

Generalized Planar Turán Numbers

Ervin Gyóri*

Alfréd Rényi Institute of Mathematics
Budapest, Hungary
Central European University
Budapest, Hungary
gyori.ervin@renyi.mta.hu

Addisu Paulos

Addis Ababa University
Addis Ababa Ethiopia
Central European University
Budapest, Hungary
addisu_2004@yahoo.com

Nika Salia*[†]

Alfréd Rényi Institute of Mathematics
Budapest, Hungary
Central European University
Budapest, Hungary
salia.nika@renyi.hu

Casey Tompkins[‡]

Institute for Basic Science (IBS)
Daejeon, Republic of Korea
Karlsruhe Institute of Technology
Karlsruhe, Germany
ctompkins496@gmail.com

Oscar Zamora*

Central European University
Budapest, Hungary
Universidad de Costa Rica
San José, Costa Rica
oscar.zamoraluna@ucr.ac.cr

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Abstract

In a generalized Turán problem, we are given graphs H and F and seek to maximize the number of copies of H in an n -vertex graph not containing F as a subgraph. We consider generalized Turán problems where the host graph is planar. In particular, we obtain the order of magnitude of the maximum number of copies of a fixed tree in a planar graph containing no even cycle of length at most 2ℓ , for all ℓ , $\ell \geq 1$. We also determine the order of magnitude of the maximum number of cycles of a given length in a planar C_4 -free graph. An exact result is given for the maximum number of 5-cycles in a C_4 -free planar graph. Multiple conjectures are also introduced.

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1 Introduction

For a given family of graphs \mathcal{F} , we say that a graph is \mathcal{F} -free if it contains no $F \in \mathcal{F}$ as a subgraph. The classical extremal function $\text{ex}(n, \mathcal{F})$ is defined to be the maximum number of edges possible in an n -vertex \mathcal{F} -free graph. This function naturally generalizes to a setting where, rather than edges, we maximize the number of copies of a given graph H in an n -vertex \mathcal{F} -free graph. Following Alon and Shikhelman [2] (see also [3]), we denote this more general function by $\text{ex}(n, H, \mathcal{F})$. In the case that $\mathcal{F} = \{F\}$, we denote the generalized extremal function $\text{ex}(n, H, \mathcal{F})$ by $\text{ex}(n, H, F)$ and we refer to \mathcal{F} -free graphs simply as F -free.

Problems of this type have a long history beginning with a result of Zykov [38] (and later independently Erdős [11]) who determined the value of $\text{ex}(n, K_r, K_t)$ for any pair of cliques. After these initial results a variety of other results of this type were obtained, perhaps the most well-known of which being the determination of $\text{ex}(n, C_5, C_3)$ by Hatami, Hladký, Král, Norine and Razborov [22] and independently by Grzesik [17]. Many other results about generalized extremal numbers have also been obtained (see, for example [4, 14, 32, 33]).

Another natural generalization of the Turán extremal function is to the setting where the host graph is taken to be something other than K_n . Examples include the Zarankiewicz problem [37] wherein the host graph is taken to be a complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$, or extremal problems on the hypercube Q_n initiated by Erdős [12].

More recently, extremal problems have been considered where the host graph is taken to be a planar graph. For a given graph F , let us denote the maximum number of edges in an n -vertex F -free planar graph by $\text{ex}_{\mathcal{P}}(n, F)$. This topic was initiated by Dowden in [9] who determined $\text{ex}_{\mathcal{P}}(n, C_4)$ and $\text{ex}_{\mathcal{P}}(n, C_5)$. A variety of other forbidden graphs F including stars, wheels and fans were considered by Lan, Shi and Song [29]. The case of theta graphs was considered in Lan, Shi and Song [30], and the case of short paths was considered by Lan and Shi in [28]. Some closely related anti-Ramsey problems were considered in [31] and [6].

Another direction of research which has been considered is maximizing the number of copies of a given graph in an n -vertex planar graph. Hakimi and Schmeichel [21] determined the maximum number of triangle and C_4 copies possible in a planar graph. These results were extended by Alon and Caro [1] who determined the maximum number of $K_{1,t}$ and $K_{2,t}$ copies. Alon and Caro [1] and independently Wood [35] proved that the maximum number of K_4 copies in a planar graph is $n-3$. Resolving a conjecture attributed to Perles in [1], Wormald [36] proved that every 3-connected graph H occurs at most $c_H n$ times in an n -vertex planar graph for some constant c_H depending on H (this result was proved again in a different way by Eppstein [10]). A simple argument shows that graphs with at least 3 vertices which are not 3-connected can occur at least quadratically many times in a planar graph. Thus the preceding result of Wormald and Eppstein provides a characterization of graphs which can occur at most $O(n)$ times in a planar graph. Recently, resolving a conjecture of Hakimi and Schmeichel [21], the maximum number of 5-cycles in a planar graph was determined in [18]. Similarly, the maximum number of

paths of length three was determined in [19]. More recently, asymptotic results have been obtained for some longer paths and cycles by Ghosh *et al.* [16] and Cox and Martin [7, 8]. Matolcsi and Nagy [34] have also obtained results in the outerplanar setting.

