Improved packings of $n(n-1)$ unit squares in a square

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Abstract

Let $s(n)$ be the side of the smallest square into which we can pack n unit squares. The purpose of this paper is to prove that $s(n^2-n) < n$ for all $n \geqslant 12$. Besides, we show that $s(18^2-17) < 18, s(17^2-16) < 17$, and $s(16^2-15) < 16$.

Mathematics Subject Classifications: 05B40, 52C15

1 Introduction

The problem of packing equal squares in a square has been around for some 40 years [1]. Let $s(n)$ be the side of the smallest square into which we can pack n unit squares. Nagamochi [3] proved that $s(n^2-2) = s(n^2-1) = n$. It follows from [1] that $s(n^2-O(n^{\frac{7}{11}})) < n$ for big n. From [4] it follows that the 7/11 degree can be reduced to 5/8.

An important question is to find the minimum n for which $s(n^2-n) < n$. For small n, only $s(2) = 2$ and $s(6) = 3$ have been proved, but we dont even know the proof of $s(12) = 4$. It was proved in [2] that $s(n^2-n-1) < n$ for $3 < n < 11$. Due to Lars Cleemann it was known that $s(17^2-17) < 17$ [2]. Nagamochi in [3] mistakenly says that the following is proved in [2]

$$s(n^2-n) < n \quad \forall n \geqslant 17. \tag{1}$$

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The truth is that in [2] a sporadic squeezable packing of 272 unit squares in a square (17,17) is given, proving that $s(17^2 - 17) < 17$, but from this it does not follow that $s(18^2 - 18) < 18$ etc. Thus, Nagamochi's implicit conjecture (1) needs a proof.

We prove the conjecture and even more: $s(n^2 - n) < n \;\; \forall n \geqslant 12$, and, moreover,

$$s(18^2 - 17) < 18, \quad s(17^2 - 16) < 17, \quad s(16^2 - 15) < 16.$$

2 Some squeezable packing of rectangles

Let a packing of m unit squares in a rectangle $R = (R_x, R_y)$ be given. We assume that $(R_x - 1)(R_y = 1) < m < R_x R_y$ and we can't pack a unit square in the waste area. This packing is called *squeezable* if both sides of a rectangle can be reduced, i.e., for some $\delta > 0$ there exists a packing of m unit squares in a rectangle $(R_x - \delta, R_y - \delta)$. The maximum of such $\delta > 0$ is called *the value of squeezing* and is denoted by $\delta(R, m)$. We write $\delta(R, m) = 0$ if the packing is not squeezable.

The property of squeezability of a packing for small parameters can be proved rather simply. However proving this property for large parameters is a non-trivial mathematical problem. The following obvious formula connects $\delta(R, m)$ and $s(n)$:

$$s(n) = \lceil s(n) \rceil - \delta((\lceil s(n) \rceil, \lceil s(n) \rceil), n).$$

If $\delta((R_x, R_y), m) < 1$ then the fact that for integer R_x, R_y

$$\delta((R_x, R_y), m) \leqslant \delta((R_x + 1, R_y), m + R_y - 1) \tag{2}$$

can be proved by adding $R_y - 1$ unit squares to the x-side of a rectangle (R_x, R_y). Figure 1 shows the basic idea for efficiently packing unit squares in a square S, where rectangles C and D are integer and the waste is in rectangles A and B. It is easy to see that if the packing of unit squares in rectangles A, B is squeezable, then the packing of unit squares in S is squeezable and

$$\delta(S, \cdot) \geqslant \min(\delta(A, \cdot), \delta(B, \cdot)). \tag{3}$$

This bound can be increased if we note that after squeezing there is a little space between rectangles A, B. We can give this space to a rectangle with minimal squeezing value in order to increase that value and thus to increase the evaluation of $\delta(S, \cdot)$.

Let us consider a packing of 26 unit squares in a rectangle $(4, 8)$ (see Figure 2). This packing is centrally symmetric and the waste is equal to 6.

