# Matching complexes of $3 \times \mathrm{n}$ grid graphs 

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#### Abstract

The matching complex of a graph $G$ is a simplicial complex whose simplices are matchings in $G$. In the last few years the matching complexes of grid graphs have gained much attention among the topological combinatorists. In 2017, Braun and Hough obtained homological results related to the matching complexes of $2 \times n$ grid graphs. Further in 2019, Matsushita showed that the matching complexes of $2 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. In this article we prove that the matching complexes of $3 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. We also give the comprehensive list of the dimensions of spheres appearing in the wedge.


Mathematics Subject Classifications: 05E45, 55P15

## 1 Introduction

A matching in a (simple) graph $G$ is a collection of pairwise disjoint edges of $G$. The matching complex of $G$, denoted $M(G)$, is a simplicial complex whose vertex set is the edge set of $G$ and simplices are all the matchings in $G$. The matching complexes first appeared in the 1979 work of Garst [4], where the matching complexes of complete bipartite graphs (also known as the chessboard complexes) were studied while dealing with the Tits coset complexes. In 1992, Bouc [1] studied the matching complexes of complete graphs
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in connection with the Brown complexes and Quillen complexes. Thereafter, these complexes arose in connection with several areas of mathematics. For a broader perspective, see the 2003 survey article of Wachs [10].

Let $[n]$ denotes the set $\{1,2, \ldots, n\}$. For two positive integers $m, n$, the $m \times n$ (rectangular) grid graph $\Gamma_{m, n}$ is a graph with vertex set $V\left(\Gamma_{m, n}\right)$ and edge set $E\left(\Gamma_{m, n}\right)$ defined as follows:

$$
\begin{aligned}
& V\left(\Gamma_{m, n}\right)=\left\{(i, j) \in \mathbb{N}^{2}: i \in[m], j \in[n]\right\}, \quad \text { and } \\
& E\left(\Gamma_{m, n}\right)=\left\{\left((i, j),\left(i^{\prime}, j^{\prime}\right)\right):\left|i-i^{\prime}\right|+\left|j-j^{\prime}\right|=1\right\} .
\end{aligned}
$$

The matching complex of $\Gamma_{1, n}$ was computed by Kozlov in [8]. In the same article, he also computed the matching complexes of cycle graphs. In 2005, Jonsson [6] studied the homotopical depth and topological connectivity of matching complexes of grid graphs and stated that "it is probably very hard to determine the homotopy type of" these complexes. In 2017, Braun and Hough [2] obtained homological results related to matching complexes of $2 \times n$ grid graphs. Matsushita [9], in 2019, extended their results by showing that the matching complexes of $2 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. In this article, we compute the homotopy type of matching complexes of $3 \times n$ grid graphs. The main results of this article are summarised below.

Theorem 1. For $n \geqslant 1$, the matching complex of $\Gamma_{3, n}$ is homotopy equivalent to a wedge of spheres. Moreover, if $n \in\{9 k, 9 k+1, \ldots, 9 k+8\}$ for some $k \geqslant 0$, then

$$
M\left(\Gamma_{3, n}\right) \simeq \bigvee_{i=n-1}^{n+k-1}\left(\bigvee_{b_{i}} \mathbb{S}^{i}\right)
$$

where $b_{i}$ 's are some positive integers and $\simeq$ denotes the homotopy equivalence of spaces.
For a graph $G$, a subset $I \subseteq V(G)$ is said to be independent if there are no edges in the induced subgraph $G[I]$, i.e., $E(G[I])=\emptyset$. The independence complex of $G$, denoted $\operatorname{Ind}(G)$, is a simplicial complex whose vertex set is $V(G)$ and simplices are all the independent subsets of $G$. The line graph of a graph $G$, denoted $L(G)$, is a graph with $V(L(G))=E(G)$ and two distinct vertices $\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right) \in V(L(G))$ are adjacent if and only if $\left\{a_{1}, b_{1}\right\} \cap\left\{a_{2}, b_{2}\right\} \neq \emptyset$. Note that the matching complex of $G$ is same as the independence complex of its line graph, i.e., $M(G)=\operatorname{Ind}(L(G))$.

Let $G_{n}$ denotes the line graph of the grid graph $\Gamma_{3, n}$. To compute the homotopy type of $M\left(\Gamma_{3, n}\right)$, we determine the homotopy type of $\operatorname{Ind}\left(G_{n}\right)$. The main idea used in this article for the computation of $\operatorname{Ind}\left(G_{n}\right)$ is to make a step by step careful choice to reduce the graph $G_{n}$ and arrive at different classes of graphs (a total of nine). All these new nine classes of graphs have been defined in Section 3.1. To obtain the Theorem 1, we use simultaneous inductive arguments on the independence complexes of these ten classes of graphs. For a quick overview of the relations between all these ten classes of graphs, we refer the reader to see Figure 3.1.

Flow of the article: In the following section, we list out various definitions and results that are used in this article. Section 3 is subdivided into three major subsections. The first two subsections deal with the base cases for the graph $G_{n}$ along with nine more associated classes of graphs. In the next subsection, Section 3.3, we provide and prove
recursive formulae to compute the homotopy type of the independence complexes of these ten classes of graphs. The main result of Section 4 is Theorem 19, which gives the exact dimensions of the spheres occurring in the homotopy type of the independence complexes of the above mentioned ten classes of graphs.

## 2 Preliminaries

An (abstract) simplicial complex $\mathcal{K}$ is a collection of finite sets such that if $\tau \in \mathcal{K}$ and $\sigma \subset \tau$, then $\sigma \in \mathcal{K}$. The elements of $\mathcal{K}$ are called the simplices (or faces) of $\mathcal{K}$. If $\sigma \in \mathcal{K}$ and $|\sigma|=k+1$, then $\sigma$ is said to be $k$-dimensional. The set of 0 -dimensional simplices of $\mathcal{K}$ is denoted by $V(\mathcal{K})$, and its elements are called vertices of $\mathcal{K}$. A subcomplex of a simplicial complex $\mathcal{K}$ is a simplicial complex whose simplices are contained in $\mathcal{K}$. In this article, we always assume empty set as a simplex of any simplicial complex and we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer to Kozlov's book [7].

For a simplex $\sigma \in \mathcal{K}$, define

$$
\begin{aligned}
\operatorname{lk}(\sigma, \mathcal{K}) & :=\{\tau \in \mathcal{K}: \sigma \cap \tau=\emptyset, \sigma \cup \tau \in \mathcal{K}\}, \\
\operatorname{del}(\sigma, \mathcal{K}) & :=\{\tau \in \mathcal{K}: \sigma \nsubseteq \tau\} .
\end{aligned}
$$

The simplicial complexes $\operatorname{lk}(\sigma, \mathcal{K})$ and $\operatorname{del}(\sigma, \mathcal{K})$ are called link of $\sigma$ in $\mathcal{K}$ and (face) deletion of $\sigma$ in $\mathcal{K}$ respectively. The join of two simplicial complexes $\mathcal{K}_{1}$ and $\mathcal{K}_{2}$, denoted as $\mathcal{K}_{1} * \mathcal{K}_{2}$, is a simplicial complex whose simplices are disjoint union of simplices of $\mathcal{K}_{1}$ and $\mathcal{K}_{2}$. Let $\Delta^{S}$ denotes a $(|S|-1)$-dimensional simplex with vertex set $S$. The cone on $\mathcal{K}$ with apex $a$, denoted as $C_{a}(\mathcal{K})$, is defined as

$$
C_{a}(\mathcal{K}):=\mathcal{K} * \Delta^{\{a\}} .
$$

For $a, b \notin V(\mathcal{K})$, the suspension of $\mathcal{K}$, denoted as $\Sigma(\mathcal{K})$, is defined as

$$
\Sigma(\mathcal{K}):=\mathcal{K} *\{a\} \cup \mathcal{K} *\{b\}) .
$$

Observe that for any vertex $v \in V(\mathcal{K})$, we have

$$
\mathcal{K}=C_{v}(\operatorname{lk}(v, \mathcal{K})) \cup \operatorname{del}(v, \mathcal{K}) \text { and } C_{v}(\operatorname{lk}(v, \mathcal{K})) \cap \operatorname{del}(v, \mathcal{K})=\operatorname{lk}(v, \mathcal{K}) .
$$

Clearly, $C_{v}(\operatorname{lk}(v, \mathcal{K}))$ is contractible. Therefore, from [5, Example 0.14], we have the following.

Lemma 2. Let $\mathcal{K}$ be a simplicial complex and $v$ be a vertex of $\mathcal{K}$. If $\operatorname{lk}(v, \mathcal{K})$ is contractible in $\operatorname{del}(v, \mathcal{K})$ then

$$
\mathcal{K} \simeq \operatorname{del}(v, \mathcal{K}) \vee \Sigma(\operatorname{lk}(v, \mathcal{K})),
$$

where $\vee$ denotes the wedge of spaces.
A (simple) graph is an ordered pair $G=(V(G), E(G))$, where $V(G)$ is called the set of vertices and $E(G) \subseteq\binom{V(G)}{2}$, the set of (unordered) edges of $G$. The vertices $v_{1}, v_{2} \in V(G)$ are said to be adjacent, if $\left(v_{1}, v_{2}\right) \in E(G)$.

The following observation directly follows from the definition of independence complexes of graphs.

Lemma 3. Let $G_{1} \sqcup G_{2}$ denotes the disjoint union of two graphs $G_{1}$ and $G_{2}$. Then

$$
\operatorname{Ind}\left(G_{1} \sqcup G_{2}\right) \simeq \operatorname{Ind}\left(G_{1}\right) * \operatorname{Ind}\left(G_{2}\right)
$$

Let $G$ and $H$ be two graphs. A map $f: V(G) \rightarrow V(H)$ is said to be a graph homomorphism if $(f(v), f(w)) \in E(H)$ for all $(v, w) \in E(G)$. A graph homomorphism is called an isomorphism if it is bijective and its inverse map is also a graph homomorphism. Two graphs $G$ and $H$ are said to be isomorphic if there is an isomorphism between them and we denote it by $G \cong H$.

For a subset $A \subseteq V(G)$, the set of neighbours of $A$ is $N_{G}(A)=\{x \in V(G):(x, a) \in$ $E(G)$ for some $a \in A\}$. The closed neighbourhood set of $A \subseteq V(G)$, is $N_{G}[A]=N_{G}(A) \cup$ $A$. If $A=\{v\}$ is a singleton set, then we write $N_{G}(v)$ (resp. $\left.N_{G}[v]\right)$ for $N_{G}(\{v\})$ (resp. $\left.N_{G}[\{v\}]\right)$. A graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a subgraph of the graph $G$. For a nonempty subset $U \subseteq V(G)$, the induced subgraph $G[U]$, is the subgraph of $G$ with $V(G[U])=U$ and $E(G[U])=\{(a, b) \in E(G): a, b \in U\}$. The graph $G[V(G) \backslash A]$ is denoted by $G-A$, for $A \subsetneq V(G)$. For a subset $B \subseteq E(G)$, we let $G-B$ to be the graph with the vertex set $V(G-B)=V(G)$ and the edge set $E(G-B)=E(G) \backslash B$.

Lemma 4. Let $G$ be a graph and $\{a, b\}$ be a 1-simplex in $\operatorname{Ind}(G)$. If $\operatorname{Ind}\left(G-N_{G}[\{a, b\}]\right)$ is contractible, then

$$
\operatorname{Ind}(G) \simeq \operatorname{Ind}(\tilde{G})
$$

where $V(\tilde{G})=V(G)$ and $E(\tilde{G})=E(G) \cup\{(a, b)\}$.
Proof. Let $\sigma=\{a, b\}$. Observe that $\operatorname{del}(\sigma, \operatorname{Ind}(G))=\operatorname{Ind}(\tilde{G})$ and $\operatorname{lk}(\sigma, \operatorname{Ind}(G))=$ $\operatorname{Ind}(\tilde{G})=\operatorname{Ind}\left(G-N_{G}[\{a, b\}]\right)$. Since $\operatorname{Ind}\left(G-N_{G}[\{a, b\}]\right)$ is contractible, the result follows from Lemma 2.

Lemma 5. [3, Lemma 2.4] Let $G$ be a graph and $u, u^{\prime} \in V(G), u \neq u^{\prime}$ such that $N_{G}(u) \subseteq$ $N_{G}\left(u^{\prime}\right)$. Then

$$
\operatorname{Ind}(G) \simeq \operatorname{Ind}\left(G-\left\{u^{\prime}\right\}\right)
$$

Lemma 6. [3, Lemma 2.5] Let $G$ be graph and $v$ be a simplicial vertex ${ }^{1}$ of $G$. Let $N_{G}(v)=\left\{w_{1}, w_{2}, \ldots, w_{k}\right\}$. Then

$$
\operatorname{Ind}(G) \simeq \bigvee_{i=1}^{k} \Sigma\left(\operatorname{Ind}\left(G-N_{G}\left[w_{i}\right]\right)\right)
$$

For $r \geqslant 1$, the path graph $P_{r}$ is a graph with $V\left(P_{r}\right)=[r]$ and $E\left(P_{r}\right)=\{(i, i+1): i \in$ $[r-1]\}$.

Lemma 7. [8, Proposition 4.6] For $r \geqslant 1$,

$$
\operatorname{Ind}\left(P_{r}\right) \simeq \begin{cases}\mathbb{S}^{k-1} & \text { if } r=3 k \\ \mathrm{pt} & \text { if } r=3 k+1 \\ \mathbb{S}^{k} & \text { if } r=3 k+2\end{cases}
$$

[^0]For $r \geqslant 3$, the cycle graph $C_{r}$ is the graph with $V\left(C_{r}\right)=[r]$ and $E\left(C_{r}\right)=\{(i, i+1)$ : $i \in[r-1]\} \cup\{(1, r)\}$.