It is interesting to note that the problem of maximizing the number of H copies in a planar graph is in some sense a special case of the problem of Alon and Shikelman [2]. Indeed, for a given graph H , and the collection \mathcal{F} of subdivisions of K_5 and $K_{3,3}$, it follows from Kuratowski's [26] theorem that $\text{ex}(n, H, \mathcal{F})$ is equal to the maximum number of H -copies in an n -vertex planar graph.

In this paper we will consider a common generalization of the types of problems mentioned above. To this end, let $\text{ex}_{\mathcal{P}}(n, H, \mathcal{F})$ denote the maximum number of copies of H possible in an n -vertex planar graph containing no graph $F \in \mathcal{F}$ as a subgraph (we write simply $\text{ex}_{\mathcal{P}}(n, H, F)$ in the case $\mathcal{F} = \{F\}$). In particular, the problems considered in the preceding paragraphs correspond to the special cases of $\text{ex}_{\mathcal{P}}(n, K_2, F)$ and $\text{ex}_{\mathcal{P}}(n, H, \emptyset)$, for given graphs F and H .

2 Notation and Results

For a graph G , we denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. Also we let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a vertex $v \in V(G)$, the degree of v is denoted by $d(v)$, and the minimum degree of the graph G is denoted by $\delta(G)$. We use the notation C_t and K_t respectively for the cycle and complete graph of order t . We denote the path on t vertices (t -vertex path) by P_t . For a graph G and an independent set $S \subseteq V(G)$, the graph obtained by blowing S up by t is the graph formed by replacing each vertex $v \in S$ with t vertices each with the same neighbors as v . Note that in all of the results and conjectures we discuss the constants hidden in the asymptotic notation are allowed to depend on the graphs appearing in the $\text{ex}_{\mathcal{P}}$ function.

In the general case, Eppstein [10] asked whether for all H , the maximum number of copies of H possible in a minor-closed family of n -vertex graphs is an integer power of n . We state a restricted version of his problem as a conjecture in the planar case, and later generalize it further.

Conjecture 1. For every graph H , there exists a non-negative integer k , such that

$$\text{ex}_{\mathcal{P}}(n, H, \emptyset) = \Theta(n^k).$$

Remark 2. Conjecture 1 has now been resolved in a more general setting of subgraphs in an arbitrary surface by Huynh, Joret and Wood [23] (see also [24] for further work in general settings).

We verify this conjecture in the case of trees. Moreover, we prove a generalization in the case the graph is assumed to have no short even cycles. To state our results we require some notation.

Definition 3. For a graph H and an integer $i, i \geq 1$, let $\beta_i(H)$ be the maximum number of components in an induced subgraph of H containing only two types of components: (1) isolated vertices which have degree one in H and (2) paths of length $i - 1$ consisting only of vertices of degree two from H .

In particular in the case when $i = 1$, we are interested in the maximum size of an independent set in the graph consisting of vertices of degree at most 2. For simplicity, we let $\beta(G) := \beta_1(G)$ for any graph G . Then we have the following result for trees.

Theorem 4. *Let T be a tree, then*

$$\text{exp}(n, T, \emptyset) = \Theta(n^{\beta(T)}).$$

For any graph H , let $\alpha(H)$ be the independence number of H . In the general case we have the following upper bound.

Theorem 5. *Let H be any graph, then*

$$\text{exp}(n, H, \emptyset) = O(n^{\alpha(H)}).$$

This theorem will follow as an immediate consequence of results we prove about degenerate graphs in Section 3. As a corollary of Theorem 5 we obtain the order of magnitude of the maximum number of cycles (note that this result was already obtained by Hakimi and Schmeichel in [21]).

Corollary 6. *For all $k \geq 3$, we have*

$$\text{exp}(n, C_k, \emptyset) = \Theta(n^{\lfloor k/2 \rfloor}).$$

The lower bound is attained by taking a cycle C_k and blowing up a maximum-size independent set by $\lfloor 2n/k \rfloor - 1$. Note that the constant in the asymptotic notation may depend on k , and this construction contains asymptotically $\left(\frac{2n}{k}\right)^{\lfloor k/2 \rfloor}$ copies of C_k .

Next we consider the case when the set of forbidden graphs is nonempty. In this case, we pose a conjecture which generalizes Conjecture 1.

Conjecture 7. For all finite sets of graphs \mathcal{F} and for all graphs H , we have

$$\text{exp}(n, H, \mathcal{F}) = \Theta(n^k),$$

for some integer k .

We consider a variation of Theorem 4 for the case when $C_4, C_6, \dots, C_{2\ell}$ are forbidden. We prove the following.

Theorem 8. *For any tree T , we have*

$$\text{exp}(n, T, \{C_4, C_6, \dots, C_{2\ell}\}) = \Theta(n^{\beta_\ell(T)}).$$

The lower bound in Theorem 8 is given as follows. Take an induced subgraph of T consisting of $\beta_\ell(T)$ components as described in Definition 3. Replace each path (including the ones of length 0) by $\Omega(n)$ paths of the same length with endpoints joined to the same neighbors as the corresponding paths in T and number of vertices summing to n . The resulting graph has $\Omega(n^{\beta_\ell(T)})$ copies of T and contains no cycle in the set $\{C_4, C_6, \dots, C_{2\ell}\}$. In fact, we believe that a construction of this form should yield the correct asymptotic value of $\text{ex}_{\mathcal{P}}(n, T, \{C_4, C_6, \dots, C_{2\ell}\})$, but our proof only yields the order of magnitude. We also have the following exact result for maximizing the number of C_5 copies in a C_4 -free planar graph.

