A characteristic polynomial for the transition probability matrix of correlated random walks on a graph

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Abstract

We define a correlated random walk (CRW) induced from the time evolution matrix (the Grover matrix) of the Grover walk on a graph G, and present a formula for the characteristic polynomial of the transition probability matrix of this CRW by using a determinant expression for the generalized weighted zeta function of G. As an application, we give the spectrum of the transition probability matrices for the CRWs induced from the Grover matrices of regular graphs and semiregular bipartite graphs. Furthermore, we consider another type of the CRW on a graph.

Mathematics Subject Classifications: 05C50, 15A15

1 Introduction

Zeta functions of graphs started from the Ihara zeta functions of regular graphs by Ihara [7]. In [7], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [16, 17]. Hashimoto [5] generalized Ihara's result on the Ihara zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [1] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

Cooper [2] treated various properties of a graph G determined by the Ihara zeta function of G. Morita [13] defined a generalized weighted zeta function of a digraph which contains various zeta functions of a graph or a digraph. Ide et al. [6] presented a determinant expression for the generalized weighted zeta function of a graph.

The time evolution matrix of a discrete-time quantum walk in a graph is closely related to the Ihara zeta function of a graph. A discrete-time quantum walk is a quantum analog of the classical random walk on a graph whose state vector is governed by a matrix called the time evolution matrix(see [9]). Ren et al. [14] gave a relationship between the discrete-time quantum walk and the Ihara zeta function of a graph. Konno and Sato [11] obtained a formula of the characteristic polynomial of the Grover matrix by using the determinant expression for the second weighted zeta function of a graph. Thus, the relation between the Grover walk and a simple random walk on a graph was established. Konno [10] treated the one-dimensional correlated random walk derived from one-dimensional quantum walk.

In this paper, we present an analogue of the above relation for the correlated random walk derived from the Grover walk on a graph. We introduce a new correlated random walk induced from the time evolution matrix (the Grover matrix) of the Grover walk on a graph, and present a formula for the characteristic polynomial of its transition probability matrix.

In Section 2, we review the Ihara zeta function and the generalized weighted zeta functions of a graph. In Section 3, we review the Grover walk on a graph. In Section 4, we define a correlated random walk (CRW) induced from the time evolution matrix (the Grover matrix) of the Grover walk on a graph G, and present a formula for the characteristic polynomial of the transition probability matrix of this CRW. In Section 5, we give the spectrum of the transition probability matrix for this CRW of a regular graph. In Section 6, we present the spectrum for the transition probability matrix of this CRW of a semiregular bipartite graph. In Section 7, we present formulas for the characteristic polynomials of the transition probability matrices of another type of the CRW on a graph, and give the spectrum of its transition probability matrix.

2 Preliminaries

2.1 Zeta functions of graphs

Graphs and digraphs treated here are finite. Let G be a connected graph and D_G the symmetric digraph corresponding to G. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$. For $v \in V(G)$, the *degree* $\deg_G v = \deg v = d_v$ is the number of vertices adjacent to v in G. A graph G is called *k-regular* if $\deg v = k$ for each $v \in V(G)$.

A *walk P of length n* in G is a sequence $P = (e_1, \cdots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leqslant i \leqslant n-1)$(see [2]). If $e_i = (v_{i-1}, v_i)$ for $i = 1, \cdots, n$, then we write $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$. Set $\mid P \mid = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an *$(o(P), t(P))$-walk*. We say that a walk $P = (e_1, \cdots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some i $(1 \leqslant i \leqslant n-1)$. A (v, w)-walk is called a *closed walk* if $v = w$. The *inverse closed walk* of a closed walk $C = (e_1, \cdots, e_n)$ is the closed walk $C^{-1} = (e_n^{-1}, \cdots, e_1^{-1})$.

We introduce an equivalence relation between closed walks. Two closed walks $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called *equivalent* if there exists a positive number k such that $f_j = e_{j+k}$ for all j, where the subscripts are considered by modulo m. The inverse closed walk of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a closed walk C. Let B^r be the closed walk obtained by going r times around a closed walk B. Such a closed walk is called a *multiple* of B. A closed walk C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cclosed walk C is *prime* if it is not a multiple of a strictly smaller closed walk. Note that each equivalence class of prime, reduced closed walks of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *Ihara(-Selberg) zeta function* of G is defined by

$$\mathbf{Z}(G, u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced closed walks of G.

Let G be a connected graph with n vertices and m edges. Then two $2m \times 2m$ matrices $\mathbf{B} = \mathbf{B}(G) = (\mathbf{B}_{e,f})_{e,f \in D(G)}$ and $\mathbf{J}_0 = \mathbf{J}_0(G) = (\mathbf{J}_{e,f})_{e,f \in D(G)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise}, \end{cases} \qquad \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$

The matrix $\mathbf{B} - \mathbf{J}_0$ is called the *edge matrix* of G.

Theorem 1 (Ihara; Hashimoto; Bass). *Let G be a connected graph with n vertices and m edges. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G, u)^{-1} = \det(\mathbf{I}_{2m} - u(\mathbf{B} - \mathbf{J}_0)) = (1 - u^2)^{m-n} \det(\mathbf{I}_n - u\mathbf{A}(G) + u^2(\mathbf{D}_G - \mathbf{I}_n)),$$

where $\mathbf{D}_G = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg_G v_i$ $(V(G) = \{v_1, \cdots, v_n\})$.

The first identity in Theorem 1 was obtained by Hashimoto [5]. Also, Bass [1] proved the second identity by using a linear algebraic method.

Stark and Terras [15] gave an elementary proof of this formula, and discussed three different zeta functions of any graph. Various proofs of Bass' Theorem were given by Kotani and Sunada [12], and Foata and Zeilberger [4].