Lemma 8. [8, Proposition 5.2] For $r \geqslant 3$,

$$
\operatorname{Ind}\left(C_{r}\right) \simeq \begin{cases}\mathbb{S}^{k-1} \bigvee \mathbb{S}^{k-1} & \text { if } r=3 k \\ \mathbb{S}^{k-1} & \text { if } r=3 k \pm 1\end{cases}
$$

We now proceed towards the main graph of this article. Recall that for $m, n \in \mathbb{N}$, the $m \times n$ grid graph is denoted by $\Gamma_{m, n}$ and $G_{n}=L\left(\Gamma_{3, n}\right)$ denotes the line graph of $\Gamma_{3, n}$ (see Figure 2.1 for example). Formally we define $G_{n}$ with,

$$
\begin{aligned}
V\left(G_{n}\right)= & \left\{u_{i}, v_{j}, w_{i}, x_{j}, y_{i}: i \in[n-1], j \in[n]\right\}, \\
E\left(G_{1}\right)= & \left\{\left(v_{1}, x_{1}\right)\right\}, \\
E\left(G_{2}\right)= & \left\{\left(v_{1}, x_{1}\right),\left(w_{1}, v_{1}\right),\left(w_{1}, v_{1}\right),\left(w_{1}, x_{1}\right),\left(w_{1}, x_{2}\right),\left(u_{1}, v_{1}\right),\right. \\
& \left.\left(u_{1}, v_{2}\right),\left(v_{2}, x_{2}\right),\left(x_{1}, y_{1}\right),\left(y_{1}, x_{2}\right)\right\}, \text { and } \\
E\left(G_{n}\right)= & \left\{\left(u_{i}, v_{i}\right),\left(v_{i}, w_{i}\right),\left(w_{i}, x_{i}\right),\left(x_{i}, y_{i}\right),\left(y_{i}, x_{i+1}\right),\left(x_{i+1}, w_{i}\right),\left(w_{i}, v_{i+1}\right),\right. \\
& \left.\left(v_{i+1}, u_{i}\right): i \in[n-1]\right\} \sqcup\left\{\left(u_{i}, u_{i+1}\right),\left(w_{i}, w_{i+1}\right),\left(y_{i}, y_{i+1}\right): i \in[n-2]\right\} \sqcup \\
& \left\{\left(v_{i}, x_{i}\right): i \in[n]\right\} .
\end{aligned}
$$

for $n \geqslant 3$.


Figure 2.1

## 3 Homotopy type of independence complexes of $G_{n}$ and associated classes of graphs

In this section, we define nine new graph classes viz. $\left\{B_{n}\right\}_{n \in \mathbb{N}},\left\{A_{n}\right\}_{n \in \mathbb{N}},\left\{D_{n}\right\}_{n \in \mathbb{N}}$, $\left\{J_{n}\right\}_{n \in \mathbb{N}},\left\{O_{n}\right\}_{n \in \mathbb{N}},\left\{M_{n}\right\}_{n \in \mathbb{N}},\left\{Q_{n}\right\}_{n \in \mathbb{N}},\left\{F_{n}\right\}_{n \in \mathbb{N}},\left\{H_{n}\right\}_{n \in \mathbb{N}}$ and compute the homotopy type of their independence complexes along with that of $G_{n}$. The $n$-th member of each of these graph classes contains a copy of $G_{n}$ as an induced subgraph but not of $G_{n+1}$.

We divide this section into three subsections. In the first subsection, we define the above said nine classes of graphs and compute the homotopy type of independence complexes of these graphs along with $G_{n}$ for $n=1$. In the next subsection, we compute the homotopy type of independence complexes of all these graphs for $n=2$. The final subsection is devoted towards proving recursive formulae for the independence complexes
of all ten graph classes, thereby computing their homotopy types. In particular, we show that the independence complex of each of the graphs from these graph classes is a wedge of spheres up to homotopy.

For a better understanding of these inductive results, the recursive formulae of the homotopy equivalences obtained in Section 3.3 are depicted in Figure 3.1. Each node in Figure 3.1 denotes a graph class defined in Section 3.1 and an edge $X \xrightarrow{r \times\left(\Sigma^{m},-k\right)}$ $Y$ indicates that $\vee_{r} \Sigma^{m}\left(\operatorname{Ind}\left(Y_{n-k}\right)\right)$ appears in the homotopy type formula of $\operatorname{Ind}\left(X_{n}\right)$ obtained in Section 3.3. For simplicity of notations, $1 \times\left(\Sigma^{m},-k\right)$ is denoted by ( $\Sigma^{m},-k$ ) and $\Sigma^{1}$ is denoted by $\Sigma$.


Figure 3.1: Summary of homotopy type formulae obtained in Section 3.3

### 3.1 Graph definitions and $n=1$ computations

### 3.1.1 $G_{n}$

Since $G_{1}$ is an edge (see Figure 2.1a), $\operatorname{Ind}\left(G_{1}\right) \simeq \mathbb{S}^{0}$.

### 3.1.2 $B_{n}$

For $n \geqslant 1$, we define the graph $B_{n}$ as follows:

```
\(V\left(B_{n}\right)=V\left(G_{n}\right) \sqcup\left\{b_{1}, b_{2}, b_{3}, b_{4}\right\}\),
\(E\left(B_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(b_{1}, v_{1}\right),\left(b_{1}, b_{2}\right),\left(b_{2}, b_{3}\right),\left(b_{3}, b_{4}\right),\left(b_{4}, x_{1}\right)\right\}\) and for \(n \geqslant 2\),
```

$$
E\left(B_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(b_{1}, u_{1}\right),\left(b_{1}, v_{1}\right),\left(b_{1}, b_{2}\right),\left(b_{2}, b_{3}\right),\left(b_{3}, b_{4}\right),\left(b_{4}, x_{1}\right),\left(b_{4}, y_{1}\right)\right\} .
$$


(a) $B_{1}$

(b) $B_{n}, n \geqslant 2$

Since $B_{1} \cong C_{6}$, Lemma 8 implies that $\operatorname{Ind}\left(B_{1}\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}$.

### 3.1.3 $A_{n}$

For $n \geqslant 1$, we define the graph $A_{n}$ as follows: $V\left(A_{n}\right)=V\left(G_{n}\right) \sqcup\{a\}$,
$E\left(A_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(a, x_{1}\right),\left(a, v_{1}\right)\right\}$ and for $n \geqslant 2$,
$E\left(A_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(a, x_{1}\right),\left(a, v_{1}\right),\left(a, w_{1}\right)\right\}$.

(a) $A_{1}$

(b) $A_{n}, n \geqslant 2$

Since $A_{1} \simeq C_{3}$, Lemma 8 implies that $\operatorname{Ind}\left(A_{1}\right) \simeq \mathbb{S}^{0} \vee \mathbb{S}^{0}$.

### 3.1.4 $D_{n}$

For $n \geqslant 1$, we define the graph $D_{n}$ as follows:

$$
\begin{aligned}
& V\left(D_{n}\right)=V\left(G_{n}\right) \sqcup\{d\}, \\
& E\left(D_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(v_{1}, d\right)\right\} \text { and for } n \geqslant 2, \\
& E\left(D_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(d, v_{1}\right),\left(d, u_{1}\right)\right\} .
\end{aligned}
$$


(a) $D_{1}$

(b) $D_{n}, n \geqslant 2$

Clearly, $\operatorname{Ind}\left(D_{1}\right) \simeq \mathbb{S}^{0}$.

### 3.1.5 $J_{n}$

For $n \geqslant 1$, we define the graph $J_{n}$ as follows:

$$
\begin{aligned}
& V\left(J_{n}\right)=V\left(G_{n}\right) \sqcup\left\{j_{1}, j_{2}, j_{3}, j_{4}, j_{5}, j_{6}\right\}, \\
& E\left(J_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(j_{1}, v_{1}\right),\left(j_{1}, x_{1}\right),\left(j_{1}, j_{3}\right),\left(j_{1}, j_{4}\right),\left(j_{2}, x_{1}\right),\left(j_{2}, j_{3}\right),\left(j_{2}, j_{5}\right),\left(j_{3}, j_{4}\right),\left(j_{3},\right.\right.
\end{aligned}
$$ $\left.\left.j_{5}\right),\left(j_{4}, j_{6}\right),\left(j_{5}, j_{6}\right)\right\}$ and for $n \geqslant 2$,

$E\left(J_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(j_{1}, v_{1}\right),\left(j_{1}, x_{1}\right),\left(j_{1}, j_{3}\right),\left(j_{1}, j_{4}\right),\left(j_{2}, x_{1}\right),\left(j_{2}, j_{3}\right),\left(j_{2}, j_{5}\right),\left(j_{3}, j_{4}\right),\left(j_{3}\right.\right.$, $\left.\left.j_{5}\right),\left(j_{4}, j_{6}\right),\left(j_{5}, j_{6}\right),\left(j_{2}, y_{1}\right),\left(j_{1}, w_{1}\right)\right\}$.

(a) $J_{1}$

(b) $J_{n}, \quad n \geqslant 2$

Since $N_{J_{1}}\left(j_{6}\right) \subseteq N_{J_{1}}\left(j_{3}\right)$, Lemma 5 implies that $\operatorname{Ind}\left(J_{1}\right) \simeq \operatorname{Ind}\left(J_{1}-\left\{j_{3}\right\}\right)$. Let $J_{1}^{\prime}$ be the graph $J_{1}-\left\{j_{3}\right\}$. Using the fact that $v_{1}$ is a simplicial vertex in $J_{1}^{\prime}$ and $N_{J_{1}^{\prime}}\left(v_{1}\right)=\left\{j_{1}, x_{1}\right\}$, from Lemma 6 we get that

$$
\operatorname{Ind}\left(J_{1}^{\prime}\right) \simeq \Sigma\left(\operatorname{Ind}\left(J_{1}^{\prime}-N_{J_{1}^{\prime}}\left[j_{1}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(J_{1}^{\prime}-N_{J_{1}^{\prime}}\left[x_{1}\right]\right)\right)
$$

Observe that $J_{1}^{\prime}-N_{J_{1}^{\prime}}\left[j_{1}\right] \cong P_{3} \cong J_{1}^{\prime}-N_{J_{1}^{\prime}}\left[x_{1}\right]$. Therefore using Lemma 7, we conclude that $\operatorname{Ind}\left(J_{1}\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}$.

### 3.1.6 $O_{n}$

For $n \geqslant 1$, we define the graph $J_{n}$ as follows:

$$
\begin{aligned}
V\left(O_{n}\right)= & V\left(G_{n}\right) \sqcup\left\{o_{1}, o_{2}, o_{3}, o_{4}, o_{5}, o_{6}, o_{7}, o_{8}, o_{9}\right\}, \\
E\left(O_{1}\right)= & E\left(G_{1}\right) \sqcup\left\{\left(o_{1}, v_{1}\right),\left(o_{1}, o_{4}\right),\left(o_{2}, v_{1}\right),\left(o_{2}, x_{1}\right),\left(o_{2}, o_{4}\right),\left(o_{2}, o_{5}\right),\left(o_{3}, x_{1}\right),\left(o_{3}, o_{6}\right),\right. \\
& \left.\left(o_{4}, o_{5}\right),\left(o_{5}, o_{7}\right),\left(o_{6}, o_{8}\right),\left(o_{7}, o_{9}\right),\left(o_{8}, o_{9}\right)\right\} \text { and } \\
E\left(O_{n}\right)= & E\left(G_{n}\right) \sqcup\left\{\left(o_{1}, u_{1}\right),\left(o_{1}, v_{1}\right),\left(o_{1}, v_{4}\right),\left(o_{2}, v_{1}\right),\left(o_{2}, x_{1}\right),\left(o_{2}, o_{4}\right),\left(o_{2}, o_{5}\right),\left(o_{2}, w_{1}\right),\right. \\
& \left.\left(o_{3}, x_{1}\right),\left(o_{3}, y_{1}\right),\left(o_{3}, o_{6}\right),\left(o_{4}, o_{5}\right),\left(o_{5} o_{8}\right),\left(o_{7}, o_{9}\right),\left(o_{8}, o_{9}\right)\right\}
\end{aligned}
$$

for $n \geqslant 2$.


Since $N_{O_{1}}\left(o_{1}\right) \subseteq N_{O_{1}}\left(o_{2}\right)$, Lemma 5 implies that $\operatorname{Ind}\left(O_{1}\right) \simeq \operatorname{Ind}\left(O_{1}-\left\{o_{2}\right\}\right)$. Observe that $O_{1}-\left\{o_{2}\right\} \cong C_{10}$, thus by Lemma 8 we get that $\operatorname{Ind}\left(O_{1}\right) \simeq \mathbb{S}^{2}$.

### 3.1.7 $M_{n}$

For $n \geqslant 1$, we define the graph $M_{n}$ as follows:

(a) $M_{1}$

(b) $M_{n}, n \geqslant 2$
$V\left(M_{n}\right)=V\left(G_{n}\right) \sqcup\left\{m_{1}, m_{2}, m_{3}\right\}$,
$E\left(M_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(m_{1}, v_{1}\right),\left(m_{1}, x_{1}\right),\left(m_{1}, m_{2}\right),\left(m_{2}, m_{3}\right),\left(m_{3}, x_{1}\right)\right\}$ and for $n \geqslant 2$, $E\left(M_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(m_{1}, v_{1}\right),\left(m_{1}, x_{1}\right),\left(m_{1}, w_{1}\right),\left(m_{1}, m_{2}\right),\left(m_{2}, m_{3}\right),\left(m_{3}, x_{1}\right),\left(m_{3}, y_{1}\right)\right\}$.
Since $N_{M_{1}}\left(m_{2}\right) \subseteq N_{M_{1}}\left(x_{1}\right)$ and $M_{1}-\left\{x_{1}\right\} \cong P_{4}$, Lemma 5 and Lemma 7 imply that

$$
\operatorname{Ind}\left(M_{1}\right) \simeq \operatorname{Ind}\left(M_{1}-\left\{x_{1}\right\}\right) \simeq \operatorname{Ind}\left(P_{4}\right) \simeq \mathrm{pt}
$$

### 3.1.8 $\quad Q_{n}$

For $n \geqslant 1$, we define the graph $Q_{n}$ as follows:

$$
\begin{aligned}
& V\left(Q_{n}\right)=V\left(G_{n}\right) \sqcup\left\{q_{1}, q_{2}, q_{3}, q_{4}, q_{5}, q_{6}, q_{7}\right\}, \\
& E\left(Q_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(q_{1}, v_{1}\right),\left(q_{1}, q_{3}\right),\left(q_{1}, q_{4}\right),\left(q_{2}, v_{1}\right),\left(q_{2}, x_{1}\right),\left(q_{2}, q_{3}\right),\left(q_{2}, q_{5}\right),\right. \\
& \left.\quad\left(q_{3}, q_{4}\right),\left(q_{3}, q_{5}\right),\left(q_{4}, q_{6}\right),\left(q_{5}, q_{6}\right),\left(q_{5}, q_{7}\right),\left(q_{6}, q_{7}\right)\right\} \text { and } \\
& E\left(Q_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(q_{1}, u_{1}\right),\left(q_{1}, v_{1}\right),\left(q_{1}, q_{3}\right),\left(q_{1}, q_{4}\right),\left(q_{2}, v_{1}\right),\left(q_{2}, x_{1}\right),\right. \\
& \left.\quad\left(q_{2}, w_{1}\right),\left(q_{2}, q_{3}\right),\left(q_{2}, q_{5}\right),\left(q_{3}, q_{4}\right),\left(q_{3}, q_{5}\right),\left(q_{4}, q_{6}\right),\left(q_{5}, q_{6}\right),\left(q_{5}, q_{7}\right),\left(q_{6}, q_{7}\right)\right\}
\end{aligned}
$$

for $n \geqslant 2$.


