *y is incident with $t_i + 1$ lines containing a unique point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma''_i(x)$;*
- *y is incident with $(t - t_i)\widetilde{t_2}$ lines containing y as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma''_{i+1}(x)$;*
- *y is incident with $1 + (t_i + 1)(\widetilde{t_2} - 1)$ lines containing y as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.*

Proof. The line L_y contains y as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$. Note that $\mathrm{d}(L_x, L_y) = i$.

Let Q be one of the $t_i + 1$ quads through L_y containing a line $L \in S$ at distance $i - 1$ from L_x. Let u denote the unique point of Q nearest to x. Then $u \in L$ and $\mathrm{d}(u, y) = 1$. By Lemma 7, the line yu contains a unique point of $\Gamma''_{i-1}(x)$ (namely u) and $\widetilde{s}$ points of $\Gamma''_i(x)$. Also by Lemma 7, each of the $\widetilde{t_2} - 1$ lines of Q through y distinct from yu and L_y contains y as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.

Let Q be one of the $t-t_i$ quads through L_y not containing a line of S at distance $i-1$ from L_x. By Lemma 7, each of the $\widetilde{t_2}$ lines of Q through y distinct from L_y contains y as unique point of $\Gamma''_i(x)$ and $\widetilde{s}$ points of $\Gamma''_{i+1}(x)$. $\square$

Lemma 9. *Let $y \in \Gamma'_i(x)$ with $i \in \{1, 2, \ldots, d+1\}$. Then:*

- *y is incident with $t_{i-1}+1$ lines containing a unique point of $\Gamma'_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$;*
- *y is incident with $(t_{i-1}+1)(\widetilde{t_2}-1)+1$ lines containing a unique point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$;*
- *y is incident with $\widetilde{t_2}(t-t_{i-1})$ lines containing y as unique point of $\Gamma'_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$.*

Proof. The line L_y contains a point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$. Note that $\mathrm{d}(L_x, L_y) = i-1$.

Let Q be one of the $t_{i-1}+1$ quads through L_y containing a line of S at distance $i-2$ from L_x. Denote by u the unique point of Q nearest to x. Then $u \in \Gamma''_{i-2}(x)$ and L_u is the unique line of Q nearest to L_x. By Lemma 7, the unique line U through y meeting L_u contains a unique point of $\Gamma'_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$. Also by Lemma 7, each of the $\widetilde{t_2}-1$ lines of Q through y distinct from U and L_y contains a unique point of $\Gamma''_{i-1}(x)$ and $\widetilde{s}$ points of $\Gamma'_i(x)$.

Let Q be one of the $t-t_{i-1}$ quads through L_y not containing a line of S at distance $i-2$ from L_x. By Lemma 7, each line of Q through y distinct from L_y contains y as unique point of $\Gamma'_i(x)$ and $\widetilde{s}$ points of $\Gamma'_{i+1}(x)$. $\square$

Proof of Theorem 4. Consider now the special case that $\mathcal{S}'$ is a generalized hexagon of order (s,t). Then $t_0=-1$, $t_1=t_2=0$, $t_3=t$ and $\mathcal{S}$ is a near octagon. Above we already remarked that

(A1) $\Gamma'_1(x) = L_x \setminus \{x\}$.

By Lemmas 5 and 9, we know the following:

(A2) $\Gamma'_i(x)$ with $i \in \{2,3\}$ consists of all points of $\Gamma_i(x)$ that are collinear with a point of $\Gamma'_{i-1}(x)$.

By the above, we also know that

(A3) $\Gamma''_i(x) = \Gamma_i(x) \setminus \Gamma'_i(x)$ for every $i \in \{1,2,3\}$.

By Lemmas 8 and 9, we also know:

(P1) Every point of $\Gamma'_2(x)$ is incident with $(t_1+1)(\widetilde{t_2}-1)+1 = \widetilde{t_2}$ lines meeting $\Gamma''_1(x)$.

(P2) Every point of $\Gamma''_2(x)$ is incident with $t_2+1=1$ lines meeting $\Gamma''_1(x)$.

(P3) Every point of $\Gamma_3'(x)$ is incident with $(t_2+1)(\widetilde{t_2}-1)+1=\widetilde{t_2}$ lines meeting $\Gamma_2''(x)$.

(P4) Every point of $\Gamma_3''(x)$ is incident with $t_3+1=t+1=\frac{\widetilde{t_2}(t+1)}{\widetilde{t_2}}$ lines meeting $\Gamma_2''(x)$.

Noting that $\mathcal{S}$ has order $(\widetilde{s},\widetilde{t_2}(t+1))$, we thus see $(\mathcal{S},S)$ is an octagonal pair with parameters $(\widetilde{s},\widetilde{t_2}(t+1),\widetilde{t_2})$.

4 Eigenvalues and multiplicities of near polygons admitting a polygonal triple

Suppose $\mathcal{T}=(\mathcal{S},S,\mathcal{Q})$ is a polygonal triple for which $\mathcal{S}$ is a finite near octagon of order (s,t) and such that every quad of $\mathcal{Q}$ has order (s,t_2). Suppose also that the near hexagon $\mathcal{S}'$ associated with $\mathcal{T}$ is a generalized hexagon, necessarily of order $(st_2,\frac{t}{t_2}-1)$. Since $t>t_2$, we can put $\frac{t}{t_2}-1=\frac{\alpha^2}{st_2}$, where α is some real positive number.

Let x be a point of $\mathcal{S}$. By Theorem 4 proved at the end of Section 3, we know that $(\mathcal{S},S)$ is an octagonal pair and so $\mathcal{S}$ belongs to the family $\mathcal{F}$ discussed in [3]. With respect to the line spread S, the point set $\mathcal{P}$ of $\mathcal{S}$ can thus be written as a disjoint union $\Gamma_0(x)\cup\Gamma_1'(x)\cup\Gamma_1''(x)\cup\Gamma_2'(x)\cup\Gamma_2''(x)\cup\Gamma_3'(x)\cup\Gamma_3''(x)\cup\Gamma_4(x)$. This expression naturally gives rise to relations $R_0,R_1',\dots,R_3'',R_4$ on $\mathcal{P}$ that partition $\mathcal{P}\times\mathcal{P}$ (e.g., $(x,y)\in R_3''\Leftrightarrow y\in\Gamma_3''(x)$). These relations are symmetric. Indeed, if $i\in\{1,2,3\}$, then $(x,y)\in R_i''\Leftrightarrow \mathrm{d}(x,y)=\mathrm{d}(L_x,L_y)=i$, with L_u, $u\in\mathcal{P}$, again denoting the unique line of S containing u. With each relation $R\in\{R_0,R_1',R_1'',\dots,R_4\}$, there is associated a symmetric matrix U whose rows and columns are indexed by the points. Specifically, we put U_{xy} equal to 1 if $(x,y)\in R$ and equal to 0 otherwise. In this way, we obtain symmetric $v\times v$-matrices $A_0,A_1',A_1'',\dots,A_4$, where v is the total number of points. Here, A_0 is the $v\times v$ identity matrix I_v and $A:=A_1'+A_1''$ is the collinearity matrix of $\mathcal{S}$. From Lemmas 8 and 9, we deduce