In Figure 2 we see one of the main ideas for packing unit squares: using of stacks $(4, 1)$ tilted by an angle $\alpha = \arcsin(8/17)$. The main idea for squeezing a packing follows from it: tilting stacks $(4, 1)$ by an angle $\alpha + \varepsilon$ so that the stack $(4, 1)$ is located in a vertical strip of width $4 - \delta$, where ε and δ are sufficiently small. Hereinafter we determine the orientation of a unit square by a unit vector (x, y) with $x > 0, y \geqslant 0, x^2 + y^2 = 1$ directed along the side of this unit square. If the bottom vertex of the unit square is at the origin then the three other vertices have coordinates $(x, y), (x - y, y + x), (-y, x)$. Note that if

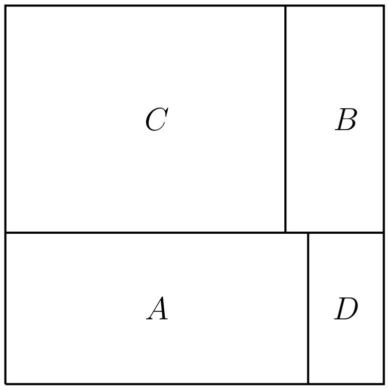

Figure 1: Scheme of squeezable packing

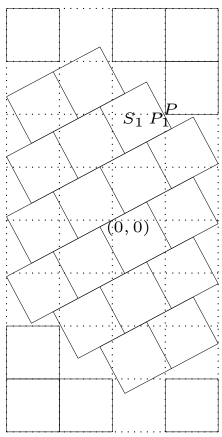

Figure 2: Squeezable packing of 26 unit squares in a rectangle (4,8)

two points P_t, P_b are taken on the top side and the bottom side of this unit square then the scalar product $\langle P_t - P_b, (x, y)\rangle$ is equal to 1.

Continuing with the example in Figure 2, after increasing the tilt the stack $(4, 1)$ in a vertical strip of width $4 - \delta$ has orientation $(x_1, y_1), x_1 > 0, y_1 \geqslant 0$ satisfying the system of equations

$$4x_1 + y_1 = 4 - \delta, x_1^2 + y_1^2 = 1.$$

To evaluate the squeezing value $\delta((4, 8), 26)$, we use the bisection method. The packing remains centrally symmetric. The distance between the point $P = (P_x, P_y) = (1 - \delta/2, 2 - \delta/2)$ and the upper side of the square S_2 intersecting the line $x = 1 - \delta/2$ in the point $P_1 = (P_{1x}, P_{1y}) = (1 - \frac{\delta}{2}, (1 - \frac{\delta}{2})\frac{y_1}{x_1} + \frac{1}{x_1} + \frac{1-x_1}{x_1 y_1})$ is critical. For $\delta = 0.01$ we have $x_1 = .877695..., y_1 = .479219..., P_y - P_{1y} = 0.021604 > 0$. For $\delta = 0.02$ $x_1 = .87312663..., y_! = .48749347..., P_y - P_{1y} = -0.0061309... < 0$. The bisection method gives evaluation $\delta((4, 8), 26) > 0.0177702$.

Figure 3 shows a more complex example, a centrally symmetric squeezable packing of