2.2 The generalized weighted zeta functions of a graph

Let G be a connected graph with n vertices and m edges, and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}(e_{m+i} = e_i^{-1}(1 \leqslant i \leqslant m))$. Furthermore, we consider two functions $\tau : D(G) \longrightarrow \mathbb{C}$ and $\mu : D(G) \longrightarrow \mathbb{C}$. Let $\theta : D(G) \times D(G) \longrightarrow \mathbb{C}$ be a function such that

$$\theta(e, f) = \tau(f)\delta_{t(e)o(f)} - \mu(f)\delta_{e^{-1}f}.$$

We introduce a $2m \times 2m$ matrix $\mathbf{M}(\theta) = (M_{ef})_{e,f \in D(G)}$ as follows:

$$M_{ef} = \theta(e, f).$$

Then the *generalized weighted zeta function* $\mathbf{Z}_G(u, \theta)$ of G is defined as follows(see [13]):

$$\mathbf{Z}_G(u, \theta) = \det(\mathbf{I}_{2m} - u\mathbf{M}(\theta))^{-1}.$$

We consider two $n \times n$ matrices $\mathbf{A}_G(\theta) = (a_{uv})_{u,v \in V(G)}$ and $\mathbf{D}_G(\theta) = (d_{uv})_{u,v \in V(G)}$ as follows:

$$a_{uv} = \begin{cases} \tau(e)/(1 - u^2\mu(e)\mu(e^{-1})) & \text{if } e = (u, v) \in D(G), \\ 0 & \text{otherwise,} \end{cases}$$

$$d_{uv} = \begin{cases} \sum_{o(e)=u} \tau(e)\mu(e^{-1})/(1 - u^2\mu(e)\mu(e^{-1})) & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

A determinant expression for the generalized weighted zeta function of a graph is given as follows(see [6]):

Theorem 2 (Ide, Ishikawa, Morita, Sato and Segawa). *Let G be a connected graph with n vertices and m edges, and let $\tau : D(G) \longrightarrow \mathbb{C}$ and $\mu : D(G) \longrightarrow \mathbb{C}$ be two functions. Then*

$$\mathbf{Z}_G(u, \theta)^{-1} = \prod_{j=1}^{m} (1 - u^2\mu(e_j)\mu(e_j^{-1})) \det(\mathbf{I}_n - u\mathbf{A}_G(\theta) + u^2\mathbf{D}_G(\theta)),$$

where $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ $(e_{m+j} = e_j^{-1}(1 \leqslant j \leqslant m))$.

Proof. We give a sketch of proof along Ide et al [6].

Let $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$. Arrange arcs of G as follows: $e_1, e_1^{-1}, \ldots, e_m, e_m^{-1}$. Furthermore, arrange vertices of G as follows: $v_1, \ldots, v_n$.

Now, we define two $2m \times n$ matrices $\mathbf{K} = (K_{ev})_{e \in D(G); v \in V(G)}$ and $\mathbf{L} = (L_{ev})_{e \in D(G); v \in V(G)}$ as follows:

$$K_{ev} := \begin{cases} 1 & \text{if } t(e) = v, \\ 0 & \text{otherwise,} \end{cases} \quad L_{ev} := \begin{cases} \tau(e) & \text{if } o(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Here we consider two matrices $\mathbf{K}$ and $\mathbf{L}$ under the above order. Furthermore, we define a $2m \times 2m$ matrix $\mathbf{J} = (J_{ef})_{e,f \in D(G)}$ as follows:

$$J_{ef} := \begin{cases} \mu(e) & \text{if } f = e^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

Then we have

$$\mathbf{M}(\theta) = \mathbf{K}\ {}^t\mathbf{L} - \mathbf{J}.$$

Thus,

$$\begin{aligned}
\mathbf{Z}_G(u,\theta)^{-1} &= \det(\mathbf{I}_{2m} - u\mathbf{M}(\theta)) \\
&= \det(\mathbf{I}_{2m} - u(\mathbf{K}\ {}^t\mathbf{L} - \mathbf{J})) \\
&= \det(\mathbf{I}_{2m} + u\mathbf{J} - u\mathbf{K}\ {}^t\mathbf{L}) \\
&= \det(\mathbf{I}_{2m} - u\mathbf{K}\ {}^t\mathbf{L}(\mathbf{I}_{2m} + u\mathbf{J})^{-1})\det(\mathbf{I}_{2m} + u\mathbf{J}).
\end{aligned}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be an $m \times n$ and $n \times m$ matrix, respectively. Then we have

$$\det(\mathbf{I}_m - \mathbf{AB}) = \det(\mathbf{I}_n - \mathbf{BA}).$$

Thus, we have

$$\mathbf{Z}_G(u,\theta)^{-1} = \det(\mathbf{I}_n - u\ {}^t\mathbf{L}(\mathbf{I}_{2m} + u\mathbf{J})^{-1}\mathbf{K})\det(\mathbf{I}_{2m} + u\mathbf{J}).$$

But, we have

$$\det(\mathbf{I}_{2m} + u\mathbf{J}) = \prod_{j=1}^{m}(1 - u^2\mu(e_j)\mu(e_j^{-1})).$$

Let

$$x_{e_j} = x_j = 1 - u^2\mu(e_j)\mu(e_j^{-1})\ (1 \leqslant j \leqslant n).$$

Then we have

$$(\mathbf{I}_{2m} + u\mathbf{J})^{-1} = \begin{bmatrix} 1/x_1 & -u\mu(e_1^{-1})/x_1 & 0 \\ -u\mu(e_1)/x_1 & 1/x_1 & 0 \\ 0 & & \ddots \end{bmatrix}.$$

Thus, for $(u,v) \in D(G)$,

$$({}^t\mathbf{L}(\mathbf{I}_{2m} + u\mathbf{J})^{-1}\mathbf{K})_{uv} = \tau(u,v)/(1 - u^2\mu(u,v)\mu(v,u)).$$

Furthermore, for each $v \in V(G)$,

$$({}^t\mathbf{L}(\mathbf{I}_{2m} + u\mathbf{J})^{-1}\mathbf{K})_{vv} = -u\sum_{o(e)=v}\tau(e)\mu(e^{-1})/(1 - u^2\mu(e)\mu(e^{-1})).$$

Hence,

$$\mathbf{Z}_G(u,\theta)^{-1} = \prod_{j=1}^{m}(1 - u^2\mu(e_j)\mu(e_j^{-1}))\det(\mathbf{I}_n - u\mathbf{A}_G(\theta) + u^2\mathbf{D}_G(\theta)).$$

□

3 The Grover walk on a graph

Let G be a connected graph with n vertices and m edges, $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$. Set $d_j = d_{v_j} = \deg v_j$ for $i = 1, \ldots, n$. The *Grover matrix* $\mathbf{U} = \mathbf{U}(G) = (U_{ef})_{e,f \in D(G)}$ of G is defined by

$$U_{ef} = \begin{cases} 2/d_{t(f)} (= 2/d_{o(e)}) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ 2/d_{t(f)} - 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The discrete-time quantum walk with the matrix $\mathbf{U}$ as a time evolution matrix is called the *Grover walk* on G.

Let G be a connected graph with n vertices and m edges. Then the $n \times n$ matrix $\mathbf{T}(G) = (T_{uv})_{u,v \in V(G)}$ is given as follows:

$$T_{uv} = \begin{cases} 1/(\deg_G u) & \text{if } (u, v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Note that the matrix $\mathbf{T}(G)$ is the transition matrix of the simple random walk on G(see [11]).