$$\begin{aligned}
A\cdot A_0 &= A_1'+A_1'',\\
A\cdot A_1' &= sA_0+(s-1)A_1'+A_2',\\
A\cdot A_1'' &= stA_0+(s-1)A_1''+t_2A_2'+A_2'',\\
A\cdot A_2' &= stA_1'+st_2A_1''+(s-1)(t_2+1)A_2'+A_3',\\
A\cdot A_2'' &= s(t-t_2)A_1''+(s-1)A_2''+t_2A_3'+\frac{t}{t_2}A_3'',\\
A\cdot A_3' &= s(t-t_2)A_2'+st_2A_2''+(s-1)(t_2+1)A_3'+\frac{t}{t_2}A_4,\\
A\cdot A_3'' &= s(t-t_2)A_2''+\frac{(s-1)t}{t_2}A_3''+(t+1-\frac{t}{t_2})A_4,\\
A\cdot A_4 &= s(t-t_2)A_3'+s(t+1-\frac{t}{t_2})A_3''+(s-1)(t+1)A_4.
\end{aligned}\tag{1}$$

As all involved matrices are symmetric, these equations can be written as

$$[A_0\ A_1'\ A_1''\ \cdots\ A_4]^T\cdot A=(B\otimes I_v)\cdot[A_0\ A_1'\ A_1''\ \cdots\ A_4]^T,$$

where $B \otimes I_v$ denotes the Kronecker product [11, Section 4.2] of

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ s & s-1 & 0 & 1 & 0 & 0 & 0 & 0 \\ st & 0 & s-1 & t_2 & 1 & 0 & 0 & 0 \\ 0 & st & st_2 & (s-1)(t_2+1) & 0 & 1 & 0 & 0 \\ 0 & 0 & s(t-t_2) & 0 & s-1 & t_2 & \frac{t}{t_2} & 0 \\ 0 & 0 & 0 & s(t-t_2) & st_2 & (s-1)(t_2+1) & 0 & \frac{t}{t_2} \\ 0 & 0 & 0 & 0 & s(t-t_2) & 0 & \frac{(s-1)t}{t_2} & t+1-\frac{t}{t_2} \\ 0 & 0 & 0 & 0 & 0 & s(t-t_2) & s(t+1-\frac{t}{t_2}) & (s-1)(t+1) \end{bmatrix}$$

and I_v. Let $\mathcal{A}$ be the subalgebra of $\mathbb{R}^{v\times v}$ generated by A, and let $\mathcal{B}$ be the subalgebra of $\mathbb{R}^{8\times 8}$ generated by B. For every $M \in \mathcal{A}$, there exists a unique $M^\theta \in \mathbb{R}^{8\times 8}$ such that

$$[A_0\ A_1'\ A_1''\ \cdots\ A_4]^T \cdot M = (M^\theta \otimes I_v) \cdot [A_0\ A_1'\ A_1''\ \cdots\ A_4]^T.$$

In fact, if $M = p(A)$ for a certain polynomial $p(X) \in \mathbb{R}[X]$, then we can take $M^\theta = p(B)$ by [11, Lemma 4.2.10]. The uniqueness of $M^\theta \in \mathbb{R}^{8\times 8}$ follows from the fact that the matrices $A_0, A_1', A_1'', \ldots, A_4$ are linearly independent in $\mathbb{R}^{v\times v}$. We conclude that θ defines an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Taken into account that $t = \frac{\alpha^2}{s} + t_2$, we can compute that the eigenvalues of B are equal to

$$\lambda_1 = \alpha^2 + st_2 + s,\ \lambda_2 = s + \alpha - t_2 - 1,\ \lambda_3 = s - \alpha - t_2 - 1,\ \lambda_4 = st_2 + s - \alpha - 1,$$

$$\lambda_5 = st_2 + s + \alpha - 1,\ \lambda_6 = -\frac{\alpha^2 + st_2 + s}{s}, \lambda_7 = \frac{\alpha^2 + st_2 - t_2}{t_2},\ \lambda_8 = \frac{s^2t_2 - st_2 - \alpha^2}{st_2}.$$

Hence, these are also the eigenvalues of A. Let m_i with $i \in \{1, 2, \ldots, 8\}$ denote the multiplicity of the eigenvalue λ_i of A.

If $j \in \mathbb{N}$, then by (1), we can write

$$A^j = a_jA_0 + b_jA_1' + c_jA_1'' + d_jA_2' + e_jA_2'' + f_jA_3' + g_jA_3'' + h_jA_4 \tag{2}$$

for certain (necessarily unique) $a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j \in \mathbb{N}$. We then have $Tr(A^j) = va_j$.

Suppose now that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_8$ are mutually distinct. Then we have $\sum_{i=1}^{8} m_i\lambda_i^j = Tr(A^j) = va_j$ for every $j \in \{0, 1, \ldots, 7\}$. These eight equations determine a system of linear equations which can be solved for the unknowns $m_1, m_2, \ldots, m_8$ as the matrix of the system is a nonsingular Vandermonde matrix. We find (e.g. with Maple, see [10]) that

$$\begin{aligned}
m_1 &= 1,\\
m_2 &= \frac{\alpha^2 s(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(\alpha^2+\alpha s+s^2)(\alpha^2-\alpha t_2+t_2^2)},\\
m_3 &= \frac{\alpha^2 s(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(\alpha^2-\alpha s+s^2)(\alpha^2+\alpha t_2+t_2^2)},\\
m_4 &= \frac{\alpha^2(\alpha^2-\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(s^2t_2^2-\alpha st_2+\alpha^2)},\\
m_5 &= \frac{\alpha^2(\alpha^2+\alpha+1)(st_2+1)(\alpha^2+st_2)}{2(s^2t_2^2+\alpha st_2+\alpha^2)},\\
m_6 &= \frac{s^6(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)}{(\alpha^2+\alpha s+s^2)(\alpha^2-\alpha s+s^2)(s+t_2)},\\
m_7 &= \frac{st_2^5(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)(st_2+1)}{(\alpha^2+\alpha t_2+t_2^2)(\alpha^2-\alpha t_2+t_2^2)(s+t_2)},\\
m_8 &= \frac{s^5t_2^5(\alpha^2+\alpha+1)(\alpha^2-\alpha+1)}{(s^2t_2^2+\alpha st_2+\alpha^2)(s^2t_2^2-\alpha st_2+\alpha^2)}.
\end{aligned}$$

These multiplicities need to be integers. In case $\alpha \in \mathbb{N}$, this leads to a number of divisibility conditions that need to be satisfied by the parameters s, t_2 and α.

