Saturation number of $tK_{l,l,l}$ in the complete tripartite graph

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Abstract

For fixed graphs F and H, a graph $G \subseteq F$ is H-saturated if there is no copy of H in G, but for any edge $e \in E(F) \setminus E(G)$, there is a copy of H in $G + e$. The saturation number of H in F, denoted $sat(F, H)$, is the minimum number of edges in an H-saturated subgraph of F. In this paper, we study saturation numbers of $tK_{l,l,l}$ in complete tripartite graph K_{n_1,n_2,n_3}. For $t \geqslant 1$, $l \geqslant 1$ and n_1, n_2 and n_3 sufficiently large, we determine $sat(K_{n_1,n_2,n_3}, tK_{l,l,l})$ exactly.

Mathematics Subject Classifications: 05C35

1 Introduction

In this paper, we only consider finite, simple and undirected graphs. Let $G = (V, E)$ be a graph, where V is the vertex set and E is the edge set of G. For a subset S of V, $G[S]$ is a subgraph of G induced by S. Let H be a graph. We will use tH to denote t pairwise disjoint copies of H. Let K_{n_1,n_2,n_3} be a complete tripartite graph with n_i vertices in the i^{th} partite, where $1 \leqslant i \leqslant 3$.

A graph G is said to be H-saturated if it does not contain H as a subgraph, but the addition of any new edge from $E(\overline{G})$ forms a copy of H, where $\overline{G}$ is the complement of G. Let $sat(n, H)$ denote the minimal size of an H-saturated n-vertex graph. Erdős, Hajnal and Moon [5] initiated the study of saturation numbers by determining $sat(n, K_r) = (k-2)n - \binom{k-1}{2}$. Since then, there are plentiful results in this field. Kászonyi and Tuza [6] gave a general upper bound for $sat(n, H)$ and determined $sat(n, P_k)$, $sat(n, K_{1,k})$ and

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$sat(n, kP_2)$. Cycle saturation numbers were studied in [17, 4, 10]. See Faudree, Faudree, and Schmitt [8] for an abundant survey. Among these results, almost all of the considered graphs are connected graphs; only a few unconnected graphs are considered, including matchings [6] and vertex-disjoint cliques [7].

Generalizing further, a subgraph G of host graph F is *H-saturated relative to F* if G does not contain H as a subgraph but adding any edge of $E(F)\setminus E(G)$ to G forms a copy of H. The *saturation number* of F in H is the minimum number of edges in an F-saturated subgraph of H, and is denoted by $sat(F, H)$. With this notation, $sat(n, H) = sat(K_n, H)$. The first result on saturation numbers in host graphs that are not complete is from a related problem in bipartite graphs. Bollobás [2, 3] and Wessel [18, 19] independently determined the saturation number $sat(K_{a,b}, K_{c,d})$. Results about the saturation number when the host graphs are not complete can be foud in [1], [11]-[15]. In [16], Sullivan and Wenger studied saturation numbers in tripartite graphs and determined $sat(K_{n_1,n_2,n_3}, K_{l,l,l})$. In this paper, we generalize Sullivan and Wenger's result and determine $sat(K_{n_1,n_2,n_3}, tK_{l,l,l})$ exactly for $t \geqslant 1$.

Throughout this paper, we assume $n_1 \geqslant n_2 \geqslant n_3$ and the partite sets of K_{n_1,n_2,n_3} are V_1, V_2 and V_3 with $|V_i| = n_i$. When G is a subgraph of K_{n_1,n_2,n_3}, let $\delta_i(G)$ denote the minimum degree of the vertices of V_i in G. When the graph is clear we simply write δ_i. For a vertex $v \in V(G)$, we let $N_i(v) = N(v) \cap V_i$. Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(G)$. Denote $[S_1, S_2] = \{uv \in E(G) | u \in S_1, v \in S_2\}$. Then $[S_1, S_2] = [S_2, S_1]$. If $S_1 = \{u\}$, we will denote $[\{u\}, S_2]$ by $[u, S_2]$. In the following sections, all subscripts are modulo 3.

2 The construction of $tK_{l,l,l}$-saturated graph of K_{n_1,n_2,n_3}

In this Section, we construct a $tK_{l,l,l}$-saturated graph of K_{n_1,n_2,n_3}. We use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. We label the vertices in the partite sets V_i of K_{n_1,n_2,n_3} as $V_i = \{v_i^1, v_i^2, \ldots, v_i^{n_i}\}$, $i \in [3]$. For $0 \leqslant j \leqslant t-1$ and $i \in [3]$, V_i^j are t pairwise disjoint subsets of V_i with $|V_i^j| = l$. We label the vertices in V_i^j as $\{v_i^{lj+1}, v_i^{lj+2}, \ldots, v_i^{(j+1)l}\}$. We begin our construction of a $tK_{l,l,l}$-saturated graph, denoted by H, of K_{n_1,n_2,n_3}.

Construction Let t, l, n_1, n_2 and n_3 be positive integers such that $n_1 \geqslant n_2 \geqslant n_3 \geqslant tl + 1$. Let $V(H) = V_1 \cup V_2 \cup V_3$ and

$$\begin{aligned} E(H) \;\; = \;\; & (\cup_{j=0}^{t-1} \cup_{i=1}^{3} \{uv | u \in V_i^j, v \in V_{i+1}^j\}) \setminus \{v_1^1 v_2^1, v_2^1 v_3^1, v_1^1 v_3^1\} \\ & \cup \cup_{i=1}^{3} \{uv | u \in V_i^0, v \in (V_{i+1} \cup V_{i+2}) \setminus (V_{i+1}^0 \cup V_{i+2}^0)\}. \end{aligned}$$

Obviously, H is a subgraph of K_{n_1,n_2,n_3} and $|E(H)| = 2l(n_1+n_2+n_3) - 3 + 3(t-2)l^2$. Our construction is illustrated in Figure 1. Let $U = \cup_{i=1}^{t-1}(V_1^i \cup V_2^i \cup V_3^i)$ and $V^0 = V_1^0 \cup V_2^0 \cup V_3^0$. About the properties of H, we have the following results.

Property 1 *H is $tK_{l,l,l}$-free.*

Proof of Property 1 Suppose $K_1, \ldots, K_t$ are pairwise disjoint copies of $K_{l,l,l}$ in H. If there is $v \in \cup_{i=1}^{t} V(K_i) \setminus (U \cup V^0)$, say $v \in V(K_1) \cap V_1$, then $N(v) = V_2^0 \cup V_3^0$ by Construction. Since $v_2^1 v_3^1 \notin E(H)$, we have $v_2^1 \notin V(K_1)$ or $v_3^1 \notin V(K_1)$, a contradiction. Hence $\cup_{i=1}^{t} V(K_i) = U \cup V^0$. Then $v_1^1 \in V(K_j)$, say $V(K_1)$. Then $v_2^1, v_3^1 \notin V(K_1)$ by

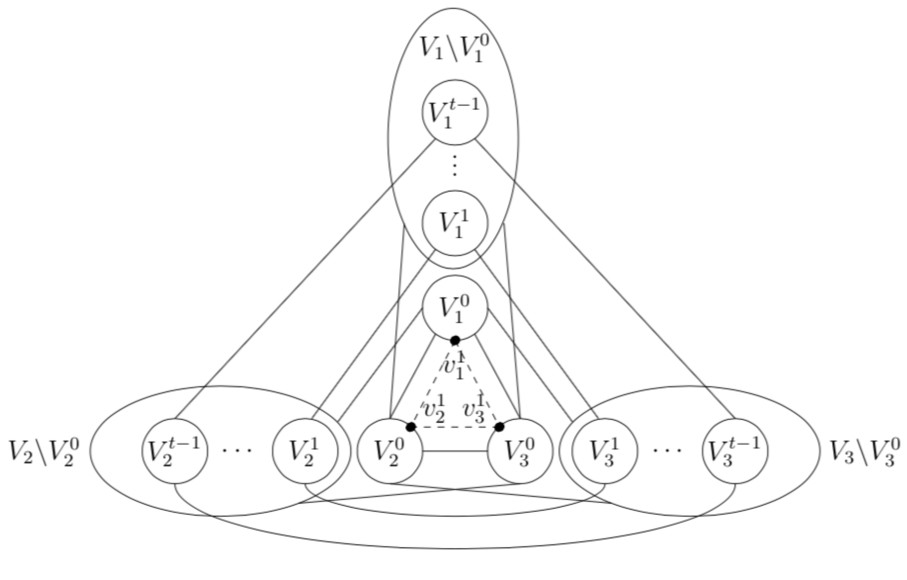

Figure 1: A $tK_{l,l,l}$-saturated subgraph of K_{n_1,n_2,n_3}. Solid lines denote complete joins between sets, and dotted lines denote edges that have been removed.

$v_1^1v_2^1, v_1^1v_3^1 \notin E(H)$. So there are $a \in V_2^i$ and $b \in V_3^i$, say $i = 1$, such that $a, b \in V(K_1)$. By Construction, $V(K_1) \cap V_i \subseteq V_i^0 \cup V_i^1$ for $i \in [3]$. Since $v_1^1 \in V(K_1) \cap V_1^0$, there is $c \in V_1^1 \setminus V(K_1)$. Assume $c \in V(K_2)$. Then $V(K_2) \cap V_i \subseteq V_i^0 \cup V_i^1$ for $i = 2, 3$. Thus $(V(K_1) \cup V(K_2)) \cap V_i = V_i^0 \cup V_i^1$ for $i = 2, 3$. Since $v_2^1v_3^1 \notin E(H)$, $V(K_2) \cap (V_2 \cup V_3) \neq V_2^0 \cup V_3^0$ which implies $(V(K_1) \cup V(K_2)) \cap V_1 = V_1^0 \cup V_1^1$. Thus $V(K_1) \cup V(K_2) = V^0 \cup (\cup_{i=1}^3 V_i^1)$. Since $v_1^1v_2^1, v_2^1v_3^1, v_1^1v_3^1 \notin E(H)$, we have a contradiction. ■

Property 2 *H is a $tK_{l,l,l}$-saturated graph of K_{n_2,n_2,n_3}.*

Proof of Property 2 Let $uv \in E(K_{n_2,n_2,n_3}) \setminus E(H)$. We will show that $H + uv$ contains $tK_{l,l,l}$ by considering the following four cases.

Case 1 $uv \in \{v_1^1v_2^1, v_2^1v_3^1, v_1^1v_3^1\}$.

Assume, without loss of generality, that $uv = v_1^1v_2^1$. By Construction and $n_3 \geqslant tl + 1$, there is $w \in V_3 \setminus \cup_{j=0}^{t-1} V_3^j$ such that $xw \in E(H)$ for all $x \in V_1^0 \cup V_2^0$. Now $H[V_1^i \cup V_2^i \cup V_3^i]$ for all $i \in [t-1]$ and $H[V_1^0 \cup V_2^0 \cup (V_3^0 \setminus \{v_3^1\}) \cup \{w\}] + uv$ form $tK_{l,l,l}$ in $H + uv$.

Case 2 $u, v \in U$.

Assume, without loss of generality, that $u \in V_1^1$ and $v \in V_2^j$, where $2 \leqslant j \leqslant t-1$. Then $V_1^1 \cup V_2^1 \cup V_3^1 \cup \{v_1^1\} \setminus \{u\}$, $V_1^j \cup V_2^j \cup V_3^j \cup \{v_2^1\} \setminus \{v\}$ and $V^0 \cup \{u, v\} \setminus \{v_1^1, v_2^1\}$ induce three pairwise disjoint copies of $K_{l,l,l}$ in $H + uv$, together with $t-3$ pairwise disjoint copies of $K_{l,l,l}$ induced by $\cup_{i=2}^{t-1}(V_1^i \cup V_2^i \cup V_3^i) \setminus (V_1^j \cup V_2^j \cup V_3^j)$, we get $tK_{l,l,l}$ in $H + uv$.

Case 3 $u \in U$ and $v \in V(H) \setminus (U \cup V^0)$.

Assume, without loss of generality, that $u \in V_1^1$ and $v \in V_2 \setminus (U \cup V^0)$. Then $V_1^1 \cup V_2^1 \cup V_3^1 \cup \{v_1^1\} \setminus \{u\}$ and $V^0 \cup \{u, v\} \setminus \{v_1^1, v_2^1\}$ induce two disjoint copies of $K_{l,l,l}$ in $H + uv$. Together with $t-2$ pairwise disjoint copies of $K_{l,l,l}$ induced by $U \setminus (V_1^1 \cup V_2^1 \cup V_3^1)$, we get $tK_{l,l,l}$ in $H + uv$.