Since $q_{7}$ is a simplicial vertex in $Q_{1}$ and $N_{Q_{1}}\left(q_{7}\right)=\left\{q_{5}, q_{6}\right\}$, Lemma 6 implies that

$$
\operatorname{Ind}\left(Q_{1}\right) \simeq \Sigma\left(\operatorname{Ind}\left(Q_{1}-N_{Q_{1}}\left[q_{5}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(Q_{1}-N_{Q_{1}}\left[q_{6}\right]\right)\right)
$$

Observe that $Q_{1}-N_{Q_{1}}\left[q_{5}\right] \cong P_{4}$, therefore $\operatorname{Ind}\left(Q_{1}-N_{Q_{1}}\left[q_{5}\right]\right)$ is contractible by Lemma 7 .
Since $N_{Q_{1}-N_{Q_{1}}\left[q_{6}\right]}\left(q_{3}\right) \subseteq N_{Q_{1}-N_{Q_{1}}\left[q_{6}\right]}\left(v_{1}\right)$, Lemma 5 implies that

$$
\operatorname{Ind}\left(Q_{1}-N_{Q_{1}}\left[q_{6}\right]\right) \simeq \operatorname{Ind}\left(Q_{1}-N_{Q_{1}}\left[q_{6}\right]-\left\{v_{1}\right\}\right)
$$

Since $Q_{1}-N_{Q_{1}}\left[q_{6}\right]-\left\{v_{1}\right\} \cong P_{4}$, we conclude that $\operatorname{Ind}\left(Q_{1}\right)$ is contractible.

### 3.1.9 $\quad F_{n}$

For $n \geqslant 1$, we define the graph $F_{n}$ as follows:

$$
\begin{aligned}
& V\left(F_{n}\right)=V\left(G_{n}\right) \sqcup\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}, \\
& E\left(F_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(f_{1}, v_{1}\right),\left(f_{1}, x_{1}\right),\left(f_{1}, f_{2}\right),\left(f_{2}, f_{3}\right),\left(f_{2}, f_{4}\right),\left(f_{3}, f_{4}\right),\left(f_{4}, x_{1}\right)\right\} \text { and } \\
& E\left(F_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(f_{1}, v_{1}\right),\left(f_{1}, x_{1}\right),\left(f_{1}, f_{2}\right),\left(f_{1}, w_{1}\right),\left(f_{2}, f_{3}\right),\left(f_{2}, f_{4}\right),\left(f_{3}, f_{4}\right),\right. \\
& \left.\quad\left(f_{4}, x_{1}\right),\left(f_{4}, y_{1}\right)\right\}
\end{aligned}
$$

for $n \geqslant 2$.

(a) $F_{1}$

(b) $F_{n}, n \geqslant 2$

Observe that $f_{3}$ is a simplicial vertex in $F_{1}$ and $N_{F_{1}}\left(f_{3}\right)=\left\{f_{2}, f_{4}\right\}$. Using Lemma 6, we get that

$$
\operatorname{Ind}\left(F_{1}\right) \simeq \Sigma\left(\operatorname{Ind}\left(F_{1}-N_{F_{1}}\left[f_{2}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(F_{1}-N_{F_{1}}\left[f_{4}\right]\right)\right) .
$$

Since $F_{1}-N_{F_{1}}\left[f_{2}\right]=F_{1}-\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\} \cong P_{2} \cong F_{1}-\left\{x_{1}, f_{2}, f_{3}, f_{4}\right\}=F_{1}-N_{F_{1}}\left[f_{4}\right]$, $\operatorname{Ind}\left(F_{1}\right) \simeq \Sigma\left(\operatorname{Ind}\left(P_{2}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(P_{2}\right)\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}$.

### 3.1.10 $H_{n}$

For $n \geqslant 1$, we define the graph $H_{n}$ as follows:

$$
\begin{aligned}
& V\left(H_{n}\right)=V\left(G_{n}\right) \sqcup\left\{h_{1}, h_{2}, h_{3}, h_{4}\right\}, \\
& E\left(H_{1}\right)=E\left(G_{1}\right) \sqcup\left\{\left(h_{1}, v_{1}\right),\left(h_{1}, h_{2}\right),\left(h_{2}, h_{3}\right),\left(h_{2}, h_{4}\right),\left(h_{3}, h_{4}\right),\left(h_{4}, v_{1}\right),\left(h_{4}, x_{1}\right)\right\} \text { and for }
\end{aligned}
$$ $n \geqslant 2$,

$E\left(H_{n}\right)=E\left(G_{n}\right) \sqcup\left\{\left(h_{1}, v_{1}\right),\left(h_{1}, u_{1}\right),\left(h_{1}, h_{2}\right),\left(h_{2}, h_{3}\right),\left(h_{2}, h_{4}\right),\left(h_{3}, h_{4}\right),\left(h_{4}, v_{1}\right),\left(h_{4}, x_{1}\right)\right.$, $\left.\left(h_{4}, w_{1}\right)\right\}$.

(a) $H_{1}$

(b) $H_{n}, n \geqslant 2$

Since $h_{3}$ is a simplicial vertex in $H_{1}$ and $N_{H_{1}}\left(h_{3}\right)=\left\{h_{2}, h_{4}\right\}$, Lemma 6 implies that

$$
\operatorname{Ind}\left(H_{1}\right) \simeq \Sigma\left(\operatorname{Ind}\left(H_{1}-N_{H_{1}}\left[h_{2}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(H_{1}-N_{H_{1}}\left[h_{4}\right]\right)\right)
$$

Note that $H_{1}-N_{H_{1}}\left[h_{2}\right]=H_{1}-\left\{h_{1}, h_{2}, h_{3}, h_{4}\right\} \cong P_{2}$ and $H_{1}-N_{H_{1}}\left[h_{4}\right]=H_{1}-$ $\left\{v_{1}, x_{1}, h_{2}, h_{3}, h_{4}\right\} \cong P_{1}$, we get that $\operatorname{Ind}\left(H_{1}\right) \simeq \Sigma\left(\operatorname{Ind}\left(P_{2}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(P_{1}\right)\right) \simeq \mathbb{S}^{1} \vee \mathrm{pt} \simeq \mathbb{S}^{1}$.

### 3.2 Case $n=2$ computation

### 3.2.1 $\quad G_{2}$

In $G_{2}, N_{G_{2}}\left(u_{1}\right) \subseteq N_{G_{2}}\left(w_{1}\right)$ (see figure on the right), thus by $\operatorname{Lemma} 5, \operatorname{Ind}\left(G_{2}\right) \simeq \operatorname{Ind}\left(G_{2}-\left\{w_{1}\right\}\right)$. Since $G_{2}-\left\{w_{1}\right\} \cong C_{6}$, Lemma 8 implies that $\operatorname{Ind}\left(G_{2}-\left\{w_{1}\right\}\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}$. Therefore $\operatorname{Ind}\left(G_{2}\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}$.


### 3.2.2 $\quad B_{2}$

We consider the vertex $b_{4}$ in $B_{2}$ (see figure on the right) and analyse $\operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)$ and $\operatorname{lk}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)$.

First note that $\operatorname{lk}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)=\operatorname{Ind}\left(B_{2}-N_{B_{2}}\left[b_{4}\right]\right)$. Let $B_{2}^{\prime}$ denote the graph $B_{2}-N_{B_{2}}\left[b_{4}\right]$ (see Figure 3.11c). Since $N_{B_{2}^{\prime}}\left(b_{2}\right) \subseteq N_{B_{2}^{\prime}}\left(v_{1}\right)$, Lemma 5 gives that $\operatorname{lk}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right) \simeq$ $\operatorname{Ind}\left(B_{2}^{\prime}-\left\{v_{1}\right\}\right)$. Let $B_{2}^{\prime \prime}$ denote the graph $B_{2}^{\prime}-\left\{v_{1}\right\}-$
 $\left\{\left(u_{1}, v_{2}\right)\right\}$.
Claim: $\operatorname{Ind}\left(B_{2}^{\prime \prime}\right) \simeq \operatorname{Ind}\left(B_{2}^{\prime}-\left\{v_{1}\right\}\right)$.
Since $\left(u_{1}, v_{2}\right) \notin E\left(B_{2}^{\prime \prime}\right),\left\{u_{1}, v_{2}\right\} \in \operatorname{Ind}\left(B_{2}^{\prime \prime}\right)$. Observe that $b_{2}$ is an isolated vertex in $B_{2}^{\prime \prime}-N_{B_{2}^{\prime \prime}}\left[\left\{u_{1}, v_{2}\right\}\right]$ and hence Lemma 3 implies that $\operatorname{Ind}\left(B_{2}^{\prime \prime}-N_{B_{2}^{\prime \prime}}\left[\left\{u_{1}, v_{2}\right\}\right]\right)$ is contractible. Therefore by Lemma $4, \operatorname{Ind}\left(B_{2}^{\prime \prime}\right) \simeq \operatorname{Ind}\left(B_{2}^{\prime \prime} \cup\left\{\left(u_{1}, v_{2}\right)\right\}\right)=\operatorname{Ind}\left(B_{2}^{\prime}-\left\{v_{1}\right\}\right)$.

Observe that, $N_{B_{2}^{\prime \prime}}\left(u_{1}\right)=N_{B_{2}^{\prime \prime}}\left(b_{2}\right)$, so Lemma 5 implies that $\operatorname{Ind}\left(B_{2}^{\prime \prime}\right) \simeq \operatorname{Ind}\left(B_{2}^{\prime \prime}-\left\{b_{2}\right\}\right)$. Since $V\left(B_{2}^{\prime \prime}-\left\{b_{2}\right\}\right) \cap N_{B_{2}-\left\{b_{4}\right\}}\left(b_{3}\right)=\emptyset, \operatorname{Ind}\left(B_{2}^{\prime \prime}-\left\{b_{2}\right\}\right) *\left\{b_{3}\right\} \subseteq \operatorname{Ind}\left(B_{2}-\left\{b_{4}\right\}\right)=$ $\operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)$. Hence the inclusion map $\operatorname{Ind}\left(B_{2}^{\prime \prime}-\left\{b_{2}\right\}\right) \hookrightarrow \operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)$ is null homotopic. Therefore the following composition of maps is null homotopic

$$
\operatorname{lk}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right) \xrightarrow{\simeq} \operatorname{Ind}\left(B_{2}^{\prime}-\left\{v_{1}\right\}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(B_{2}^{\prime \prime}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(B_{2}^{\prime \prime}-\left\{b_{2}\right\}\right) \hookrightarrow \operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right) .
$$

Hence $\operatorname{lk}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)$ is contractible in $\operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)$ and therefore by Lemma 2,

$$
\begin{equation*}
\operatorname{Ind}\left(B_{2}\right) \simeq \operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right) \vee \Sigma\left(\operatorname{lk}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)\right) \tag{1}
\end{equation*}
$$

Note that $B_{2}^{\prime \prime}-\left\{b_{2}\right\} \cong P_{2} \sqcup A_{1}$, hence $\operatorname{lk}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right) \simeq \operatorname{Ind}\left(B_{2}^{\prime \prime}-\left\{b_{2}\right\}\right) \simeq \operatorname{Ind}\left(P_{2} \sqcup A_{1}\right)$. Also, $\operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right)=\operatorname{Ind}\left(B_{2}-\left\{b_{4}\right\}\right)$ and $N_{B_{2}-\left\{b_{4}\right\}}\left(b_{3}\right) \subseteq N_{B_{2}-\left\{b_{4}\right\}}\left(b_{1}\right)$, Lemma 5 implies that $\operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right) \simeq \operatorname{Ind}\left(B_{2}-\left\{b_{4}, b_{1}\right\}\right)$. However, $B_{2}-\left\{b_{4}, b_{1}\right\}$ is isomorphic to $P_{2} \sqcup G_{2}$ (see Figure 3.11b), therefore $\operatorname{del}\left(b_{4}, \operatorname{Ind}\left(B_{2}\right)\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{2}\right)\right)$. Hence from Equation (1), we get the following homotopy equivalence.

$$
\begin{equation*}
\operatorname{Ind}\left(B_{2}\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{2}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(A_{1}\right)\right) \tag{2}
\end{equation*}
$$

Since $\operatorname{Ind}\left(A_{1}\right) \simeq \mathbb{S}^{0} \vee \mathbb{S}^{0}($ see Section 3.1.3 $)$ and $\operatorname{Ind}\left(G_{2}\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}$ (see Section 3.2.1), we get that $\operatorname{Ind}\left(B_{2}\right) \simeq\left(\mathbb{S}^{2} \vee \mathbb{S}^{2}\right) \vee \Sigma^{2}\left(\mathbb{S}^{0} \vee \mathbb{S}^{0}\right) \simeq \vee_{4} \mathbb{S}^{2}$.