If a particular eigenvalue λ occurs more than once in the collection $\lambda_1, \lambda_2, \cdots, \lambda_8$, then its multiplicity is equal to $\sum m_i$, where the summation ranges over all $i \in \{1, 2, \ldots, 8\}$ for which $\lambda_i = \lambda$.

Recall that $\mathcal{S}'$ is a generalized hexagon of order $(st_2, \frac{t}{t_2} - 1)$. By Section 2, we know that if $t_2 = 1$, then $\mathcal{S} \cong \mathcal{S}' \times \mathbb{L}_{s+1}$ where $\mathbb{L}_{s+1}$ is some line of size $s+1$ (Proposition 1), and if $s = 1$, then $\mathcal{S} \cong \overline{\mathcal{S}'}$. In the sequel, we may therefore assume that $s, t_2 > 1$.

Put $q := st_2 \geqslant 4$ and as before let α be the positive real number such that $\alpha^2 = q \cdot (\frac{t}{t_2} - 1)$. If $\frac{t}{t_2} - 1 > 1$, then $\alpha \in \mathbb{N}$, see e.g. [19, Theorem 1.7.1]. The eigenvalues and multiplicities can thus be computed with the aid of the above formulas. Note that all the multiplicities must be integral.

Using the information provided in [7, Section 5.8], [12, Chapter 3], [15, Sections 3, 4 and 6] and [19, Chapter 2], we can easily find all triples (s, t_2, t) of natural numbers distinct from 0 and 1 satisfying the following, with $d = 3$:

(A) There exists a known generalized quadrangle Q of order (s, t_2) that is moreover known to have a line spread.

(B) There exists a known generalized $2d$-gon of order $(st_2, \frac{t}{t_2} - 1)$.

We find that there exists a prime power r such that (s, t_2, t) is equal to either $(r, r, 2r)$, $(r, r^2, 2r^2)$, $(r, r, r(r^2+1))$, $(r, r^2, r^2(r^3+1))$, $(r, r, r(r^6+1))$ or $(r, r^2, r^2(r^9+1))$. The multiplicities of the eigenvalues are only integral when $(s, t_2, t) \in \{(r, r, 2r), (r, r, r(r^2+1))\}$, see [10].

In case $(s, t_2, t) = (r, r, 2r)$, the associated generalized hexagon $\mathcal{S}'$ has order $(r^2, 1)$ and all quads of $\mathcal{S}$ have order (r, r). One example is known, namely the polygonal triple corresponding to the $L_3(4)$ near octagon $(r = 2)$.

In case $(s, t_2, t) = (r, r, r(r^2 + 1))$, the associated generalized hexagon $\mathcal{S}'$ has order (r^2, r^2) and all quads of $\mathcal{S}$ have order (r, r). One example is known, namely the polygonal triple corresponding to the $G_2(4)$ near octagon $(r = 2)$.

We can give a similar treatment for polygonal triples $(\mathcal{S}, S, \mathcal{Q})$, where $\mathcal{S}$ is a finite near decagon of order (s, t) with quads of order (s, t_2) such that the associated near polygon $\mathcal{S}'$ is a generalized octagon of order $(st_2, \frac{t}{t_2} - 1)$. Then we must find all triples (s, t_2, t) of natural numbers distinct from 0 and 1 such that properties (A) and (B) hold with $d = 4$. Using again the information provided in [7, 12, 15, 19], we find the following possibilities for (s, t_2, t):

(a) $(q, q, 2q)$, where q is some prime power;
(b) $(q, q, (q + 1)q)$, where q is a power of 2 with odd exponent;
(c) $(q, q^2, 2q^2)$, where q is some prime power;
(d) $(q, q^2, (q^6 + 1)q^2)$, where q is a power of 2 with odd exponent;
(e) $(q^2, q^4, (q^3 + 1)q^4)$, where q is a power of 2 with odd exponent.

With similar techniques as above, we can compute all eigenvalues and multiplicities, see [10]. The cases (d) and (e) cannot occur since not all multiplicities are integral. In cases (b) and (c), the multiplicities are always integral. In case (a), all multiplicities are integral if and only if q is odd or a multiple of 8.

Lemma 10. *There are no polygonal triples whose quads are isomorphic to $W(2)$ and whose associated near polygons are isomorphic to either $T(4, 64)$ or $GO(4, 1)$.*

Proof. As $W(2)$ has order $(2, 2)$, the triple (s, t_2, t) is equal to $(2, 2, 130)$ for the generalized hexagon $T(4, 64)$ (with $\alpha = 16$) and equal to $(2, 2, 4)$ for the generalized octagon $GO(4, 1)$. By the discussion preceding this lemma, these cases cannot occur as some of the multiplicities of the eigenvalues are then not integral. □

5 Algorithms to classify polygonal triples

5.1 Polygonal data

In this section, $\mathcal{S}$ denotes a near polygon, S is a line spread of $\mathcal{S}$ and $\mathcal{Q}$ is a set of quads of $\mathcal{S}$ such that $(\mathcal{S}, S, \mathcal{Q})$ is a polygonal triple with associated near polygon $\mathcal{S}'$. For every point x of $\mathcal{S}$, L_x denotes the unique line of S containing x. Recall that every quad $Q \in \mathcal{Q}$ is classical in $\mathcal{S}$.

If $L_1, L_2 \in S$, then L_1 and L_2 are parallel and we denote by Π_{L_1,L_2} the bijection between L_1 and L_2 that sends each point x of L_1 to the unique point of L_2 nearest to x. If $L_1, L_2, L_3 \in S$, then we define $\Phi_{L_1,L_2,L_3} = \Pi_{L_3,L_1} \circ \Pi_{L_2,L_3} \circ \Pi_{L_1,L_2}$.

Let L^* be some specific line of S. For every point x of $\mathcal{S}$, let $\pi(x)$ denote the unique point of L^* nearest to x. We coordinatize $\mathcal{S}$ as follows. With each point x of $\mathcal{S}$, we associate the pair $(L_x, \pi(x))$. If L_1 and L_2 are two distinct lines of S, then we denote by Φ_{L_1,L_2} the permutation Φ_{L^*,L_1,L_2} of L^*.

Lemma 11.

(1) $\Phi_{L^*,L}$ *is the identical permutation* 1_{L^*} *of* L^* *for every line* L *of* S.

(2) *For any two distinct lines* $L_1, L_2 \in S$, *we have* $\Phi_{L_2,L_1} = \Phi^{-1}_{L_1,L_2}$.

(3) *If* L_1, L_2 *and* L_3 *are three lines of* S *such that* L_2 *is contained on a shortest path from* L_1 *to* L_3 *in the near polygon* $\mathcal{S}'$, *then* $\Phi_{L_1,L_3} = \Phi_{L_2,L_3} \circ \Phi_{L_1,L_2}$.