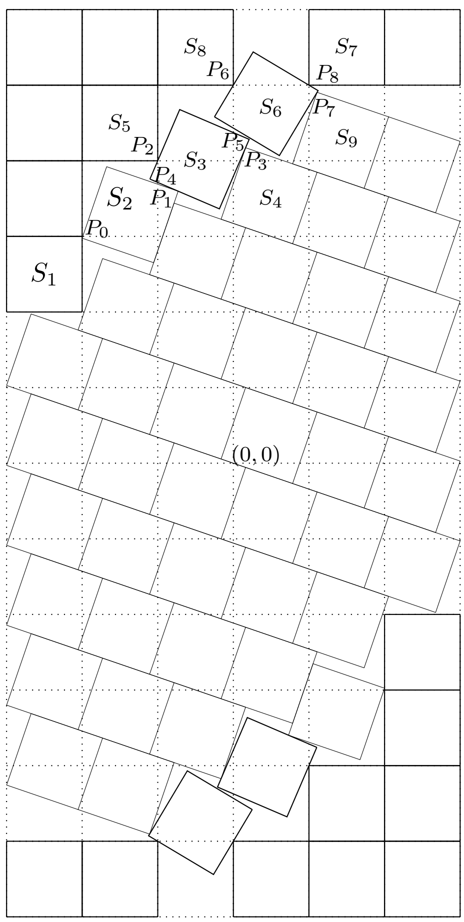

Figure 3: Squeezable packing of 64 unit squares in a rectangle (6,12)

64 unit squares in a rectangle (6,12). Four unit squares: S_3, S_6 and their symmetric ones have not the orientation $(\frac{35}{37}, \frac{12}{37})$ nor $(1,0)$. Hereinafter we denote points and squares by the same indices in different figures without losing accuracy.

In this packing the left vertex of S_2 is on a side of S_1. The square S_3 is placed so that the right vertices of squares S_2, S_5, and the top vertex of S_4 are on the sides of S_3. Vertices of the squares S_8, S_7, S_9 are on sides of S_6. Calculations show that there is a small distance between S_3 and S_6, which guarantees squeezability of the given packing.

To calculate the squeezing value $\delta((6,12),64)$, take $\delta = 0.004$ and define the existence of a packing 64 unit squares in a rectangle $(6-\delta, 12-\delta)$. The distance between the right vertex of S_3 and the top side of S_6 should be not less than 1.

Table 1 contains calculations with $\delta = 0.004$.

Calculations with $\delta = 0.005$ give $\langle P_8 - P_5, (x_2, y_2)\rangle = 0.999617371807702270$, i.e., the squares S_3, S_6 intersect. The bisection method gives evaluation $\delta((6,12),64) > .00490823$.

A packing of 58 unit squares in a rectangle (6,11-2/35) can be obtained by removing one stack (6,1) in Figure 3 and lifting up by 37/35 all the squares that are below this

Object	Formulae or system of equations	Numerical value
δ		0.004
Orientation (x_1, y_1) of stack (6,1)	$y_1^2 + x_1^2 = 1, 6y_1 + x_1 = 6 - \delta$	(.328061226490, .94465646225)
P_0	$P_0 = (-2 + \delta/2, (2 - \delta/2)\frac{x_1}{y_1} + \frac{2}{y_1} + \frac{1-y_1}{x_1 y_1})$	(-1.998,2.989621361)
P_1	$P_1 = P_0 + (x_1 + y_1, y_1 - x_1)$	(-0.725282311,3.6062165968)
P_2	$P_2 = (\delta/2 - 1, 4 - \delta/2)$	(-0.998,3.998)
P_3	$P_3 = (3 - 3y_1 - \frac{\delta}{2}, -\frac{(3-3y_1-\delta/2)x_1}{y_1} + \frac{4}{y_1})$	(.1640306130, 4.177378839)
Orientation (x_2, y_2) of S_3	$x_2^2 + y_2^2 = 1.$, $\langle P_2 - P_3, (-y_2, x_2)\rangle = 1$	(.390085325,.92077871336)
P_4	$P_4 = \langle P_1, (x_2, y_2)\rangle \cdot (x_2, y_2)+$ $+\langle P_2, (y_2, -x_2)\rangle \cdot (y_2, -x_2)$	(-1.0972231,3.76378828)
P_5	$P_5 = P_4 + (x_2 + y_2, y_2 - x_2)$	(0.213640902,4.29448167498)
P_6	$P_6 = (\frac{1}{2}\delta, 5 - \frac{1}{2}\delta)$	(0.002,4.998)
P_7	$P_7 = (3 - \delta/2, -(3 - \delta/2)x_1/y_1)+$ $+5(0, 1/y_1) + 2(-y_1, x_1)$	(1.108687,4.9079035)
P_8	$P_8 = (1 - \delta/2, 5 - \delta/2)$	(0.998,4.998)
Orientation (x_3, y_3) of S_6	$x_3^2 + y_3^2 = 1.$, $\langle P_6 - P_7, (-y_3, x_3)\rangle = 1$	(.5062565099,.862382946)
Distance between P_5 and top side of S_6	$\langle P_8 - P_5, (x_3, y_3)\rangle$	1.00378910536129684

Table 1: Calculations with $\delta = 0.004$.

stack. Similar calculations give the evaluation of the squeezing value $\delta((6, 11), 58) > 0.01681735886$.