Case 4 $u, v \in V(H) \setminus (U \cup V^0)$.

Assume, without loss of generality, that $u \in V_1 \setminus (U \cup V^0)$ and $v \in V_2 \setminus (U \cup V^0)$. Then $V^0 \cup \{u, v\} \setminus \{v_1^1, v_2^1\}$ induces a $K_{l,l,l}$ in $H + uv$. Together with the $t-1$ pairwise disjoint copies of $K_{l,l,l}$ induced by U, we get $tK_{l,l,l}$ in $H + uv$. ■

By Properties 1 and 2, we have our first main result in this Section.

Theorem 1. *Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant tl + 1$. For all $t \geqslant 1$ and $l \geqslant 1$,*

$$sat(K_{n_1,n_2,n_3}, tK_{l,l,l}) \leqslant 2l(n_1 + n_2 + n_3) - 3 + 3(t-2)l^2.$$

3 The saturation number of $tK_{l,l,l}$ in tripartite graphs

In this Section, we prove our main result on saturation number in tripartite graphs.

Theorem 2. *Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant 24l^3 + 44l^2 + 12l + 3(t-1)l^2$. For all $t \geqslant 1$ and $l \geqslant 1$,*

$$sat(K_{n_1,n_2,n_3}, tK_{l,l,l}) = 2l(n_1 + n_2 + n_3) - 3 + 3(t-2)l^2.$$

Since we already have Theorem 1, we just need to prove the lower bound. Before that, we need some lemmas. The idea of the proofs of the following two lemmas comes from [16]. Let

$$k = 2l(n_1 + n_2 + n_3) - 3 + 3(t-2)l^2$$

for short. In the following, we will show that if G is a $tK_{l,l,l}$-saturated subgraph of K_{n_1,n_2,n_3}, then $|E(G)| \geqslant k$. Note that if G is a $tK_{l,l,l}$-saturated subgraph of K_{n_1,n_2,n_3}, then there is a new $K_{l,l,l}$ containing e in $G + e$, where $e \in E(K_{n_1,n_2,n_3}) \setminus E(G)$.

Lemma 3. *Let $i \in [3]$ and assume that $n_i \geqslant (3l+1)(\delta_{i+1} + \delta_{i+2}) + (3t-3)l^2$. If G is a $tK_{l,l,l}$-saturated subgraph of K_{n_1,n_2,n_3} such that $\delta_i > 2l$, then $|E(G)| \geqslant k$.*

Proof. For each $i \in [3]$, let v_i be a vertex of degree δ_i in V_i, respectively. Since $G+e$ forms a new $K_{l,l,l}$ contained e for any edge $e \in E(K_{n_1,n_2,n_3}) \setminus E(G)$, $|N(v_i) \cap N(x)| \geqslant l$ for any $x \in V_{i+1} \cup V_{i+2}$ with $xv_i \notin E(G)$. Therefore there are at least $l(n_{i+1} + n_{i+2} - \delta_i)$ edges joining V_{i+1} and V_{i+2}. Similarly there are at least $l(n_{i+1} - \delta_{i+2})$ edges joining V_{i+1} and $N_i(v_{i+2})$ and at least $l(n_{i+2} - \delta_{i+1})$ edges joining V_{i+2} and $N_i(v_{i+1})$. Finally, for the other vertices in V_i, there are at least $\delta_i(n_i - \delta_{i+1} - \delta_{i+2})$ edges incident to $V_i \setminus (N_i(v_{i+1}) \cup N_i(v_{i+2}))$. Sum these edges, and we have

$$|E(G)| \geqslant l(2n_{i+1} + 2n_{i+2} - \delta_{i+1} - \delta_{i+2}) + \delta_i(n_i - \delta_{i+1} - \delta_{i+2} - l).$$

Note that $n_i > \delta_{i+1} + \delta_{i+2} + l$. With $\delta_i > 2l$, we have

$$\begin{aligned} |E(G)| \geqslant & l(2n_{i+1} + 2n_{i+2} - \delta_{i+1} - \delta_{i+2}) + (2l+1)(n_i - \delta_{i+1} - \delta_{i+2} - l) \\ = & 2l(n_1 + n_2 + n_3) + n_i - [(3l+1)(\delta_{i+1} + \delta_{i+2}) + 2l^2 + l] \geqslant k. \end{aligned}$$

□

Lemma 4. *Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant 24l^3 + 44l^2 + 12l + (3t-3)l^2$. If G is a $tK_{l,l,l}$-saturated subgraph of K_{n_1,n_2,n_3} such that $\delta_i > 2l$ for some $i \in \{1, 2, 3\}$, then $|E(G)| \geqslant k$.*

Proof. Since G is a $tK_{l,l,l}$-saturated subgraph of K_{n_1,n_2,n_3}, $G+e$ forms a new $K_{l,l,l}$ contained e for any edge $e \in E(K_{n_1,n_2,n_3}) \setminus E(G)$ which implies each vertex in V_i has at least l neighbors in both V_{i+1} and V_{i+2} or is completely joined to V_{i+1} or V_{i+2}. Thus $\delta(G) \geqslant 2l$. We distinguish two cases.

Case 1 $n_1 < (4l+1)n_2$.

If $\delta_1 \geqslant 6l+1$, then $|E(G)| \geqslant (6l+1)n_1 \geqslant 2l(n_1+n_2+n_3)+n_1 > k$ and we are done. So we assume that $\delta_1 < 6l+1$. If $\delta_2 \geqslant 8l^2+6l+1$, then $|E(G)| \geqslant (8l^2+6l+1)n_2 \geqslant 2l(n_1+n_2+n_3)+n_2 > k$ and we are done, so we assume that $\delta_2 < 8l^2+6l+1$. Since $n_3 \geqslant 24l^3+44l^2+12l+(3t-3)l^2 \geqslant (3l+1)(\delta_1+\delta_2)+(3t-3)l^2$, Lemma 3 implies that if $\delta_3 > 2l$, then $|E(G)| \geqslant k$ and we are done, so we assume $\delta_3 = 2l$. Lemma 3 implies that if $\delta_1 > 2l$ or $\delta_2 > 2l$, then $|E(G)| \geqslant k$.

Case 2 $n_1 \geqslant (4l+1)n_2$.

If $\delta_1 > 2l$, then $|E(G)| \geqslant (2l+1)n_1 \geqslant 2l(n_1+n_2+n_3)+n_2 > k$, so we assume $\delta_1 = 2l$. Let $R = \{v \in V_1 | d(v) = 2l\}$. If $|V_1 - R| \geqslant 2l(n_2+n_3)+(3t-3)l^2$, then $|E(G)| \geqslant k$, so we assume $|V_1 - R| < 2l(n_2+n_3)+(3t-3)l^2$.

If $v \in R$, then each vertex in $N_2(v)$ is adjacent to every vertex in $V_3 \setminus N_3(v)$. Thus each vertex in $N_2(R)$ has at least $n_3 - l$ neighbors in V_3. If $|N_2(R)| \geqslant \frac{(4l+1)n_2}{n_3-l}$, there are at least $(4l+1)n_2$ edges joining V_2 and V_3, then $|E(G)| \geqslant k$ and we are done, so we assume $|N_2(R)| < \frac{(4l+1)n_2}{n_3-l}$.

There are at least $\delta_2(n_2 - \frac{(4l+1)n_2}{n_3-l})$ edges incident to $V_2 - N_2(R)$. There are at least $2l(n_1 - 2l(n_2+n_3) - (3t-3)l^2)$ edges incident to R. When $\delta_2 \geqslant 8l^2+8l+1$,

$$\delta_2\left(n_2 - \frac{(4l+1)n_2}{n_3-l}\right) \geqslant n_2(8l^2+8l+1)\left(1 - \frac{4l+1}{24l^3+44l^2+11l}\right) \geqslant n_2(8l^2+6l+1).$$

Then we have

$$\begin{aligned}|E(G)| &\geqslant \delta_2\left(n_2 - \frac{(4l+1)n_2}{n_3-l}\right) + 2l[n_1 - 2l(n_2+n_3) - (3t-3)l^2] \\ &\geqslant (8l^2+6l+1)n_2 + 2ln_1 - 4l^2(n_2+n_3) - 2(3t-3)l^3 \\ &\geqslant 2l(n_1+n_2+n_3) + (3t-3)l^2 \geqslant k\end{aligned}$$

and we are done. So we assume $\delta_2 \leqslant 8l^2+8l$. Since $\delta_1 = 2l$, $\delta_2 \leqslant 8l^2+8l$, and $n_3 \geqslant 24l^3+44l^2+12l+(3t-3)l^2 > (3l+1)(\delta_1+\delta_2)+(3t-3)l^2$, Lemma 3 implies that if $\delta_3 > 2l$, then $|E(G)| \geqslant k$ and we are done, so we assume that $\delta_3 = 2l$. By Lemma 3 we know if $\delta_2 > 2l$, then $|E(G)| \geqslant k$. □

Lemma 5. *Let $S \subseteq V(K_{l,l,l})$ and $\overline{S} = V(K_{l,l,l}) \setminus S$. If $|S|, |\overline{S}| \geqslant 1$, then $|[S,\overline{S}]| \geqslant 2l$.*

Proof. Let $V(K_{l,l,l}) = U_1 \cup U_2 \cup U_3$, $S_i = S \cap U_i$ and $\overline{S_i} = \overline{S} \cap U_i$. Let $|S_i| = a_i$ for $i \in [3]$. Then $|\overline{S_i}| = l - a_i$. Assume $a_1 \geqslant a_2 \geqslant a_3$. Then

$$\begin{aligned}|[S,\overline{S}]| &= a_1(2l-a_2-a_3) + a_2(2l-a_1-a_3) + a_3(2l-a_1-a_2) \\ &= 2(a_1l + a_2l + a_3l - a_1a_2 - a_2a_3 - a_1a_3) \\ &= 2(a_1(l-a_2-a_3) + a_2l + a_3l - a_2a_3).\end{aligned}$$

When $a_2 + a_3 > l$, the lower bound of $|[S, \overline{S}]|$ is decreases as a_1 increases. Since $a_1 \leqslant l$, we have $|[S, \overline{S}]| \geqslant 2(l(l - a_2 - a_3) + a_2 l + a_3 l - a_2 a_3) = 2(l^2 - a_2 a_3)$. Note that $a_3 \leqslant l - 1$. Thus $|[S, \overline{S}]| \geqslant 2l$.

Suppose $a_2 + a_3 \leqslant l$. If $a_2 = a_3 = 0$, then $|[S, \overline{S}]| = 2a_1 l \geqslant 2l$. If $a_2 \geqslant 1$, the lower bound of $|[S, \overline{S}]|$ is increases as a_1 increases. So

$$
\begin{aligned}
|[S, \overline{S}]| &\geqslant 2(2a_2 l + a_3 l - a_2^2 - 2a_2 a_3) \\
&= 2(-(a_2 + (a_3 - l))^2 + a_3 l + (a_3 - l)^2) \\
&\geqslant 2(-(1 + (a_3 - l))^2 + a_3 l + (a_3 - l)^2) \\
&= 2(2l - 1 + a_3(l - 2)) \geqslant 4l - 2 \geqslant 2l.
\end{aligned}
$$

□

Now we are going to prove Theorem 2. Let G be a $tK_{l,l,l}$-saturated graph of K_{n_1,n_2,n_3}. We will show that $|E(G)| \geqslant k = 2l(n_1 + n_2 + n_3) - 3 + 3(t - 2)l^2$. From Lemma 4, we assume that $\delta_1 = \delta_2 = \delta_3 = 2l$.