(a) $B_{2}-\left\{b_{4}\right\}$

(d) $B_{2}^{\prime}-\left\{v_{1}\right\}$

(b) $B_{2}-\left\{b_{4}, b_{1}\right\}$

(e) $B_{2}^{\prime \prime}$

(c) $B_{2}^{\prime}=B_{2}-N_{B_{2}}\left[b_{4}\right]$

(f) $B_{2}^{\prime \prime}-\left\{b_{2}\right\}$

Figure 3.11

### 3.2.3 $\quad A_{2}$

Note that $N_{A_{2}}\left(u_{1}\right) \subseteq N_{A_{2}}\left(w_{1}\right)$ (see figure on the right), therefore by Lemma $5, \operatorname{Ind}\left(A_{2}\right) \simeq \operatorname{Ind}\left(A_{2}-\left\{w_{1}\right\}\right)$. Let $A_{2}^{\prime}=A_{2}-\left\{w_{1}\right\}$. Since $a$ is a simplicial vertex in $A_{2}^{\prime}$ and $N_{A_{2}^{\prime}}(a)=\left\{x_{1}, v_{1}\right\}$, Lemma 6 implies that $\operatorname{Ind}\left(A_{2}^{\prime}\right) \simeq \Sigma\left(\operatorname{Ind}\left(A_{2}^{\prime}-N_{A_{2}^{\prime}}\left[x_{1}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(A_{2}^{\prime}-N_{A_{2}^{\prime}}\left[v_{1}\right]\right)\right.$. The graph $A_{2}^{\prime}-N_{A_{2}^{\prime}}\left[x_{1}\right]=A_{2}-\left\{w_{1}, a, x_{1}, v_{1}, y_{1}\right\} \cong P_{3}$. Therefore $\operatorname{Ind}\left(A_{2}^{\prime}-N\left[x_{1}\right]\right) \simeq \mathbb{S}^{0}$. Also, $A_{2}^{\prime}-N_{A_{2}^{\prime}}\left[x_{1}\right] \cong A_{2}^{\prime}-N_{A_{2}^{\prime}}\left[v_{1}\right]$. Hence $\operatorname{Ind}\left(A_{2}\right) \simeq \operatorname{Ind}\left(A_{2}^{\prime}\right) \simeq \Sigma\left(\mathbb{S}^{0}\right) \vee \Sigma\left(\mathbb{S}^{0}\right)=\mathbb{S}^{1} \vee \mathbb{S}^{1}$.


### 3.2.4 $\quad D_{2}$

Note that $N_{D_{2}}\left(y_{1}\right) \subseteq N_{D_{2}}\left(w_{1}\right)$ (see figure on the right), therefore by Lemma $5, \operatorname{Ind}\left(D_{2}\right) \simeq \operatorname{Ind}\left(D_{2}-\left\{w_{1}\right\}\right)$. Let $D_{2}^{\prime}=D_{2}-\left\{w_{1}\right\}$. Since $d$ is a simplicial vertex in $D_{2}^{\prime}$ and $N_{D_{2}^{\prime}}(d)=\left\{u_{1}, v_{1}\right\}$, Lemma 6 implies that $\operatorname{Ind}\left(D_{2}^{\prime}\right) \simeq \Sigma\left(\operatorname{Ind}\left(D_{2}^{\prime}-N_{D_{2}^{\prime}}\left[u_{1}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(D_{2}^{\prime}-N_{D_{2}^{\prime}}\left[v_{1}\right]\right)\right.$. The graph $D_{2}^{\prime}-N_{D_{2}^{\prime}}\left[u_{1}\right]=D_{2}-\left\{w_{1}, d, u_{1}, v_{1}, v_{2}\right\} \cong P_{3}$. Therefore $\operatorname{Ind}\left(D_{2}^{\prime}-N_{D_{2}^{\prime}}\left[u_{1}\right]\right) \simeq \mathbb{S}^{0}$. Also, $D_{2}^{\prime}-N_{D_{2}^{\prime}}\left[u_{1}\right] \cong D_{2}^{\prime}-N_{D_{2}^{\prime}}\left[v_{1}\right]$, therefore $\operatorname{Ind}\left(D_{2}^{\prime}-N_{D_{2}^{\prime}}\left[v_{1}\right]\right) \simeq \mathbb{S}^{0}$. Hence $\operatorname{Ind}\left(D_{2}\right) \simeq \operatorname{Ind}\left(D_{2}^{\prime}\right) \simeq \Sigma\left(\mathbb{S}^{0}\right) \vee \Sigma\left(\mathbb{S}^{0}\right)=$
 $\mathbb{S}^{1} \vee \mathbb{S}^{1}$.

### 3.2.5 $\quad J_{2}$

Since $N_{J_{2}}\left(j_{6}\right) \subseteq N_{J_{2}}\left(j_{3}\right)$ (see figure on the right), from Lemma 5, $\operatorname{Ind}\left(J_{2}\right) \simeq \operatorname{Ind}\left(J_{2}-\left\{j_{3}\right\}\right)$. Observe that $J_{2}-\left\{j_{3}, x_{1}\right\} \cong O_{1}$ and therefore from Section 3.1.6 we see that $\operatorname{del}\left(x_{1}, \operatorname{Ind}\left(J_{2}-\left\{j_{3}\right\}\right)\right) \simeq \operatorname{Ind}\left(O_{1}\right) \simeq \mathbb{S}^{2}$.

Note that, $\operatorname{lk}\left(x_{1}, \operatorname{Ind}\left(J_{2}-\left\{j_{3}\right\}\right)\right) \simeq \operatorname{Ind}\left(J_{2}-\right.$ $\left.\left\{x_{1}, j_{3}, j_{1}, j_{2}, v_{1}, w_{1}, y_{1}\right\}\right) \cong \operatorname{Ind}\left(P_{2} \sqcup P_{3}\right) \simeq \mathbb{S}^{1}$. Since the fundamental group of $\mathbb{S}^{2}$ is trivial, $\operatorname{lk}\left(x_{1}, \operatorname{Ind}\left(J_{2}\right.\right.$ -
 $\left.\left.\left\{j_{3}\right\}\right)\right)$ is contractible in $\operatorname{del}\left(x_{1}, \operatorname{Ind}\left(J_{2}-\left\{j_{3}\right\}\right)\right)$. $\operatorname{Ind}\left(J_{2}\right) \simeq \mathbb{S}^{2} \vee \Sigma\left(\mathbb{S}^{1}\right) \simeq \mathbb{S}^{2} \vee \mathbb{S}^{2}$.

### 3.2.6 $\quad O_{2}$

Since $N_{O_{2}-\left\{o_{9}\right\}}\left(o_{7}\right)=\left\{o_{5}\right\} \subseteq N_{O_{2}-\left\{o_{9}\right\}}\left(o_{2}\right) \cap N_{O_{2}-\left\{o_{9}\right\}}\left(o_{4}\right)$ and $N_{O_{2}-\left\{o_{9}\right\}}\left(o_{8}\right)=\left\{o_{6}\right\} \subseteq$ $N_{O_{2}-\left\{o_{9}\right\}}\left(o_{3}\right)$, from Lemma 5 we get that $\operatorname{del}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right)=\operatorname{Ind}\left(O_{2}-\left\{o_{9}\right\}\right) \simeq \operatorname{Ind}\left(O_{2}-\right.$ $\left\{o_{9}, o_{2}, o_{4}, o_{3}\right\}$ ). Observe that $O_{2}-\left\{o_{9}, o_{2}, o_{4}, o_{3}\right\} \cong D_{2} \sqcup P_{2} \sqcup P_{2}$ (see Figure 3.12b), and therefore del $\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(D_{2}\right)\right)$.

We see that $\operatorname{lk}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right)=\operatorname{Ind}\left(O_{2}-N_{O_{2}}\left[o_{9}\right]\right)=\operatorname{Ind}\left(O_{2}-\left\{o_{7}, o_{8}, o_{9}\right\}\right)$. Let $O_{2}^{\prime}$ be the graph $O_{2}-\left\{o_{7}, o_{8}, o_{9}\right\}$ (see Figure 3.12c). Since $N_{O_{2}^{\prime}}\left(o_{6}\right) \subseteq N_{O_{2}^{\prime}}\left(x_{1}\right)$, from Lemma 5, $\operatorname{Ind}\left(O_{2}^{\prime}\right) \simeq \operatorname{Ind}\left(O_{2}^{\prime}-\left\{x_{1}\right\}\right)$. Denote the graph $O_{2}^{\prime}-\left\{x_{1}\right\}$ by $O_{2}^{\prime \prime}$ (see Figure 3.12d). Since $O_{2}^{\prime \prime}-N_{O_{2}^{\prime \prime}}\left[\left\{y_{1}, x_{2}\right\}\right]$ contains an isolated vertex $o_{6}$, we see that $\operatorname{Ind}\left(O_{2}^{\prime \prime}-N_{O_{2}^{\prime \prime}}\left[\left\{y_{1}, x_{2}\right\}\right]\right)$ is a cone over $o_{6}$ and therefore contractible. Hence from Lemma 4, $\operatorname{Ind}\left(O_{2}^{\prime \prime}\right) \simeq \operatorname{Ind}\left(O_{2}^{\prime \prime}-\right.$ $\left.\left\{\left(y_{1}, x_{2}\right)\right\}\right)$.


Figure 3.12

Denote the graph $O_{2}^{\prime \prime}-\left\{\left(y_{1}, x_{2}\right)\right\}$ by $O_{2}^{\prime \prime \prime}$. Since $N_{O_{2}^{\prime \prime}}\left(o_{6}\right)=\left\{o_{3}\right\}=N_{O_{2}^{\prime \prime \prime}}\left(y_{1}\right)$, from Lemma 5 we get that $\operatorname{Ind}\left(O_{2}^{\prime \prime \prime}\right) \simeq \operatorname{Ind}\left(O_{2}^{\prime \prime \prime}-\left\{o_{6}\right\}\right)$. Clearly $O_{2}^{\prime \prime \prime}-\left\{o_{6}\right\} \cong Q_{1} \sqcup P_{2}$, and therefore $\operatorname{lk}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right) \simeq \Sigma\left(\operatorname{Ind}\left(Q_{1}\right)\right)$. Note that $V\left(O_{2}^{\prime \prime \prime}-\left\{o_{6}\right\}\right) \cap N_{O_{2}-\{o 9\}}\left(o_{8}\right)=\emptyset$ and therefore $\operatorname{Ind}\left(O_{2}^{\prime \prime \prime}-\left\{o_{6}\right\}\right) *\left\{o_{8}\right\} \subseteq \operatorname{Ind}\left(O_{2}-\left\{o_{9}\right\}\right)=\operatorname{del}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right)$. Hence the inclusion map $\operatorname{Ind}\left(O_{2}^{\prime \prime \prime}-\left\{o_{6}\right\}\right) \hookrightarrow \operatorname{del}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right)$ is null homotopic. Thus the composite map
$\mathrm{lk}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right)=\operatorname{Ind}\left(O_{2}^{\prime}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(O_{2}^{\prime \prime}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(O_{2}^{\prime \prime \prime}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(O_{2}^{\prime \prime \prime}-\left\{o_{6}\right\}\right) \hookrightarrow \operatorname{del}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right)$ is null homotopic.

Therefore Lemma 2 implies that $\operatorname{Ind}\left(O_{2}\right) \simeq \operatorname{del}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right) \vee \Sigma\left(\operatorname{lk}\left(o_{9}, \operatorname{Ind}\left(O_{2}\right)\right)\right) \simeq$ $\Sigma^{2}\left(\operatorname{Ind}\left(D_{2}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(Q_{1}\right)\right)$. Since $\operatorname{Ind}\left(D_{2}\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}($ cf. Section 3.2.4 $)$ and $\operatorname{Ind}\left(Q_{1}\right)$ is contractible (cf. Section 3.1.8), $\operatorname{Ind}\left(O_{2}\right) \simeq \mathbb{S}^{3} \vee \mathbb{S}^{3}$.

### 3.2.7 $\quad M_{2}$

Since $N_{M_{2}}\left(m_{2}\right) \subseteq N_{M_{2}}\left(x_{1}\right)$ (see figure on the right), Lemma 5 implies that $\operatorname{Ind}\left(M_{2}\right) \simeq \operatorname{Ind}\left(M_{2}-\left\{x_{1}\right\}\right)$. Since $N_{M_{2}-\left\{x_{1}\right\}}\left(u_{1}\right) \subseteq$ $N_{M_{2}-\left\{x_{1}\right\}}\left(w_{1}\right)$ and $M_{2}-\left\{x_{1}\right\} \cong C_{8}$, Lemma 5 and Lemma 8 implies that $\operatorname{Ind}\left(M_{2}\right) \simeq \mathbb{S}^{2}$.


### 3.2.8 $\quad Q_{2}$

Since $q_{7}$ is a simplicial vertex in $Q_{2}$ (see Figure 3.13a), Lemma 6 implies that $\operatorname{Ind}\left(Q_{2}\right) \simeq$ $\Sigma\left(\operatorname{Ind}\left(Q_{2}-N_{Q_{2}}\left[q_{5}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(Q_{2}-N_{Q_{2}}\left[q_{6}\right]\right)\right)$.


Figure 3.13

Note that

$$
N_{Q_{2}-N_{Q_{2}}\left[q_{5}\right]}\left(q_{4}\right)=\left\{q_{1}\right\} \subseteq N_{Q_{2}-N_{Q_{2}}\left[q_{5}\right]}\left(u_{1}\right) \cap N_{Q_{2}-N_{Q_{2}}\left[q_{5}\right]}\left(u_{2}\right),
$$

therefore Lemma 5 implies that $\operatorname{Ind}\left(Q_{2}-N_{Q_{2}}\left[q_{5}\right]\right) \simeq \operatorname{Ind}\left(Q_{2}-N_{Q_{2}}\left[q_{5}\right]-\left\{u_{1}, v_{1}\right\}\right)$. See that $Q_{2}-N_{Q_{2}}\left[q_{5}\right]-\left\{u_{1}, v_{1}\right\} \cong M_{1} \sqcup P_{2}$ (see Figure 3.13b). Hence $\operatorname{Ind}\left(Q_{2}-\right.$ $\left.N_{Q_{2}}\left[q_{5}\right]\right) \simeq \Sigma\left(\operatorname{Ind}\left(M_{1}\right)\right)$. Also $Q_{2}-N_{Q_{2}}\left[q_{6}\right] \cong M_{2}$ (see Figure 3.13c), and therefore $\operatorname{Ind}\left(Q_{2}\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(M_{1}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(M_{2}\right)\right)$. Since $\operatorname{Ind}\left(M_{1}\right)$ is contractible (cf. Section 3.1.7) and $\operatorname{Ind}\left(M_{2}\right) \simeq \mathbb{S}^{2}$ (cf. Section 3.2.7, we get that $\operatorname{Ind}\left(Q_{2}\right) \simeq \mathbb{S}^{3}$.