Consider a more difficult problem of a squeezable packing of 43 unit squares in a rectangle (5,10). In Figure 4 six unit squares $S_1, S_4, S_9, S_{10}, S_{11}, S_{12}$ have not the orientation $(\frac{5}{13}, \frac{12}{13})$ nor $(1, 0)$.

The square S_1 has a vertex on the side of the rectangle (5,10), one on a side of S_2, and one on a side of S_3. The right vertex of S_1 is on the bottom side of S_4. S_4 is tilted so that the bottom right vertex of S_3 is on the left side of S_4 and the top vertex of the stack $(3, 1)$ is on the right side of S_4. The left vertex of S_5 is on the side of S_6. The squares S_9, S_{10} are tilted by the same angle so that the vertex of S_8 is on the side of S_9, the vertex of S_5 is on the bottom side of S_9, and the vertex of S_7 is on the bottom side of S_{10}. The squares S_{11}, S_{12} form a rectangle (2,1). The right vertex of S_{12} is on the right side of a rectangle (5,10). The vertex of S_{13} is on the top side of S_{11}. The bottom sides of S_{11} and S_{12} are parallel to the line connecting the right vertices of S_9 and S_{10}. The vertex of S_{14} is on the bottom side of S_{15}. Calculations show that there is a small distance 0.0055111... between the bottom side of the rectangle $(2, 1) = S_{11} \cup S_{12}$ and the line connecting the

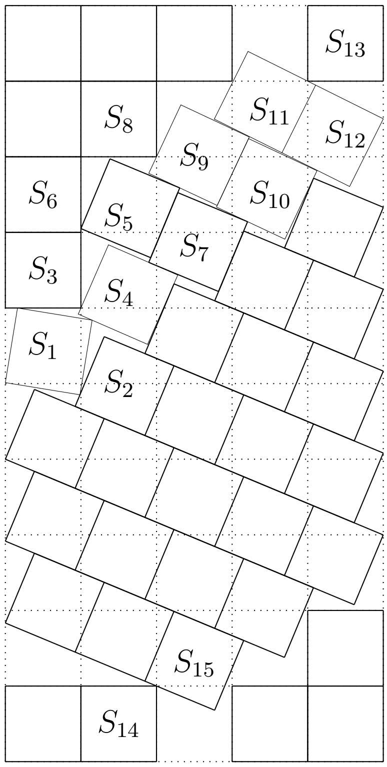

Figure 4: Squeezable packing of 43 unit squares in a rectangle (5,10)

right vertices of S_9 and S_{10}. This guarantees squeezability of the given packing.

Calculation of the squeezing value $\delta((5,10),43)$ gives the evaluation $\delta((5,10),43) > 0.0009652493$. This packing plays an important role in the squeezable packing of 132 unit squares in a square (12,12). Below we show the evaluation of $\delta((12,12),132)$. From this evaluation one can obtain the evaluation of $\delta((5,10),43)$. Analogous calculations give the evaluation of the squeezing value $\delta((5,9),38) > 0.020403$.

Table 2 contains the evaluations of the squeezing values of some rectangles.

Rectangle R	n	$\delta(R,n)$
(4,8)	26	> 0.01777021751
(5,10)	43	> 0.0009652493
(5,9)	38	> 0.020403
(6,12)	64	> 0.004908231774819
(6,11)	58	> 0.01681735886

Table 2. Evaluations of squeezing value of some rectangles

To prove conjecture (1), we need the following lemma.