For $i \in [3]$, let $v_i \in V_i$ such that $d(v_i) = \delta_i = 2l$. Thus $|N_{i+1}(v_i)| = |N_{i+2}(v_i)| = l$ and G contains all edges joining $N_{i+1}(v_i)$ to $V_{i+2} \setminus N_{i+2}(v_i)$ and all edges joining $N_{i+2}(v_i)$ to $V_{i+1} \setminus N_{i+1}(v_i)$. Therefore, the vertices of degree $2l$ in G form an independent set. Let $V^0 = N(v_1) \cup N(v_2) \cup N(v_3)$ and let $V_i^0 = V^0 \cap V_i$. Since $|N(v_{i+1}) \cap N(v_{i+2})| = l$, we conclude that $N_i(v_{i+1}) = N_i(v_{i+2})$ and therefore $V_i^0 = N_i(v_{i+1}) = N_i(v_{i+2})$ and $|V_i^0| = l$. Denote $G_0 = G[V^0]$, $E_i = [V_i^0, V_{i+1} \setminus V_{i+1}^0]$ and $E_i' = [V_i^0, V_{i+2} \setminus V_{i+2}^0]$ for $i \in [3]$. Then $|E_i| = l(n_{i+1} - l)$ and $|E_i'| = l(n_{i+2} - l)$. Let $\overline{E}_1 = \cup_{i=1}^3 (E_i \cup E_i')$. Then $|\overline{E}_1| = 2l(n_1 + n_2 + n_3) - 6l^2$. Since $G + v_i x$ completes a copy of $K_{l,l,l}$ containing v_i for any $x \in V_{i+1} \setminus N(v_i)$, there is a complete bipartite graph joining $l - 1$ vertices in V_{i+1}^0 and l vertices in V_{i+2}^0. Also there is a complete bipartite graph joining $l - 1$ vertices in V_{i+2}^0 and l vertices in V_{i+1}^0. Thus G_0 is a complete tripartite graph minus at most three edges, implying that $|E(G_0)| \geqslant 3l^2 - 3$.

Proof of Theorem 2 (in the case $l = 1$) In this case, $K_{1,1,1} = K_3$ and $k = 2(n_1 + n_2 + n_3) + 3t - 9$. Denote $V_i^0 = \{x_i\}$ for $i \in [3]$. Let $G' = G[V \setminus \{x_1, x_2, x_3\}]$ and $K_1, \ldots, K_s$ be all pairwise disjoint copies of K_3 in G'. Since G contains $t - 1$ pairwise disjoint copies of K_3, $t - 4 \leqslant s \leqslant t - 1$. Note that $N(v_1) = \{x_2, x_3\}$, $N(v_2) = \{x_1, x_3\}$ and $N(v_3) = \{x_1, x_2\}$. So $v_1, v_2, v_3 \notin \cup_{i=1}^s V(K_i)$. Then $|E_i| = n_{i+1} - 1$, $|E_i'| = n_{i+2} - 1$ and $|\overline{E}_1| = 2(n_1 + n_2 + n_3) - 6$. If $s = t - 1$, then $|E(G)| \geqslant |\overline{E}_1| + 3(t - 1) = 2(n_1 + n_2 + n_3) + 3t - 9 = k$ and we are done. If $s = t - 4$, $G + v_1 v_2$ contains at most $t - 1$ pairwise disjoint copies of K_3, a contradiction with G being tK_3-saturated. So we just consider the following two cases.

Case 1 $s = t - 3$.

If there is i, say $i = 1$, such that $x_1 x_2 \notin E(G)$, then there are at most $(t - 3) + 2$ pairwise disjoint copies of K_3 in $G + x_1 x_2$, a contradiction. Hence $|E(G_0)| = 3$ and then $|E(G)| \geqslant |\overline{E}_1| + |E(G_0)| + 3(t - 3) = k - 3$. Since there are t pairwise disjoint copies of K_3 in $G + v_2 v_3$, we can assume there are $u_1, u_1' \in V_1 \setminus (\cup_{i=1}^{t-3} V(K_i) \cup \{x_1\})$ with $u_1 \neq u_1'$, $u_2 \in V_2 \setminus (\cup_{i=1}^{t-3} V(K_i) \cup \{x_2\})$ and $u_3 \in V_3 \setminus (\cup_{i=1}^{t-3} V(K_i) \cup \{x_3\})$ such that $x_2 u_1, x_2 u_3, u_1 u_3 \in E(G)$ and $x_3 u_1', x_3 u_2, u_1' u_2 \in E(G)$. So $|E(G)| \geqslant k - 1$. If

$|E(G)| = k-1$, then $G+v_1v_3$ contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction.

Case 2 $s = t-2$.

In this case, we have

$$\begin{aligned}|E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + 3(t-2)\\ &= 2(n_1+n_2+n_3) - 6 + |E(G_0)| + 3(t-2) = k + |E(G_0)| - 3.\end{aligned}$$

So we can assume $|E(G_0)| \leqslant 2$. Denote $\overline{E}_2 = \cup_{i=1}^{t-2} E(K_i)$ and $V' = \cup_{i=1}^{t-2} V(K_i)$. Then $|V'| = 3t-6$. We first consider the case $|E(G_0)| = 2$. Then $|E(G)| \geqslant k-1$. Assume $x_1x_2, x_1x_3 \in E(G)$. If $|E(G)| = k-1$, then

$$E(G) = \overline{E}_1 \cup \overline{E}_2 \cup \{x_1x_2, x_1x_3\}.$$

But $G+v_2v_3$ contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction.

Suppose $|E(G_0)| = 1$, say $x_1x_2 \in E(G)$. Then $|E(G)| \geqslant k-2$. Suppose $|E(G)| = k-2$. Let $G' = G + x_1x_3$. Then $|E(G')| = k-1$. By the discussion above, G' contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction. Hence $|E(G)| \geqslant k-1$. Suppose $|E(G)| = k-1$. Then there is $e \notin \overline{E}_1 \cup \overline{E}_2 \cup \{x_1x_2\}$ such that $E(G) = \overline{E}_1 \cup \overline{E}_2 \cup \{x_1x_2, e\}$. Let $e = uv$. Suppose $\{u, v\} \subseteq V'$. Since G is a tK_3-saturated graph, there are t pairwise disjoint copies of K_3, say $K_0^{v_1v_3}, \ldots, K_{t-1}^{v_1v_3}$, in $G + v_1v_3$. Denote $V^{v_1v_3} = \cup_{i=0}^{t-1} V(K_i^{v_1v_3})$. Then $V^{v_1v_3} \subseteq V' \cup \{x_1, x_2, x_3, v_1, v_3\}$ which implies $|V^{v_1v_3}| \leqslant 3t-1$, a contradiction. Suppose $u \in V'$ and $v \in V \setminus (V' \cup \{x_1, x_2, x_3, v_1, v_2, v_3\})$. Then $N(v) \subseteq \{x_1, x_2, x_3, u\}$ which implies $G+v_1v_3$ (resp. $G+v_2v_3$) contains at most $t-1$ pairwise disjoint copies of K_3 if $v \in V_1 \cup V_3$ (resp. $v \in V_2$), a contradiction. Suppose $u, v \in V \setminus (V' \cup \{x_1, x_2, x_3, v_1, v_2, v_3\})$. Then $N(u) \cup N(v) \subseteq \{u, v, x_1, x_2, x_3\}$. If $u, v \notin V_3$, then $x_1x_2v_3x_1, ux_3vu, K_1, \ldots, K_{t-2}$ form tK_3 of G, a contradiction. So we assume $u \in V_3$. Then $G+v_1v_3$ (resp. $G+v_2v_3$) contains at most $t-1$ pairwise disjoint copies of K_3 if $v \in V_1$ (resp. $v \in V_2$), a contradiction.

Suppose $|E(G_0)| = 0$. Then $|E(G)| \geqslant k-3$. If $|E(G)| = k-3$, then $G+x_1x_2$ contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction. Suppose $|E(G)| = k-2$. Then there is $e \notin \overline{E}_1 \cup \overline{E}_2$ such that $E(G) = \overline{E}_1 \cup \overline{E}_2 \cup \{e\}$. Let $e = uv$. Since there are t pairwise disjoint copies of K_3 in $G + x_1x_2$, by the discussion in the case $|E(G_0)| = 1$, we have $u, v \in V \setminus (V' \cup \{x_1, x_2, x_3, v_1, v_2, v_3\})$ and $u, v \notin V_3$. Assume $u \in V_1$ and $v \in V_2$. Since $n_3 \geqslant 3(t-1)+80$, there is a vertex $w \in V_3 \setminus (V' \cup \{x_3, v_3\})$. But $G+uw$ contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction. Hence $|E(G)| \geqslant k-1$. Suppose $|E(G)| = k-1$. Then there are $e_1, e_2 \notin \overline{E}_1 \cup \overline{E}_2$ such that $E(G) = \overline{E}_1 \cup \overline{E}_2 \cup \{e_1, e_2\}$. Let $e_i = u_iw_i$, $i = 1, 2$. Suppose $u_1, w_1 \in V'$, say $u_1 \in V(K_1)$ and $w_1 \in V(K_2)$. Then there are $q_1 \in V(K_1)$ and $q_2 \in V(K_2)$ such that $q_1q_2 \notin E(G)$. Thus, there are t pairwise disjoint copies of K_3, say $K_0^{q_1q_2}, \ldots, K_{t-1}^{q_1q_2}$, in $G + q_1q_2$. Denote $V^{q_1q_2} = \cup_{i=0}^{t-1} V(K_i^{q_1q_2})$. Then $V^{q_1q_2} \subseteq V' \cup \{x_1, x_2, x_3, u_2, w_2\}$ which implies $|V^{q_1q_2}| \leqslant 3t-1$, a contradiction. So we can assume $w_1, w_2 \in V \setminus (V' \cup \{x_1, x_2, x_3, v_1, v_2, v_3\})$. Suppose $u_1, u_2 \in V'$. Then $G + x_1x_2$ contains at most $t-1$ pairwise disjoint copies of K_3 by $N(w_i) \subseteq \{u_i, x_1, x_2, x_3\}$ for $i = 1, 2$, a contradiction. Suppose $u_1, u_2 \in V \setminus (V' \cup \{x_1, x_2, x_3, v_1, v_2, v_3\})$. Assume that $u_1, u_2 \in V_1$. If $w_1, w_2 \in V_i$, then $i \neq 1$ and $G + v_1v_i$ contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction. Now we assume that $w_1 \in V_2$ and $w_2 \in V_3$. In this case, we claim that $u_1 = u_2$; otherwise $w_1u_1x_3w_1, w_2u_2x_2w_2, K_1, \ldots, K_{t-2}$ form tK_3 of G, a contradiction. When $u_1 = u_2$, $G + w_1w_2$ contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction. Suppose $u_1 \in V \setminus (V' \cup \{x_1, x_2, x_3, v_1, v_2, v_3\})$ and $u_2 \in V'$, say $u_2 \in V(K_1)$. Let $V(K_1) = \{q_1, q_2, q_3\}$, where $q_i \in V_i$ for $i \in [3]$. Assume that $u_1 \in V_1$ and $w_1 \in V_2$. Then $N(u_1) = \{w_1, x_2, x_3\}$ and $N(w_1) = \{u_1, x_1, x_3\}$. If $u_2 = q_3$ (resp. $u_2 \in \{q_1, q_2\}$ and $w_2 \in V_1 \cap V_2$), then $N(q_1) \cap N(q_2) \cap V(G_0) = \{x_3\}$ (resp. $N(u_2) \cap N(w_2) \cap V(G_0) = \{x_3\}$). In these cases, $G + x_1x_3$ contains at most $t-1$ pairwise disjoint copies of K_3, a contradiction. If $u_2 \in \{q_1, q_2\}$ and $w_2 \in V_3$, say $u_2 = q_1$, then $u_2w_2x_2u_2, u_1w_1x_3u_1, q_2q_3x_1q_2, K_2, \ldots, K_{t-2}$ form tK_3 of G, a contradiction. □

Proof of Theorem 2 (in the case $l \geqslant 2$) Now we are going to prove Theorem 2 where $l \geqslant 2$. Recall that for $i \in [3]$, $d(v_i) = \delta_i = 2l$, where $v_i \in V_i$. Denote $V^0 = N(v_1) \cup N(v_2) \cup N(v_3)$, $V_i^0 = V^0 \cap V_i$, $G_0 = G[V^0]$, $E_i = [V_i^0, V_{i+1} \setminus V_{i+1}^0]$ and $E_i' = [V_i^0, V_{i+2} \setminus V_{i+2}^0]$ for $i \in [3]$. Then $|E_i| = l(n_{i+1} - l)$ and $|E_i'| = l(n_{i+2} - l)$. Let $\overline{E}_1 = \cup_{i=1}^3 (E_i \cup E_i')$. Then $|\overline{E}_1| = 2l(n_1 + n_2 + n_3) - 6l^2$ and $|E(G_0)| \geqslant 3l^2 - 3$. We first have the following claim.