### 3.2.9 $\quad F_{2}$

Since $f_{3}$ is a simplicial vertex in $F_{2}$, from Lemma 6 we have

$$
\operatorname{Ind}\left(F_{2}\right) \simeq \Sigma\left(\operatorname{Ind}\left(F_{2}-N_{F_{2}}\left[f_{2}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(F_{2}-N_{F_{2}}\left[f_{4}\right]\right)\right)
$$

Observe that $F_{2}-N_{F_{2}}\left[f_{2}\right]=F_{2}-\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\} \cong G_{2}$ and $F_{2}-N_{F_{2}}\left[f_{4}\right]=F_{2}-$ $\left\{x_{1}, y_{1}, f_{2}, f_{3}, f_{4}\right\} \cong H_{1}$ (see Figure 3.14a). Since $\operatorname{Ind}\left(G_{2}\right) \simeq \mathbb{S}^{1} \vee \mathbb{S}^{1}$ (cf. Section 3.2.1) and $\operatorname{Ind}\left(H_{1}\right) \simeq \mathbb{S}^{1}(c f$. Section 3.1.10 $), \operatorname{Ind}\left(F_{2}\right) \simeq \Sigma\left(\mathbb{S}^{1} \vee \mathbb{S}^{1}\right) \vee \Sigma\left(\mathbb{S}^{1}\right) \simeq \vee_{3} \mathbb{S}^{2}$.


Figure 3.14

### 3.2.10 $\quad \mathrm{H}_{2}$

Observe that $h_{3}$ is a simplicial vertex in $H_{2}$ (see Figure 3.14b). Thus using Lemma 6, we get

$$
\operatorname{Ind}\left(H_{2}\right) \simeq \Sigma\left(\operatorname{Ind}\left(H_{2}-N_{H_{2}}\left[h_{2}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(H_{2}-N_{H_{2}}\left[h_{4}\right]\right)\right) .
$$

Observe that $H_{2}-N_{H_{2}}\left[h_{2}\right] \cong G_{2}$ and $H_{2}-N_{H_{2}}\left[h_{4}\right] \cong P_{5}$. Therefore, $\operatorname{Ind}\left(H_{2}\right) \simeq$ $\Sigma\left(\operatorname{Ind}\left(G_{2}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(P_{5}\right)\right) \simeq \Sigma\left(\mathbb{S}^{1} \vee \mathbb{S}^{1}\right) \vee \Sigma\left(\mathbb{S}^{1}\right)=\vee_{3} \mathbb{S}^{2}$.

### 3.3 General case computation

The main outcome of this subsection is that the independence complex of any graph among the ten classes of graphs (defined in Section 3.1) is a wedge of spheres up to homotopy. We prove this by induction on the subscript of the graphs, i.e., $n$. The cases $n=1,2$ follow from the Section 3.1 and Section 3.2. Fix $n \geqslant 3$, and inductively assume that for any $k<n$, the independence complex of any graph, among the ten classes of graphs, with subscript $k$ is a wedge of spheres up to homotopy.

### 3.3.1 $G_{n}$

For $n \geqslant 3$, we show that

$$
\operatorname{Ind}\left(G_{n}\right) \simeq \begin{cases}\vee_{5} \mathbb{S}^{2} & \text { if } n=3,  \tag{3}\\ \operatorname{Ind}\left(B_{n-1}\right) \vee \Sigma^{3}\left(\operatorname{Ind}\left(A_{n-3}\right)\right) & \text { if } n \geqslant 4\end{cases}
$$

We do this by analysing $\operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)$ and $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)$. As $\operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)=$ $\operatorname{Ind}\left(G_{n}-\left\{w_{1}\right\}\right)$ and $G_{n}-\left\{w_{1}\right\} \cong B_{n-1}$ (cf. Figure 3.15a), we have $\operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) \simeq$ $\operatorname{Ind}\left(B_{n-1}\right)$, for $n \geqslant 3$. Further note that, $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)=\operatorname{Ind}\left(G_{n}-N_{G_{n}}\left[w_{1}\right]\right)$.

For $n=3, \operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{3}\right)\right) \simeq \operatorname{Ind}\left(B_{2}\right)$ thus by Section 3.2.2, $\operatorname{Ind}\left(B_{2}\right) \simeq \vee_{4} \mathbb{S}^{2}$. Also, $G_{3}-N_{G_{3}}\left[w_{1}\right] \cong P_{6}$, therefore Lemma 7 implies that $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{3}\right)\right) \simeq \mathbb{S}^{1}$. Since the fundamental group of $\vee_{4} \mathbb{S}^{2}$ is trivial, $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{3}\right)\right)$ is contractible in $\operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{3}\right)\right)$. Hence, from Lemma 2, $\operatorname{Ind}\left(G_{3}\right) \simeq \operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{3}\right)\right) \vee \Sigma\left(\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{3}\right)\right)\right) \simeq \vee_{4} \mathbb{S}^{2} \vee \Sigma\left(\mathbb{S}^{1}\right) \simeq$ $\vee_{5} \mathbb{S}^{2}$.

We now analyse $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)$ for $n \geqslant 4$. Let $G_{n}^{\prime}$ be the graph $G_{n}-N_{G_{n}}\left[w_{1}\right]$ (cf. Figure 3.15b). Since $N_{G_{n}^{\prime}}\left(y_{1}\right) \subseteq N_{G_{n}^{\prime}}\left(x_{3}\right)$, Lemma 5 implies that $\operatorname{Ind}\left(G_{n}^{\prime}\right) \simeq$ $\operatorname{Ind}\left(G_{n}^{\prime}-\left\{x_{3}\right\}\right)$. Observe that both the graphs $G_{n}^{\prime}-\left\{x_{3}\right\}-N_{G_{n}^{\prime}-\left\{x_{3}\right\}}\left[\left\{y_{3}, x_{4}\right\}\right]$ and $G_{n}^{\prime}-\left\{x_{3}\right\}-N_{G_{n}^{\prime}-\left\{x_{3}\right\}}\left[\left\{y_{3}, y_{4}\right\}\right]$ have $y_{1}$ as an isolated vertex (see Figure 3.15c), therefore


Figure 3.15
the independence complexes of these graphs are contractible. Hence, Lemma 4 implies that $\operatorname{Ind}\left(G_{n}^{\prime}-\left\{x_{3}\right\}\right) \simeq \operatorname{Ind}\left(G_{n}^{\prime}-\left\{x_{3}\right\}-\left\{\left(y_{3}, x_{4}\right),\left(y_{3}, y_{4}\right)\right\}\right)$.

Let $G_{n}^{\prime \prime}=G_{n}^{\prime}-\left\{x_{3}\right\}-\left\{\left(y_{3}, x_{4}\right),\left(y_{3}, y_{4}\right)\right\}$. Since $N_{G_{n}^{\prime \prime}}\left(y_{3}\right) \subseteq N_{G_{n}^{\prime \prime}}\left(y_{1}\right)$ (see Figure 3.15d), Lemma 5 implies $\operatorname{Ind}\left(G_{n}^{\prime \prime}\right) \simeq \operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}\right)$. Note that $V\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}\right) \cap N_{G_{n}-\left\{w_{1}\right\}}\left(x_{1}\right)=$ $\emptyset$, therefore $\operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}\right) *\left\{w_{1}\right\} \subseteq \operatorname{Ind}\left(G_{n}-\left\{w_{1}\right\}\right)=\operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)$. Hence the inclusion map $\operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}\right) \hookrightarrow \operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)$ is null homotopic. Thus the following composition of maps is null homotopic:

$$
\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) \xrightarrow{\simeq} \operatorname{Ind}\left(G_{n}^{\prime}-\left\{x_{3}\right\}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(G_{n}^{\prime \prime}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}\right) \hookrightarrow \operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) .
$$

Therefore by Lemma 2,

$$
\operatorname{Ind}\left(G_{n}\right) \simeq \operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) \vee \Sigma\left(\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right)\right)
$$

As shown earlier, $\operatorname{del}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) \simeq \operatorname{Ind}\left(B_{n-1}\right)$, therefore to prove Equation (3), it suffices to show that $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(A_{n-3}\right)\right)$. From the above discussion, we know that $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) \simeq \operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}\right)$. Since $N_{G_{n}^{\prime \prime}-\left\{y_{1}\right\}}\left(u_{1}\right) \subseteq N_{G_{n}^{\prime \prime}-\left\{y_{1}\right\}}\left(u_{3}\right) \cap$ $N_{G_{n}^{\prime \prime}-\left\{y_{1}\right\}}\left(v_{3}\right)$, Lemma 5 implies $\operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}\right) \simeq \operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}\right\}-\left\{u_{3}, v_{3}\right\}\right)$. Moreover, $G_{n}^{\prime \prime}-\left\{y_{1}, u_{3}, v_{3}\right\}$ is isomorphic to $P_{2} \sqcup P_{2} \sqcup A_{n-3}$ (see Figure 3.15f). Thus, by Lemma 3, $\operatorname{lk}\left(w_{1}, \operatorname{Ind}\left(G_{n}\right)\right) \simeq \operatorname{Ind}\left(G_{n}^{\prime \prime}-\left\{y_{1}, u_{3}, v_{3}\right\}\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(A_{n-3}\right)\right)$.

Corollary 9. For $n \geqslant 1$, $\operatorname{Ind}\left(G_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.1, Section 3.2.1, induction hypothesis, and Equation (3).

### 3.3.2 $B_{n}$

Let $n \geqslant 3$ and consider the vertex $b_{4}$ in $B_{n}$. Let $B_{n}^{\prime}=B_{n}-N_{B_{n}}\left[b_{4}\right]$. Since both the graphs $B_{n}^{\prime}-\left\{v_{1}\right\}-N_{B_{n}^{\prime}-\left\{v_{1}\right\}}\left[\left\{u_{1}, u_{2}\right\}\right]$ and $B_{n}^{\prime}-\left\{v_{1}\right\}-N_{B_{n}^{\prime}-\left\{v_{1}\right\}}\left[\left\{u_{1}, v_{2}\right\}\right]$ have $b_{2}$ as an
isolated vertex, their independence complexes are contractible. Let $B_{n}^{\prime \prime}=B_{n}^{\prime}-\left\{v_{1}\right\}-$ $\left\{\left(u_{1}, u_{2}\right),\left(u_{1}, v_{2}\right)\right\}$. Now using the similar arguments as in the case of $B_{2}$, we get that $\operatorname{Ind}\left(B_{n}\right) \simeq \operatorname{Ind}\left(B_{n}^{\prime \prime}\right)$. Moreover, the similar arguments imply that

$$
\begin{equation*}
\operatorname{Ind}\left(B_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(A_{n-1}\right)\right) \tag{4}
\end{equation*}
$$

Corollary 10. For $n \geqslant 1, \operatorname{Ind}\left(B_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.2, Section 3.2.2, Corollary 9, induction hypothesis, and Equation (4).

### 3.3.3 $\quad A_{n}$

For $n \geqslant 3$, we show that

$$
\operatorname{Ind}\left(A_{n}\right) \simeq \begin{cases}\vee_{5} \mathbb{S}^{2} & \text { if } n=3,  \tag{5}\\ \vee_{2} \Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \vee \Sigma^{3}\left(\operatorname{Ind}\left(A_{n-3}\right)\right) & \text { if } n \geqslant 4 .\end{cases}
$$

In $A_{n}, a$ is a simplicial vertex with $N_{A_{n}}(a)=\left\{v_{1}, w_{1}, x_{1}\right\}$. Therefore by Lemma 6 ,

$$
\operatorname{Ind}\left(A_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(A_{n}-N_{A_{n}}\left[v_{1}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(A_{n}-N_{A_{n}}\left[w_{1}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(A_{n}-N_{A_{n}}\left[x_{1}\right]\right)\right) .
$$

Clearly $A_{n}-N_{A_{n}}\left[x_{1}\right] \cong D_{n-1} \cong A_{n}-N_{A_{n}}\left[v_{1}\right]$ (see Figure 3.16a for $n=3$ case). Therefore $\operatorname{Ind}\left(A_{n}\right) \simeq \vee_{2} \operatorname{Ind}\left(D_{n-1}\right) \vee \operatorname{Ind}\left(A_{n}-N_{A_{n}}\left[w_{1}\right]\right)$. Since $A_{3}-N_{A_{3}}\left[w_{1}\right] \cong P_{6}$, Lemma 7 and Section 3.2.4 implies $\operatorname{Ind}\left(A_{3}\right) \simeq \Sigma\left(\mathbb{S}^{1} \vee \mathbb{S}^{1}\right) \vee \Sigma\left(\mathbb{S}^{1} \vee \mathbb{S}^{1}\right) \vee \Sigma\left(\mathbb{S}^{1}\right)=\vee_{5} \mathbb{S}^{2}$.