Proof. (1) We have $\Phi_{L^*,L} = \Pi_{L,L^*} \circ \Pi_{L^*,L} \circ \Pi_{L^*,L^*} = \Pi_{L,L^*} \circ \Pi_{L^*,L} = 1_{L^*}$.

(2) We have $\Phi^{-1}_{L_1,L_2} = \Big(\Pi_{L_2,L^*} \circ \Pi_{L_1,L_2} \circ \Pi_{L^*,L_1}\Big)^{-1} = \Pi^{-1}_{L^*,L_1} \circ \Pi^{-1}_{L_1,L_2} \circ \Pi^{-1}_{L_2,L^*} = \Pi_{L_1,L^*} \circ \Pi_{L_2,L_1} \circ \Pi_{L^*,L_2} = \Phi_{L_2,L_1}$.

(3) If L_2 is on a shortest path from L_1 to L_3, then $\Pi_{L_1,L_3} = \Pi_{L_2,L_3} \circ \Pi_{L_1,L_2}$ and this implies that $\Phi_{L_1,L_3} = \Phi_{L_2,L_3} \circ \Phi_{L_1,L_2}$. □

Lemma 12.

(1) *If* $L_1, L_2 \in S$ *such that* L_1 *is contained on a shortest path from* L^* *to* L_2 *in the near polygon* $\mathcal{S}'$, *then* Φ_{L_1,L_2} *is the identical permutation of* L^*. *In particular,* $\Phi_{L,L}$ *is the identical permutation of* L^* *for every line* $L \in S$.

(2) *Let* Q *and* Q' *be two opposite quads of* $\mathcal{Q}$ *(i.e. two opposite lines of* $\mathcal{S}'$*). Let* L_1 *and* L_2 *be two lines of* S *contained in* Q, *and put* $L_1' := \pi_{Q'}(L_1) \in S$ *and* $L_2' := \pi_{Q'}(L_2) \in S$. *Then* $\Phi_{L_1,L_2'} = \Phi_{L_1',L_2'} \circ \Phi_{L_1,L_1'} = \Phi_{L_2,L_2'} \circ \Phi_{L_1,L_2}$.

Proof. (1) From Lemma 11(3), we know that $\Phi_{L^*,L_2} = \Phi_{L_1,L_2} \circ \Phi_{L^*,L_1}$. From Lemma 11(1), we know that $\Phi_{L^*,L_1} = \Phi_{L^*,L_2} = 1_{L^*}$. Hence, $\Phi_{L_1,L_2} = 1_{L^*}$.

(2) Since every quad of $\mathcal{Q}$ is classical, we observe that L_1' and L_2 are on shortest path from L_1 to L_2' in the near polygon $\mathcal{S}'$. The remaining part of (2) then follows from Lemma 11(3). □

Lemma 13. $\mathcal{S}$ *can be completely described in terms of* $\mathcal{S}'$ *and the maps* Φ_{L_1,L_2}, *where* L_1 *and* L_2 *are two lines of* S *at distance* 1 *from each other.*

Proof. Consider two points with labels (K_1, x) and (K_2, y). If $K_1 = K_2$, then the points (K_1, x) and (K_2, y) are collinear. If K_1, K_2 are distinct collinear points of $\mathcal{S}'$, then (K_1, x) and (K_2, y) are collinear if and only if $y = \Phi_{K_1,K_2}(x)$. If K_1, K_2 are distinct noncollinear points in $\mathcal{S}'$, then (K_1, x) and (K_2, y) are not collinear. So, the collinearity graph of $\mathcal{S}$ and hence also $\mathcal{S}$ itself can be completely described in terms of $\mathcal{S}'$ and the maps Φ_{L_1,L_2}, where L_1 and L_2 are two points of $\mathcal{S}'$ at distance 1 from each other. □

Suppose $\widetilde{\mathcal{S}}$ is a near polygon isomorphic to $\mathcal{S}'$, $\widetilde{X}$ is a set of the same cardinality of L^*, and $\widetilde{\Phi}$ is a map which associates with each pair (x_1, x_2) of distinct collinear points of $\widetilde{\mathcal{S}}$ a permutation $\widetilde{\Phi}(x_1, x_2)$ of $\widetilde{X}$. Suppose θ is an isomorphism from $\widetilde{\mathcal{S}}$ to $\mathcal{S}'$ and ϕ is a bijection of $\widetilde{X}$ to L^* such that $\widetilde{\Phi}(x_1, x_2) = \phi^{-1} \circ \Phi_{x_1^\theta, x_2^\theta} \circ \phi$ for every pair (x_1, x_2) of distinct collinear points of $\widetilde{\mathcal{S}}$. If $\widetilde{p}$ is the unique point of $\widetilde{\mathcal{S}}$ for which $\widetilde{p}^\theta = L^*$, then we call the quadruple $(\widetilde{\mathcal{S}}, \widetilde{X}, \widetilde{p}, \widetilde{\Phi})$ *polygonal data* for the polygonal triple $(\mathcal{S}, S, \mathcal{Q})$. Note that $(\mathcal{S}', L^*, L^*, \Phi')$ is polygonal data for $(\mathcal{S}, S, \mathcal{Q})$, where Φ' is the restriction of Φ to distinct collinear points of $\mathcal{S}'$, the former L^* is regarded as set of points of $\mathcal{S}$ and the latter is regarded as point of $\mathcal{S}'$.

Having this polygonal data $(\widetilde{\mathcal{S}}, \widetilde{X}, \widetilde{p}, \widetilde{\Phi})$, it is possible (by using Lemma 13) to reconstruct an isomorphic copy[1] $(\mathcal{S}_1, S_1, \mathcal{Q}_1)$ of $(\mathcal{S}, S, \mathcal{Q})$ in the following way.

- $\mathcal{S}_1$ is the near polygon with points the pairs (a, x), with a a point of $\widetilde{\mathcal{S}}$ and $x \in \widetilde{X}$, where two distinct points (a_1, x_1) and (a_2, x_2) are collinear whenever either $a_1 = a_2$ or $(\mathrm{d}_{\widetilde{\mathcal{S}}}(a_1, a_2) = 1$ and $x_2 = x_1^{\widetilde{\Phi}(a_1,a_2)})$.
- S_1 consists of all lines of the form $\{(a, x) \mid x \in \widetilde{X}\}$, with a a point of $\widetilde{\mathcal{S}}$.
- $\mathcal{Q}_1$ consists of all quads of $\mathcal{S}_1$ containing a line of S_1.

Classifying particular polygonal triples is thus equivalent with determining the corresponding polygonal data. The proof of Theorem 2 will make use of this observation.

5.2 Properties of polygonal data

The following lemma is a consequence of Lemmas 11 and 12.

Lemma 14. *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data for a polygonal triple. Then the following hold:*

(a) $\Phi(p, x)$ is the identical permutation 1_X of X for every point x of $\mathcal{S}$ at distance 1 from p.