Lemma 1. *For any $k \geqslant 3$ there exists a squeezable packing of $4k^2+6k-2$ unit squares in a rectangle $(2k, 2k+4)$ (the waste is equal to $2k+2$).*

The proof is technically simple and can be understood from Figure 5, showing a centrally symmetric squeezable packing of 86 unit squares in a rectangle $(8, 12)$. For an arbitrary $k \geqslant 3$, the centrally symmetric packing in the upper half of a rectangle $(2k, 2k+4)$ consists of 2 staircases. A staircase with orientation (1,0) having $\frac{k(k+1)}{2}$ unit squares, and a staircase with orientation $(x_1, y_1) = (\frac{4k^2-1}{4k^2+1}, \frac{4k}{4k^2+1})$ that has $\frac{(3k-1)(k+2)}{2}$ unit squares. The top vertex of S_{k+1} has ordinate

$$y_{k+1} = -\frac{4k^2}{4k^2-1} + (k+2)\frac{4k^2+1}{4k^2-1} + (k-1)\frac{4k}{4k^2+1} <$$

$$< -\frac{4k^2}{4k^2-1} + (k+2)\frac{4k^2+1}{4k^2-1} + (k-1)\frac{4k}{4k^2-1} = k+2 - \frac{2(k-2)}{4k^2-1} < k+2,$$

i.e., S_{k+1} is in rectangle $(2k, 2k+4)$. The top vertex of S_0 has ordinate

$$\frac{4k^2}{4k^2-1} + \frac{4k^2-1}{4k^2+1} = 2 + \frac{1}{4k^2-1} - \frac{2}{4k^2+1} < 2,$$

i.e., S_0 does not intersect the staircase with orientation (1,0). Each square $S_j, 1 \leqslant j \leqslant k$ intersects the vertical line $x = k - j$ in the point

$$(k-j, j \cdot \frac{1-x_1}{x_1 y_1} + (k-j)\frac{y_1}{x_1} + \frac{j}{x_1}).$$

The ordinate of this point satisfies

$$j \cdot \frac{1-x_1}{x_1 y_1} + (k-j)\frac{y_1}{x_1} + \frac{j}{x_1} = 1 + j + \frac{1}{2} \cdot \frac{j \cdot (-4k^2+4k+1) + 2k}{k(4k^2-1)} < 1 + j,$$

i.e., none of the $S_j, 1 \leqslant j \leqslant k$ intersects the staircase with orientation (1,0). We see that there is a positive distance between the two staircases. Therefore, this packing is squeezable. The lemma is proved.

3 Improved squeezable packing of some squares

As mentioned in the introduction, in [3] Nagamochi mistakebly says that in [2] it is proved that

$$s(n^2 - n) < n \quad \forall n \geqslant 17. \tag{4}$$

Thus he implicitly formulates the conjecture (4). For the proof of this conjecture we use lemma 1 as follows.

For even $n \geqslant 14$ we use Figure 1 with rectangles $A = (12, 6),\ B = (n-10, n-6),\ C = (10, n-6),\ D = (n-12, 6)$.

For odd $n \geqslant 13$ we use Figure 1 with rectangles $A = (10, 5),\ B = (n-9, n-5),\ C = (9, n-5),\ D = (n-10, 5)$.

Thus the conjecture (4) is proved for $n \geqslant 13$.

For the proof of this conjecture for $n = 12$ see Figure 6.