(a) $A_{3}-N_{A_{3}}\left[v_{1}\right]$

(b) $A_{5}^{\prime}=A_{5}-N_{A_{5}}\left[w_{1}\right]$

(c) $A_{5}^{\prime}-\left\{v_{3}, u_{3}\right\}-\left\{x_{3}, y_{3}\right\}$

Figure 3.16

It now suffices to show that $\operatorname{Ind}\left(A_{n}-N_{A_{n}}\left[w_{1}\right]\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(A_{n-3}\right)\right)$, for $n \geqslant 4$. Let $A_{n}^{\prime}=A_{n}-N_{A_{n}}\left[w_{1}\right]$. Since $N_{A_{n}^{\prime}}\left(u_{1}\right)=\left\{u_{2}\right\} \subseteq N_{A_{n}^{\prime}}\left(u_{3}\right) \cap N_{A_{n}^{\prime}}\left(v_{3}\right)$ and $N_{A_{n}^{\prime}}\left(y_{1}\right)=$ $\left\{y_{2}\right\} \subseteq N_{A_{n}^{\prime}}\left(x_{3}\right) \cap N_{A_{n}^{\prime}}\left(y_{3}\right)$ (see Figure 3.16b), Lemma 5 implies that $\operatorname{Ind}\left(A_{n}^{\prime}\right) \simeq \operatorname{Ind}\left(A_{n}^{\prime}-\right.$ $\left\{v_{3}, u_{3}, x_{3}, y_{3}\right\}$ ). Moreover, $A_{n}^{\prime}-\left\{v_{3}, u_{3}, x_{3}, y_{3}\right\} \cong P_{2} \sqcup P_{2} \sqcup A_{n-3}$ (see Figure 3.16c). Therefore, $\operatorname{Ind}\left(A_{n}-N_{A_{n}}\left[w_{1}\right]\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(A_{n-3}\right)\right)$.

Corollary 11. For $n \geqslant 1$, $\operatorname{Ind}\left(A_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.3, Section 3.2.3, induction hypothesis, and Equation (5).

### 3.3.4 $D_{n}$

Let $n \geqslant 3$. Clearly $d$ is a simplicial vertex of $D_{n}$ and $N_{D_{n}}(d)=\left\{u_{1}, v_{1}\right\}$. Therefore by Lemma 6, $\operatorname{Ind}\left(D_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(D_{n}-N_{D_{n}}\left[u_{1}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(D_{n}-N_{D_{n}}\left[v_{1}\right]\right)\right)$. Since $D_{n}-N_{D_{n}}\left[v_{1}\right] \cong$ $D_{n-1}$ and $D_{n}-N_{D_{n}}\left[u_{1}\right] \cong J_{n-2}$ (see Figure 3.17), we have the following homotopy equivalence

$$
\begin{equation*}
\operatorname{Ind}\left(D_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(J_{n-2}\right)\right) . \tag{6}
\end{equation*}
$$



Figure 3.17

Corollary 12. For $n \geqslant 1$, $\operatorname{Ind}\left(D_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.4, Section 3.2.4, induction hypothesis, and Equation (6).

### 3.3.5 $J_{n}$

For $n \geqslant 3$, we show that

$$
\begin{equation*}
\operatorname{Ind}\left(J_{n}\right) \simeq \operatorname{Ind}\left(O_{n-1}\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(D_{n-1}\right)\right) . \tag{7}
\end{equation*}
$$

Since $N_{J_{n}}\left(j_{6}\right) \subseteq N_{J_{n}}\left(j_{3}\right)$, Lemma 5 implies that $\operatorname{Ind}\left(J_{n}\right) \simeq \operatorname{Ind}\left(J_{n}-\left\{j_{3}\right\}\right)$. Observe that $J_{n}-\left\{j_{3}, x_{1}\right\} \cong O_{n-1}$ (see Figure 3.18a) and therefore $\operatorname{del}\left(x_{1}, \operatorname{Ind}\left(J_{n}-\left\{j_{3}\right\}\right)\right)=$ $\operatorname{Ind}\left(J_{n}-\left\{j_{3}, x_{1}\right\}\right) \simeq \operatorname{Ind}\left(O_{n-1}\right)$.

Let $J_{n}^{\prime}$ be the graph $J_{n}-\left\{j_{3}\right\}$. We analyse $\operatorname{lk}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right)$. Clearly $\operatorname{lk}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right)=$ $\operatorname{Ind}\left(J_{n}^{\prime}-N_{J_{n}^{\prime}}\left[x_{1}\right]\right)$. Let $J_{n}^{\prime \prime}=J_{n}^{\prime}-N_{J_{n}^{\prime}}\left[x_{1}\right]$. Observe that the graph $J_{n}^{\prime \prime} \cong D_{n-1} \sqcup P_{3}$ (see Figure 3.18b). Since $N_{J_{n}^{\prime \prime}}\left(j_{4}\right)=\left\{j_{6}\right\}=N_{J_{n}^{\prime \prime}}\left(j_{5}\right)$, from Lemma 5 we get that $\operatorname{Ind}\left(J_{n}^{\prime \prime}\right) \simeq$ $\operatorname{Ind}\left(J_{n}^{\prime \prime}-\left\{j_{5}\right\}\right)$. Clearly $J_{n}^{\prime \prime}-\left\{j_{5}\right\}$ is isomorphic to $D_{n-1} \sqcup P_{2}$. Hence $\operatorname{lk}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right) \simeq$ $\operatorname{Ind}\left(D_{n-1} \sqcup P_{2}\right) \simeq \Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right)$.

(a) $J_{3}-\left\{j_{3}, x_{1}\right\} \cong O_{2}$

(b) $J_{3}^{\prime \prime} \cong D_{2} \sqcup P_{3}$

Figure 3.18

Note that $V\left(J_{n}^{\prime \prime}-\left\{j_{5}\right\}\right) \cap N_{J_{n}-\left\{j_{3}, x_{1}\right\}}\left(j_{2}\right)=\emptyset$ and therefore $\operatorname{Ind}\left(J_{n}^{\prime \prime}-\left\{j_{5}\right\}\right) *\left\{j_{2}\right\} \subseteq$ $\operatorname{Ind}\left(J_{n}-\left\{j_{3}, x_{1}\right\}\right)=\operatorname{del}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right)$. Hence the inclusion map $\operatorname{Ind}\left(J_{n}^{\prime \prime}-\left\{j_{5}\right\}\right) \hookrightarrow$ $\operatorname{del}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right)$ is null homotopic. Thus the composite map

$$
\operatorname{lk}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right)=\operatorname{Ind}\left(J_{n}^{\prime \prime}\right) \xrightarrow{\simeq} \operatorname{Ind}\left(J_{n}^{\prime \prime}-\left\{j_{5}\right\}\right) \hookrightarrow \operatorname{del}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right)
$$

is null homotopic. Thus from Lemma 2 we get that $\operatorname{Ind}\left(J_{n}^{\prime}\right) \simeq \operatorname{del}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right) \vee$ $\Sigma\left(\operatorname{lk}\left(x_{1}, \operatorname{Ind}\left(J_{n}^{\prime}\right)\right)\right) \simeq \operatorname{Ind}\left(O_{n-1}\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(D_{n-1}\right)\right)$.

Corollary 13. For $n \geqslant 1, \operatorname{Ind}\left(J_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.5, Section 3.2.5, induction hypothesis, and Equation (7).

### 3.3.6 $O_{n}$

For $n \geqslant 3$, we show that

$$
\begin{equation*}
\operatorname{Ind}\left(O_{n}\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(D_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(Q_{n-1}\right)\right) \tag{8}
\end{equation*}
$$

Let $O_{n}^{\prime}=O_{n}-\left\{o_{7}, o_{8}, o_{9}\right\}$ (see Figure 3.19a). Using similar arguments as in the case of $O_{2}$, we get that $\operatorname{del}\left(o_{9}, \operatorname{Ind}\left(O_{n}\right)\right)=\Sigma^{2}\left(\operatorname{Ind}\left(D_{n}\right)\right)$ and $\left.\operatorname{lk}\left(o_{9}, \operatorname{Ind}\left(O_{n}\right)\right)\right) \simeq \operatorname{Ind}\left(O_{n}^{\prime \prime}\right)$, where $O_{n}^{\prime \prime}=O_{n}^{\prime}-\left\{x_{1}\right\}$ (see Figure 3.19b). Since $O_{n}^{\prime \prime}-N_{O_{n}^{\prime \prime}}\left[\left\{y_{1}, x_{2}\right\}\right]$ and $O_{n}^{\prime \prime}-N_{O_{n}^{\prime \prime}}\left[\left\{y_{1}, y_{2}\right\}\right]$ both contain $o_{6}$ as an isolated vertex, $\operatorname{Ind}\left(O_{n}^{\prime \prime}-N_{O_{n}^{\prime \prime}}\left[\left\{y_{1}, x_{2}\right\}\right]\right)$ and $\operatorname{Ind}\left(O_{n}^{\prime \prime}-N_{O_{n}^{\prime \prime}}\left[\left\{y_{1}, y_{2}\right\}\right]\right)$ both are cones with apex $o_{6}$ and therefore contractible. Hence from Lemma 4, $\operatorname{Ind}\left(O_{n}^{\prime \prime}\right) \simeq$ $\operatorname{Ind}\left(O_{n}^{\prime \prime}-\left\{\left(y_{1}, x_{2}\right),\left(y_{1}, y_{2}\right)\right\}\right)$. Denote the graph $O_{n}^{\prime \prime}-\left\{\left(y_{1}, x_{2}\right),\left(y_{1}, y_{2}\right)\right\}$ by $O_{n}^{\prime \prime \prime}$. Observe that the graph $O_{n}^{\prime \prime \prime}$ is isomorphic to $Q_{n-1} \sqcup P_{3}$ (see Figure 3.19c) implying that $\operatorname{lk}\left(o_{9}, \operatorname{Ind}\left(O_{n}\right)\right) \simeq \Sigma\left(\operatorname{Ind}\left(Q_{n-1}\right)\right)$.

Now similar arguments as in the case of $O_{2}$ imply that $\operatorname{lk}\left(o_{9}, \operatorname{Ind}\left(O_{n}\right)\right)$ is contractible in $\operatorname{del}\left(o_{9}, \operatorname{Ind}\left(O_{n}\right)\right)$. The Equation (8) then follows from Lemma 2.


Figure 3.19

Corollary 14. For $n \geqslant 1, \operatorname{Ind}\left(O_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.6, Section 3.2.6, Corollary 12, induction hypothesis, and Equation (8).

### 3.3.7 $M_{n}$

For $n \geqslant 3$, we show that

$$
\begin{equation*}
\operatorname{Ind}\left(M_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(M_{n-1}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(F_{n-2}\right)\right) \tag{9}
\end{equation*}
$$

Since $N_{M_{n}}\left(m_{2}\right) \subseteq N_{M_{n}}\left(x_{1}\right)$, Lemma 5 implies that $\operatorname{Ind}\left(M_{n}\right) \simeq \operatorname{Ind}\left(M_{n}-\left\{x_{1}\right\}\right)$. Let $M_{n}^{\prime}$ be the graph $M_{n}-\left\{x_{1}\right\}$. We compute the link and deletion of the vertex $m_{1}$ in $\operatorname{Ind}\left(M_{n}^{\prime}\right)$.


Figure 3.20
Note that $\operatorname{lk}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right)=\operatorname{Ind}\left(M_{n}^{\prime}-N_{M_{n}^{\prime}}\left[m_{1}\right]\right)$. Let $M_{n}^{\prime \prime}$ be the graph $M_{n}^{\prime}-N_{M_{n}^{\prime}}\left[m_{1}\right]$ (see Figure 3.20a). Since $N_{M_{n}^{\prime \prime}}\left(m_{3}\right) \subseteq N_{M_{n}^{\prime \prime}}\left(x_{2}\right)$, from Lemma 5, we get that $\operatorname{Ind}\left(M_{n}^{\prime \prime}\right) \simeq$ $\operatorname{Ind}\left(M_{n}^{\prime \prime}-\left\{x_{2}\right\}\right)$. Observe that both the graphs $\left.\left.M_{n}^{\prime \prime}-\left\{x_{2}\right\}-N_{M_{n}^{\prime \prime}-\left\{x_{2}\right\}}\right\}\left\{y_{2}, x_{3}\right\}\right]$ and $M_{n}^{\prime \prime}-\left\{x_{2}\right\}-N_{M_{n}^{\prime \prime}-\left\{x_{2}\right\}}\left[\left\{y_{2}, y_{3}\right\}\right]$ contain an isolated vertex $m_{3}$ and therefore their independence complexes are contractible. Thus, using Lemma 4 we get that $\operatorname{Ind}\left(M_{n}^{\prime \prime}-\right.$ $\left.\left\{x_{2}\right\}\right) \simeq \operatorname{Ind}\left(M_{n}^{\prime \prime}-\left\{x_{2}\right\}-\left\{\left(y_{2}, x_{3}\right),\left(y_{2}, y_{3}\right)\right\}\right)$ (see Figure 3.20b). Denote the graph $\left.M_{n}^{\prime \prime}-\left\{x_{2}\right\}-\left\{\left(y_{2}, x_{3}\right),\left(y_{2}, y_{3}\right)\right\}\right)$ by $M_{n}^{\prime \prime \prime}$. Since $N_{M_{n}^{\prime \prime \prime}}\left(y_{2}\right) \subseteq N_{M_{n}^{\prime \prime \prime}}\left(m_{3}\right), \operatorname{Ind}\left(M_{n}^{\prime \prime \prime}\right) \simeq$ $\operatorname{Ind}\left(M_{n}^{\prime \prime \prime}-\left\{m_{3}\right\}\right)$. Observe that $M_{n}^{\prime \prime \prime}-\left\{m_{3}\right\} \cong P_{2} \sqcup F_{n-2}$ (see Figure 3.20c), therefore $\operatorname{lk}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right) \simeq \operatorname{Ind}\left(M_{n}^{\prime \prime \prime}\right) \simeq \operatorname{Ind}\left(P_{2} \sqcup F_{n-2}\right) \simeq \Sigma\left(\operatorname{Ind}\left(F_{n-2}\right)\right)$.

We now compute the homotopy type of $\operatorname{del}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right)=\operatorname{Ind}\left(M_{n}^{\prime}-\left\{m_{1}\right\}\right)$ (see Figure 3.20d). Since $N_{M_{n}^{\prime}-\left\{m_{1}\right\}}\left(m_{2}\right) \subseteq N_{M_{n}^{\prime}-\left\{m_{1}\right\}}\left(y_{1}\right)$ and $M_{n}^{\prime}-\left\{m_{1}, y_{1}\right\} \cong P_{2} \sqcup M_{n-1}$, we get that $\operatorname{del}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right) \simeq \operatorname{Ind}\left(P_{2} \sqcup M_{n-1}\right) \simeq \Sigma\left(\operatorname{Ind}\left(M_{n-1}\right)\right)$.