(b) If x_1 and x_2 are two distinct collinear points of $\mathcal{S}$, then $\Phi(x_1, x_2) = \Phi(x_2, x_1)^{-1}$.

(c) If x_1 and x_2 are two distinct collinear points of $\mathcal{S}$ such that the unique point of x_1x_2 nearest to p coincides with either x_1 or x_2, then $\Phi(x_1, x_2) = 1_X$.

(d) If $u = x_1, x_2, \ldots, x_k = v$ and $u = y_1, y_2, \ldots, y_k = v$ are two shortest paths connecting the points u and v, then

$$\Phi(x_{k-1}, x_k)\circ\Phi(x_{k-2}, x_{k-1})\circ\cdots\circ\Phi(x_1, x_2) = \Phi(y_{k-1}, y_k)\circ\Phi(y_{k-2}, y_{k-1})\circ\cdots\circ\Phi(y_1, y_2).$$

[1]This means that there is an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}$ mapping S_1 to S and $\mathcal{Q}_1$ to $\mathcal{Q}$.

Lemma 15. *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data and L_1, L_2 are two parallel lines of $\mathcal{S}$ such that $p \in L_1$. Then $\Phi(u,v) = \Phi(\pi_{L_1}(u), \pi_{L_1}(v))$ for any two distinct points u and v of L_2.*

Proof. Put $\pi_{L_1}(u) = u'$ and $\pi_{L_1}(v) = v'$. If w_1 and w_2 are two consecutive points on a shortest path from u' to u (or from v' to v), then $\Phi(w_1, w_2) = 1_X$ by Lemma 14(c). Now, consider two shortest paths connecting u' and v, one that passes through the point u and another that passes through v'. If we apply Lemma 14(d) to these two paths, then we find $\Phi(u,v) = \Phi(u', v')$ if we take into account that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points w_1 and w_2 on these paths for which the line $w_1 w_2 \notin \{L_1, L_2\}$. □

We now prove a number of useful lemmas. In these lemmas, the following notation is used. If L is a line of a near polygon, then $\mathcal{C}_L$ denotes the set of all ordered pairs of distinct points of L.

Lemma 16. *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is a generalized hexagon with at least three lines through each point and at least three points on each line. Let L be an arbitrary line through p. Then Φ is uniquely determined by the values it takes on $\mathcal{C}_L$.*

Proof. Let $\mathcal{L}$ denote the line set of $\mathcal{S}$. We write $\mathcal{L} \setminus \{L\}$ as a disjoint union $\mathcal{L}_{0,0} \cup \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1} \cup \mathcal{L}_{2,1} \cup \mathcal{L}_{2,2}$, where $\mathcal{L}_{i,j}$ denotes the set of lines of $\mathcal{L} \setminus \{L\}$ at distance i from p and at distance j from L. Suppose we know all values of Φ on the set $\mathcal{C}_L$. We gradually show how these values uniquely determine Φ on each $\mathcal{C}_K$, $K \in \mathcal{L} \setminus \{L\}$.

Step 1. If $K \in \mathcal{L}_{2,2}$, then for any $(x,y) \in \mathcal{C}_K$, we have $\Phi(x,y) = \Phi(\pi_L(x), \pi_L(y))$ by Lemma 15.

Step 2. If $K \in \mathcal{L}_{0,0}$, then Lemma 15 implies that for any $(x,y) \in \mathcal{C}_K$, we have $\Phi(x,y) = \Phi(\pi_M(x), \pi_M(y))$, where M is a line of $\mathcal{L}_{2,2}$ opposite to K and L. Such a line M can be constructed in the following way. Let M'' be a line through p distinct from K and L, let M' be a line intersecting M'' in a singleton distinct from $\{p\}$ and let M be a line meeting M' in a singleton distinct from $M' \cap M''$.

Step 3. If $K \in \mathcal{L}_{2,1}$, then Lemma 15 implies that for any $(x,y) \in \mathcal{C}_K$, we have $\Phi(x,y) = \Phi(\pi_M(x), \pi_M(y))$, where M is any line of $\mathcal{L}_{0,0}$ opposite to K.

Step 4. Suppose $K \in \mathcal{L}_{1,1}$. Let u_1 denote the unique point of K collinear with p and let u_2 denote a point at distance 1 from pu_1 such that the unique point of pu_1 collinear u_2 is distinct from p and u_1. Such a point exists as there are at least three points on the line pu_1. Let M be a line through u_2 opposite to K. Let $(x,y) \in \mathcal{C}_K$, put $x'' := \pi_M(x)$, $y'' := \pi_M(y)$, $\{x'\} := \Gamma_1(x) \cap \Gamma_1(x'')$ and $\{y'\} := \Gamma_1(y) \cap \Gamma_1(y'')$. If one of x, y coincides with u_1, then $\Phi(x,y) = 1_X$ by Lemma 14(c). Suppose therefore that $x \neq u_1 \neq y$. Then $\Phi(x', x) = \Phi(y, y') = 1_X$ by Lemma 14(c). Lemma 14(d) then implies that $\Phi(x,y) = \Phi(y,y') \circ \Phi(x,y) \circ \Phi(x',x) = \Phi(y'',y') \circ \Phi(x'',y'') \circ \Phi(x',x'')$. Note that the lines $y''y'$, $M = x''y''$ and $x'x''$ belong to $\mathcal{L}_{2,2} \cup \mathcal{L}_{2,1}$ since $y', x' \in \Gamma_3(p)$ and $\mathrm{d}(p, M) = 2$.

Step 5. Suppose $K \in \mathcal{L}_{1,0}$. In an ordinary 6-gon containing p and K, we can take a line M opposite to K. Then $M \in \mathcal{L}_{1,1}$ and for every $(x,y) \in \mathcal{C}_K$, we have $\Phi(x,y) =$

$\Phi(\pi_M(x), \pi_M(y))$. Indeed, if $\Gamma_1(x) \cap \Gamma_1(\pi_M(x)) = \{x'\}$ and $\Gamma_1(y) \cap \Gamma_1(\pi_M(y)) = \{y'\}$, then Lemma 14(c) implies that $\Phi(x', x) = \Phi(x', \pi_M(x)) = \Phi(y', y) = \Phi(y', \pi_M(y)) = 1_X$. Lemma 14(d) then implies that $\Phi(x, y) = \Phi(y, y') \circ \Phi(x, y) \circ \Phi(x', x) = \Phi(\pi_M(y), y') \circ \Phi(\pi_M(x), \pi_M(y)) \circ \Phi(x', \pi_M(x)) = \Phi(\pi_M(x), \pi_M(y))$. □

Lemma 17. *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is a generalized octagon with at least three lines through each point and at least three points on each line. Let L be an arbitrary line through p. Then Φ is uniquely determined by the values it takes on $\mathcal{C}_L$.*

Proof. We write $\mathcal{L} \setminus \{L\}$ as a disjoint union $\mathcal{L}_{0,0} \cup \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1} \cup \mathcal{L}_{2,1} \cup \mathcal{L}_{2,2} \cup \mathcal{L}_{3,2} \cup \mathcal{L}_{3,3}$, where $\mathcal{L}_{i,j}$ denotes the set of lines of $\mathcal{L} \setminus \{L\}$ at distance i from p and at distance j from L. Suppose we know all values of Φ on the set $\mathcal{C}_L$. We gradually show how these values uniquely determine Φ on each $\mathcal{C}_K$, $K \in \mathcal{L} \setminus \{L\}$.