Theorem 9. *For all $n \geq 4$, $n \neq 6$, we have*

$$\text{ex}_{\mathcal{P}}(n, C_5, \{C_4\}) = n - 4.$$

Moreover, we determine the order of magnitude of $\text{ex}_{\mathcal{P}}(n, C_k, \{C_4\})$ for every k . We obtain the following result.

Theorem 10. *For all $k \geq 5$, we have*

$$\text{ex}_{\mathcal{P}}(n, C_k, \{C_4\}) = \Theta(n^{\lfloor k/3 \rfloor}).$$

We conjecture that in fact a much more general result holds.

Conjecture 11. *For sufficiently large k , we have*

$$\text{ex}_{\mathcal{P}}(n, C_k, \{C_4, C_6, \dots, C_{2\ell}\}) = \Theta(n^{\lfloor \frac{k}{\ell+1} \rfloor}).$$

A construction for a lower bound in Conjecture 11 is similar to that of Theorem 8. Namely, we note that $\beta_\ell(C_k) = \lfloor \frac{k}{\ell+1} \rfloor$, and replace each of the $\beta_\ell(C_k)$ paths with $\Omega(n)$ paths of the same length joined to the corresponding pair of vertices. The proof of Theorem 10 can be adapted to resolve Conjecture 11 in the cases when k is congruent to 0, 1 or 2 modulo $\ell + 1$.

We conclude this section by contrasting our results in the planar case with the known results in the general case. It was shown in [15] and [13] that $\text{ex}(n, C_k, C_4) = \Theta(n^{k/2})$. This result is in stark contrast to our results in the planar case in two ways. First, in the planar case the order of magnitude is always an integer power of n , and second in the planar case we have $k/3$ rather than $k/2$ in the exponent.

3 General Upper Bounds for Degenerate Graph Classes

Alon and Shikhelman [2] proved that for any bipartite graph H and tree T we have $\text{ex}(n, H, T) = O(n^{\alpha(H)})$, where $\alpha(H)$ is the independence number of H . This result was extended to all graphs H in [20]. Since the extremal number, $\text{ex}(n, T)$, of a any tree T is linear in n , it follows that any T -free graph has a vertex of degree at most c_T , a constant depending on T . We say that a graph G is c -degenerate if every subgraph of G contains

a vertex of degree at most c . The proof from [20] can easily be extended to work for the class of c -degenerate graphs. We now present a proof of this theorem for completeness.

First we introduce some notation. For given graphs G and H , let $\mathcal{N}(H, G)$ denote the number of copies of H in G . Let \mathcal{G}_c denote the class of c -degenerate graphs, and let

$$f_c(n, H) := \max\{\mathcal{N}(H, G) : G \in \mathcal{G}_c, v(G) = n\}.$$

Proposition 12. $f_c(n, K_r) = O(n)$, where the hidden constant depends only on r and c .

Proof. We proceed by induction on r . For $r = 1$ the result is clear, so assume $r > 1$ and that $f_c(n, K_{r-1}) \leq C_{r-1}n$ for a constant C_{r-1} . Let G be an n -vertex graph in \mathcal{G}_c , then we have that

$$r\mathcal{N}(K_r, G) = \sum_{v \in V(G)} \mathcal{N}(K_{r-1}, G[N(v)]) \leq \sum_{v \in V(G)} C_{r-1}d(v) = O(e(G)) = O(n). \quad \square$$

We will also make use of the following well-known fact.

Proposition 13. Let H be a graph, and let u be a vertex of H . If H' is the graph obtained from H by removing u together with its neighborhood, then $\alpha(H') \leq \alpha(H) - 1$.

Proof. If X is a maximal independent set in H' , then since no neighbor of u is in X , the set $X \cup \{u\}$ is independent in H and so $\alpha(H') + 1 \leq \alpha(H)$. \square

To prove Theorem 5 we proceed by induction on $\alpha(H)$. The base case when $\alpha(H) = 1$ follows from Proposition 12 (or by the results in [1, 35]). Now consider an arbitrary graph H and assume the theorem holds for graphs with independence number less than $\alpha(H)$. The theorem will follow immediately from the following claim.

Claim 14.

$$f_c(n+1, H) - f_c(n, H) = O(n^{\alpha(H)-1}).$$

Here, the constant hidden in the O notation depends only on H and c .

Proof of Claim 14. We proceed by induction on $\alpha(H)$, the case when $\alpha(H) = 1$ is trivial. To estimate $f_c(n+1, H) - f_c(n, H)$, we will start with a graph $G \in \mathcal{G}_c$ on $n+1$ vertices with the maximum number of copies of H . We know that $\delta(G) \leq c$. Let v be a vertex of minimum degree in G . We will estimate the number of copies of H in G containing v .