From Lemma 2, it is now enough to show that the inclusion map $\operatorname{lk}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right) \hookrightarrow$ $\operatorname{del}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right)$ is null homotopic. We know that $\operatorname{lk}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right)=\operatorname{Ind}\left(M_{n}^{\prime \prime}\right) \simeq$ $\operatorname{Ind}\left(M_{n}^{\prime \prime \prime}\right) \simeq \operatorname{Ind}\left(M_{n}^{\prime \prime \prime}-\left\{m_{3}\right\}\right)$ and $\operatorname{del}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right)=\operatorname{Ind}\left(M_{n}^{\prime}-\left\{m_{1}\right\}\right)$. Note that $V\left(M_{n}^{\prime \prime \prime}-\left\{m_{3}\right\}\right) \cap N_{M_{n}^{\prime}-\left\{m_{1}\right\}}\left(m_{2}\right)=\emptyset$. Thus $\operatorname{Ind}\left(M_{n}^{\prime \prime \prime}-\left\{m_{3}\right\}\right) *\left\{m_{2}\right\} \subseteq \operatorname{Ind}\left(M_{n}^{\prime}-\left\{m_{1}\right\}\right)$ implying that the map $\operatorname{Ind}\left(M_{n}^{\prime \prime \prime}-\left\{m_{3}\right\}\right) \hookrightarrow \operatorname{Ind}\left(M_{n}^{\prime}-\left\{m_{1}\right\}\right)$ is null homotopic. Hence the composition map

$$
\operatorname{lk}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right) \xrightarrow{\simeq} \operatorname{Ind}\left(M_{n}^{\prime \prime \prime}-\left\{m_{3}\right\}\right) \hookrightarrow \operatorname{del}\left(m_{1}, \operatorname{Ind}\left(M_{n}^{\prime}\right)\right)
$$

is null homotopic.
Corollary 15. For $n \geqslant 1$, $\operatorname{Ind}\left(M_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.7, Section 3.2.7, induction hypothesis, and Equation (9).

### 3.3.8 $Q_{n}$

For $n \geqslant 3$, using the same arguments along the lines as in the case of $Q_{2}$ we get that,

$$
\begin{equation*}
\operatorname{Ind}\left(Q_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(M_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(M_{n-1}\right)\right) . \tag{10}
\end{equation*}
$$

Corollary 16. For $n \geqslant 1$, $\operatorname{Ind}\left(Q_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.8, Section 3.2.8, Corollary 15, induction hypothesis, and Equation (10).

### 3.3.9 $\quad \boldsymbol{F}_{\boldsymbol{n}}$

Observe that $f_{3}$ is a simplicial vertex in $F_{n}$ with $N_{F_{n}}\left(f_{3}\right)=\left\{f_{2}, f_{4}\right\}$, and therefore, $\operatorname{Ind}\left(F_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(F_{n}-N_{F_{n}}\left[f_{2}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(F_{n}-N_{F_{n}}\left[f_{4}\right]\right)\right)$ by Lemma 6. Since $F_{n}-N_{F_{n}}\left[f_{2}\right] \simeq$ $G_{n}$ and $F_{n}-N_{F_{n}}\left[f_{4}\right] \simeq H_{n-1}$ (see Figure 3.21), we get that

$$
\begin{equation*}
\operatorname{Ind}\left(F_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(H_{n-1}\right)\right) \tag{11}
\end{equation*}
$$

Corollary 17. For $n \geqslant 1, \operatorname{Ind}\left(F_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.9, Section 3.2.9, Corollary 9, induction hypothesis, and Equation (11).

(a) $F_{3}$

(b) $F_{3}-N_{F_{3}}\left[f_{2}\right]$

(c) $F_{3}-N_{F_{3}}\left[f_{4}\right]$

Figure 3.21

### 3.3.10 $H_{n}$

Observe that $h_{3}$ is a simplicial vertex in $H_{n}$ with $N_{H_{n}}\left(h_{3}\right)=\left\{h_{2}, h_{4}\right\}$ (see Figure 3.22a). Therefore from Lemma 6,

$$
\operatorname{Ind}\left(H_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(H_{n}-N_{H_{n}}\left[h_{2}\right]\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(H_{n}-N_{H_{n}}\left[h_{4}\right]\right)\right) .
$$



Figure 3.22

We see that $N_{H_{n}-N_{H_{n}}\left[h_{4}\right]}\left(h_{1}\right) \subseteq N_{H_{n}-N_{H_{n}}\left[h_{4}\right]}\left(u_{2}\right) \cap N_{H_{n}-N_{H_{n}}\left[h_{4}\right]}\left(v_{2}\right)$, therefore Lemma 5 implies that $\operatorname{Ind}\left(H_{n}-N_{H_{n}}\left[h_{4}\right]\right) \simeq \operatorname{Ind}\left(H_{n}-N_{H_{n}}\left[h_{4}\right]-\left\{u_{2}, v_{2}\right\}\right)$. Since $H_{n}-N_{H_{n}}\left[h_{4}\right]-$ $\left\{u_{2}, v_{2}\right\} \cong P_{2} \sqcup F_{n-2}$ (see Figure 3.22c) and $H_{n}-N_{H_{n}}\left[h_{2}\right] \cong G_{n}$, we get that

$$
\begin{equation*}
\operatorname{Ind}\left(H_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(F_{n-2}\right)\right) \tag{12}
\end{equation*}
$$

Corollary 18. For $n \geqslant 1$, $\operatorname{Ind}\left(H_{n}\right)$ is homotopy equivalent to a wedge of spheres.
Proof. The result follows from Section 3.1.10, Section 3.2.10, Corollary 9, induction hypothesis, and Equation (12).

## 4 Dimension of the spheres occurring in the homotopy type

In this section we determine the dimensions of all the spheres occurring in the homotopy type of the independence complexes of all ten classes of graphs defined in Section 3.1. For any $m \geqslant n \geqslant 1$, let $[n, m]=\{a \in \mathbb{Z}: n \leqslant a \leqslant m\}$ and

$$
\mathcal{S}^{[n, m]}=\left\{X: X \simeq \bigvee_{d_{1}} \mathbb{S}^{n} \vee \ldots \vee \bigvee_{d_{m+1}} \mathbb{S}^{n+m} \text { for some } d_{1}, \ldots, d_{m+1}>0\right\}
$$

Theorem 19. Let $n \geqslant 1$.

1. If $n \in[9 k, 9 k+8]$, then $\operatorname{Ind}\left(G_{n}\right) \in \mathcal{S}^{[n-1, n+k-1]}$.
2. If $n \in[9 k+8,9 k+16]$, then $\operatorname{Ind}\left(B_{n}\right) \in \mathcal{S}^{[n, n+k+1]}$.
3. If $n \in[9 k+7,9 k+15]$, then $\operatorname{Ind}\left(A_{n}\right) \in \mathcal{S}^{[n-1, n+k]}$.
4. If $n \in[9 k+6,9 k+14]$, then $\operatorname{Ind}\left(D_{n}\right) \in \mathcal{S}^{[n-1, n+k]}$.
5. If $n \in[9 k+4,9 k+12]$, then $\operatorname{Ind}\left(J_{n}\right) \in \mathcal{S}^{[n, n+k+1]}$.
6. If $n \in[9 k+3,9 k+11]$, then $\operatorname{Ind}\left(O_{n}\right) \in \mathcal{S}^{[n+1, n+k+2]}$.
7. If $1 \neq n \in[9 k+2,9 k+10]$, then $\operatorname{Ind}\left(M_{n}\right) \in \mathcal{S}^{[n, n+k]}$, and $\operatorname{Ind}\left(M_{1}\right) \simeq \mathrm{pt}$.
8. If $1 \neq n \in[9 k+2,9 k+10]$, then $\operatorname{Ind}\left(Q_{n}\right) \in \mathcal{S}^{[n+1, n+k+1]}$, and $\operatorname{Ind}\left(Q_{1}\right) \simeq \mathrm{pt}$.
9. If $n \in[9 k, 9 k+8]$, then $\operatorname{Ind}\left(F_{n}\right) \in \mathcal{S}^{[n, n+k]}$.
10. If $n \in[9 k, 9 k+8]$, then $\operatorname{Ind}\left(H_{n}\right) \in \mathcal{S}^{[n, n+k]}$.

Proof. The proof is by induction on $n$. For $n \leqslant 8$, the explicit homotopy types can be computed using the results from Section 3 and are listed in the Table 1.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\operatorname{Ind}\left(G_{n}\right)$ | $\mathbb{S}^{0}$ | $\mathbb{S}^{1} \vee \mathbb{S}^{1}$ | $\vee_{5} \mathbb{S}^{2}$ | $\vee_{9} \mathbb{S}^{3}$ | $\vee_{16} \mathbb{S}^{4}$ | $\vee_{31} \mathbb{S}^{5}$ | $\vee_{55} \mathbb{S}^{6}$ | $\vee_{94} \mathbb{S}^{7}$ |
| $\operatorname{Ind}\left(B_{n}\right)$ | $\vee_{2} \mathbb{S}^{1}$ | $\vee_{4} \mathbb{S}^{2}$ | $\vee_{7} \mathbb{S}^{3}$ | $\vee_{14} \mathbb{S}^{4}$ | $\vee_{26} \mathbb{S}^{5}$ | $\vee_{45} \mathbb{S}^{6}$ | $\vee_{76} \mathbb{S}^{7}$ | $\vee_{136} \mathbb{S}^{8} \vee_{16} \mathbb{S}^{9}$ |
| $\operatorname{Ind}\left(A_{n}\right)$ | $\vee_{2} \mathbb{S}^{0}$ | $\mathbb{S}^{1} \vee \mathbb{S}^{1}$ | $\vee_{5} \mathbb{S}^{2}$ | $\vee_{10} \mathbb{S}^{3}$ | $\vee_{14} \mathbb{S}^{4}$ | $\vee_{21} \mathbb{S}^{5}$ | $\vee_{42} \mathbb{S}^{6} \vee_{16} \mathbb{S}^{7}$ | $\vee_{70} \mathbb{S}^{7} \vee_{22} \mathbb{S}^{8}$ |
| $\operatorname{Ind}\left(D_{n}\right)$ | $\mathbb{S}^{0}$ | $\mathbb{S}^{1} \vee \mathbb{S}^{1}$ | $\vee_{4} \mathbb{S}^{2}$ | $\vee_{6} \mathbb{S}^{3}$ | $\vee_{8} \mathbb{S}^{4}$ | $\vee_{16} \mathbb{S}^{5} \vee_{8} \mathbb{S}^{6}$ | $\vee_{28} \mathbb{S}^{6} \vee_{11} \mathbb{S}^{7}$ | $\vee_{44} \mathbb{S}^{7} \vee_{20} \mathbb{S}^{8}$ |
| $\operatorname{Ind}\left(J_{n}\right)$ | $\vee_{2} \mathbb{S}^{1}$ | $\vee_{2} \mathbb{S}^{2}$ | $\vee_{4} \mathbb{S}^{3}$ | $\vee_{8} \mathbb{S}^{4} \vee_{8} \mathbb{S}^{5}$ | $\vee_{12} \mathbb{S}^{5} \vee_{3} \mathbb{S}^{6}$ | $\vee_{16} \mathbb{S}^{6} \vee_{9} \mathbb{S}^{7}$ | $\vee_{32} \mathbb{S}^{7} \vee_{36} \mathbb{S}^{8}$ | $\vee_{56} \mathbb{S}^{8} \vee_{66} \mathbb{S}^{9}$ |
| $\operatorname{Ind}\left(O_{n}\right)$ | $\mathbb{S}^{2}$ | $\vee_{2} \mathbb{S}^{3}$ | $\vee_{4} \mathbb{S}^{4} \vee^{5} \mathbb{S}^{5}$ | $\vee_{6} \mathbb{S}^{5} \vee_{3} \mathbb{S}^{6}$ | $\vee_{8} \mathbb{S}^{6} \vee_{9} \mathbb{S}^{7}$ | $\vee_{16} \mathbb{S}^{7} \vee_{28} \mathbb{S}^{8}$ | $\vee_{28} \mathbb{S}^{8} \vee_{55} \mathbb{S}^{9}$ | $\vee_{44} \mathbb{S}^{9} \vee_{108} \mathbb{S}^{10}$ |
| $\operatorname{Ind}\left(M_{n}\right)$ | pt | $\mathbb{S}^{2}$ | $\vee_{3} \mathbb{S}^{3}$ | $\vee_{6} \mathbb{S}^{4}$ | $\vee_{14} \mathbb{S}^{5}$ | $\vee_{30} \mathbb{S}^{6}$ | $\vee_{58} \mathbb{S}^{7}$ | $\vee_{93} \mathbb{S}^{8}$ |
| $\operatorname{Ind}\left(Q_{n}\right)$ | pt | $\mathbb{S}^{3}$ | $\vee_{3} \mathbb{S}^{4}$ | $\vee_{9} \mathbb{S}^{5}$ | $\vee_{20} \mathbb{S}^{6}$ | $\vee_{44} \mathbb{S}^{7}$ | $\vee_{88} \mathbb{S}^{8}$ | $\vee_{151} \mathbb{S}^{9}$ |
| $\operatorname{Ind}\left(F_{n}\right)$ | $\vee_{2} \mathbb{S}^{1}$ | $\vee_{3} \mathbb{S}^{2}$ | $\vee_{8} \mathbb{S}^{3}$ | $\vee_{16} \mathbb{S}^{4}$ | $\vee_{28} \mathbb{S}^{5}$ | $\vee_{35} \mathbb{S}^{6}$ | $\vee_{102} \mathbb{S}^{7}$ | $\vee_{177} \mathbb{S}^{8}$ |
| $\operatorname{Ind}\left(H_{n}\right)$ | $\mathbb{S}^{1}$ | $\vee_{3} \mathbb{S}^{2}$ | $\vee_{7} \mathbb{S}^{3}$ | $\vee_{12} \mathbb{S}^{4}$ | $\vee_{24} \mathbb{S}^{5}$ | $\vee_{47} \mathbb{S}^{6}$ | $\vee_{83} \mathbb{S}^{7}$ | $\vee_{129} \mathbb{S}^{8}$ |

Table 1: Independence complexes of all ten classes of graphs for $n \leqslant 8$

For $n \geqslant 9$, let us assume that the result holds for all $m<n$.