Theorem 3 (Konno and Sato)**.** *Let G be a connected graph with n vertices $v_1, \ldots, v_n$ and m edges. Then the characteristic polynomial for the Grover matrix $\mathbf{U}$ of G is given by*

$$\begin{aligned} \det(\lambda \mathbf{I}_{2m} - \mathbf{U}) &= (\lambda^2 - 1)^{m-n} \det((\lambda^2 + 1)\mathbf{I}_n - 2\lambda \mathbf{T}(G)) \\ &= \tfrac{(\lambda^2 - 1)^{m-n} \det((\lambda^2+1)\mathbf{D} - 2\lambda \mathbf{A}(G))}{d_{v_1} \cdots d_{v_n}}. \end{aligned}$$

From this Theorem, the spectra of the Grover matrix on a graph is obtained by means of those of $\mathbf{T}(G)$ (see [3]). Let $Spec(\mathbf{F})$ be the spectra of a square matrix $\mathbf{F}$.

Corollary 4 (Emms, Hancock, Severini and Wilson)**.** *Let G be a connected graph with n vertices and m edges. The Grover matrix $\mathbf{U}$ has $2n$ eigenvalues of the form*

$$\lambda = \lambda_T \pm i\sqrt{1 - \lambda_T^2},$$

where λ_T is an eigenvalue of the matrix $\mathbf{T}(G)$. The remaining $2(m-n)$ eigenvalues of $\mathbf{U}$ are ± 1 with equal multiplicities.

4 A correlated random walk on a graph

Let G be a connected graph with n vertices and m edges, and $\mathbf{U}$ be the Grover matrix of G. Then we define a $2m \times 2m$ matrix $\mathbf{P} = (P_{ef})_{e,f \in D(G)}$ as follows:

$$P_{ef} = |U_{ef}|^2.$$

Note that

$$P_{ef} = \begin{cases} 4/d^2_{t(f)}(= 4/d^2_{o(e)}) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ (2/d_{t(f)} - 1)^2 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The random walk with the matrix $\mathbf{P}$ as a transition probability matrix is called the *correlated random walk (CRW)* (with respect to the Grover matrix) on G(see [8, 10]).

Let $\mathbf{R} = (R_{ef})_{e,f \in D(G)}$ be a $2m \times 2m$ matrix such that

$$R_{ef} = \begin{cases} 4/d^2_{o(f)}(= 4/d^2_{o(e)}) & \text{if } o(e) = o(f) \text{ and } f \neq e, \\ (2/d_{o(f)} - 1)^2 & \text{if } f = e, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{P} = \mathbf{J}_0\mathbf{R}.$$

Let G be a d-regular graph. In the case of $d = 4$, we consider $\mathbf{P} = (P_{ef})_{e,f \in D(G)}$ be the transition probability matrix of the CRW with respect to the Grover matrix on a d-regular graph G. If $t(f) = o(e)$ and $f \neq e^{-1}$, then $P_{ef} = 4/d^2 = 4/4^2 = 1/4$. If $f = e^{-1}$, then $P_{ef} = 4/d^2 - (4/d - 1) = 4/4^2 - (4/4 - 1) = 1/4$. Thus, this CRW is considered to be a simple random walk on G which the particle moves over each arc in terms of the same probability.

By Theorem 2, we obtain the following formula for $\mathbf{P}$.

Theorem 5. *Let G be a connected graph with n vertices and m edges, and let $\mathbf{P}$ be the transition probability matrix of the CRW with respect to the Grover matrix. Then*

$$\det(\mathbf{I}_{2m} - u\mathbf{P}) = \prod_{j=1}^{m}(1 - u^2(\frac{4}{d_{o(e_j)}} - 1)(\frac{4}{d_{t(e_j)}} - 1)) \det(\mathbf{I}_n - u\mathbf{A}_{CRW} + u^2\mathbf{D}_{CRW}),$$

where

$$(\mathbf{A}_{CRW})_{xy} = \begin{cases} \frac{4/d_x^2}{1-u^2(4/d_x-1)(4/d_y-1)} & \textit{if } (x,y) \in D(G), \\ 0 & \textit{otherwise,} \end{cases}$$

$$(\mathbf{D}_{CRW})_{xy} = \begin{cases} \sum_{o(e)=x} \frac{4/d_x^2(4/d_{t(e)}-1)}{1-u^2(4/d_x-1)(4/d_{t(e)}-1)} & \textit{if } x = y, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For the matrix $\mathbf{P}$, we have

$$P_{ef} = \frac{4}{d^2_{o(e)}}\delta_{t(f)o(e)} - (\frac{4}{d_{o(e)}} - 1)\delta_{f^{-1}e}.$$

The we let two functions $\tau : D(G) \longrightarrow \mathbb{C}$ and $\mu : D(G) \longrightarrow \mathbb{C}.$ as follows:

$$\tau(e) = \frac{4}{d^2_{o(e)}} \textit{ and } \mu(e) = \frac{4}{d_{o(e)}} - 1.$$

Furthermore, let

$$\theta(e,f)=\frac{4}{d_{o(f)}^2}\delta_{t(e)o(f)}-(\frac{4}{d_{o(f)}}-1)\delta_{e^{-1}f}.$$

Then we have

$$\mathbf{P}={}^t\mathbf{M}(\theta).$$

Thus, we obtain

$$\det(\mathbf{I}_{2m}-u\mathbf{P})=\det(\mathbf{I}_{2m}-u\,{}^t\mathbf{M}(\theta))=\det(\mathbf{I}_{2m}-u\mathbf{M}(\theta))=\mathbf{Z}_G(u,\theta)^{-1}.$$

By Theorem 2, we have

$$\det(\mathbf{I}_{2m}-u\mathbf{P})=\prod_{j=1}^{m}(1-u^2(\frac{4}{d_{o(e_j)}}-1)(\frac{4}{d_{t(e_j)}}-1))\det(\mathbf{I}_n-u\mathbf{A}_{CRW}+u^2\mathbf{D}_{CRW}),$$

where

$$(\mathbf{A}_{CRW})_{xy}=\begin{cases}\frac{4/d_x^2}{1-u^2(4/d_x-1)(4/d_y-1)} & \text{if } (x,y)\in D(G),\\ 0 & \text{otherwise,}\end{cases}$$

$$(\mathbf{D}_{CRW})_{xy}=\begin{cases}\sum_{o(e)=x}\frac{4/d_x^2(4/d_{t(e)}-1)}{1-u^2(4/d_x-1)(4/d_{t(e)}-1)} & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

□

By Theorem 4, we obtain the spectrum of the transition probability matrices for the CRWs induced from the Grover matrices of regular graphs and semiregular bipartite graphs.