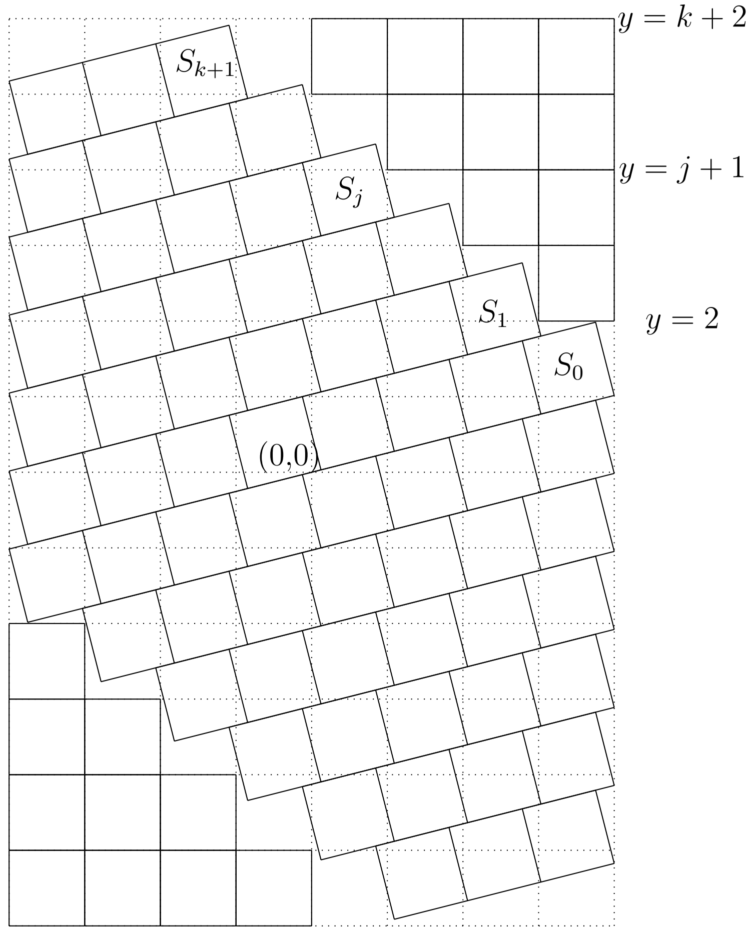

Figure 5: Squeezable packing of $4k^2+6k-2$ unit squares in a rectangle $(2k,2k+4)$

The packing in Figure 6 is obtained from the squeezable packing in rectangles (8,4), (5,10). In the packing in (5,10) we tilt the angular squares S_1, S_2 by an angle arcsin(10/26) so that the bottom vertex of S_1 has an integer y-coordinate and S_2 has intruded space in the rectangle (8,4). From the packing in (8,4) we remove two right top squares and move to the left by 1/20 unit squares tilted by an angle arcsin(8/17) so that the bottom vertex of S_3 is on the side of S_4. The small distance between S_2 and S_5 makes the packing in Figure 6 squeezable.

Thus we have proved that

$$s(n^2-n) < n \ \forall n \geqslant 12.$$

To evaluate $\delta((12,12),132)$, take $\delta = 0.002$. The origin is in the right bottom vertex of the integer rectangle $(7,8)$. The bottom side of (12,12) has y-coordinate $-4+\delta$, the right side of (12,12) has x-coordinate $5-\delta$.

Table 2 contains the calculations.