Step 1. If $K \in \mathcal{L}_{3,3}$, then for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_L(x), \pi_L(y))$ by Lemma 15.

Step 2. If $K \in \mathcal{L}_{0,0}$, then for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$, where M is a line of $\mathcal{L}_{3,3}$ opposite to K and L. Similarly as in Step 2 of Lemma 16, we can construct such a line opposite to K and L, by starting from a line through p distinct from K and L.

Step 3. If $K \in \mathcal{L}_{3,2}$, then Lemma 15 implies that for any $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$, where M is any line of $\mathcal{L}_{0,0}$ opposite to K.

Step 4. Suppose $K \in \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1}$. Let u_1 denote the unique point of K collinear with p and let u_2 denote a point at distance 2 from pu_1 such that the unique point of pu_1 nearest to u_2 is distinct from p and u_1. Such a point exists since the line pu_1 contains at least three points. Let M be a line through u_2 opposite to K. Let $(x, y) \in \mathcal{C}_K$, put $x''' := \pi_M(x)$, $y''' := \pi_M(y)$, $\{x'\} := \Gamma_1(x) \cap \Gamma_2(x''')$, $\{y'\} := \Gamma_1(y) \cap \Gamma_2(y''')$, $\{x''\} := \Gamma_2(x) \cap \Gamma_1(x''')$ and $\{y''\} := \Gamma_2(y) \cap \Gamma_1(y''')$. If one of x, y coincides with u_1, then $\Phi(x, y) = 1_X$ by Lemma 14(c). Suppose therefore that $x \neq u_1 \neq y$. By Lemma 14(c), we know that $\Phi(y, y') = \Phi(y', y'') = \Phi(x, x') = \Phi(x', x'') = 1_X$. With a similar reasoning as in Step 4 of Lemma 16, this allows to conclude that $\Phi(x, y) = \Phi(y''', y'') \circ \Phi(x''', y''') \circ \Phi(x'', x''')$. Note that the lines $y'''y''$, $x'''y''' = M$ and $x''x'''$ belong to $\mathcal{L}_{3,3} \cup \mathcal{L}_{3,2}$ since $y'', x'' \in \Gamma_4(p)$ and $\mathrm{d}(p, M) = 3$.

Step 5. Suppose $K \in \mathcal{L}_{2,1} \cup \mathcal{L}_{2,2}$. In an ordinary 8-gon containing p, K and L, we can take a line M opposite to K. Then $M \in \mathcal{L}_{1,0} \cup \mathcal{L}_{1,1}$ for every $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$. The proof of that claim is similar to the proof of Step 5 in Lemma 16, taking into account that Lemma 14(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points w_1 and w_2 on a shortest path connecting x with $\pi_M(x)$ (or y with $\pi_M(y)$). □

Lemma 18. *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is a generalized $2d$-gon, $d \in \{3, 4\}$, with exactly two lines through each point and at least three points on each line. Let L_1 and L_2 be the two lines through p. Then Φ is uniquely determined by the values it takes on $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$.*

Proof. We follow the same notational convention as in the proofs of Lemmas 16 and 17. In particular, we write $\mathcal{L} \setminus \{L\}$ as the disjoint union of the mentioned sets, where $L := L_1$. Note that $\mathcal{L}_{0,0}$ is the singleton $\{L_2\}$. We observe now that the proofs of Lemmas 16 and 17 only break down at one point, namely in Step 2, where it is no longer possible to choose a line $M \in \mathcal{L}_{d-1,d-1}$ opposite to K and L. However, by assuming that the values of Φ on the set $\mathcal{C}_{L_2}$ are also known, the rest of the proof can remain without any changes. □

Lemma 19. *Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is a generalized dodecagon with exactly two lines through each point and at least three points on each line. Let L_1 and L_2 be the two lines through p. Then Φ is uniquely determined by the values it takes on $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$.*

Proof. We write $\mathcal{L} \setminus \{L_1, L_2\}$ as a disjoint union $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4 \cup \mathcal{L}_5$, where $\mathcal{L}_i$ with $i \in \{1, 2, \ldots, 5\}$ is the set of lines of $\mathcal{L} \setminus \{L_1, L_2\}$ at distance i from p. Suppose we know all values of Φ on the set $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$. We gradually show how these values uniquely determine Φ on each $\mathcal{C}_K$, $K \in \mathcal{L} \setminus \{L_1, L_2\}$.

Step 1. Suppose $K \in \mathcal{L}_5$. Let $i \in \{1, 2\}$ such that K and L_i are opposite lines. Then $\Phi(x, y) = \Phi(\pi_{L_i}(x), \pi_{L_i}(y))$ by Lemma 15.

Step 2. Suppose $K \in \mathcal{L}_1$. Then K meets one of L_1, L_2, say L_i, in a point u_1. Let u_2 denote a point at distance 4 from pu_1 such that the unique point of pu_1 nearest to u_2 is distinct from p and u_1. Such a point exists since the line $pu_1 = L_i$ contains at least three points. Let M be a line through u_2 opposite to K. Let $(x, y) \in \mathcal{C}_K$, put $x'' := \pi_M(x)$, $y'' := \pi_M(y)$, $\{x'\} := \Gamma_1(x'') \cap \Gamma_4(x)$, $\{y'\} := \Gamma_1(y'') \cap \Gamma_4(y)$. If one of x, y coincides with u_1, then $\Phi(x, y) = 1_X$ by Lemma 14(c). Suppose therefore that $x \neq u_1 \neq y$. With a similar reasoning as in the proof of Step 4 in Lemma 16, we have $\Phi(x, y) = \Phi(y'', y') \circ \Phi(x'', y'') \circ \Phi(x', x'')$. Indeed, Lemma 14(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from x to x' (or from y to y'). Note that the lines $y''y'$, $x''y'' = M$ and $x'x''$ belong to $\mathcal{L}_5$ since $y', x' \in \Gamma_6(p)$ and $\mathrm{d}(p, M) = 5$.

Step 3. Suppose $K \in \mathcal{L}_4$. In an ordinary 12-gon containing p and K, we can take a line M opposite to K. Then $M \in \mathcal{L}_1$ and for every $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$. The proof of that claim is similar to the proof of Step 5 in Lemma 16, taking into account that Lemma 14(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from x to $\pi_M(x)$ (or from y to $\pi_M(y)$).