5 An application to the correlated random walk on a regular graph

We present spectra for the transition probability matrix of the correlated random walk on a regular graph with respect to the Grover matrix.

Theorem 6. *Let G be a connected d-regular graph with n vertices and m edges, where $d \geqslant 2$. Furthermore, let $\mathbf{P}$ be the transition probability matrix of the CRW with respect to the Grover matrix. Then*

$$\det(\mathbf{I}_{2m}-u\mathbf{P})=\frac{(d^2-u^2(4-d)^2)^{m-n}}{d^{2m}}\det(d(d+(4-d)u^2)\mathbf{I}_n-4u\mathbf{A}(G)).$$

Proof. Let G be a connected d-regular graph with n vertices and m edges, where $d \geqslant 2$. Then we have

$$d_{o(e)}=d_{t(e)}=d \; for\ each\ e\in D(G).$$

Thus, we have

$$1-u^2(\frac{4}{d_{o(e)}}-1)(\frac{4}{d_{t(e)}}-1)=\frac{d^2-u^2(4-d)^2}{d^2},$$

$$(\mathbf{A}_{CRW})_{xy} = \frac{4/d_x^2}{1-u^2(4/d_x-1)(4/d_y-1)} = \frac{4}{d^2-u^2(4-d)^2} \ if \ (x,y) \in D(G)$$

and

$$(\mathbf{D}_{CRW})_{xy} = \sum_{o(e)=x} \frac{4/d_x^2(4/d_{t(e)}-1)}{1-u^2(4/d_x-1)(4/d_{t(e)}-1)} = \frac{4(4-d)}{d^2-u^2(4-d)^2}.$$

Therefore, it follows that

$$\mathbf{A}_{CRW} = \frac{4}{d^2-u^2(4-d)^2}\mathbf{A}(G) \ and \ \mathbf{D}_{CRW} = \frac{4(4-d)}{d^2-u^2(4-d)^2}\mathbf{I}_n.$$

By Theorem 4, we have

$$\begin{aligned}
&\det(\mathbf{I}_{2m} - u\mathbf{P}) \\
&= \frac{(d^2-u^2(4-d)^2)^m}{d^{2m}} \det(\mathbf{I}_n - u\frac{4}{d^2-u^2(4-d)^2}\mathbf{A}(G) + u^2\frac{4(4-d)}{d^2-u^2(4-d)^2}\mathbf{I}_n) \\
&= \frac{(d^2-u^2(4-d)^2)^{m-n}}{d^{2m}} \det((d^2-u^2(4-d)^2)\mathbf{I}_n - 4u\mathbf{A}(G) + 4(4-d)u^2\mathbf{I}_n) \\
&= \frac{(d^2-u^2(4-d)^2)^{m-n}}{d^{2m}} \det(d(d+(4-d)u^2)\mathbf{I}_n - 4u\mathbf{A}(G)).
\end{aligned}$$

□

By substituting $u = 1/\lambda$, we obtain the following result.

Corollary 7. *Let G be a connected d-regular graph with n vertices and m edges, where $d \geqslant 2$. Furthermore, let $\mathbf{P}$ be the transition probability matrix of the CRW with respect to the Grover matrix. Then*

$$\begin{aligned}
\det(\lambda\mathbf{I}_{2m} - \mathbf{P}) &= \tfrac{(d^2\lambda^2-(4-d)^2)^{m-n}}{d^{2m}} \det(d(d\lambda^2+(4-d))\mathbf{I}_n - 4\lambda\mathbf{A}(G)) \\
&= (\lambda^2 - (\tfrac{4}{d}-1)^2)^{m-n}\lambda^n \det((\lambda + (\tfrac{4}{d}-1))\tfrac{1}{\lambda})\mathbf{I}_n - \tfrac{4}{d^2}\mathbf{A}(G)).
\end{aligned}$$

The second identity of Corollary 2 is considered as the spectral mapping theorem for $\mathbf{P}$.

By Corollary 2, we obtain the spectra for the transition matrix $\mathbf{P}$ of the CRW with respect to the Grover matrix on a regular graph.

Corollary 8. *Let G be a connected $d(\geqslant 2)$-regular graph with n vertices and m edges. Then the transition probability matrix $\mathbf{P}$ has $2n$ eigenvalues of the form*

$$\lambda = \frac{2\lambda_A \pm \sqrt{4{\lambda_A}^2 - d^3(4-d)}}{d^2},$$

where λ_A is an eigenvalue of the matrix $\mathbf{A}(G)$. The remaining $2(m-n)$ eigenvalues of $\mathbf{P}$ are $\pm(4-d)/d$ with equal multiplicities $m-n$.

Proof. By Corollary 2, we have

$$\begin{aligned}\det(\lambda \mathbf{I}_{2m}-\mathbf{P}) &= (d^2\lambda^2-(4-d)^2)^{m-n}/d^{2m}\textstyle\prod_{\lambda_A\in Spec(\mathbf{A}(G))}(d(d\lambda^2+4-d)-4\lambda_A\lambda) \\ &= (\lambda^2-(\frac{4-d}{d})^2)^{m-n}/d^{2n}\textstyle\prod_{\lambda_A\in Spec(\mathbf{A}(G))}(d^2\lambda^2-4\lambda_A\lambda+d(4-d)).\end{aligned}$$

Thus, solving

$$d^2\lambda^2-4\lambda_A\lambda+d(4-d)=0,$$

we obtain

$$\lambda=\frac{2\lambda_A\pm\sqrt{4{\lambda_A}^2-d^3(4-d)}}{d^2}.$$

□

6 An application to the correlated random walk on a semiregular bipartite graph

We present spectra for the transition probability matrix of the correlated random walk on a semiregular bipartite graph. Hashimoto [5] presented a determinant expression for the Ihara zeta function of a semiregular bipartite graph. We use an analogue of the method in the proof of Hashimoto's result.

A bipartite graph $G=(V_1,V_2)$ is called (q_1,q_2)-*semiregular* if $\deg_G v=q_i$ for each $v\in V_i (i=1,2)$. For a (q_1+1,q_2+1)-semiregular bipartite graph $G=(V_1,V_2)$, let $G^{[i]}$ be the graph with vertex set V_i and edge set $\{P:$ reduced walk $\mid\,\mid P\mid=2; o(P),t(P)\in V_i\}$ for $i=1,2$. Then $G^{[1]}$ is $(q_1+1)q_2$-regular, and $G^{[2]}$ is $(q_2+1)q_1$-regular.