1. From Equation (3), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(G_{n}\right) \simeq \operatorname{Ind}\left(B_{n-1}\right) \vee \Sigma^{3}\left(\operatorname{Ind}\left(A_{n-3}\right)\right) \tag{13}
\end{equation*}
$$

Let $n \in[9 k, 9 k+8]$, then $n-1 \in[9(k-1)+8,9(k-1)+16]$. By induction, $\operatorname{Ind}\left(B_{n-1}\right) \in \mathcal{S}^{[n-1, n+k-1]}$. Clearly $n-3 \in[9(k-1)+6,9(k-1)+14]$. If $n-3=$ $9(k-1)+6=9(k-2)+15$, then by induction $\operatorname{Ind}\left(A_{n-3}\right) \in \mathcal{S}^{[n-4, n-3+k-2]}$ and for $n-3 \in[9(k-1)+7,9(k-1)+14]$, $\operatorname{Ind}\left(A_{n-3}\right) \in \mathcal{S}^{[n-4, n-3+k-1]}$. Hence from Equation (13), $\operatorname{Ind}\left(G_{n}\right) \in \mathcal{S}^{[n-1, n+k-1]}$.
2. From Equation (4), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(B_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(A_{n-1}\right)\right) \tag{14}
\end{equation*}
$$

Let $n \in[9 k+8,9 k+16]$. From part (1), we know that for $n=9 k+8, \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \in$ $\mathcal{S}^{[n, n+k]}$ and for $n \in[9 k+9,9 k+16]=[9(k+1), 9(k+1)+7], \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \in$ $\mathcal{S}^{[n, n+k+1]}$. Clearly $n \in[9 k+8,9 k+16]$ implies $n-1 \in[9 k+7,9 k+15]$. By induction, $\operatorname{Ind}\left(A_{n-1}\right) \in \mathcal{S}^{[n-2, n+k-1]}$ and therefore $\Sigma^{2}\left(\operatorname{Ind}\left(A_{n-1}\right)\right) \in \mathcal{S}^{[n, n+k+1]}$. Thus Equation (14) implies that $\operatorname{Ind}\left(B_{n}\right) \in \mathcal{S}^{[n, n+k+1]}$.
3. From Equation (5), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(A_{n}\right) \simeq \vee_{2} \Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \vee \Sigma^{3}\left(\operatorname{Ind}\left(A_{n-3}\right)\right) \tag{15}
\end{equation*}
$$

Let $n \in[9 k+7,9 k+15]$. If $n-1 \in[9 k+6,9 k+14]$, then by induction $\operatorname{Ind}\left(D_{n-1}\right) \in$ $\mathcal{S}^{[n-2, n+k-1]}$ and therefore $\Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \in \mathcal{S}^{[n-1, n+k]}$. Clearly $n \in[9 k+7,9 k+15]$
implies that $n-3 \in[9 k+4,9 k+12]$. If $n-3 \in[9 k+4,9 k+6]=[9(k-1)+13,9(k-$ 1) +15 ], then by induction $\operatorname{Ind}\left(A_{n-3}\right) \in \mathcal{S}^{[n-4, n+k-4]}$ and therefore $\Sigma^{3}\left(\operatorname{Ind}\left(A_{n-3}\right)\right) \in$ $\mathcal{S}^{[n-1, n+k-1]}$. For $n-3 \in[9 k+7,9 k+12], \operatorname{Ind}\left(A_{n-3}\right) \in \mathcal{S}^{[n-4, n-3+k]}$ and therefore $\Sigma^{3}\left(\operatorname{Ind}\left(A_{n-3}\right)\right) \in \mathcal{S}^{[n-1, n+k]}$. The result now follows from Equation (15).
4. From Equation (6), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(D_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(J_{n-2}\right)\right) . \tag{16}
\end{equation*}
$$

Let $n \in[9 k+6,9 k+14]$, then $n-1 \in[9 k+5,9 k+13]$. If $n-1=9 k+5=9(k-$ 1) +14 , then by induction $\operatorname{Ind}\left(D_{n-1}\right) \in \mathcal{S}^{[n-2, n+k-2]}$ and therefore $\Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \in$ $\mathcal{S}^{[n-1, n+k-1]}$. Otherwise $n-1 \in[9 k+7,9 k+14]$ and by induction $\operatorname{Ind}\left(D_{n-1}\right) \in$ $\mathcal{S}^{[n-2, n+k-1]}$ and thus $\Sigma\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \in \mathcal{S}^{[n-1, n+k]}$. Also $n \in[9 k+6,9 k+14]$ implies $n-2 \in[9 k+4,9 k+12]$. By induction $\left.\operatorname{Ind}\left(J_{n-2}\right)\right) \in \mathcal{S}^{[n-2, n-2+k+1]}$, thereby implying that $\Sigma\left(\operatorname{Ind}\left(J_{n-2}\right)\right) \in \mathcal{S}^{[n-1, n+k]}$. Result then follows from Equation (16).
5. From Equation (7), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(J_{n}\right) \simeq \operatorname{Ind}\left(O_{n-1}\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(D_{n-1}\right)\right) . \tag{17}
\end{equation*}
$$

Let $n \in[9 k+4,9 k+12]$, then $n-1 \in[9 k+3,9 k+11]$. By induction, we have $\operatorname{Ind}\left(O_{n-1}\right) \in \mathcal{S}^{[n, n+k+1]}$. Moreover, if $n-1 \in[9 k+3,9 k+5]=[9(k-1)+12,9(k-$ 1) +14$]$, then by induction $\Sigma^{2}\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \in \mathcal{S}^{[n, n+k]}$. For $n-1 \in[9 k+6,9 k+11]$, $\Sigma^{2}\left(\operatorname{Ind}\left(D_{n-1}\right)\right) \in \mathcal{S}^{[n, n+k+1]}$. Therefore the result follows from Equation (17).
6. From Equation (8), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(O_{n}\right) \simeq \Sigma^{2}\left(\operatorname{Ind}\left(D_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(Q_{n-1}\right)\right) . \tag{18}
\end{equation*}
$$

Let $n \in[9 k+3,9 k+11]$. If $n \in[9 k+3,9 k+5]=[9(k-1)+12,9(k-1)+14]$, then by induction $\Sigma^{2}\left(\operatorname{Ind}\left(D_{n}\right)\right) \in \mathcal{S}^{[n+1, n+k+1]}$. Further, if $n \in[9 k+6,9 k+11]$, then again by induction $\Sigma^{2}\left(\operatorname{Ind}\left(D_{n}\right)\right) \in \mathcal{S}^{[n+1, n+k+2]}$. Observe that $n \in[9 k+3,9 k+11]$ implies $n-1 \in[9 k+2,9 k+10]$ and therefore by induction we have $\Sigma^{2}\left(\operatorname{Ind}\left(Q_{n-1}\right)\right) \in$ $\mathcal{S}^{[n+2, n+k+2]}$. The result then follows from Equation (18).
7. From Equation (9), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(M_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(M_{n-1}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(F_{n-2}\right)\right) . \tag{19}
\end{equation*}
$$

Let $n \in[9 k+2,9 k+10]$, then $n-1 \in[9 k+1,9 k+9]$. If $n-1=9 k+1=9(k-1)+10$, then $\Sigma\left(\operatorname{Ind}\left(M_{n-1}\right)\right) \in \mathcal{S}^{[n, n+k-1]}$. For $n-1 \in[9 k+2,9 k+9], \Sigma\left(\operatorname{Ind}\left(M_{n-1}\right)\right) \in$ $\mathcal{S}^{[n, n+k]}$ by induction. Clearly $n \in[9 k+2,9 k+10]$ implies $n-2 \in[9 k, 9 k+$ 8]. Therefore by induction, $\operatorname{Ind}\left(F_{n-2}\right) \in \mathcal{S}^{[n-2, n+k-2]}$ and hence $\Sigma^{2}\left(\operatorname{Ind}\left(F_{n-2}\right)\right) \in$ $\mathcal{S}^{[n, n+k]}$. Equation (19) then implies that for $n \in[9 k+2,9 k+10], \operatorname{Ind}\left(M_{n}\right) \in \mathcal{S}^{[n, n+k]}$.
8. From Equation (10), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(Q_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(M_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(M_{n-1}\right)\right) . \tag{20}
\end{equation*}
$$

Let $n \in[9 k+2,9 k+10]$, then $\Sigma\left(\operatorname{Ind}\left(M_{n}\right)\right) \in \mathcal{S}^{[n+1, n+k+1]}$. Clearly $n-1 \in[9 k+1,9 k+$ 9]. If $n-1=9 k+1=9(k-1)+10$, then $\left.\operatorname{Ind}\left(M_{n-1}\right)\right) \in \mathcal{S}^{[n-1, n+k-2]}$ thereby implying that $\Sigma^{2}\left(\operatorname{Ind}\left(M_{n-1}\right)\right) \in \mathcal{S}^{[n+1, n+k]}$. For $n-1 \in[9 k+2,9 k+9], \Sigma^{2}\left(\operatorname{Ind}\left(M_{n-1}\right)\right) \in$ $\mathcal{S}^{[n+1, n+k+1]}$ by induction. The result now follows from Equation (20).
9. From Equation (11), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(F_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \vee \Sigma\left(\operatorname{Ind}\left(H_{n-1}\right)\right) \tag{21}
\end{equation*}
$$

Let $n \in[9 k, 9 k+8]$. From part (1), we have $\operatorname{Ind}\left(G_{n}\right) \in \mathcal{S}^{[n-1, n+k-1]}$ and therefore $\Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \in \mathcal{S}^{[n, n+k]}$. Clearly $n-1 \in[9 k-1,9 k+7]$. If $n-1=9 k-$ $1=9(k-1)+8$, then by induction $\operatorname{Ind}\left(H_{n-1}\right) \in \mathcal{S}^{[n-1, n+k-2]}$ and therefore $\Sigma\left(\operatorname{Ind}\left(H_{n-1}\right)\right) \in \mathcal{S}^{[n, n+k-1]}$. For $n-1 \in[9 k, 9 k+7], \operatorname{Ind}\left(H_{n-1}\right) \in \mathcal{S}^{[n-1, n+k-1]}$ and hence $\Sigma\left(\operatorname{Ind}\left(H_{n-1}\right)\right) \in \mathcal{S}^{[n, n+k]}$ by induction. The result now follows from Equation (21).
10. From Equation (12), we have the following:

$$
\begin{equation*}
\operatorname{Ind}\left(H_{n}\right) \simeq \Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \vee \Sigma^{2}\left(\operatorname{Ind}\left(F_{n-2}\right)\right) \tag{22}
\end{equation*}
$$

Let $n \in[9 k, 9 k+8]$. From part (1), we have $\operatorname{Ind}\left(G_{n}\right) \in \mathcal{S}^{[n-1, n+k-1]}$ and therefore $\Sigma\left(\operatorname{Ind}\left(G_{n}\right)\right) \in \mathcal{S}^{[n, n+k]}$. Clearly $n-2 \in[9 k-2,9 k+6]$. If $n-2 \in[9 k-2,9 k-1]=$ $[9(k-1)+7,9(k-1)+8]$, then by induction $\Sigma^{2}\left(\operatorname{Ind}\left(F_{n-2}\right)\right) \in \mathcal{S}^{[n, n+k-1]}$. Further, if $n-2 \in[9 k, 9 k+6]$, then again from induction $\Sigma^{2}\left(\operatorname{Ind}\left(F_{n-2}\right)\right) \in \mathcal{S}^{[n, n+k]}$. The result then follows from Equation (22).

## 5 Future directions

From Theorem 19, we see that the number 9 plays an important role in determining the dimensions of spheres that occur in the homotopy type of $M\left(\Gamma_{3, n}\right)$. It would be interesting to see if there is any relation between the number or the dimension of spheres in the homotopy type of $M\left(\Gamma_{3, n}\right)$ and the combinatorial description of $\Gamma_{3, n}$. Another interesting enumerative problem is to calculate the Betti numbers of $M\left(\Gamma_{3, n}\right)$. More precisely,

Question 20. Can we determine the closed form formula for the homotopy type of $M\left(\Gamma_{3, n}\right)$ ?

Based on the main result of this article, our computer-based computations for various general grid graphs, and [9], we propose the following.

Conjecture 21. The complex $M\left(\Gamma_{m, n}\right)$ is homotopy equivalent to a wedge of spheres for any grid graph $\Gamma_{m, n}$.

## References

[1] S. Bouc. Homologie de certains ensembles de 2-sous-groupes des groupes symétriques. J. Algebra, 150(1):158-186, 1992.
[2] B. Braun and W. K. Hough. Matching and independence complexes related to small grids. Electron. J. Combin., 24(4): \#P4.18, 2017.
[3] A. Engström. Independence complexes of claw-free graphs. European J. Combin., 29(1):234-241, 2008.
[4] P. F. Garst. Cohen-Macaulay complexes and group actions. ProQuest LLC, Ann Arbor, MI, 1979. Thesis (Ph.D.)-The University of Wisconsin - Madison.
[5] A. Hatcher. Algebraic Topology. Cambridge University Press, 2002.
[6] J. Jonsson. Matching complexes on grids. unpublished manuscript available at https: //people.kth.se/~jakobj/doc/thesis/grid.pdf, 2005.
[7] D. Kozlov. Combinatorial algebraic topology, volume 21 of Algorithms and Computation in Mathematics. Springer, Berlin, 2008.
[8] D. N. Kozlov. Complexes of directed trees. J. Combin. Theory Ser. A, 88(1):112-122, 1999.
[9] T. Matsushita. Matching complexes of small grids. Electron. J. Combin., 26(3): \#P3.1, 2019.
[10] M. L. Wachs. Topology of matching, chessboard, and general bounded degree graph complexes. Algebra Universalis, 49(4):345-385, 2003.


[^0]:    ${ }^{1}$ A vertex $v$ of $G$ is called simplicial if $G\left[N_{G}(v)\right]$ is a complete graph, i.e. any two distinct vertices are adjacent.