Step 4. Suppose $K \in \mathcal{L}_2$. Then there is a unique line meeting one of L_1, L_2 in a point u_1 and K in a point u_1'. Let u_2 denote a point at distance 4 from u_1u_1' such that the unique point of u_1u_1' nearest to u_2 is distinct from u_1 and u_1'. Such a point exists since the line u_1u_1' contains at least three points. Let M be a line through u_2 opposite to K. Let $(x, y) \in \mathcal{C}_K$. Put $x''' := \pi_M(x)$, $y''' := \pi_M(y)$, $\{x''\} := \Gamma_1(x''') \cap \Gamma_4(x)$, $\{x'\} := \Gamma_2(x''') \cap \Gamma_3(x)$, $\{y''\} := \Gamma_1(y''') \cap \Gamma_4(y)$, $\{y'\} := \Gamma_2(y''') \cap \Gamma_3(y)$. If one of x, y coincides with u_1', then $\Phi(x, y) = 1_X$. Suppose therefore that $x \neq u_1' \neq y$. With a similar reasoning as in the proof of Step 4 in Lemma 16, we have $\Phi(x, y) = \Phi(y'', y') \circ \Phi(y''', y'') \circ \Phi(x''', y''') \circ \Phi(x'', x''') \circ \Phi(x', x'')$.

Indeed, Lemma 14(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from x to x' (or from y to y'). Note that the lines $y''y'$, $y'''y''$, $x'''y''' = M$, $x''x'''$, $x'x''$ belong to $\mathcal{L}_4 \cup \mathcal{L}_5$ since $u_2, x', y' \in \Gamma_6(p)$ and $x'', y'' \in \Gamma_5(p) \cup \Gamma_6(p)$.

Step 5. Suppose $K \in \mathcal{L}_3$. In an ordinary 12-gon containing p and K, we can take a line M opposite to K. Then $M \in \mathcal{L}_2$ and for every $(x, y) \in \mathcal{C}_K$, we have $\Phi(x, y) = \Phi(\pi_M(x), \pi_M(y))$. The proof of that claim is similar to the proof of Step 5 in Lemma 16, taking into account that Lemma 14(c) implies that $\Phi(w_1, w_2) = 1_X$ for any two consecutive points on a shortest path from x to $\pi_M(x)$ (or from y to $\pi_M(y)$). □

Suppose $(\mathcal{S}, X, p, \Phi)$ is polygonal data, where $\mathcal{S}$ is some generalized polygon. If all points of $\mathcal{S}$ are incident with at least three lines and L is a line through p, then the quadruple $(\mathcal{S}, X, p, \Phi')$, where Φ' is the restriction of Φ to the set $\mathcal{C}_L$, is called *partial polygonal data*. If every point of $\mathcal{S}$ is incident with precisely two lines and L_1, L_2 are the two lines through the point p, then the quadruple $(\mathcal{S}, X, p, \Phi')$, where Φ' is the restriction of Φ to the set $\mathcal{C}_{L_1} \cup \mathcal{C}_{L_2}$, is called *partial polygonal data*.

Suppose $(\mathcal{S}, X, p, \Phi')$ is partial polygonal data, where $\mathcal{S}$ is one of the generalized polygons under consideration in Lemmas 16, 17, 18 and 19. Using the algorithms exposed in the proofs of these lemmas, it is possible to reconstruct the whole polygonal data, and hence to construct an isomorphic copy of the polygonal triple from which $(\mathcal{S}, X, p, \Phi')$ arose. This approach will be followed during the proof of Theorem 2.

6 Proof of Theorem 2

The intention of this section is to prove Theorem 2. During this proof, we will use the following notation. Suppose η is a function on two arguments belonging to the same set $\{x_1, x_2, x_3, x_4, x_5\}$ of size 5. Then $\mathcal{T}(\eta, (x_1, x_2, x_3, x_4, x_5))$ denotes the 5×5 table whose entry in the i-th row and the j-th column is equal to $\eta(x_i, x_j)$, where we put a "–" if $\eta(x_i, x_j)$ is not defined.

By Section 4, we already know that there exists no polygonal triple whose quads are isomorphic to $W(2)$ and whose associated near polygon is isomorphic to either $T(4, 64)$ and $GO(4, 1)$.

Using the algorithms involving (partial) polygonal data discussed in Section 5, we now classify all polygonal triples whose quads are isomorphic to $W(2)$ and whose associated near polygons $\mathcal{A}$ are isomorphic to either $H^D(4)$, $GH(4, 1)$, $H(4)$, $GO(4, 1)$, $RT(4, 2)$, $\mathcal{F}(H(4))$ or $\mathcal{F}(H^D(4))$. The case where $\mathcal{A}$ is isomorphic to $T(4, 64)$ will not be treated here as it seems to be out of reach of our computer computations. Note that in each case, $\mathcal{A}$ is either a generalized hexagon, octagon or dodecagon for which one of the lemmas 16, 17, 18, 19 is applicable.

For each generalized polygon $\mathcal{A} \in \{H^D(4), H(4), RT(4, 2)\}$, the following lemma shows that there is essentially one quadruple that can serve as potential[2] partial polygonal data

[2]We are not sure in advance whether there is a polygonal triple associated with the quadruple; if there is one, then the quadruple should be polygonal data.

for the problem. Note also that the flag-transitivity of $\mathcal{A}$ implies that there is essential one choice for the pair (p, L) in the lemma.

Lemma 20. *Suppose $(\mathcal{A}, X, p, \Phi)$ is polygonal data for a polygonal triple $(\mathcal{S}, S, \mathcal{Q})$, where each quad of $\mathcal{Q}$ is isomorphic to $W(2)$ and*

$$\mathcal{A} \in \{H^D(4), H(4), RT(4,2), GH(4,1), GO(4,1), \mathcal{F}(H(4)), \mathcal{F}(H^D(4))\}.$$

If $L = \{p, x_1, x_2, x_3, x_4\}$ is a line of $\mathcal{A}$ through p, then the elements of X can be labeled with a, b and c such that $\mathcal{T}(\Phi, (p, x_1, x_2, x_3, x_4))$ is the following table:

–	1_X	1_X	1_X	1_X
1_X	–	(a,b)	(a,c)	(b,c)
1_X	(a,b)	–	(b,c)	(a,c)
1_X	(a,c)	(b,c)	–	(a,b)
1_X	(b,c)	(a,c)	(a,b)	–

Proof. The point p corresponds to a line $L^* \in S$ and the line L with a quad $Q \in \mathcal{Q}$ through L^*. Let L^*, L_1, L_2, L_3 and L_4 denote the five lines of S contained in Q. For two distinct $K, M \in \{L^*, L_1, L_2, L_3, L_4\}$, let $\Phi_{K,M}$ be as defined in Section 5.1. Obviously, $\Phi_{K,M}$ is the trivial permutation if $L^* \in \{K, M\}$.