Theorem 9. *Let $G=(V,W)$ be a connected (r,s)-semiregular bipartite graph with ν vertices and ϵ edges. Set $\mid V\mid=m$ and $\mid W\mid=n(m\leqslant n)$. Furthermore, let $\mathbf{P}$ be the transition probability matrix of the CRW with respect to the Grover matrix of G, and*

$$Spec(\mathbf{A}(G)=\{\pm\lambda_1,\cdots,\pm\lambda_m,0,\ldots,0\}.$$

Then

$$\det(\mathbf{I}_{2\epsilon}-u\mathbf{P})=(1-u^2(4/r-1)(4/s-1))^{\epsilon-\nu}(1-u^2(4/r-1))^{n-m}$$
$$\times\prod_{j=1}^{m}((1-u^2(4/s-1))(1-u^2(4/r-1))-16\frac{\lambda_j^2}{r^2s^2}u^2).$$

Proof. Let $e\in D(G)$. If $o(e)\in V$, then

$$d_{o(e)}=r,\ d_{t(e)}=s.$$

Thus, we have

$$1-u^2(\frac{4}{d_{o(e)}}-1)(\frac{4}{d_{t(e)}}-1)=\frac{rs-u^2(4-r)(4-s)}{rs},$$

$$(\mathbf{A}_{CRW})_{xy} = \frac{4/d_x^2}{1-u^2(4/d_x-1)(4/d_y-1)}$$

$$= \begin{cases} \frac{4s}{rs-u^2(4-r)(4-s)}\frac{1}{r} & \text{if } (x,y) \in D(G) \text{ and } x \in V, \\ \frac{4r}{rs-u^2(4-r)(4-s)}\frac{1}{s} & \text{if } (x,y) \in D(G) \text{ and } x \in W, \end{cases}$$

and

$$(\mathbf{D}_{CRW})_{xx} = \sum_{o(e)=x} \frac{4/d_x^2(4/d_{t(e)}-1)}{1-u^2(4/d_x-1)(4/d_{t(e)}-1)}$$

$$= \begin{cases} r \cdot \frac{4(4-s)}{r(rs-u^2(4-r)(4-s))} = \frac{4(4-s)}{rs-u^2(4-r)(4-s))} & \text{if } x \in V, \\ s \cdot \frac{4(4-r)}{s(rs-u^2(4-r)(4-s))} = \frac{4(4-r)}{rs-u^2(4-r)(4-s))} & \text{if } x \in W. \end{cases}$$

Next, let $V = \{v_1, \cdots, v_m\}$ and $W = \{w_1, \cdots, w_n\}$. Arrange vertices of G as follows: $v_1, \cdots, v_m; w_1, \cdots, w_n$. We consider the matrix $\mathbf{A} = \mathbf{A}(G)$ under this order. Then, let

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ {}^t\mathbf{E} & \mathbf{0} \end{bmatrix}.$$

By the Gram-Schmidt orthogonalization, there exists an orthogonal matrix $\mathbf{F} \in O(n)$ such that

$$\mathbf{EF} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mu_1 & & & 0 & 0 & \cdots & 0 \\ & \ddots & & & \vdots & & \vdots \\ \star & & & \mu_m & 0 & \cdots & 0 \end{bmatrix}.$$

Now, let

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}.$$

Then we have

$${}^t\mathbf{HAH} = \begin{bmatrix} \mathbf{0} & \mathbf{R} & \mathbf{0} \\ {}^t\mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Furthermore, let

$$\alpha = 4/(rs - u^2(4-r)(4-s)).$$

Then we have

$$\mathbf{A}_{CRW} = \begin{bmatrix} \mathbf{0} & \alpha s/r\mathbf{E} \\ \alpha r/s\, {}^t\mathbf{E} & \mathbf{0} \end{bmatrix},$$

and

$$\mathbf{D}_{CRW} = \begin{bmatrix} \alpha(4-s)\mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \alpha(4-r)\mathbf{I}_n \end{bmatrix}.$$

Thus, we have

$${}^t\mathbf{H}\mathbf{A}_{CRW}\mathbf{H} = \begin{bmatrix} \mathbf{0} & \alpha s/r\mathbf{R} & \mathbf{0} \\ \alpha r/s\, {}^t\mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and

$$ {}^t\mathbf{H}\mathbf{D}_{CRW}\mathbf{H} = \left[\begin{array}{cc} \alpha(4-s)\mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \alpha(4-r)\mathbf{I}_n \end{array} \right]. $$

By Theorem 2,

$$ \det(\mathbf{I}_{2\epsilon} - u\mathbf{P}) = \frac{(rs-u^2(4-r)(4-s))^{\epsilon}}{r^{\epsilon}s^{\epsilon}} \det(\mathbf{I}_{\nu} - u\mathbf{A}_{CRW} + u^2\mathbf{D}_{CRW}) $$

$$ = \frac{(rs-u^2(4-r)(4-s))^{\epsilon}}{r^{\epsilon}s^{\epsilon}} $$

$$ \times \det\left(\left[\begin{array}{ccc} \mathbf{I}_m + \alpha(4-s)u^2\mathbf{I}_m & -\alpha su/r\mathbf{R} & \mathbf{0} \\ -\alpha ru/s\ {}^t\mathbf{R} & \mathbf{I}_m + \alpha(4-r)u^2\mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-m} + \alpha(4-r)u^2\mathbf{I}_{n-m} \end{array} \right] \right) $$

$$ = \frac{(rs-u^2(4-r)(4-s))^{\epsilon}}{r^{\epsilon}s^{\epsilon}} (1+\alpha(4-r)u^2)^{n-m} $$

$$ \times \det\left(\left[\begin{array}{cc} (1+\alpha(4-s)u^2)\mathbf{I}_m & -\alpha su/r\mathbf{R} \\ -\alpha ru/s\ {}^t\mathbf{R} & (1+\alpha(4-r)u^2)\mathbf{I}_m \end{array} \right] \right) $$

$$ \times \det\left(\left[\begin{array}{cc} \mathbf{I}_m & \frac{1}{1+\alpha(4-s)u^2}\frac{\alpha su}{r}\mathbf{R} \\ \mathbf{0} & \mathbf{I}_m \end{array} \right] \right) $$

$$ = \frac{(rs-u^2(4-r)(4-s))^{\epsilon+m-n}}{r^{\epsilon}s^{\epsilon}} (rs+u^2(4-r)s)^{n-m} $$

$$ \times \det\left(\left[\begin{array}{cc} (1+\alpha(4-s)u^2)\mathbf{I}_m & \mathbf{0} \\ -\alpha ru/s\ {}^t\mathbf{R} & (1+\alpha(4-r)u^2)\mathbf{I}_m - \frac{\alpha^2u^2}{1+\alpha(4-s)u^2}\ {}^t\mathbf{R}\mathbf{R} \end{array} \right] \right) $$

$$ = \frac{(rs-u^2(4-r)(4-s))^{\epsilon+m-n}}{r^{\epsilon}s^{\epsilon}} (rs+u^2(4-r)s)^{n-m} $$

$$ \times \ (1+\alpha(4-s)u^2)^m \det((1+\alpha(4-r)u^2)\mathbf{I}_m - \frac{\alpha^2u^2}{1+\alpha(4-s)u^2}\ {}^t\mathbf{R}\mathbf{R}) $$

$$ = \frac{(rs-u^2(4-r)(4-s))^{\epsilon+m-n}}{r^{\epsilon}s^{\epsilon}} (rs+u^2(4-r)s)^{n-m} $$

$$ \times \ \det((1+\alpha(4-s)u^2)(1+\alpha(4-r)u^2)\mathbf{I}_m - \alpha^2u^2\ {}^t\mathbf{R}\mathbf{R}). $$