Claim 1 *Let $x_i, y_i \in V_i^0$ for $i \in [3]$ such that $x_1x_2, y_2y_3, x_3y_1 \notin E(G_0)$. Then there is $i \in [3]$ such that $x_i = y_i$ and $x_{i+1} = y_{i+1}$.*

Proof of Claim 1 Suppose $x_1 \neq y_1$ and $x_2 \neq y_2$. Then there is no copy of $K_{l,l,l}$ in $G + v_1v_2$ containing v_1v_2, a contradiction with G being a $tK_{l,l,l}$-saturated graph of K_{n_1,n_2,n_3}. Now we suppose $x_1 = y_1$, but $x_2 \neq y_2$ and $x_3 \neq y_3$. Then there is no copy of $K_{l,l,l}$ in $G + v_2v_3$ containing v_2v_3, a contradiction. ■

Since G is a $tK_{l,l,l}$-saturated graph and $v_iv_{i+1} \notin E(G)$ for all $i \in [3]$, there are t pairwise disjoint copies of $K_{l,l,l}$ in $G + v_iv_{i+1}$ and one of them, denote by $K_0^{v_iv_{i+1}}$, contains v_iv_{i+1}. Since $V_i^0 = N_i(v_{i+1}) = N_i(v_{i+2})$ for $i \in [3]$, $V(K_0^{v_iv_{i+1}}) = (V^0 \cup \{v_i, v_{i+1}\}) \setminus \{x_{v_iv_{i+1}}, y_{v_iv_{i+1}}\}$, where $x_{v_iv_{i+1}} \in V_i^0$ and $y_{v_iv_{i+1}} \in V_{i+1}^0$. Let $K_1^{v_iv_{i+1}}, K_2^{v_iv_{i+1}}, \ldots, K_{t-1}^{v_iv_{i+1}}$ be the other $t-1$ copies of $K_{l,l,l}$ in $G + v_iv_{i+1}$. Then $(\cup_{j=1}^{t-1} V(K_j^{v_iv_{i+1}})) \cap V^0 \subseteq \{x_{v_iv_{i+1}}, y_{v_iv_{i+1}}\}$. In each case, we choose $K_1^{v_iv_{i+1}}, K_2^{v_iv_{i+1}}, \ldots, K_{t-1}^{v_iv_{i+1}}$ such that $|(\cup_{j=1}^{t-1} V(K_j^{v_iv_{i+1}})) \cap \{x_{v_iv_{i+1}}, y_{v_iv_{i+1}}\}|$ is as small as possible. If there is i, say $i = 1$, such that $|(\cup_{j=1}^{t-1} V(K_j^{v_1v_2})) \cap \{x_{v_1v_2}, y_{v_1v_2}\}| = 0$, then

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + \textstyle\sum_{i=1}^{t-1} |E(K_i^{v_1v_2})| \\ &\geqslant 2l(n_1 + n_2 + n_3) - 3l^2 - 3 + 3(t-1)l^2 = k, \end{aligned}$$

and we are done. So we will assume that $|(\cup_{j=1}^{t-1} V(K_j^{v_iv_{i+1}})) \cap \{x_{v_iv_{i+1}}, y_{v_iv_{i+1}}\}| \geqslant 1$ for all $i \in [3]$.

In the following, we will denote $V^{v_iv_{i+1}} = \cup_{j=1}^{t-1} V(K_j^{v_iv_{i+1}}) \setminus \{x_{v_iv_{i+1}}, y_{v_iv_{i+1}}\}$, $i \in [3]$. Let $u \in V$. Denote $N_j^{v_iv_{i+1}}(u) = (N(u) \cap V(K_j^{v_iv_{i+1}})) \setminus V^0$ and $\tau_j^{v_iv_{i+1}}(u) = [u, N_j^{v_iv_{i+1}}(u)]$, where $i \in [3]$ and $j \in [t-1]$. Let $\uplus$ denote the disjoint union of sets. We consider the following three cases.

Case 1 There is i, say $i = 1$, such that $|(\cup_{j=1}^{t-1} V(K_j^{v_1v_2})) \cap \{x_{v_1v_2}, y_{v_1v_2}\}| = 1$.

Assume, without loss of generality, that $x_{v_1v_2} \in V(K_1^{v_1v_2})$. Set $K_1 = G[V(K_1^{v_1v_2}) \setminus \{x_{v_1v_2}\}]$. Then $|E(K_1)| = 3l^2 - 2l$. Since $x_{v_1v_2} \in V(K_1^{v_1v_2}) \cap V_1^0$ and $V_1^0 \subseteq V(K_0^{v_2v_3})$ in $G + v_2v_3$, $|V^{v_1v_2} \cap V_1| < |V^{v_2v_3} \cap V_1|$ which implies there is $u \in V_1 \setminus V_1^0$ such that

$u \in V^{v_2v_3} \setminus V^{v_1v_2}$. Then $|N(u) \setminus V^0| \geqslant 2l-2$. If $|E(G_0)| \geqslant 3l^2-1$, then

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \textstyle\sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + |N(u) \setminus V^0| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + 3l^2 - 1 + 3(t-1)l^2 - 2l + (2l-2) \\ &\geqslant 2l(n_1+n_2+n_3) - 3l^2 - 3 + 3(t-1)l^2 = k, \end{aligned}$$

and we are done. If $|N(u) \setminus V^0| \geqslant 2l$, then

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \textstyle\sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + 2l \\ &\geqslant 2l(n_1+n_2+n_3) - 3l^2 - 3 + 3(t-1)l^2 = k, \end{aligned}$$

and we are done. So we will assume $3l^2-3 \leqslant |E(G_0)| \leqslant 3l^2-2$, $2l-2 \leqslant |N(u) \setminus V^0| \leqslant 2l-1$ and consider the following two subcases.

Case 1.1 $|E(G_0)| = 3l^2-2$.

In this case, if $|N(u) \setminus V^0| = 2l-1$, then

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \textstyle\sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + |N(u) \setminus V^0| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - 2l + (2l-1) \\ &= 2l(n_1+n_2+n_3) - 3l^2 - 3 + 3(t-1)l^2 = k, \end{aligned}$$

and we are done. So we assume $|N(u) \setminus V^0| = 2l-2$ and $u \in V(K_1^{v_2v_3})$. Since $V^0 \setminus \{x_{v_2v_3}, y_{v_2v_3}\} \subseteq V(K_0^{v_2v_3})$ and $|N(u) \setminus V^0| = 2l-2$, we have $x_{v_2v_3}, y_{v_2v_3} \in V(K_1^{v_2v_3})$. Now we have

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \textstyle\sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + (2l-2) \\ &= 2l(n_1+n_2+n_3) - 3l^2 - 4 + 3(t-1)l^2 = k-1. \end{aligned}$$

If $|E(G)| = k-1$, then all inequalities given above are tight. So $N(u) \setminus V^0 = N_1^{v_2v_3}(u) = V(K_1^{v_2v_3}) \setminus (V_1 \cup \{x_{v_2v_3}, y_{v_2v_3}\})$ and

$$E(G) = \overline{E}_1 \uplus E(G_0) \uplus E(K_1) \uplus (\uplus_{i=2}^{t-1} E(K_i^{v_1v_2})) \uplus \tau_1^{v_2v_3}(u),$$

which implies $\uplus_{i=2}^{t-1} E(K_i^{v_2v_3}) \uplus (E(K_1^{v_2v_3}) \setminus (\tau_1^{v_2v_3}(u) \cup \tau_1^{v_2v_3}(x_{v_2v_3}) \cup \tau_1^{v_2v_3}(y_{v_2v_3}) \cup \{x_{v_2v_3}y_{v_2v_3}\}) \subseteq E(K_1) \uplus \uplus_{i=2}^{t-1} E(K_i^{v_1v_2})$. Thus $V(K_1^{v_2v_3}) \setminus \{u, x_{v_2v_3}, y_{v_2v_3}\} \subseteq V(K_1)$. Since $l \geqslant 2$, there are $b_2, b_3 \in V(K_1) \setminus V^{v_2v_3}$ such that $b_2 \in V_2$ and $b_3 \in V_3$. Then $G[(V(K_0^{v_2v_3}) \cup \{b_2, b_3\}) \setminus \{v_2, v_3\}]$ and $K_i^{v_2v_3}$ for $1 \leqslant i \leqslant t-1$ form $tK_{l,l,l}$ in G, a contradiction.

Case 1.2 $|E(G_0)| = 3l^2-3$.

By Claim 1, we assume there are $x, x', y, z \in V^0$ such that $xy, x'z, yz \notin E(G)$ (possibly $x = x'$). If $x, x' \in V_1^0$, then $x_{v_2v_3} = y$ and $y_{v_2v_3} = z$, where we assume $y \in V_2^0$ and $z \in V_3^0$. Assume $u \in V(K_1^{v_2v_3})$. Since $yz \notin E(G)$ and $2l-2 \leqslant |N(u) \setminus V^0| \leqslant 2l-1$, we can assume $y \in V(K_1^{v_2v_3})$ but $z \notin V(K_1^{v_2v_3})$. Thus we have

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \textstyle\sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + |N_1^{v_2v_3}(u)| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - 2l + (2l-1) \\ &= 2l(n_1+n_2+n_3) - 3l^2 - 4 + 3(t-1)l^2 = k-1. \end{aligned}$$

If $|E(G)| = k-1$, then all inequalities given above are tight. So $N(u) \setminus V^0 = N_1^{v_2v_3}(u) = V(K_1^{v_2v_3}) \setminus (V_1 \cup \{y\})$ and

$$E(G) = \overline{E}_1 \uplus E(G_0) \uplus E(K_1) \uplus \uplus_{i=2}^{t-1} E(K_i^{v_1v_2}) \uplus \tau_1^{v_2v_3}(u).$$

By the same argument as above, we have $V(K_1^{v_2v_3}) \setminus \{u, y\} \subseteq V(K_1)$. Since $l \geqslant 2$, there are $b_2 \in (V(K_1) \cap V_2) \setminus V(K_1^{v_2v_3})$ and $b_1 \in V(K_1) \cap V_1 \cap V(K_1^{v_2v_3})$. But $G[(V_0 \cup \{b_1, b_2\}) \setminus \{y, x'\}]$, $G[(V(K_1^{v_2v_3}) \cup \{x'\}) \setminus \{b_1\}]$ and $K_i^{v_1v_2}$ for $2 \leqslant i \leqslant t-1$ form $tK_{l,l,l}$ in G, a contradiction.

Now we will assume $x, x' \in V_2^0 \cup V_3^0$, say $x, x' \in V_2^0$. Suppose $y \in V_3^0$ and $z \in V_1^0$. Then $x_{v_2v_3} = x'$ and $y_{v_2v_3} = y$. As the discussion above, we assume $|N_1^{v_2v_3}(u)| = 2l-2$ and $u, x', y \in V(K_1^{v_2v_3})$. Then

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \textstyle\sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + |N_1^{v_2v_3}(u)| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - 2l + (2l-2) \\ &= 2l(n_1+n_2+n_3) - 3l^2 - 5 + 3(t-1)l^2 = k-2. \end{aligned}$$

If there is $u' \in V^{v_2v_3} \setminus V^{v_1v_2}$ and $u \neq u'$, then $|N(u') \setminus V^0| \geqslant 2l-2$. So $|E(G)| \geqslant k-2+(2l-2) \geqslant k$ and we are done. If there is no j $(1 \leqslant j \leqslant t-1)$ such that $V(K_1^{v_2v_3}) \setminus \{u, x', y\} \subseteq V(K_j^{v_1v_2})$, then $|E(G)| \geqslant k-2+(2l-2) \geqslant k$ by Lemma 5 and we are done. So we assume there is j $(1 \leqslant j \leqslant t-1)$ such that $V(K_1^{v_2v_3}) \setminus \{u, x', y\} \subseteq V(K_j^{v_1v_2})$. By the same argument, there is j_i $(1 \leqslant j_i \leqslant t-1)$ such that $V(K_i^{v_2v_3}) \subseteq V(K_{j_i}^{v_1v_2})$ for all $2 \leqslant i \leqslant t-1$. Since $l \geqslant 2$, we have $V(K_1^{v_2v_3}) \setminus \{u, x', y\} \subseteq V(K_1)$. Then there are $b_2, b_3 \in V(K_1) \setminus V(K_1^{v_2v_3})$ such that $b_2 \in V_2$ and $b_3 \in V_3$. Thus $G[(V^0 \cup \{b_2, b_3\}) \setminus \{x', y\}]$, $K_1^{v_2v_3}$ and $K_i^{v_1v_2}$ for $2 \leqslant i \leqslant t-1$ form $tK_{l,l,l}$ in G, a contradiction.