Let $V(H) = \{u_1, u_2, \dots, u_{v(H)}\}$, and let H_i be the graph obtained by removing u_i together with its neighbors. By Proposition 13, we know that $\alpha(H_i) \leq \alpha(H) - 1$. Now for each copy of H using v , the vertex v must play the role of some u_i , and the neighbors of u_i must be embedded in the neighborhood of v . It follows that the other vertices of H , that is the vertices of H_i , must be embedded in some way in the remaining vertices of G . We have to choose $d_H(u_i)$ vertices in $N(v)$, so the number of copies of H using v is at most

$$\sum_{i=1}^{v(H)} d(v)^{d_H(u_i)} \mathcal{N}(H_i, G) \leq \sum_{i=1}^{v(H)} c^{d_H(u_i)} \mathcal{N}(H_i, G) = \sum_{i=1}^{v(H)} O_{H_i}(n^{\alpha(H_i)}) = O(n^{\alpha(H)-1}).$$

Here we have used the notation O_{H_i} to emphasize that the constant may depend on the graph H_i . The second to last equality follows by induction and the last one follows from Proposition 13.

Thus, if G' is the graph obtained from G by removing v , we have that

$$f_c(n+1, H) = \mathcal{N}(H, G) = \mathcal{N}(H, G') + O(n^{\alpha(H)-1}) \leq f_c(n, H) + O(n^{\alpha(H)-1}). \quad \square$$

4 The Number of Trees in Planar Graphs

In this section we prove Theorem 4. First we provide the lower bound, $\text{exp}_{\mathcal{P}}(n, T, \emptyset) = \Omega(n^{\beta(T)})$. Observe that, if a given graph is planar and one blows up a set of independent vertices each of degree at most two, then resulting graph is also planar. Therefore the following construction provides the desired lower bound. Given a tree T , fix an independent set S of size $\beta(T)$ which contains vertices of degree at most two (as in Definition 3) and blow up this set by $\lfloor \frac{n}{2\beta(T)} \rfloor$. The resulting graph is planar with at most n vertices, when n is sufficiently large, and contains $\Omega(n^{\beta(T)})$ copies of the tree T .

Observe that we have $\text{exp}_{\mathcal{P}}(n, P_k, \emptyset) = O(n^{\alpha(P_k)})$ from Theorem 5, where P_k is a k -vertex path. Even more we have $\alpha(P_k) = \beta(P_k)$ from Definition 3. Therefore we have the following simple proposition.

Proposition 15. $\text{exp}_{\mathcal{P}}(n, P_k, \emptyset) = \Theta(n^{\beta(P_k)})$.

Hence Theorem 4 holds for paths. To show that Theorem 4 holds for any tree, we are going to use the following lemma.

Lemma 16. *For a given planar graph G , let v , u and w be fixed vertices in G and let n_1 , n_2 and n_3 be nonnegative integers. The number of vertices x , such that, there are three internally vertex disjoint paths from x to v , from x to u and from x to w of length n_1 , n_2 and n_3 , respectively, is bounded by a constant $C := C(n_1, n_2, n_3)$.*

Proof. Suppose we have a planar embedding of G .

The proof will be by induction on $n_1 + n_2$. The result is trivial if either n_1 or n_2 is equal to 0. So suppose that $n_1 + n_2 \geq 2$, and that the result holds for any pair with smaller sum. Consider a maximal set \mathcal{P} , of internally vertex disjoint paths $v, v_2^i, \dots, v_{n_1}^i, a_i, u_2^i, \dots, u_{n_2}^i, u$, where each a_i is such that there exists a length n_3 path from a_i to w which does not contain any of the vertices $v, v_2^i, \dots, v_{n_1}^i, u_{n_2}^i, \dots, u_2^i, u$. Let us denote the set of a_i in these paths by A . Observe that the paths from \mathcal{P} divide the plane into $|A|$ regions $R_1, R_2, \dots, R_{|A|}$. Since the vertex w is fixed, it is in one of the regions, and there is a path of length n_3 from w to each vertex of A , not using the vertices v and u . Thus $|A| \leq 2n_3 + 1$.

Now let Y to be the set of $|A|(n_1 + n_2 - 1) + 2$ vertices that appear in some path from \mathcal{P} , and let X be the set of those vertices x in G which are not in Y such that there exist three internally vertex disjoint paths from x to v , from x to u and from x to w of length n_1 , n_2 and n_3 , respectively. It is sufficient to bound $|X| + |Y|$ by a constant depending

on n_1, n_2 and n_3 . If $X = \emptyset$ we immediately have the required bound. Suppose X is nonempty and let $x \in X$, and let $P_1 = v, v_2, \dots, v_{n_1}, x$ and $P_2 = x, u_{n_2}, \dots, u_2, u$ be two of the three internally vertex disjoint paths from x . Let v' and u' be the first vertex (closest to x in P_i) in the intersection of Y with P_1 and P_2 , respectively. Note that it is possible for v' to be v or u' to be u , but by the definition of Y and \mathcal{P} , it is not possible for both to happen simultaneously. Then the vertex x is such that there exist three internally vertex disjoint paths from x to v' , from x to u' and from x to w of length n'_1, n'_2 and n_3 respectively, where $1 \leq n'_i \leq n_i$ for $i = 1, 2$ and $1 \leq n_3$, with the additional property that $n'_1 + n'_2 < n_1 + n_2$. Therefore, setting $C' = \max_{n'_1+n'_2 < n_1+n_2} C(n'_1, n'_2, n_3)$, we have that

$$|X| \leq \binom{|A|(n_1 + n_2 - 1) + 2}{2} C' \leq \binom{(2n_3 + 1)(n_1 + n_2 - 1) + 2}{2} C'.$$