Since $\mathbf{A}$ is symmetric, ${}^t\mathbf{R}\mathbf{R}$ is symmetric and positive semi-definite, i.e., the eigenvalues of ${}^t\mathbf{R}\mathbf{R}$ are of form:

$$ \lambda_1^2, \cdots, \lambda_m^2 (\lambda_1, \cdots, \lambda_m \geqslant 0). $$

Furthermore, we have

$$ \det(\lambda\mathbf{I}_{\nu} - \mathbf{A}(G)) = \lambda^{n-m} \det(\lambda^2 - {}^t\mathbf{R}\mathbf{R}), $$

and so,

$$ Spec(\mathbf{A}(G)) = \{\pm\lambda_1, \cdots, \pm\lambda_m, 0, \ldots, 0\}. $$

Therefore it follows that

$$\det(\mathbf{I}_{2\epsilon} - u\mathbf{P})$$

$$= \frac{(rs-u^2(4-r)(4-s))^{\epsilon+m-n}}{r^{\epsilon}s^{\epsilon}}(rs+u^2(4-r)s)^{n-m}$$

$$\times \prod_{j=1}^{m}((1+\alpha(4-s)u^2)(1+\alpha(4-r)u^2)-\alpha^2\lambda_j^2u^2)$$

$$= \frac{(rs-u^2(4-r)(4-s))^{\epsilon+m-n}}{r^{\epsilon}s^{\epsilon}}(rs+u^2(4-r)s)^{n-m}$$

$$\times \prod_{j=1}^{m}(\frac{rs+u^2(4-s)r}{rs-u^2(4-r)(4-s)}\frac{rs+u^2(4-r)s}{rs-u^2(4-r)(4-s)}$$

$$-\lambda_j^2\frac{16u^2}{(rs-u^2(4-r)(4-s))^2})$$

$$= \frac{(rs-u^2(4-r)(4-s))^{\epsilon-m-n}}{r^{\epsilon}s^{\epsilon}}(rs+u^2(4-r)s)^{n-m}$$

$$\times \prod_{j=1}^{m}(rs(s+u^2(4-s))(r+u^2(4-r))-16\lambda_j^2u^2)$$

$$= (1-u^2(4/r-1)(4/s-1))^{\epsilon-\nu}(1+u^2(4/r-1))^{n-m}$$

$$\times \prod_{j=1}^{m}((1+u^2(4/s-1))(1+u^2(4/r-1))-16\frac{\lambda_j^2}{r^2s^2}u^2). \qquad \square$$

Now, let $u = 1/\lambda$. Then we obtain the following result.

Corollary 10. *Let $G = (V, W)$ be a connected (r, s)-semiregular bipartite graph with ν vertices and ϵ edges. Set $\mid V \mid = m$ and $\mid W \mid = n (m \leqslant n)$. Furthermore, let $\mathbf{P}$ be the transition probability matrix of the CRW with respect to the Grover matrix and*

$$Spec(\mathbf{A}(G)) = \{\pm\lambda_1, \cdots, \pm\lambda_m, 0, \ldots, 0\}.$$

Then

$$\det(\lambda\mathbf{I}_{2\epsilon} - \mathbf{P}) = (\lambda^2 - (4/r-1)(4/s-1))^{\epsilon-\nu}(\lambda^2 + (4/r-1))^{n-m}$$

$$\times \prod_{j=1}^{m}((\lambda^2+(4/s-1))(\lambda^2+(4/r-1))-16\frac{\lambda_j^2}{r^2s^2}\lambda^2).$$

By Corollary 4, we obtain the spectra for the transition probability matrix $\mathbf{P}$ of the CRW with respect to the Grover matrix of a semiregular bipartite graph.

Corollary 11. *Let $G=(V,W)$ be a connected (r,s)-semiregular bipartite graph with ν vertices and ϵ edges. Set $\mid V \mid = m$ and $\mid W \mid = n (m \leqslant n)$. Furthermore, let $\mathbf{P}$ be the transition probability matrix of the CRW with respect to the Grover matrix and*

$$Spec(\mathbf{A}(G)) = \{\pm\lambda_1, \cdots, \pm\lambda_m, 0, \ldots, 0\}.$$

Then the transition matrix $\mathbf{P}$ has 2ϵ eigenvalues of the form

1. *$4m$ eigenvalues: $\lambda =$*

$$\pm\sqrt{\frac{2r^2s^2-4rs^2-4r^2s+16\lambda_j^2 \pm \sqrt{(2r^2s^2-4rs^2-4r^2s+16\lambda_j^2)^2-4r^3s^3(4-r)(4-s)}}{2r^2s^2}};$$

2. *$2n-2m$ eigenvalues:*

$$\lambda = \pm i\sqrt{\frac{4}{r}-1};$$

3. *$2(\epsilon-\nu)$ eigenvalues:*

$$\lambda = \pm\sqrt{(\frac{4}{r}-1)(\frac{4}{s}-1)}.$$

Proof. Solving

$$(\lambda^2+(4/s-1))(\lambda^2+(4/r-1))-16\frac{\lambda_j^2}{r^2s^2}\lambda^2=0,$$

i.e.,

$$\lambda^4+(\frac{4}{r}+\frac{4}{s}-2-\frac{16\lambda_j^2}{r^2s^2})\lambda^2+(\frac{4}{r}-1)(\frac{4}{s}-1)=0,$$

we obtain

$$\lambda = \pm\sqrt{\frac{1}{2}((2-\frac{4}{r}-\frac{4}{s}+\frac{16\lambda_j^2}{r^2s^2}) \pm \sqrt{(2-\frac{4}{r}-\frac{4}{s}+\frac{16\lambda_j^2}{r^2s^2})^2-4(\frac{4}{r}-1)(\frac{4}{s}-1)}},$$

and so the result follows. □

7 Another type of the correlated random walk on a cycle graph

The CRW is defined by the following transition probability matrix $\mathbf{P}$ on the one dimensional lattice:

$$\mathbf{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where

$$a+c=b+d=1,\ a,b,c,d \in [0,1].$$

As for the CRW, see [8, 10], for example.