By Case 1, we assume that $|(\cup_{j=1}^{t-1} V(K_j^{v_iv_{i+1}})) \cap \{x_{v_iv_{i+1}}, y_{v_iv_{i+1}}\}| = 2$ for all $i \in [3]$.

Case 2 There is i, say $i=1$, such that $x_{v_1v_2}, y_{v_1v_2} \in V(K_j^{v_1v_2})$, where $1 \leqslant j \leqslant t-1$.

Assume that $j=1$, that is $x_{v_1v_2}, y_{v_1v_2} \in V(K_1^{v_1v_2})$. Recall that $x_{v_1v_2} \in V_1^0$ and $y_{v_1v_2} \in V_2^0$. Since $V_i^0 \subseteq V(K_0^{v_{i+1}v_{i+2}})$ in $G + v_{i+1}v_{i+2}$ for $i = 1, 2$, there is $u_{v_{i+1}v_{i+2}} \in V_i \setminus V_i^0$ such that $u_{v_{i+1}v_{i+2}} \in V^{v_{i+1}v_{i+2}} \setminus V^{v_1v_2}$. If $u_{v_2v_3}u_{v_3v_1} \in E(G)$, then $G[(V(K_0^{v_1v_2}) \cup \{u_{v_2v_3}, u_{v_3v_1}\}) \setminus \{v_1, v_2\}]$ and $K_i^{v_1v_2}$ for $1 \leqslant i \leqslant t-1$ form $tK_{l,l,l}$ in G, a contradiction. Thus $u_{v_2v_3}u_{v_3v_1} \notin E(G)$. In the following, we assume $u_{v_{i+1}v_{i+2}} \in V(K_1^{v_{i+1}v_{i+2}})$ for $i = 1, 2$. Let $K_1 = G[V(K_1^{v_1v_2}) \setminus \{x_{v_1v_2}, y_{v_1v_2}\}]$. Then $|E(K_1)| = 3l^2 - (4l-1)$. If $|N_1^{v_2v_3}(u_{v_2v_3})| = 2l$ or $|N_1^{v_3v_1}(u_{v_3v_1})| = 2l$, say $|N_1^{v_2v_3}(u_{v_2v_3})| = 2l$, then

$$\begin{aligned} |E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + |N_1^{v_2v_3}(u_{v_2v_3})| + |N_1^{v_3v_1}(u_{v_3v_1})| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - (4l-1) + 2l + (2l-2) \\ &\geqslant 2l(n_1+n_2+n_3) - 3l^2 - 3 + 3(t-1)l^2 = k-1. \end{aligned}$$

If $|E(G)| = k-1$, then all inequalities given above are tight. So $N(u_{v_2v_3}) \setminus V^0 = N_1^{v_2v_3}(u_{v_2v_3}) = V(K_1^{v_2v_3}) \setminus V_1$ which implies $y_{v_1v_2} \notin V(K_1^{v_2v_3})$. Also $N(u_{v_3v_1}) \setminus V^0 = N_1^{v_3v_1}(u_{v_3v_1}) = V(K_1^{v_3v_1}) \setminus (V_2 \cup \{x_{v_3v_1}, y_{v_3v_1}\})$ and

$$E(G) = \overline{E}_1 \uplus E(G_0) \uplus E(K_1) \uplus_{i=1}^{t-1} E(K_i^{v_1v_2}) \uplus \tau_1^{v_2v_3}(u_{v_2v_3}) \uplus \tau_1^{v_3v_1}(u_{v_3v_1}).$$

Hence there is i_0, $1 \leqslant i_0 \leqslant t-1$ such that $V(K_1^{v_2v_3}) \setminus \{u_{v_2v_3}\} \subseteq V(K_{i_0}^{v_1v_2})$. Since $y_{v_1v_2} \in V(K_1^{v_1v_2})$ but $y_{v_1v_2} \notin V(K_1^{v_2v_3})$, $i_0 \neq 1$, say $i_0 = 2$. Thus there is $u \in V(K_2^{v_1v_2}) \cap V_1$ and $u \notin V^{v_2v_3}$. Since $N_2^{v_1v_2}(u) = V(K_2^{v_1v_2}) \setminus V_1$ and $x_{v_1v_2} \in (V(K_1^{v_1v_2}) \cap V_1^0) \setminus \cup_{i=1}^{t-1} V(K_i^{v_2v_3})$, we have a contradiction with $\cup_{i=2}^{t-1} E(K_i^{v_2v_3}) \subseteq \cup_{i=1}^{t-1} E(K_i^{v_1v_2})$. So we have $|\{x_{v_2v_3}, y_{v_2v_3}\} \cap V(K_1^{v_2v_3})| \geqslant 1$ and $|\{x_{v_3v_1}, y_{v_3v_1}\} \cap V(K_1^{v_3v_1})| \geqslant 1$. We first have the following claim.

Claim 2 *For any $i \in \{2,3\}$, $x_{v_iv_{i+1}}, y_{v_iv_{i+1}} \in V(K_1^{v_iv_{i+1}})$.*

Proof of Claim 2 Suppose $|\{x_{v_2v_3}, y_{v_2v_3}\} \cap V(K_1^{v_2v_3})| = 1$. Then

$$\begin{aligned}
|E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1) + \sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + |N_1^{v_2v_3}(u_{v_2v_3})| + |N_1^{v_3v_1}(u_{v_3v_1})| \\
&\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - (4l-1) + 2l - 1 + 2l - 2 \\
&\geqslant 2l(n_1+n_2+n_3) - 3l^2 - 5 + 3(t-1)l^2 = k-2.
\end{aligned}$$

If there is $u' \in V^{v_2v_3} \setminus (V^{v_1v_2} \cup \{u_{v_2v_3}, u_{v_3v_1}\})$, then $|E(G)| \geqslant k - 2 + (2l-2) \geqslant k$ and we are done. If there is no j $(1 \leqslant j \leqslant t-1)$ such that $V(K_i^{v_2v_3}) \setminus \{u_{v_2v_3}, x_{v_2v_3}, y_{v_2v_3}, u_{v_3v_1}\} \subseteq V(K_j^{v_1v_2})$ for some $i \in [t-1]$, then $|E(G)| \geqslant k - 2 + (2l-2) \geqslant k$ by Lemma 5 and we are done. So we assume that there is j_i $(1 \leqslant j_i \leqslant t-1)$ such that $V(K_1^{v_2v_3}) \setminus \{u_{v_2v_3}, x_{v_2v_3}, y_{v_2v_3}\} \subseteq V(K_{j_1}^{v_1v_2})$ and $V(K_i^{v_2v_3}) \setminus \{u_{v_3v_1}, x_{v_2v_3}, y_{v_2v_3}\} \subseteq V(K_{j_i}^{v_1v_2})$ for $2 \leqslant i \leqslant t-1$. Hence $j_1 = 1$, which implies $x_{v_2v_3} \in V(K_1^{v_2v_3})$ by $y_{v_1v_2} \in V(K_1^{v_1v_2})$. Let $K_1' = G[V(K_1) \cup \{u_{v_2v_3}, y_{v_1v_2}\}]$. Then $K_1', K_2^{v_1v_2}, \ldots, K_{t-1}^{v_1v_2}$ are $t-1$ pairwise disjoint copies of $K_{l,l,l}$ in $G+v_1v_2$ such that $|(V(K_1') \cup \cup_{j=2}^{t-1} V(K_j^{v_1v_2})) \cap \{x_{v_1v_2}, y_{v_1v_2}\}| < |(\cup_{j=1}^{t-1} V(K_j^{v_1v_2})) \cap \{x_{v_1v_2}, y_{v_1v_2}\}|$, a contradiction. ■

By Claim 2, we have that $x_{v_iv_{i+1}}, y_{v_iv_{i+1}} \in V(K_1^{v_iv_{i+1}})$ for $i \in [3]$.

Claim 3 $|E(G_0)| \geqslant 3l^2 - 2$.

Proof of Claim 3 Recall that $|E(G_0)| \geqslant 3l^2 - 3$. Suppose $|E(G_0)| = 3l^2 - 3$. By Claim 1, there are $x, x', y, z \in V^0$ such that $xy, x'z, yz \notin E(G)$. Assume, without loss of generality, that $y \in V_2^0$ and $z \in V_3^0$. Then $x_{v_2v_3} = y$ and $y_{v_2v_3} = z$. Since $yz \notin E(G)$, $|\{x_{v_2v_3}, y_{v_2v_3}\} \cap V(K_1^{v_2v_3})| \leqslant 1$, a contradiction. ■

By Claim 3 and $l \geqslant 2$, we easily have the following claim.

Claim 4 *For any $i \in [3]$, there are $a_i \in V_i^0$ and $a_{i+1} \in V_{i+1}^0$ such that $a_ia_{i+1} \in E(G)$ and $G[V^0 \setminus \{a_i, a_{i+1}\}]$ is a complete tripartite graph.*

Note that $|N_1^{v_iv_{i+1}}(u_{v_iv_{i+1}})| = 2l - 2$ for $i = 2, 3$. So we have

$$\begin{aligned}
|E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + \sum_{i=2}^{t-1} |E(K_i^{v_1v_2})| + |N_1^{v_2v_3}(u_{v_2v_3})| + |N_1^{v_3v_1}(u_{v_3v_1})| \\
&= 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - (4l-1) + 2(2l-2) \\
&= k + |E(G_0)| - 3l^2. \qquad (*)
\end{aligned}$$

If $|E(G_0)| = 3l^2$, then we have $|E(G)| \geqslant k$ and we are done. So we assume $|E(G_0)| \leqslant 3l^2 - 1$ and then $3l^2 - 2 \leqslant |E(G_0)| \leqslant 3l^2 - 1$ by Claim 3. If there is $u \in V^{v_iv_{i+1}} \setminus \{u_{v_iv_{i+1}}, u_{v_{i+1}v_{i+2}}\}$ for some $i \in \{2,3\}$ such that $u \notin V^{v_1v_2}$, then $|E(G)| \geqslant k - 2 + 2l - 2 \geqslant k$ and we are done. So we assume

$$V^{v_iv_{i+1}} \setminus \{u_{v_iv_{i+1}}, u_{v_{i+1}v_{i+2}}\} \subseteq V^{v_1v_2} \quad \text{for } i = 2, 3.$$

By Lemma 5 and the same argument as above, we have $V(K_1^{v_iv_{i+1}})\setminus\{u_{v_iv_{i+1}}, x_{v_iv_{i+1}}, y_{v_iv_{i+1}}\} \subseteq V(K_1)$ for $i=2,3$. Since $V(K_1^{v_1v_2})\cap V_3^0=\emptyset$ and $|V(K_1^{v_2v_3})\cap V_3^0|=1$ (resp. $|V(K_1^{v_3v_1})\cap V_3^0|=1$), there is a unique vertex $b\in V(K_1)\cap V_3$ (resp. $b'\in V(K_1)\cap V_3$) such that $b\notin V(K_1^{v_2v_3})$ (resp. $b'\notin V(K_1^{v_3v_1})$). If $u_{v_2v_3}b\in E(G)$ (resp. $u_{v_3v_1}b'\in E(G)$), then $G[(V(K_1^{v_1v_2})\setminus\{x_{v_1v_2}\})\cup\{u_{v_2v_3}\}]$ (resp. $G[(V(K_1^{v_1v_2})\setminus\{y_{v_1v_2}\})\cup\{u_{v_3v_1}\}]$), $K_2^{v_1v_2},\dots,K_{t-1}^{v_1v_2}$ would be a contradiction with the choice of $K_1^{v_1v_2},K_2^{v_1v_2},\dots,K_{t-1}^{v_1v_2}$. So we have $u_{v_2v_3}b\notin E(G)$ and $u_{v_3v_1}b'\notin E(G)$.