Thus, $|X| + |Y|$ is bounded and so the lemma holds. \square

Note that Lemma 16 implies in particular that if G is a planar graph and T is a tree with $s \geq 3$ leaves x_1, x_2, \dots, x_s , then for any vertex $x \in V(T)$ of degree at least 3 and $v_1, v_2, \dots, v_s \in V(G)$, the number of vertices $v \in V(G)$, such that, there exists a copy of T where x is embedded at v and x_i is embedded at v_i , for $i = 1, \dots, s$, is bounded by a constant that does not depend on G . This follows since we are able to find three different leaves such that the paths from x to each of these leaves are internally vertex disjoint.

At this point we are ready to prove Theorem 4.

Proof of Theorem 4. We may assume T is not a path otherwise we are done, from Proposition 15.

Let G be an n -vertex planar graph. Let A be the set of vertices of degree at least 3 in T , and let T_1, T_2, \dots, T_k be the connected components of the graph induced by $V(T) \setminus A$ (the set A is non-empty since T is not a path). Observe that since A has every vertex of degree at least 3, then every vertex of T_i has degree at most 2 in both T_i and T , so we have

$$\beta(T) = \beta\left(\bigcup_{i=1}^k T_i\right) = \sum_{i=1}^k \beta(T_i).$$

Moreover the trees T_i are paths and so $\mathcal{N}(T_i, G) = O(n^{\beta(T_i)})$, from Proposition 15. Then for any embedding of the trees T_i by Lemma 16, there is a constant number of ways to complete the embedding of the set A to a copy of T . Therefore the number of copies of T is bounded by $O(n^{\sum_{i=1}^k \beta(T_i)}) = O(n^{\beta(T)})$. \square

5 The number of C_5 's in C_4 -free Planar Graphs

In this section we are going to prove Theorem 9, namely that for all $n, n \geq 4, n \neq 6$, we have

$$\text{exp}_{\mathcal{P}}(n, C_5, \{C_4\}) = n - 4.$$

Proof. We begin by providing the lower bound. Let $n = 5 + 3t + 2s$ for some nonnegative integers s and t . (Note that when $n = 6$, it is easy to verify that there can be at most one pentagon, thus $\text{ex}_{\mathcal{P}}(6, C_5, \{C_4\}) = 1$.) The construction is as follows: Take a pentagon x_1, x_2, x_3, x_4, x_5 , as well as t internally vertex disjoint paths $x_1, y_3^i, y_4^i, y_5^i, x_2$ (where $1 \leq i \leq t$ and $y_j^i \notin \{x_3, x_4, x_5\}$, $j \in [3]$) between x_1 and x_2 and add the edges so that x_4, y_4^1, \dots, y_4^t forms a path. Next take a path z_1, z_2, \dots, z_{2s} on new set of vertices. Add the edge from z_1 to x_1 and the edges from z_i to x_5 , for odd $i \equiv 0, 1 \pmod{4}$, and z_i to x_3 , for $i \equiv 2, 3 \pmod{4}$. See Figure 1 for an example of an extremal graph.

Observe that the 5-cycles in the construction are either x_1, x_2, x_3, x_4, x_5 or the 5-cycles containing y_4^i , $1 \leq i \leq t$ or the 5-cycles containing an edge $\{z_i, z_{i+1}\}$, $1 \leq i \leq s$ and $i \equiv 1, 3 \pmod{4}$.

For $1 \leq i \leq t$, we have three distinct 5-cycles containing y_3^i in the construction. These are $x_2, y_3^i, y_4^i, y_4^{i-1}, y_3^{i-1}$ or $x_1, y_5^i, y_4^i, y_4^{i-1}, y_5^{i-1}$ or $x_2, y_3^i, y_4^i, y_5^i, x_1, x_2$, where y_3^{i-1}, y_4^{i-1} and y_5^{i-1} are respectively x_3, x_4 and x_5 when $i = 1$. Thus we have $3t$ 5-cycles in the construction containing y_4^i , $1 \leq i \leq t$.

On the other hand for $1 \leq i \leq s$ and $i \equiv 1, 3 \pmod{4}$, each edge $\{z_i, z_{i+1}\}$ is contained in two distinct 5-cycles of the construction. Indeed, for an edge $\{z_i, z_{i+1}\}$ ($i \equiv 1 \pmod{4}$), notice that z_{i+1} and z_{i-2} are adjacent with x_3 and z_i is adjacent with x_5 . Hence we have the 5-cycles $x_3, z_{i+1}, z_i, x_5, x_4$ and $x_3, z_{i+1}, z_i, z_{i-1}, z_{i-2}$ which contain the edge $\{z_i, z_{i+1}\}$. In the case that $i = 1$, we take x_1 and x_2 in place of z_{i-1} and z_{i-2} respectively. For an edge $\{z_i, z_{i+1}\}$ ($i \equiv 3 \pmod{4}$), observe that z_{i+1} and z_{i-2} are adjacent with x_5 and z_i is adjacent with x_3 . In this scenario, we have two distinct 5-cycles containing the edge $\{z_i, z_{i+1}\}$, namely $x_5, z_{i+1}, z_i, z_{i-1}, z_{i-2}$ and $x_5, z_{i+1}, z_i, x_3, x_4$. Thus, we have $2s$ 5-cycles in the construction containing $\{z_i, z_{i+1}\}$, where $1 \leq i \leq s$. Therefore, the number of 5-cycles in the construction is $3t + 2s + 1 = n - 4$.