We formulate a CRW on the arc set of a graph with respect to the above matrix $\mathbf{P}$. The cycle graph is a connected 2-regular graph. Let C_n be the cycle graph with n vertices and n edges. Furthermore, let $V(C_n) = \{v_1, \dots, v_n\}$ and $e_j = (v_j, v_{j+1})(1 \leqslant j \leqslant n)$, where the subscripts are considered by modulo n. Then we introduce a $2n \times 2n$ matrix $\mathbf{U} = (U_{ef})_{e,f \in D(C_n)}$ as follows:

$$U_{ef} = \begin{cases} d & \text{if } t(f)=o(e),\ f \neq e^{-1} \text{ and } f=e_j, \\ b & \text{if } f=e^{-1} \text{ and } f=e_j, \\ a & \text{if } t(f)=o(e),\ f \neq e^{-1} \text{ and } f=e_j^{-1}, \\ c & \text{if } f=e^{-1} \text{ and } f=e_j^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbf{U}$ can be written as follows:

$$\mathbf{U} = \begin{bmatrix} d\mathbf{Q}^{-1} & c\mathbf{I}_n \\ b\mathbf{I}_n & a\mathbf{Q} \end{bmatrix},$$

where $\mathbf{Q} = \mathbf{P}_\sigma$ is the permutation matrix of $\sigma = (12\dots n)$. The CRW with $\mathbf{U}$ as a transition probability matrix is called the *second type of CRW* on C_n with respect to the above matrix $\mathbf{P}$.

Now, we define a function $w : D(C_n) \longrightarrow \mathbb{R}$ as follows:

$$w(e) = \begin{cases} d & \text{if } e=e_j \ (1 \leqslant j \leqslant n), \\ a & \text{if } e=e_j^{-1} \ (1 \leqslant j \leqslant n). \end{cases}$$

Furthermore, let an $n \times n$ matrix $\mathbf{W}(C_n) = (w_{uv})_{u,v \in V(C_n)}$ as follows:

$$w_{uv} = \begin{cases} w(u,v) & \text{if } (u,v) \in D(C_n), \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $\mathbf{U}$ is given as follows.

Theorem 12. *Let C_n be the cycle graph with n vertices, and $\mathbf{U}$ the transition probability matrix of the second type of CRW on C_n. Then*

$$\det(\lambda\mathbf{I}_{2n} - \mathbf{U}) = \det((\lambda^2 + (ad-bc))\mathbf{I}_n - \lambda\mathbf{W}(C_n)).$$

Proof. At first, we consider two $2n\times 2n$ matrices $\mathbf{B}=(B_{ef})_{e,f\in D(C_n)}$ and $\mathbf{J}=(J_{ef})_{e,f\in D(C_n)}$ as follows:

$$B_{ef}=\begin{cases} w(f) & \text{if } t(e)=o(f),\\ 0 & \text{otherwise,}\end{cases}\qquad J_{ef}=\begin{cases} b-a & \text{if } f=e^{-1} \text{ and } e=e_j,\\ c-d & \text{if } f=e^{-1} \text{ and } e=e_j^{-1},\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

$$\mathbf{U}={}^t\mathbf{B}+{}^t\mathbf{J}.$$

Now, we define two $2n\times n$ matrices $\mathbf{K}=(K_{ev})$ and $\mathbf{L}=(L_{ev})$ as follows:

$$K_{ev}=\begin{cases} 1 & \text{if } t(e)=v,\\ 0 & \text{otherwise,}\end{cases}\qquad L_{ev}=\begin{cases} w(e) & \text{if } o(e)=v,\\ 0 & \text{otherwise,}\end{cases}$$

where $e\in D(C_n), v\in V(C_n)$. Then we have

$$\mathbf{K}\,{}^t\mathbf{L}=\mathbf{B},\ {}^t\mathbf{L}\mathbf{K}=\mathbf{W}(C_n).$$

Thus,

$$\begin{aligned}\det(\mathbf{I}_{2n}-u\mathbf{U}) &= \det(\mathbf{I}_{2n}-u({}^t\mathbf{B}+{}^t\mathbf{J}))\\ &= \det(\mathbf{I}_{2n}-u(\mathbf{B}+\mathbf{J}))\\ &= \det(\mathbf{I}_{2n}-u\mathbf{J}-u\mathbf{B})\\ &= \det(\mathbf{I}_{2n}-u\mathbf{J}-u\mathbf{K}\,{}^t\mathbf{L})\\ &= \det(\mathbf{I}_{2n}-u\mathbf{K}\,{}^t\mathbf{L}(\mathbf{I}_{2n}-u\mathbf{J})^{-1})\det(\mathbf{I}_{2n}-u\mathbf{J})\\ &= \det(\mathbf{I}_{n}-u\,{}^t\mathbf{L}(\mathbf{I}_{2n}-u\mathbf{J})^{-1}\mathbf{K})\det(\mathbf{I}_{2n}-u\mathbf{J}).\end{aligned}$$

But, we have

$$\begin{aligned}&\det(\mathbf{I}_{2n}-u\mathbf{J})\\ =\ &\det\begin{bmatrix}\mathbf{I}_n & -(b-a)u\mathbf{I}_n\\ -(c-d)u\mathbf{I}_n & \mathbf{I}_n\end{bmatrix}\cdot\det\begin{bmatrix}\mathbf{I}_n & (b-a)u\mathbf{I}_n\\ \mathbf{0} & \mathbf{I}_n\end{bmatrix}\\ =\ &\det\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ -(c-d)u\mathbf{I}_n & \mathbf{I}_n-u^2(b-a)(c-d)\mathbf{I}_n\end{bmatrix}\\ =\ &(1-(a-b)(d-c)u^2)^n.\end{aligned}$$

Furthermore, we have

$$(\mathbf{I}_{2n}-u\mathbf{J})^{-1}=\frac{1}{1-(a-b)(d-c)u^2}(\mathbf{I}_{2n}+u\mathbf{J}).$$