Object	Formulae or system of equations	Numerical value
δ		0.002
Orientation (x_1, y_1) of stack (4,1)	$y_1^2 + x_1^2 = 1, y_1 + 4x_1 = 4 - \delta$	(.881413748866, 0.4723450045357421)
P_0	$P_0 = (4/x_1 - 1/y_1 + x_1/y_1 - 5, 0)$	(-.712894713,0)
Orientation (x_2, y_2) of stack (5,1)	$y_2^2 + x_2^2 = 1, 5y_2 + x_2 = 5 - \delta$	(.386451637219073..., .9223096725561...)
$P_1 = (P_{1x}, P_{1y})$	$P_1 = ((2 - 2x_2 - \delta) \cdot y_1/x_1 + 2 \cdot y_2, -2 + \delta) + P_0$	(1.788247541,-1.998)
Lower ordinate of intersection S_2 with line $x = 0$	$Y_1 = P_{1y} + P_{1x} \cdot \frac{x_2}{y_2}$	-1.248716749
Orientation (x_3, y_3) of square S_6	$x_3^2 + y_3^2 = 1$, $\frac{x_2}{y_2} = \frac{(x_2+y_2-y_3)}{Y_1+4/y_2+y_2-x_2-4+x_3+y_3}$	(.1523435..., .98832760...)
P_2	$P_2 = (x_3 + y_3, 4 - x_3)$	(1.140671137,3.847656465)
P_3	$P_3 = (x_2 + 2y_2, Y_1 + \frac{5}{y_2} + y_2 - 2x_2)$	(2.231070982,4.321862330)
Orientation (x_4, y_4) of square S_9	$x_4^2 + y_4^2 = 1$, $\langle P_3 - (1, 4), (y_4, -x_4)\rangle = 1$	(.39947627..., 0.9167435347...)
P_4	$P_4 = (1, \frac{6}{y_2} + (P_{1x} - 1)\frac{x_2}{y_2} - -2 + \delta + \frac{1-y_2}{x_2 y_2})$	(1,5.055655408)
P_5	$P_5 = P_4 + (x_2 + y_2, y_2 - x_2)$	(2.30876131, 5.5915134434)
P_6	$\langle (P_6 - P_2), (x_4, y_4)\rangle = 1$ $\langle (P_6 - P_4), (y_2, -x_2)\rangle = 1$	(1.897035430423, 4.608883990)
P_7	$P_7 = P_6 + (x_2 + y_2, y_2 - x_2)$	(3.20579674042318, 5.14474202553891)
P_8	$P_8 = P_3 + (2y_2, 2/y2 - 2x_2)$	(4.075690327,5.717428128)
Orientation (x_5, y_5) of squares S_{14}, S_{15}	$x_5^2 + y_5^2 = 1$, $\langle P_8 - (2, 6), (y_5, -x_5)\rangle = 2$	(.4235421115..., .905876415...)
$P_9 = (P_{9x}, P_{9y})$	$P_9 = \langle P_5, (x_5, y_5)\rangle \cdot (x_5, y_5)+$ $+\langle (2, 6), (-y_5, x_5)\rangle \cdot (-y_5, x_5)$ $+(x_5 + y_5, y_5 - x_5)$	3.22807975740513 6.26558985540152)
$P_{10} = (P_{10x}, P_{10y})$	$P_{10} = \langle P_7, (x_5, y_5)\rangle \cdot (x_5, y_5)+$ $+\langle (2, 6), (-y_5, x_5)\rangle \cdot (-y_5, x_5)$ $+(x_5 + 2y_5, y_5 - 2x_5)$	4.12345766036105 5.81959341362795
$P_{11} = (P_{11x}, P_{11y})$	$P_{11} = (4 - \delta, 7)$	(3.998,7)
Distance between P_{11} and segment $[P_9, P_{10}]$	$\frac{(P_{9y}-P_{10y})\cdot(P_{11x}-P_{9x})}{\sqrt{((P_{9y}-P_{10y})^2+(P_{9x}-P_{10x})^2)}} - -\frac{(P_{9x}-P_{10x})\cdot(P_{11y}-P_{9y})}{\sqrt{((P_{9y}-P_{10y})^2+(P_{9x}-P_{10x})^2)}}$	1.000648944...

Table 2: Calculations for $\delta = 0.002$.