Let K and M be distinct elements of $\{L_1, L_2, L_3, L_4\}$. Then there exists a unique 3×3 subgrid of Q containing K and M, and each of the remaining three lines of $\{L^*, L_1, L_2, L_3, L_4\}$ meets this subgrid in a unique point. So, there is a unique line meeting L^*, K and M, implying that $\Phi_{K,M}$ has a unique fixed point, i.e. $\Phi_{K,M}$ is a transposition. Put $L^* = \{x_1, x_2, x_3\}$ and $K \in \{L_1, L_2, L_3, L_4\}$. Then there is a unique line (of size 3) through each x_i meeting K, showing that the permutation (x_{i+1}, x_{i+2}) (with subindices taken modulo 3) occurs exactly once in the row and column corresponding to the line K. The lemma now follows if we take also into account that $\Phi_{K,M} = \Phi_{M,K}^{-1}$ for two distinct lines K and M of S. □

For each generalized polygon $\mathcal{A} \in \{GH(4,1), GO(4,1), \mathcal{F}(H(4)), \mathcal{F}(H^D(4))\}$, the following lemma, which is an immediate consequence of Lemma 20, shows that several quadruples might serve as potential polygonal data for the problem, namely one for each permutation τ of the set $\{1, 2, 3, 4\}$. Note also that the vertex-transitivity of $\mathcal{A}$ implies that there is essential one choice for the point p in the lemma.

Lemma 21. *Suppose $(\mathcal{A}, X, p, \Phi)$ is polygonal data for a polygonal triple $(\mathcal{S}, S, \mathcal{Q})$, where each quad of $\mathcal{Q}$ is isomorphic to $W(2)$ and $\mathcal{A} \in \{GH(4,1), GO(4,1), \mathcal{F}(H(4)), \mathcal{F}(H^D(4))\}$. If $L_1 = \{p, x_1, x_2, x_3, x_4\}$ and $L_2 = \{p, y_1, y_2, y_3, y_4\}$ are the two lines of $\mathcal{A}$ through p, then the elements of X can be labeled with a, b and c such that $\mathcal{T}(\Phi, (p, x_1, x_2, x_3, x_4))$ and $\mathcal{T}(\Phi, (p, y_{\tau(1)}, y_{\tau(2)}, y_{\tau(3)}, y_{\tau(4)}))$ are equal to the table mentioned in Lemma 20 for a certain permutation τ of $\{1, 2, 3, 4\}$.*

Now that we know all potential partial polygonal data, we can construct the potential polygonal data, following the algorithms exposed in Lemmas 16, 17, 18 and 19. For that purpose, we have implemented a computer program in GAP [18], see [10]. It should be remarked that there is no unique way to reconstruct the potential polygonal data from the potential partial polygonal data. During the reconstruction process, certain choices need to be made which are not unique. E.g., if one needs to take a line opposite a given line, there are usual several choices for that, and the “complete polygonal data” might depend on the choices made during this reconstruction process. However, if the “partial polygonal data” is associated with a polygonal triple, then we know that the complete polygonal data one obtains has to be independent of the choices made during the reconstruction process. As our purpose is to classify polygonal triples, we do not have to worry about this complication, and we can make any choices we like during the reconstruction process.

Once we have obtained the “complete polygonal data”, we have followed the algorithm exposed in Lemma 13 to build the graph which – in case of an associated polygonal triple T – must be isomorphic to the collinearity graph of the near polygon that occurs as first component of T. Subsequently, we have verified whether this graph was indeed the collinearity graph of a near polygon.

This turned out to be the case if $\mathcal{A} = H^D(4)$ (as it should be), but not if $\mathcal{A} = H(4)$ or $\mathcal{A} = RT(4,2)$. In the case $\mathcal{A} = H^D(4)$, the graph must be isomorphic to the collinearity graph of the $G_2(4)$ near octagon. In this case, there is essentially one quadruple that can serve as partial polygonal data and so we already knew in advance, without making any computer computations, that the $G_2(4)$ near octagon is the unique near octagon admitting a polygonal triple all whose quads are isomorphic to $W(2)$ and for which the associated near polygon is isomorphic to $H^D(4)$.

When $\mathcal{A}$ is equal to either $GO(4,1)$, $\mathcal{F}(H(4))$ or $\mathcal{F}(H^D(4))$, the graph was not the collinearity graph of a near polygon for any of the 24 choices of the permutation τ.

In case $\mathcal{A}$ is equal to the unique generalized hexagon $GH(4,1)$ of order $(4,1)$, this graph turned out to be the collinearity graph of a near polygon for precisely 12 of the 24 possible choices of τ. These 12 permutations turn out to have the same parity. The twelve near octagons that arise in this way are all isomorphic since the potential partial polygonal data from which they are derived are all equivalent. The latter follows from the symmetry of the generalized hexagon $GH(4,1)$ exposed in Lemma 22(b) below.

Lemma 22.

(*a*) *Let* $\{x, L\}$ *be a flag of the projective plane* $\mathrm{PG}(2,4)$ *and let* H *denote the group of automorphisms of* $\mathrm{PG}(2,4)$ *fixing each point of* L*. If* $\mathcal{L}$ *denotes the set of four lines through* x *distinct from* L*, then each* $h \in H$ *induces a permutation* $\bar{h}$ *of* $\mathcal{L}$*. The group* $\{\bar{h} \mid h \in H\}$ *of permutations of* $\mathcal{L}$ *consists of all even permutations of this set.*

(*b*) *Let* p *be a point of* $GH(4,1)$*, and* L_1, L_2 *be the two lines of* $GH(4,1)$ *through* p*. Let* H *denote the group of automorphisms of* $GH(4,1)$ *fixing each point of the line* L_1*. Then each* $h \in H$ *induces a permutation* $\bar{h}$ *of* $L_2 \setminus \{p\}$*. Then the group* $\{\bar{h} \mid h \in H\}$ *of permutations of* $L_2 \setminus \{p\}$ *consists of all even permutations of this set.*

Proof. (a) This is easily verified. Each $h \in H$ is either an elation or a homology. If h is an elation, then $\bar{h}$ is the product of two disjoint transpositions. If h is a homology, then $\bar{h}$ is a cycle of length 3.

(b) Since the automorphism group of $GH(4,1)$ acts transitively on the set of lines of the generalized hexagon, we may without loss of generality suppose that L_1 is a line of L of $\mathrm{PG}(2,4)$. Then p is a certain flag $\{x, L\}$ of $\mathrm{PG}(2,4)$. The automorphisms of $GH(4,1)$ that fix each point of L_1 bijectively correspond with the automorphisms of $\mathrm{PG}(2,4)$ that fix each point of L. The points of $L_2 \setminus \{p\}$ are the four flags $\{x, K\}$ where K is one of the four lines through x distinct from L. The lemma then follows from Claim (a). $\square$

In the case that $\mathcal{A} = GH(4,1)$, these twelve near octagons thus have to be isomorphic to the $L_3(4)$ near octagon.

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