Therefore, it follows that

$$\begin{aligned}
& \det(\mathbf{I}_{2n} - u\mathbf{U}) \\
= \; & (1-(a-b)(d-c)u^2)^n \det(\mathbf{I}_n - u/(1-(a-b)(d-c)u^2)\ {}^t\mathbf{L}(\mathbf{I}_{2n} + u\mathbf{J})\mathbf{K}) \\
= \; & \det((1-(a-b)(d-c)u^2)\mathbf{I}_n - u\ {}^t\mathbf{L}\mathbf{K} - u\ {}^t\mathbf{L}\mathbf{J}\mathbf{K}) \\
= \; & \det((1-(a-b)(d-c)u^2)\mathbf{I}_n - u\mathbf{W}(C_n) - u^2\ {}^t\mathbf{L}\mathbf{J}\mathbf{K}).
\end{aligned}$$

The matrix ${}^t\mathbf{LJK}$ is diagonal, and its (v_i, v_i) entry is equal to

$$(c-d)w(e_{i-1}^{-1}) + (b-a)w(e_i) = (c-d)a + (b-a)d = ac + bd - 2ad.$$

That is,

$${}^t\mathbf{LJK} = (ab + cd - 2ad)\mathbf{I}_n.$$

Thus,

$$\begin{aligned}
& \det(\mathbf{I}_{2n} - u\mathbf{U}) \\
= \; & \det((1-(a-b)(d-c)u^2)\mathbf{I}_n - u\mathbf{W}(C_n) - u^2(ac+bd-2ad)\mathbf{I}_n) \\
= \; & \det(((1+(ad-bc)u^2)\mathbf{I}_n - u\mathbf{W}(C_n)).
\end{aligned}$$ □

Substituting $u = 1/\lambda$, the result follows.

By Theorem 7, we obtain the spectra for the transition probability matrix $\mathbf{U}$ of the second type of the CRW on C_n. The matrix $\mathbf{W}(C_n)$ is given as follows:

$$\mathbf{W}(C_n) = \begin{bmatrix} 0 & d & 0 & \dots & & a \\ a & 0 & d & \dots & & 0 \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & \dots & 0 & d \\ d & 0 & 0 & \dots & a & 0 \end{bmatrix},$$

Corollary 13. *Let C_n be the cycle graph with n vertices, and $\mathbf{U}$ the transition probability matrix of the second type of CRW on C_n. Then the transition probability matrix $\mathbf{U}$ has $2n$ eigenvalues of the form*

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4(ad-bc)}}{2}, \ \mu \in Spec(\mathbf{W}(C_n)).$$

Proof. At first, we have

$$\det(\mathbf{I}_{2n} - u\mathbf{U}) = \prod_{\mu \in Spec(\mathbf{W}(C_n))} (\lambda^2 - \mu\lambda + (ad-bc)).$$

Solving

$$\lambda^2 - \mu\lambda + (ad - bc) = 0,$$

we obtain

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4(ad - bc)}}{2}. \qquad \square$$

Now, we consider the case of $a = b = c = d = 1/2$. Then the matrix $\mathbf{W}(C_n)$ is equal to

$$\mathbf{W}(C_n) = \frac{1}{2}\mathbf{A}(C_n).$$

By Corollary 6, we obtain the spectra for the transition probability matrix $\mathbf{U}$ of the second type of CRW on C_n.

Corollary 14. *Let C_n be the cycle graph with n vertices, and $\mathbf{U}$ the transition probability matrix of the second type of the CRW on C_n. Assume that $a = b = c = d = 1/2$. Then the transition probability matrix $\mathbf{U}$ has n eigenvalues of the form*

$$\lambda = \cos\theta_j,\ \theta_j = \frac{2\pi j}{n}(j = 0, 1, \dots, n-1) \quad (*).$$

The remaining n eigenvalues of $\mathbf{U}$ are 0 with multiplicities n.

Proof. It is known that the spectrum of $\mathbf{A}(C_n)$ are

$$2\cos\theta_j,\ \theta_j = \frac{2\pi j}{n}(j = 0, 1, \dots, n-1). \qquad \square$$

Note that the spectrum of (*) are those of the transition probability matrix of the simple random walk on a cycle graph C_n.

We can generalize the result for $a = b = c = d = 1/2$ on C_n to a d-regular graph($d \geqslant 2$). Let G be a connected d-regular graph with n vertices and m edges. Furthermore, let $\mathbf{P}$ be the $d \times d$ matrix as follows:

$$\mathbf{P} = \frac{1}{d}\mathbf{J}_d,$$

where $\mathbf{J}_d$ is the matrix whose elements are all one. Let $\mathbf{U} = (U_{ef})_{e,f\in D(G)}$ be the the transition probability matrix of a CRW on G with respect to $\mathbf{P}$. Then we have

$$U_{ef} = \begin{cases} 1/d & \text{if } t(e) = o(f), \\ 0 & \text{otherwise}, \end{cases}$$

and so,

$$\mathbf{U} = \frac{1}{d}\mathbf{B}.$$

Similarly to The proof of Theorem 7, we obtain the following result.

Theorem 15. *Let G be a connected d-regular graph with n vertices and m edges. Furthermore, let $\mathbf{U}$ the transition probability matrix of the CRW on G with respect to $\mathbf{P} = 1/d\mathbf{J}_d$. Then*

$$\det(\lambda\mathbf{I}_{2m} - \mathbf{U}) = \lambda^{2m-n}\det(\lambda\mathbf{I}_n - \frac{1}{d}\mathbf{A}(G)).$$

Thus,

Corollary 16. *Let G be a connected d-regular graph with n vertices and m edges. Furthermore, let $\mathbf{U}$ the transition probability matrix of the CRW on G with respect to $\mathbf{P} = 1/d\mathbf{J}_d$. Then the transition probability matrix $\mathbf{U}$ has n eigenvalues of the form*

$$\lambda = \frac{1}{d}\lambda_A, \ \lambda_A \in Spec(\mathbf{A}(G)).$$

The remaining $2(m-n)$ eigenvalues of $\mathbf{U}$ are 0 with multiplicities $2m-n$.

8 Future work

In this paper, we presented the spectrum of the transition probability matrix $\mathbf{P}$ of the CRW induced from the time evolution matrix $\mathbf{U}$ of the Grover walk on a regular graph and a semiregular bipartite graph by using a determinant expression for the generalized weighted zeta function of a graph. Here, the transition probability matrix $\mathbf{P}$ is the Hadamard product $\mathbf{U} \circ \mathbf{U}$ of $\mathbf{U}$ and itself.

Thus, we can propose the following problem.

Problem 17. Let a matrix $\mathbf{U}$ be the time evolution matrix of any discrete-time quantum walk on a graph. Then, what is the spectrum of the doubly stochastic matrix $\mathbf{P} = \mathbf{U} \circ \overline{\mathbf{U}}$?

From now on, we shall study the above problem.

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