Suppose $b\neq b'$. By Claim 4, there are $a_1\in V_1^0$, $a_3\in V_3^0$ and $a_1a_3\in E(G)$ such that $G[V^0\setminus\{a_1,a_3\}]$ is a complete tripartite graph. But $G[(V^0\setminus\{a_1,a_3\})\cup\{u_{v_2v_3},b'\}]$, $G[(V(K_1^{v_1v_2})\setminus\{x_{v_1v_2},y_{v_1v_2},b'\})\cup\{a_1,a_3,u_{v_3v_1}\}]$, $K_2^{v_1v_2},\dots,K_{t-1}^{v_1v_2}$ form $tK_{l,l,l}$ in G, a contradiction. Hence we have $b=b'$.

Now we complete the proof of Case 2. Note that $N_1^{v_2v_3}(u_{v_2v_3})=V(K_1)\setminus(V_1\cup\{b\})$ and $N_1^{v_3v_1}(u_{v_3v_1})=V(K_1)\setminus(V_2\cup\{b\})$. Denote

$$E'=\overline{E}_1\uplus E(G_0)\uplus E(K_1)\uplus_{i=2}^{t-1}E(K_i^{v_1v_2}).$$

Suppose $|E(G_0)|=3l^2-1$. Then $|E(G)|\geqslant k-1$ by (*). If $|E(G)|=k-1$, then $E(G)=E'\uplus\tau_1^{v_2v_3}(u_{v_2v_3})\uplus\tau_1^{v_3v_1}(u_{v_3v_1})$. So $N(w)\setminus V_0=\emptyset$ for any $w\in V\setminus(V^{v_1v_2}\cup V^0\cup\{u_{v_2v_3},u_{v_3v_1}\})$. Since $|E(G_0)|=3l^2-1$, there are $q_1\in V_i^0$ and $q_2\in V_{i+1}^0$ such that $q_1q_2\notin E(G)$ for some $i\in[3]$. Since G is a $tK_{l,l,l}$-saturated graph, there are t pairwise disjoint copies of $K_{l,l,l}$, say $K_0^{q_1q_2},\dots,K_{t-1}^{q_1q_2}$, in $G+q_1q_2$. Assume $q_1q_2\in E(K_0^{q_1q_2})$. If there is $w\in V\setminus(V^{v_1v_2}\cup V^0\cup\{u_{v_2v_3},u_{v_3v_1}\})$ such that $w\in V(K_0^{q_1q_2})$, then we have $V_2^0\subseteq V(K_0^{q_1q_2})$ or $V_3^0\subseteq V(K_0^{q_1q_2})$. Thus there are at most $t-1$ pairwise disjoint copies of $K_{l,l,l}$ in $G+q_1q_2$ by $bu_{v_3v_1},bu_{v_2v_3}\notin E(G)$, a contradiction. Thus $\cup_{i=0}^{t-1}V(K_i^{q_1q_2})=(\cup_{j=0}^{t-1}V(K_j^{v_1v_2})\setminus\{v_1,v_2\})\cup\{u_{v_2v_3},u_{v_3v_1}\}$. But there are at most $t-1$ pairwise disjoint copies of of $K_{l,l,l}$ in $G+q_1q_2$ by $u_{v_2v_3}u_{v_3v_1},u_{v_2v_3}b,bu_{v_3v_1}\notin E(G)$, a contradiction.

Suppose $|E(G_0)|=3l^2-2$. Then $|E(G)|\geqslant k-2$ by (*). By the same argument as above, we can assume $|E(G)|\geqslant k-1$.

Suppose $|E(G)|=k-1$. Then there is $e\notin E'\uplus\tau_1^{v_2v_3}(u_{v_2v_3})\uplus\tau_1^{v_3v_1}(u_{v_3v_1})$ such that $e\in E(G)$. Let $e=uv$. Then $\{u,v\}\cap V^0=\emptyset$.

Claim 5 $\{u,v\}\cap V^{v_1v_2}\neq\emptyset$.

Proof of Claim 5 Suppose $\{u,v\}\cap V^{v_1v_2}=\emptyset$. Assume $u\in V_i$ and $v\in V_{i+1}$, and a is the vertex in $\{u_{v_2v_3},u_{v_3v_1},b\}$ such that $a\in V_{i+2}$, $i\in[3]$. By Claim 4, there are $a_i\in V_i^0$, $a_{i+1}\in V_{i+1}^0$ and $a_ia_{i+1}\in E(G)$ such that $G[V^0\setminus\{a_i,a_{i+1}\}]$ is a complete tripartite graph. Then $G[(V(K_0^{v_1v_2})\cup\{x_{v_1v_2},y_{v_1v_2},u,v\})\setminus\{v_1,v_2,a_i,a_{i+1}\}]$, $G[(V(K_1)\setminus\{b\})\cup\{a,a_i,a_{i+1}\}]$ and $K_j^{v_1v_2}$ for $2\leqslant j\leqslant t-1$ form $tK_{l,l,l}$ in G, a contradiction. ∎

By Claim 5, we assume $u\in V^{v_1v_2}$. If $u=b$, then $v\notin\{u_{v_2v_3},u_{v_3v_1}\}$ and we claim that $v\in V^{v_1v_2}$. Otherwise, assume $v\in V_1$. Since $l\geqslant 2$, there is $x_b\in V_3^0$ such that $x_bx_{v_1v_2}\in E(G)$. Then $G[(V^0\cup\{b,v\})\setminus\{x_b,x_{v_1v_2}\}]$, $G[(V(K_1^{v_1v_2})\setminus\{y_{v_1v_2},b\})\cup\{u_{v_3v_1},x_b\}]$ and $K_i^{v_1v_2}$ $(2\leqslant i\leqslant t-1)$ form $tK_{l,l,l}$ in G, a contradiction.

Since $|E(G_0)|=3l^2-2$, there are $q_1\in V_i^0$ and $q_2\in V_{i+1}^0$ such that $q_1q_2\notin E(G)$ for some $i\in[3]$. Since G is a $tK_{l,l,l}$-saturated graph, there are t pairwise disjoint copies of $K_{l,l,l}$ in $G+q_1q_2$ and one of them, denote by $K_0^{q_1q_2}$, contains q_1q_2. If $e\notin E(K_0^{q_1q_2})$ or $e\in E(K_0^{q_1q_2})$ but $v\in V^{v_1v_2}\cup\{u_{v_2v_3},u_{v_3v_1}\}$, then there are at most $t-1$ copies of $K_{l,l,l}$

in $G+q_1q_2$ by $u_{v_2v_3}u_{v_3v_1}, u_{v_2v_3}b, bu_{v_3v_1} \notin E(G)$, a contradiction. Suppose $e \in E(K_0^{q_1q_2})$ and $v \notin V^{v_1v_2} \cup \{u_{v_2v_3}, u_{v_3v_1}\}$. Then $N(v) \setminus V_0 = \{u\}$ and $u \neq b$. If $u \in V_3$ or $v \in V_3$, say $u \in V_3$, then $V_1^0 \subseteq V(K_0^{q_1q_2})$ when $v \in V_2$ (resp. $V_2^0 \subseteq V(K_0^{q_1q_2})$ when $v \in V_1$). Thus there are at most $t-1$ copies of $K_{l,l,l}$ in $G+q_1q_2$ by $u_{v_2v_3}b \notin E(G)$ when $v \in V_2$ (resp. $bu_{v_3v_1} \notin E(G)$ when $v \in V_1$), a contradiction. Now we consider the case $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$, say $u \in V_1$ and $v \in V_2$. Then $(V(K_0^{q_1q_2}) \cap V_1) \setminus V_1^0 = \{u\}$ and $V_3^0 \subseteq V(K_0^{q_1q_2})$. Thus there are at most $t-1$ copies of $K_{l,l,l}$ in $G+q_1q_2$ by $u_{v_2v_3}b \notin E(G)$, a contradiction.

Case 3 For any $i \in [3]$, we can assume $x_{v_iv_{i+1}} \in V(K_1^{v_iv_{i+1}})$ and $y_{v_iv_{i+1}} \in V(K_2^{v_iv_{i+1}})$.

By the same argument as that of Case 2, there is $u_{v_{i+1}v_{i+2}} \in V_i \setminus V_i^0$ such that $u_{v_{i+1}v_{i+2}} \in V^{v_{i+1}v_{i+2}} \setminus V^{v_1v_2}$ for $i=1,2$ and $u_{v_2v_3}u_{v_3v_1} \notin E(G)$. Then $|N(u_{v_2v_3}) \setminus V_0| \geqslant 2l-1$ and $|N(u_{v_3v_1}) \setminus V_0| \geqslant 2l-1$. Let $K_1 = G[V(K_1^{v_1v_2}) \setminus \{x_{v_1v_2}\}]$ and $K_2 = G[V(K_2^{v_1v_2}) \setminus \{y_{v_1v_2}\}]$. Then $|E(K_1)| = |E(K_2)| = 3l^2-2l$. If $|E(G_0)| \geqslant 3l^2-1$, then

$$\begin{aligned}|E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + |E(K_2)| + \sum_{i=3}^{t-1} |E(K_i^{v_1v_2})| + |N(u_{v_2v_3}) \setminus V_0| \\ &\quad +|N(u_{v_3v_1}) \setminus V_0| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - 4l + (2l-1) + (2l-1) \\ &\geqslant 2l(n_1+n_2+n_3) - 3l^2 - 3 + 3(t-1)l^2 = k,\end{aligned}$$

and we are done. So we assume $3l^2-3 \leqslant |E(G_0)| \leqslant 3l^2-2$. If $|N(u_{v_2v_3}) \setminus V_0| \geqslant 2l$ and $|N(u_{v_3v_1}) \setminus V_0| \geqslant 2l$, then we have $|E(G)| \geqslant k$ and we are done. So we assume that $u_{v_3v_1} \in V(K_1^{v_3v_1}) \cup V(K_2^{v_3v_1})$. Suppose $|N(u_{v_2v_3}) \setminus V_0| \geqslant 2l$. Assume, without loss of generality, that $u_{v_3v_1} \in V(K_1^{v_3v_1})$ and $u_{v_2v_3} \in V(K_3^{v_2v_3})$. Then

$$\begin{aligned}|E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + |E(K_2)| + \textstyle\sum_{i=3}^{t-1} |E(K_i^{v_1v_2})| + |N(u_{v_3v_1}) \setminus V_0| \\ &\quad +|N(u_{v_2v_3}) \setminus V_0| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - 4l + (2l-1) + 2l \\ &\geqslant 2l(n_1+n_2+n_3) - 3l^2 - 4 + 3(t-1)l^2 = k-1.\end{aligned}$$

If $|E(G)| = k-1$, then all inequalities given above are tight. So $N(u_{v_3v_1}) \setminus V^0 = N_1^{v_3v_1}(u_{v_3v_1}) = V(K_1^{v_3v_1}) \setminus (V_2 \cup \{x_{v_3v_1}\})$, $N(u_{v_2v_3}) \setminus V^0 = N_3^{v_2v_3}(u_{v_2v_3}) = V(K_3^{v_2v_3}) \setminus V_1$ and

$$E(G) = \overline{E}_1 \uplus E(G_0) \uplus E(K_1) \uplus E(K_2) \uplus_{i=3}^{t-1} E(K_i^{v_1v_2}) \uplus \tau_1^{v_3v_1}(u_{v_3v_1}) \uplus \tau_3^{v_2v_3}(u_{v_2v_3}),$$

which implies $V(K_3^{v_2v_3}) \setminus \{u_{v_2v_3}\} \subseteq V(K_1)$. Let $K_1' = G[V(K_1) \cup \{u_{v_2v_3}\}]$. Then K_1' and $K_i^{v_1v_2}$ for $2 \leqslant i \leqslant t-1$ are $t-1$ copies of $K_{l,l,l}$ in $G+v_1v_2$ such that $|(V(K_1) \cup \cup_{j=2}^{t-1} V(K_j^{v_1v_2})) \cap \{x_{v_1v_2}, y_{v_1v_2}\}| < |(\cup_{j=1}^{t-1} V(K_j^{v_1v_2})) \cap \{x_{v_1v_2}, y_{v_1v_2}\}|$, a contradiction. Hence we can assume $u_{v_2v_3} \in V(K_1^{v_2v_3}) \cup V(K_2^{v_2v_3})$. Now we have

$$\begin{aligned}|E(G)| &\geqslant |\overline{E}_1| + |E(G_0)| + |E(K_1)| + |E(K_2)| + \textstyle\sum_{i=3}^{t-1} |E(K_i^{v_1v_2})| + |N(u_{v_2v_3}) \setminus V_0| \\ &\quad +|N(u_{v_3v_1}) \setminus V_0| \\ &\geqslant 2l(n_1+n_2+n_3) - 6l^2 + |E(G_0)| + 3(t-1)l^2 - 4l + (2l-1) + (2l-1) \\ &\geqslant 2l(n_1+n_2+n_3) - 3l^2 - 5 + 3(t-1)l^2 = k-2.\end{aligned}$$

By the same argument as that of Case 2, we can assume that $V^{v_2v_3}\setminus\{u_{v_2v_3}, u_{v_3v_1}\} \subseteq V^{v_1v_2}$, and $V(K_1^{v_2v_3})\setminus\{u_{v_2v_3}, x_{v_2v_3}\} \subseteq V(K_1)$ if $u_{v_2v_3} \in V(K_1^{v_2v_3})$ (resp. $V(K_2^{v_2v_3})\setminus\{u_{v_2v_3}, y_{v_2v_3}\} \subseteq V(K_1)$ if $u_{v_2v_3} \in V(K_2^{v_2v_3})$).