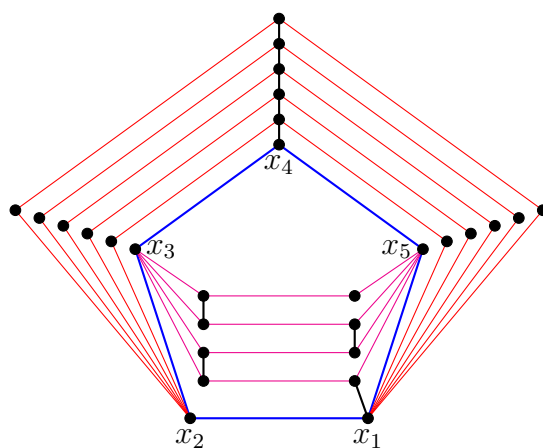


Figure 1: Example of an extremal graph for Theorem 9.

Now we are going to prove by induction that $\text{ex}_{\mathcal{P}}(n, C_5, \{C_4\}) \leq n - 4$. The proof proceeds by induction on n with the base cases $\text{ex}_{\mathcal{P}}(4, C_5, \{C_4\}) = 0$, $\text{ex}_{\mathcal{P}}(5, C_5, \{C_4\}) = 1$ and $\text{ex}_{\mathcal{P}}(6, C_5, \{C_4\}) = 1$. Let G be an n vertex, C_4 -free planar graph with $n \geq 7$. Without

loss of generality we may assume that G is connected. Consider an embedding of G on the plane. To prove an upper bound for Theorem 9, we take a planar embedding of G and consider two cases.

Case 1: *All pentagons in G are face.* We can assume that G is a connected plane graph. Otherwise, we can still join the components of G with edges so that there is no new cycle created.

Remove an edge from each triangular face. Observe that two triangular faces do not share an edge since G is C_4 -free. Let us denote the resulting graph by G' . Since there is no pentagon in G' which is not a face, the total number of faces of G' is at least the number of pentagonal faces in G .

Let f_i denote the number of faces of size i in G' , and let f denote the number of faces in G . Since G' is connected and $n \geq 7$, $f_1 = f_2 = 0$. Moreover, since G' contains no triangular face and G' contains no C_4 we have $f_3 = f_4 = 0$. Thus,

$$2e(G') = \sum_{i \geq 5} i f_i = 5f_5 + \sum_{i \geq 6} i f_i \geq 5f_5 + 5(f - f_5) = 5f.$$

This implies $5f \leq 2e(G')$. Using the Euler's formula, $f + n = e(G') + 2$, we get $f \leq \frac{2}{3}n - \frac{4}{3}$. Thus we have that the number of C_5 copies in G is at most $n - 4$, since $n \geq 7$.

Case 2: *There is a pentagon in G which is not a face.* Let P be a non-facial pentagon in G . The pentagon P cuts the plane into two regions. Let us denote the subgraph of G in the inner and outer regions of P by G_1 and G_2 , respectively, where both graphs include the vertices of the pentagon. Assume that G_1 and G_2 have n_1 and n_2 vertices respectively. Thus we have $n = n_1 + n_2 - 5$. Since n_1 and n_2 are both non-zero and less than n and both G_1 and G_2 are C_4 -free, then by induction hypothesis, the number of pentagons in G_1 and G_2 is at most $n_1 - 4$ and $n_2 - 4$, respectively.

It can be checked that there is no pentagon crossing P in G . Indeed, let $P = x_1, x_2, x_3, x_4, x_5$. Notice that since G is C_4 -free, no two non-consecutive vertices of P are adjacent. Suppose that there is a pentagon, say P' , crossing P . P' must contain a cherry x_i, y, x_j where y is a vertex either in the interior or in the exterior region of P . Without loss of generality suppose y is in the interior region of P . Notice that since G is C_4 -free, x_i and x_j can not be non-consecutive vertices of the pentagon P . Thus, we assume that x_i and x_j are adjacent vertices of P . Without loss of generality assume that $i = 1$ and $j = 2$. Since any two non-consecutive vertices of P are not adjacent, there are two vertices say, y_1 and y_2 in the exterior region of P such that $P' = x_1, y, x_2, y_1, y_2$. However, in this case we got a C_4 , namely x_1, x_2, y_1, y_2 , which is a contradiction to the fact that G is C_4 -free.