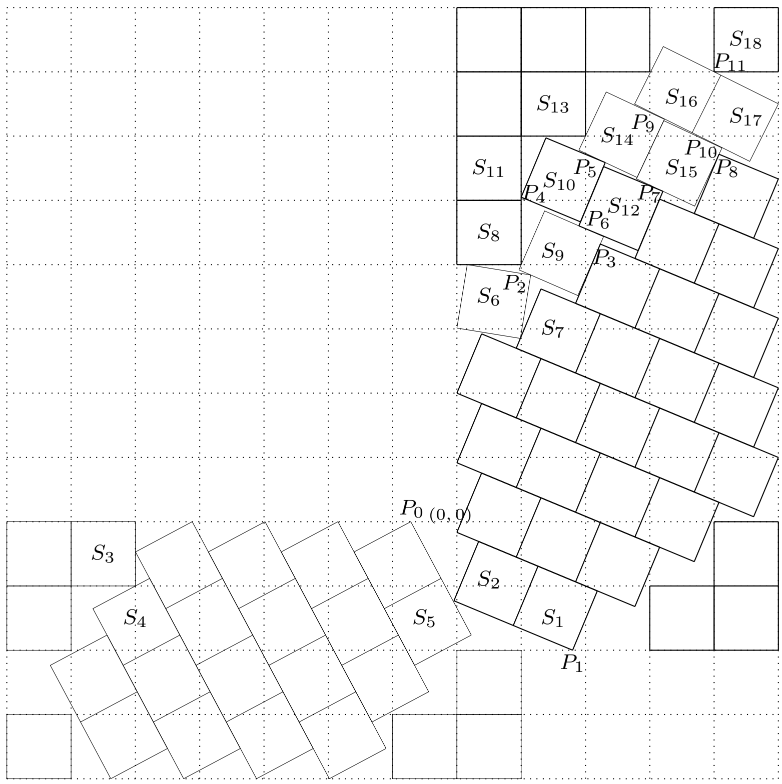

Figure 6: Squeezable packing of 132 unit squares in a square (12,12)

Calculations with $\delta = .0021$ give the distance 0.9999866543 between the bottom left vertex of S_{18} and the segment $[P_9, P_{10}]$. The bisection method gives the evaluation $\delta((12,12),132) > 0.00209798269$, i.e., $s(132) < 11.99790201731$.

Analogous calculations give evaluations

$$\delta((5,10),43) > 0.0009652493, \delta((5,9),38) > 0.020403$$

$$\delta((13,13),156) > 0.0059576, s(156) < 12.9940424.$$

Calculations with $C = (10,8), D = (3,6), A = (11,6), B = (4,8)$ in Figure 1 give

$$\delta((14,14),182) > 0.01681735886, s(14^2 - 14) < 13.98318264114.$$

For the square (15,15) we have $\delta((15,15),210) \geqslant \min(\delta((5,9),38),\ \delta((11,6),58)) > 0.01681735886$, i.e., $s(210) < 14.98318264114$.

For the square (16,16) we have $\delta((16,16),241) > \min(\delta((5,10),43), \delta((12,6),64)) > 0.0009652493$, i.e., $s(16^2 - 15) < 15.9990347507$.

More careful analysis when we use the space between rectangles (5,10) and (12,6) gives $\delta((16,16),241) > 0.00404996$, i.e., $s(16^2 - 15) < 15.99595004$.

Calculations with $A = (12,6), B = (6,11), C = (11,11), D = (5,6)$ give

$$\delta((17,17),17^2 - 16) > 0.0049082317748, s(17^2 - 16) < 16.9950917682252.$$

Notice that this squeezable packing of a square (17,17) contains one unit square more than in [2].

Calculations with $A = (13, 6), B = (6, 12), C = (12, 12), D = (5, 6)$ give

$$\delta((18, 18), 18^2 - 17) \geqslant 0.0049082317748, s(18^2 - 17) < 17.9950917682252.$$

Table 4 contains the evaluations of the squeezing values and the upper bounds of $s(n)$ for new n.

n	$s(n)$	$\delta((\lceil s(n)\rceil, \lceil s(n)\rceil), n)$
132	$s(12^2 - 12) < 11.99790201731$	$\delta((12, 12), 132) > 0.00209798269$
156	$s(13^2 - 13) < 12.9940424$	$\delta((13, 13), 156) > 0.0059576$
182	$s(14^2 - 14) < 13.98318264114$	$\delta((14, 14), 182) > 0.01681735886$
210	$s(15^2 - 15) < 14.98318264114$	$\delta((15, 15), 210) > 0.01681735886$
241	$s(16^2 - 15) < 15.99595004.$	$\delta((16, 16), 241) > 0.00404996$
273	$s(17^2 - 16) < 16.9950917682252$	$\delta((17, 17), 17^2 - 16) > 0.0049082317748$
307	$s(18^2 - 17) < 17.9950917682252$	$\delta((18, 18), 18^2 - 17) > 0.0049082317748$

Table 4. Evaluations of squeezing values and upper bounds of $s(n)$ for new n

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