Assume without loss of generality that $u_{v_2v_3} \in V(K_1^{v_2v_3})$. Then $V(K_1^{v_2v_3})\setminus\{u_{v_2v_3}, x_{v_2v_3}\} \subseteq V(K_1)$. Since $u_{v_2v_3}u_{v_3v_1} \notin E(G)$, $u_{v_3v_1} \notin V(K_1)$. Since $y_{v_1v_2} \in V(K_2^{v_1v_2}) \cap V_2^0$ and $\cup_{j=2}^{t-1}V(K_j^{v_2v_3}) \setminus \{y_{v_2v_3}, u_{v_3v_1}\} \subseteq \cup_{j=2}^{t-1}V(K_j^{v_1v_2})$, we have $u_{v_3v_1} \in \cup_{j=2}^{t-1}V(K_j^{v_2v_3})$. If $u_{v_3v_1} \notin V(K_2^{v_2v_3})$, say $u_{v_3v_1} \in V(K_3^{v_2v_3})$, then $V(K_3^{v_2v_3})\setminus\{u_{v_3v_1}\} \subseteq V(K_2)$. Thus $K_1^{v_1v_2}, G[V(K_2)\cup \{u_{v_3v_1}\}], \dots, K_{t-1}^{v_1v_2}$ would contradict with the choice of $K_i^{v_1v_2}$, $1 \leqslant i \leqslant t-1$. Hence we have $u_{v_3v_1} \in V(K_2^{v_2v_3})$ and then $V(K_2^{v_2v_3}) \setminus \{u_{v_3v_1}, y_{v_2v_3}\} \subseteq V(K_2)$. Since $x_{v_2v_3} \in V(K_1^{v_2v_3}) \cap V_2^0$ (resp. $y_{v_2v_3} \in V(K_2^{v_2v_3}) \cap V_3^0$) and $l \geqslant 2$, there is a unique vertex $a_2 \in V(K_1) \cap V_2$ (resp. $a_3 \in V(K_2) \cap V_3$) such that $a_2 \notin V(K_1^{v_2v_3})$ (resp. $a_3 \notin V(K_2^{v_2v_3})$). If $u_{v_2v_3}a_2 \in E(G)$, then $G[V(K_1)\cup\{u_{v_2v_3}\}], K_2^{v_1v_2}, \dots, K_{t-1}^{v_1v_2}$ will be a contradiction with the choice of $K_i^{v_1v_2}$, $1 \leqslant i \leqslant t-1$. Hence $u_{v_2v_3}a_2 \notin E(G)$. Similarly, $u_{v_3v_1}a_3 \notin E(G)$.

Now we have $N_1^{v_2v_3}(u_{v_2v_3}) = V(K_1)\setminus(V_1\cup\{a_2\})$ and $N_2^{v_2v_3}(u_{v_3v_1}) = V(K_2)\setminus(V_2\cup\{a_3\})$. Let

$$E' = \overline{E}_1 \uplus E(G_0) \uplus E(K_1) \uplus E(K_2) \uplus_{i=3}^{t-1} E(K_i^{v_1v_2}) \uplus \tau_1^{v_2v_3}(u_{v_2v_3}) \uplus \tau_2^{v_2v_3}(u_{v_3v_1}).$$

Then $|E(G)| \geqslant |E'| = k - 3l^2 + 1 + |E(G_0)|$.

We will complete the proof by considering the following two subcases.

Case 3.1 $|E(G_0)| = 3l^2 - 2$.

In this case, we have $|E(G)| \geqslant k - 1$. Suppose $|E(G)| = k - 1$. Then $E(G) = E'$. If $G[V_1^0 \cup V_3^0]$ is a complete bipartite graph, then we can choose $k_1, k_2 \in V_2^0$ such that $G[V^0\setminus\{k_1, k_2\}]$ is a complete tripartite graph. By $n_2 \geqslant 24l^3+44l^2+12l+3(t-1)l^2$, we can choose $w_1, w_2 \in V_2 \setminus (V^{v_1v_2} \cup V_2^0 \cup \{u_{v_3v_1}\})$. But $G[V_0 \cup \{w_1, w_2\} \setminus \{k_1, k_2\}]$, $G[(V(K_1) \cup \{k_1, u_{v_2v_3}\}) \setminus \{a_2\}]$, $G[V(K_2) \cup \{k_2\}]$ and $\cup_{i=3}^{t-1}K_i^{v_1v_2}$ form $tK_{l,l,l}$ in G, a contradiction. Hence there are $q_1 \in V_1^0$ and $q_3' \in V_3^0$ such that $q_1q_3' \notin E(G)$. Since $|E(G_0)| = 3l^2 - 2$, we can assume there is $q_2 \in V_2^0$ and $q_3 \in V_3^0$ such that $q_2q_3 \notin E(G)$.

Since G is a $tK_{l,l,l}$-saturated graph, there are t pairwise disjoint copies of $K_{l,l,l}$, say $K_0^{q_2q_3}, \dots, K_{t-1}^{q_2q_3}$, in $G + q_2q_3$. If there is $v \in (V_i \cap V(K_j^{q_2q_3})) \setminus (V^{v_1v_2} \cup V^0 \cup \{u_{v_2v_3}, u_{v_3v_1}\})$ for some $i \in [3]$ and $0 \leqslant j \leqslant t-1$, then $V_{i+1}^0, V_{i+2}^0 \subseteq V(K_j^{q_2q_3})$ and then there are at most $t-1$ pairwise disjoint $K_{l,l,l}$ in $G + q_2q_3$ by $u_{v_2v_3}a_2, u_{v_3v_1}a_3 \notin E(G)$, a contradiction. Hence $\cup_{i=0}^{t-1}V(K_i^{q_2q_3}) = V^{v_1v_2} \cup V^0 \cup \{u_{v_2v_3}, u_{v_3v_1}\}$. Assume $q_2q_3 \in E(K_0^{q_2q_3})$. Note that there is $u' \in V(K_0^{q_2q_3}) \setminus V^0$ such that $u' \in V_1 \cup V_3$ by $q_1q_3' \notin E(G)$. Since $u_{v_3v_1} \in V_2$, $u' \neq u_{v_3v_1}$. If $u' \in V^{v_1v_2}$, say $u' \in V(K_i^{v_1v_2})$, then $V(K_0^{q_2q_3}) \cap V_2 \subseteq V_2^0 \cup V(K_i^{v_1v_2})$. Thus $G + q_2q_3$ has at most $t-1$ pairwise disjoint copies of $K_{l,l,l}$ by $u_{v_2v_3}a_2, u_{v_3v_1}a_3 \notin E(G)$, a contradiction. If $u' = u_{v_2v_3}$, then $V(K_0^{q_2q_3}) \cap V_2 \subseteq V_2^0 \cup V(K_1)$ and $V(K_0^{q_2q_3}) \cap V_3 \subseteq V_3^0 \cup V(K_1)$. So $G + q_2q_3$ has at most $t-1$ pairwise disjoint copies of $K_{l,l,l}$ by $u_{v_3v_1}a_3 \notin E(G)$, a contradiction.

Case 3.2 $|E(G_0)| = 3l^2 - 3$.

In this case, we have $|E(G)| \geqslant k - 2$. If $|E(G)| = k - 2$, let $G' = G + q_1q_2$, where $q_1 \in V_1^0, q_2 \in V_2^0$ with $q_1q_2 \notin E(G_0)$. Then $|E(G')| = k - 1$. By Case 3.1, G' has at most $t-1$ pairwise disjoint copies of $K_{l,l,l}$, a contradiction. So $|E(G)| \geqslant k - 1$. Suppose $|E(G)| = k - 1$. Then there is $e = uv \in E(G)$ but $e \notin E'$, that is $E(G) = E' \cup \{e\}$. Then $\{u, v\} \cap V^0 = \emptyset$. By Claim 1, we easily have the following claim.

Claim 6 *For any $i \in [3]$, there are $b_i \in V_i^0$ and $b_{i+1} \in V_{i+1}^0$ such that $G[V^0 \setminus \{b_i, b_{i+1}\}]$ is a complete tripartite graph.*

Let $V^1 = V(K_1) \cup \{u_{v_2v_3}\}$, $V^2 = V(K_2) \cup \{u_{v_3v_1}\}$ and $V^i = V(K_i^{v_1v_2})$ for $3 \leqslant i \leqslant t-1$ for short. Denote $V_j^i = V^i \cap V_j$, where $i \in [t-1]$ and $j \in [3]$. We have the following claim.

Claim 7 $\{u, v\} \cap (\cup_{i=1}^{t-1} V^i) \neq \emptyset$.

Proof of Claim 7 Suppose $\{u, v\} \cap (\cup_{i=1}^{t-1} V^i) = \emptyset$. We first consider the case $u, v \notin V_1$, say $u \in V_2$ and $v \in V_3$. By Claim 6, there are $b_2 \in V_2^0$ and $b_3 \in V_3^0$ such that $G[V^0 \setminus \{b_2, b_3\}]$ is a complete tripartite graph. Then $G[(V^0 \cup \{u, v\}) \setminus \{b_2, b_3\}]$, $G[(V^1 \cup \{b_2\}) \setminus \{a_2\}]$, $G[(V^2 \cup \{b_3\}) \setminus \{a_3\}]$ and $K_i^{v_1v_2}$ for $3 \leqslant i \leqslant t-1$ form $tK_{l,l,l}$ in G, a contradiction. Now we assume $u \in V_1$. By Claim 6, there are $b_1 \in V_1^0$ and $b_2 \in V_2^0$ if $v \in V_2$ (resp. $b_3 \in V_3$ if $v \in V_3$) such that $G[V^0 \setminus \{b_1, b_2\}]$ (resp. $G[V^0 \setminus \{b_1, b_3\}]$) is a complete tripartite graph. Then $G[(V^0 \cup \{u, v\}) \setminus \{b_1, b_2\}]$ (resp. $G[(V^0 \cup \{u, v\}) \setminus \{b_1, b_3\}]$), $G[(V^1 \cup \{b_1\}) \setminus \{u_{v_2v_3}\}]$, $G[(V^2 \cup \{b_2\}) \setminus \{u_{v_3v_1}\}]$ (resp. $G[(V^2 \cup \{b_3\}) \setminus \{a_3\}]$) and $K_i^{v_1v_2}$ for $3 \leqslant i \leqslant t-1$ form $tK_{l,l,l}$ in G, a contradiction. ∎

By Claim 7 and $\{u, v\} \cap V^0 = \emptyset$, we assume $u \in \cup_{i=1}^{t-1} V^i$. Since $|E(G_0)| = 3l^2 - 3$, there are $q_1 \in V_1^0$ and $q_2 \in V_2^0$ such that $q_1q_2 \notin E(G)$. Since G is a $tK_{l,l,l}$-saturated graph, there are t pairwise disjoint copies of $K_{l,l,l}$, say $K_0^{q_1q_2}, \ldots, K_{t-1}^{q_1q_2}$ in $G + q_1q_2$. By Case 3.1, we know there are at most $t-1$ pairwise disjoint $K_{l,l,l}$ in $G + q_1q_2 - uv$. So $uv \in \cup_{i=0}^{k-1} E(K_i^{q_1q_2})$.