From the above observation it follows that every pentagon of G is a pentagon of G_1 or G_2 . Since the pentagon P is in both graphs, we get that the number of pentagons in G is at most $n_1 - 4 + n_2 - 4 - 1 = n + 5 - 9 = n - 4$, completing the proof. \square

6 The Order of Magnitude of the Maximum Number of Cycles of Length k in a C_4 -free Planar Graph

We have seen in Corollary 6 that $\text{ex}_{\mathcal{P}}(n, C_k, \emptyset) = \Theta(n^{\lfloor k/2 \rfloor})$ follows immediately from Theorem 5. We now consider the case when C_4 is forbidden and prove Theorem 10 which states that $\text{ex}_{\mathcal{P}}(n, C_k, \{C_4\}) = \Theta(n^{\lfloor k/3 \rfloor})$.

Proof. For the construction we take a cycle C_k and find an induced matching of size $\lfloor k/3 \rfloor$. Next we replace each edge in this matching with $\frac{n-k}{2\lfloor k/3 \rfloor}$ edges each adjacent to the same pair of vertices as the original edge. This graph clearly has $\Theta(n^{\lfloor k/3 \rfloor})$ copies of C_k and at most n vertices, when n is sufficiently large.

We will now prove the upper bound. It is well-known that every planar graph contains a vertex of degree at most 5. It was proved in [25] that a C_4 -free planar graph with minimum degree at least 2 contains an edge $\{x, y\}$ such that $d(x) + d(y)$ is at most 8. This result was improved to 7 in [5], which is best possible. Note that, in proving Theorem 10, we may delete all edges containing a vertex of degree at most one, since such vertices do not contribute to any k -cycles.

We will distinguish cases based on the value of k modulo 3. When k is equal to 0 or 1 modulo 3, the result can be proved using the fact that a planar graph contains a vertex of degree at most 5. We present here the proof in the case when $k \equiv 2 \pmod{3}$, the other cases are similar but require only the fact that there is a vertex of bounded degree.

Suppose $k = 3m + 2$ for some integer $m \geq 1$ and that G is an n -vertex, C_4 -free planar graph with no vertex of degree at most 1. Let us label the vertices of C_k by $v_1, v_2, \dots, v_{3m+2}$, consecutively. Applying the aforementioned result of [5] we find an edge $\{u, v\}$ such that $d(u) + d(v) \leq 7$. We will show that at most n^{m-1} cycles C_k can use the edge $\{u, v\}$. Then, by iteratively deleting such an edge and the resulting vertices of degree 1, we find all copies of C_k after at most $O(n)$ steps (since there are at most linearly many edges in a planar graph). Thus in total we will have shown that there are at most n^m copies of C_k in G , as required.

Let us consider a copy of C_k so that u and v correspond to v_1 and v_2 in the C_k . There is at most 7 ways to embed v_{3m+2} and v_3 as neighbors of v_1 and v_2 , respectively. Thus we now consider v_{3m+2}, v_1, v_2 and v_3 as being embedded. Next choose edges of G at which to embed the edges of the cycle $\{v_5, v_6\}, \{v_8, v_9\}, \{v_{11}, v_{12}\}$ and so on. We now consider embedding the remaining vertices. Since our graph is C_4 -free, there is at most one way to embed the vertices v_4, v_7, v_{10} and so on. It follows that we have at most on the order of n^{m-1} copies of the cycle C_k in the graph which use the edge $\{u, v\}$, and we are done. \square

7 The Number of Trees in Planar Graph with no even cycle of length at most 2ℓ

In this section we prove Theorem 8, namely that

$$\text{ex}_{\mathcal{P}}(n, T, \{C_4, C_6, \dots, C_{2\ell}\}) = \Theta(n^{\beta_{\ell}(T)}).$$

Proof of Theorem 8. First we will show that the result is true for paths. We will make use of the following theorem due to Lam and Verstraete.

Theorem 17 ([27]). *Let G be a graph containing no even cycles of length at most 2ℓ . There exists a constant D_ℓ such that for any vertices v, u of G and $k \leq \ell$ a positive integer, the number of paths from v to u of length k in G is at most D_ℓ .*

It is simple to check that $\beta_\ell(P_k) = 1 + \lfloor \frac{k+\ell-2}{\ell+1} \rfloor$. Now we will prove that

$$\text{ex}_{\mathcal{P}}(n, P_k, \{C_4, C_6, \dots, C_{2\ell}\}) = O\left(n^{1+\lfloor \frac{k+\ell-2}{\ell+1} \rfloor}\right).$$

Let G be an n -vertex planar graph containing no even cycle of length at most 2ℓ . We will bound the number of k -vertex paths in G by fixing a particular k -vertex path, v_1, v_2, \dots, v_k , and bounding the number of ways it can be embedded in G . Suppose $k \geq 3$ (otherwise we are already done), and fix the images of the edges $\{v_{(\ell+1)i+1}, v_{(\ell+1)i+2}\}$ for $i = 0, 1, \dots, \lfloor \frac{k+\ell-2}{\ell+1} \rfloor - 1$ and the edge $\{v_{k-1}, v_k\}$ in the embedding. With these edges embedded, by Theorem 17 there is a constant number of ways to embed the portion the paths between each pair of consecutive edges which we enumerated. We notice that we have a linear number of choices for each of the edges which we specified, and in total the number of choices for these edges is of order $n^{1+\lfloor \frac{k+\ell-2}{\ell+1} \rfloor}$. Therefore $\mathcal{N}(G, P_k) = O\left(n^{1+\lfloor \frac{k+\ell-2}{\ell+1} \rfloor}\right)$.