Claim 8 $\cup_{i=0}^{t-1} V(K_i^{q_1q_2}) \subseteq \cup_{i=0}^{t-1} V^i \cup \{v\}$.

Proof of Claim 8 Suppose there is $w \in \cup_{i=0}^{t-1} V(K_i^{q_1q_2}) \setminus (\cup_{i=0}^{t-1} V^i \cup \{v\})$, say $w \in V_i \cap V(K_j^{q_1q_2})$, where $i \in [3]$ and $0 \leqslant j \leqslant t-1$. Then $d(w) = 2l$, which implies $V_{i+1}^0 \cup V_{i+2}^0 \subseteq V(K_j^{q_1q_2})$. Since $|E(G_0)| = 3l^2 - 3$ and $q_1 \in V_1^0$, $q_2 \in V_2^0$, we have $i = 3$ and then $|\cup_{i=0}^{t-1} V(K_i^{q_1q_2}) \setminus (\cup_{i=0}^{t-1} V^i \cup \{v\})| \leqslant l$. Since $u_{v_2v_3}a_2 \notin E(G)$, $(V_1^1 \cup V_2^1) \cap (\cup_{i=0}^{t-1} V(K_i^{q_1q_2})) = \emptyset$. Then

$$
\begin{aligned}
|\cup_{i=0}^{t-1} V(K_i^{q_1q_2})| &\leqslant |(\cup_{i=0}^{t-1} V^i \cup \{v\}) \setminus (V_1^1 \cup V_2^1)| + |\cup_{i=0}^{t-1} V(K_i^{q_1q_2}) \setminus (\cup_{i=0}^{t-1} V^i \cup \{v\})| \\
&\leqslant 3tl - l + 1,
\end{aligned}
$$

a contradiction with $|\cup_{i=0}^{t-1} V(K_i^{q_1q_2})| = 3tl$ and $l \geqslant 2$. ∎

Claim 9 $e \neq a_2a_3$ *and* $e \neq u_{v_2v_3}a_3$.

Proof of Claim 9 Suppose $e = a_2a_3$. By Claim 6, there are $b_2 \in V_2^0$ and $b_3 \in V_3^0$ such that $G[V^0 \setminus \{b_2, b_3\}]$ is a complete tripartite graph. But $G[(V^0 \setminus \{b_2, b_3\}) \cup \{a_2, a_3\}], G[(V^1 \setminus \{a_2\}) \cup \{b_2\}], G[(V^2 \setminus \{a_3\}) \cup \{b_3\}], \ldots, K_{t-1}^{v_1v_2}$ form $tK_{l,l,l}$ in G, a contradiction.

Suppose $e = u_{v_2v_3}a_3$. By Claim 6, there are $b_1 \in V_1^0$ and $b_3 \in V_3^0$ such that $G[V^0 \setminus \{b_1, b_3\}]$ is a complete tripartite graph. But $G[(V^0 \setminus \{b_1, b_3\}) \cup \{u_{v_2v_3}, a_3\}], G[(V^1 \cup \{b_1\}) \setminus \{u_{v_2v_3}\}], G[(V^2 \cup \{b_3\}) \setminus \{a_3\}], \ldots, K_{t-1}^{v_1v_2}$ form $tK_{l,l,l}$ in G, a contradiction. ∎

Claim 10 $v \notin \cup_{i=1}^{t-1} V^i$.

Proof of Claim 10 Suppose $v \in \cup_{i=1}^{t-1} V^i$. By Claim 8, $\cup_{i=0}^{t-1} V(K_i^{q_1q_2}) = \cup_{i=0}^{t-1} V^i$. Assume $uv \in E(K_0^{q_1q_2})$, $u \in V_{j_u}^{i_u}$ and $v \in V_{j_v}^{i_v}$, where $i_u, i_v \in [t-1]$, $j_u, j_v \in [3]$, $i_u \neq i_v$ and $j_u \neq j_v$. Let $j = \{1, 2, 3\} \setminus \{j_u, j_v\}$. Then $V_j^0 \subseteq V(K_0^{q_1q_2})$. By $\cup_{i=0}^{t-1} V(K_i^{q_1q_2}) = \cup_{i=0}^{t-1} V^i$, we can assume there are $u_1 \in V_j^{i_u} \cap V(K_1^{q_1q_2})$ and $v_1 \in V_j^{i_v} \cap V(K_2^{q_1q_2})$. Then $\cup_{i=0}^{2} V(K_i^{q_1q_2}) = V^{i_u} \cup V^{i_v} \cup V^0$. Since $u_{v_2v_3}a_2, u_{v_3v_1}a_3 \notin E(G)$, we have $\{i_u, i_v\} = \{1, 2\}$. Then there is

$i \in \{0,1,2\}$ such that $|\{a_2, a_3, u_{v_2v_3}, u_{v_3v_1}\} \cap V(K_i^{q_1q_1})| \geqslant 2$. Since $u_{v_2v_3}u_{v_3v_1} \notin E(G)$, we have $uv = a_2a_3$ or $u_{v_2v_3}a_3$, a contradiction with Claim 9. ■

By Claims 8 and 10, we have $\cup_{i=0}^{t-1}V(K_i^{q_1q_2}) \subset \cup_{i=0}^{t-1}V^i \cup \{v\}$. Assume $uv \in E(K_0^{q_1q_2})$, $u \in V_{j_u}^{i_u}$ and $v \in V_{j_v}$, where $i_u \in [t-1]$, $j_u, j_v \in [3]$ and $j_u \neq j_v$. Let $j = \{1,2,3\} \setminus \{j_u, j_v\}$. Since $N(v) = V^0 \cup \{u\}$, we have $V(K_0^{q_1q_2}) \cap V_j = V_j^0$, $(V_{j_v} \cap V(K_0^{q_1q_2})) \setminus \{v\} \subseteq V_{j_v}^0 \cup V_{j_v}^{i_u}$ and $V(K_0^{q_1q_2}) \cap V_{j_u} = (V_{j_u}^0 \cup \{u\}) \setminus \{w_{j_u}\}$, where $w_{j_u} \in V_{j_u}^0$. Since $\cup_{i=0}^{t-1}V(K_i^{q_1q_2}) \subset \cup_{i=0}^{t-1}V^i \cup \{v\}$, there is $u_1 \in V_j^{i_u} \cap V(K_i^{q_1q_2})$ for some $i \neq 0$, say $i = 1$. Then $V(K_1^{q_1q_2}) \cap V_{j_u} = (V_{j_u}^{i_u} \cup \{w_{j_u}\}) \setminus \{u\}$, $V(K_1^{q_1q_2}) \cap V_j = V_j^{i_u}$ and $V(K_1^{q_1q_2}) \cap V_{j_v} \subset V_{j_v}^0 \cup V_{j_v}^{i_u}$. So $(V(K_0^{q_1q_2}) \cup V(K_1^{q_1q_2})) \setminus \{v\} \subset V^0 \cup V^{i_u}$. Since $u_{v_2v_3}a_2, u_{v_3v_1}a_3 \notin E(G)$, we have $i_u \in \{1,2\}$ and there is a unique vertex $w_{j_v} \in V_{j_v}^0$ such that $w_{j_v} \in \cup_{i=2}^{t-1}V(K_i^{q_1q_2})$.

Claim 11 $V(K_1^{q_1q_2}) \cap V_{j_v}^0 \neq \emptyset$.

Proof of Claim 11 Suppose $V(K_1^{q_1q_2}) \cap V_{j_v}^0 = \emptyset$. Then $V(K_1^{q_1q_2}) \cap V^0 = \{w_{j_u}\}$ and $V(K_0^{q_1q_2}) = (V^0 \cup \{u, v\}) \setminus \{w_{j_u}, w_{j_v}\}$. By Claim 6, there are $b_{j_u} \in V_{j_u}^0$ and $b_{j_v} \in V_{j_v}^0$ such that $G[V^0 \setminus \{b_{j_u}, b_{j_v}\}]$ is a complete tripartite graph. But $G[(V(K_0^{q_1q_2}) \cup \{w_{j_u}, w_{j_v}\}) \setminus \{b_{j_u}, b_{j_v}\}]$, $G[(V(K_1^{q_1q_2}) \setminus \{w_{j_u}\}) \cup \{b_{j_u}\}]$ and $G[(\cup_{i=2}^{t-1}V(K_i^{q_1q_2}) \setminus \{w_{j_v}\}) \cup \{b_{j_v}\}]$ form t pairwise disjoint $K_{l,l,l}$s in G, a contradiction. ■

By Claim 11, we assume $w'_{j_v} \in V(K_1^{q_1q_2}) \cap V_{j_v}^0$. Since $E(G_0) = 3l^2 - 3$, by Claim 1, there are $x, x', y, z \in V^0$ such that $xy, yz, zx' \notin E(G)$ (possibly $x = x'$). If $x = x'$, say $x \in V_{j_u}^0$ and $y \in V_{j_v}^0$, then $G[(V(K_0^{q_1q_2}) \cup \{w_{j_u}, w_{j_v}\}) \setminus \{x, y\}]$, $G[(V(K_1^{q_1q_2}) \setminus \{w_{j_u}\}) \cup \{x\}]$ and $G[(\cup_{i=2}^{t-1}V(K_i^{q_1q_2}) \setminus \{w_{j_v}\}) \cup \{y\}]$ form t pairwise disjoint $K_{l,l,l}$s in G, a contradiction. So we have $x \neq x'$. If $x, x' \in V_{j_v}$, assume $y \in V_{j_u}$, then $G[(V(K_0^{q_1q_2}) \cup \{w_{j_v}, w'_{j_v}, w_{j_u}\}) \setminus \{x, x', y\}]$, $G[(V(K_1^{q_1q_2}) \setminus \{w_{j_u}, w'_{j_v}\}) \cup \{y, x'\}]$ and $G[(\cup_{i=2}^{t-1}V(K_i^{q_1q_2}) \setminus \{w_{j_v}\}) \cup \{x\}]$ form t pairwise disjoint $K_{l,l,l}$s in G, a contradiction. If $x, x' \in V_j$, assume $y \in V_{j_u}$ and $z \in V_{j_v}$, then $G[(V(K_0^{q_1q_2}) \cup \{w_{j_u}, w_{j_v}\}) \setminus \{y, z\}]$, $G[(V(K_1^{q_1q_2}) \setminus \{w_{j_u}\}) \cup \{y\}]$ and $G[(\cup_{i=2}^{t-1}V(K_i^{q_1q_2}) \setminus \{w_{j_v}\}) \cup \{z\}]$ form t pairwise disjoint $K_{l,l,l}$s in G, a contradiction. Now we consider the case $x, x' \in V_{j_u}$. Assume $y \in V_{j_v}$. If $y \neq w_{j_v}$, then $G[(V(K_0^{q_1q_2}) \cup \{w'_{j_v}, w_{j_u}\}) \setminus \{x', y\}]$, $G[(V(K_1^{q_1q_2}) \setminus \{w_{j_u}, w'_{j_v}\}) \cup \{x', y\}]$ and $G[\cup_{i=2}^{t-1}V(K_i^{q_1q_2})]$ form t pairwise disjoint $K_{l,l,l}$s in G, a contradiction. If $y = w_{j_v}$, then $G[(V(K_0^{q_1q_2}) \cup \{w_{j_u}\}) \setminus \{x'\}]$, $G[(V(K_1^{q_1q_2}) \setminus \{w_{j_u}\}) \cup \{x'\}]$ and $G[\cup_{i=2}^{t-1}V(K_i^{q_1q_2})]$ form t pairwise disjoint $K_{l,l,l}$s in G, our final contradiction. □

Remark In [9], Ferrara, Jacobson, Pfender and Wenger determined $sat(K_k^n, K_3)$ for $k \geqslant 3$ and $n \geqslant 100$, where K_k^n is the complete balanced k-partite graph with partite sets of size n. Our result in the case $l = 1$ generalizes their conclusion if $k = 3$.

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