Now let T be any tree. Let us partition the vertex set $V(T)$ into five sets A_1, A_2, A'_2, A''_2 and $A_{\geq 3}$. First we partition the set of vertices of degree not equal to 2 as follows.

$$A_1 = \left\{v \in V(T) \mid d(v) = 1\right\} \text{ and } A_{\geq 3} = \left\{v \in V(T) \mid d(v) \geq 3\right\}.$$

In particular, A_1 is the set of leaves of the tree T . Now we will partition the set of vertices of degree equal to 2 into three sets. Let

$$A_2 = \left\{v \in V(T) \mid d(v) = 2 \text{ and there is no vertex of } A_{\geq 3} \text{ at distance less than } \ell \text{ from } v\right\}.$$

Now consider every path P in T such that:

- (i) Both end-vertices of P have degree at least 3 in T .
- (ii) The length of P is at least $\ell + 1$, but at most $2\ell - 1$.
- (iii) Every internal vertex of P has degree 2 in T .

For each such path let $f(P)$ be the middle vertex of P ; if P has odd length, take either of the two middle vertices.

Let A'_2 be the set of consisting of the vertices $f(P)$ for the paths P defined above. Finally, define

$$A''_2 = \left\{v \in V(T) \mid d(v) = 2, v \notin A_2 \cup A'_2\right\}.$$

Let F be the subgraph of T induced by the vertex set $V_1 = A_1 \cup A_2 \cup A'_2$. Note that F is a path forest and suppose $F = P_{i_1} \cup P_{i_2} \cup \dots \cup P_{i_t}$ for paths P_{i_j} , $1 \leq j \leq t$.

Now we will show that $\beta_\ell(F) = \beta_\ell(T)$. Take a set S of vertices and paths which is a witness for the value of $\beta_\ell(T)$ in T . Suppose S contains a path $P = x_1, x_2, \dots, x_\ell$ using at least one vertex from A_2'' . Note that it is not possible for this path to be fully contained in A_2'' . Indeed, if it was contained in A_2'' we would be able to find an i such that x_i is at distance $\ell - 1$ from a vertex $v \in A_{\geq 3}$ and x_{i+1} is at distance $\ell - 1$ from a vertex $u \in A_{\geq 3}$ and so P would be contained in a path of length $2\ell + 1$ from v to u . It follows that the middle vertex of this path must be a vertex of P . So we may replace the choice of P in S with a vertex (leaf) $f(P)$ in F . We obtain that $\beta_\ell(F) \geq \beta_\ell(T)$.

Next we show $\beta_\ell(F) \leq \beta_\ell(T)$. Take a set S of vertices and paths which is a witness for $\beta_\ell(F)$ in F . If P is a length ℓ path in S , by definition, no end-vertex is a leaf in F , so P is still a length ℓ path in T such that every vertex has degree 2. If v is a leaf in F , but is no longer a leaf in T , then v must have degree 2 in T , and there are two possibilities. Suppose $v \in A_2''$, then there exists a path P of length at least $\ell + 1$ with internal vertices of degree 2 containing v . In this case we may replace the choice of v in S by a subpath of P of length $\ell - 1$ without using the end-vertices of P . Suppose v is in A_2 , since one neighbor of v is not in F , it must be in $A_{\geq 3} \cup A_2''$, but by definition of A_2 , then v must have a neighbor in A_2 and so v is at distance ℓ of $A_{\geq 3}$. Let u be the closest neighbor of v in u , and let P be the length ℓ path from v to u , then we may replace the choice of v in S by P' the path obtained from P by deleting u . It follows that $\beta_2(F) \leq \beta_2(T)$.

Now let G be an n -vertex planar graph containing no even cycle of length at most 2ℓ and fix a copy of F in G . By Lemma 16 since every leaf of T is already fixed, we have a bounded number of choices to embed the vertices of $A_{\geq 3}$ in G such that together with the copy of F the embedding can be completed to a copy of T . For any given embedding of F and $A_{\geq 3}$, let $x \in A_2''$. By the definition of A_2'' , there is a vertex $a \in A_{\geq 3}$ with distance less than ℓ to x . If there are two choices for a , pick one which is closest to x , on the branch from x that does not contain a , and let b be the closest vertex of $V \setminus A_2''$. We show that the distance between a and b is at most ℓ . Suppose by contradiction b is at distance more than ℓ from a , then pick c to be the vertex between x and b at distance exactly ℓ from a . It follows that $c \in A_2''$, so there is a vertex $d \in A_{\geq 3}$ in the same branch from x as b and c , at distance at most ℓ from c . Hence the path from a to d has length at most $2\ell - 1$ and its middle vertex y is in F and it is closer to x than y , but this contradicts the choice of b since y is also in the branch from x not containing a . By Theorem 17, there is a constant number of choices to embed x in G such that the embedding can be completed to T , since for each such embedding we have a path of length at most ℓ from the vertices corresponding to a and b in G . \square

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