

Paths of Length Three are K_{r+1} -Turán-Good

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Abstract

The generalized Turán problem $\text{ex}(n, T, F)$ is to determine the maximal number of copies of a graph T that can exist in an F -free graph on n vertices. Recently, Gerbner and Palmer noted that the solution to the generalized Turán problem is often the original Turán graph. They gave the name “ F -Turán-good” to graphs T for which, for large enough n , the solution to the generalized Turán problem is realized by a Turán graph. They prove that the path graph on two edges, P_2 , is K_{r+1} -Turán-good for all $r \geq 3$, but they conjecture that the same result should hold for all P_ℓ . In this paper, using arguments based in flag algebras, we prove that the path on three edges, P_3 , is also K_{r+1} -Turán-good for all $r \geq 3$.

Mathematics Subject Classifications: 05C35, 05C38

1 Introduction

One of extremal graph theory’s most celebrated results was introduced in [27] by Turán who asked how many edges a (simple) graph on n vertices can contain if it has no clique containing $r+1$ vertices. Turán’s solution, which we denote $\text{ex}(n, K_{r+1})$, is asymptotically $(1 - \frac{1}{r})\binom{n}{2}$. Additionally, Turán showed that the unique extremal graph is the complete r -partite graph on n vertices with parts of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$ (so that no pair of parts differs in size by more than one). We call this graph the *Turán graph* and denote it $T_r(n)$.

The first extensions to Turán’s theorem considered forbidding graphs other than cliques. For any graph F , we say a graph G is F -free if it contains no (not necessarily induced) subgraph isomorphic to F . We use $\text{ex}(n, F)$ to denote the maximal number of edges in an F -free graph on n vertices. The general case is solved asymptotically by the Erdős-Stone-Simonovits Theorem [9] which proves

$$\text{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}.$$

To further generalize the problem, one may consider counting subgraphs other than edges. Let $\nu(T, G)$ denote the number of distinct, not necessarily induced subgraphs of G isomorphic to T . We denote by $\text{ex}(n, T, F)$ the maximum of $\nu(T, G)$ over all F -free graphs G on n vertices. (Here T is the “target” graph while F is “forbidden.”) The first question of this form to be resolved was due to Zykov in 1949 [28] who determined the value of the function $\text{ex}(n, K_t, K_r)$ when $t < r$ by proving that the Turán graph is the unique extremal graph.

Theorem 1 (Zykov [28]). *Let r and t be integers such that $t < r$. Then for all n , the Turán graph $T_t(n)$ is the unique K_r -free graph on n vertices containing the maximum number of K_t subgraphs.*

Several sporadic cases were investigated (see, for example, [6, 15]) before 2015 when Alon and Shikhelman introduced a systematic study in [2] in which they determine, among other results, that for forbidden graphs F with $\chi(F) = k + 1 > r$,

$$\text{ex}(n, K_r, F) = (1 + o(1)) \binom{k}{r} \left(\frac{n}{k}\right)^r.$$

A more precise result can be found in [22]. Since then, the area has been widely studied; see [8, 11, 16, 20, 21] for an (incomplete) sampling of authors and results.

As in the original Zykov result, for many choices of T and F the Turán graph emerges as the optimal graph, at least for large enough n . In [12], Gerbner and Palmer introduced the term F -Turán-good to describe such target graphs T :

Definition 2. Fix an $(r + 1)$ -chromatic graph F and a graph T that does not contain F as a subgraph. We say that T is F -Turán-good if $\text{ex}(n, T, F) = \nu(T, T_r(n))$ for every n large enough.

In the same paper, Gerbner and Palmer prove that the path graph on ℓ edges, P_ℓ , is K_{r+1} -Turán-good for $\ell = 2$ and $r \geq 3$. They conjecture that paths should be Turán-good for all choices of r and ℓ . In this paper we establish that P_3 , the path on three edges, is K_{r+1} -Turán-good for all $r \geq 3$.

To be precise, define the *density* of H in G to be

$$d(H, G) = \nu(H, G) \binom{|G|}{|H|}^{-1}$$

and let $\mathcal{F}_{n,r}$ be the family of K_{r+1} -free graphs on n vertices. We define

$$\text{OPT}_r(P_3) = \lim_{n \rightarrow \infty} \max_{G_n \in \mathcal{F}_{n,r}} d(P_3, G_n).$$

Then the following theorem is the primary result of this paper:

Theorem 3. *For any integer $r \geq 3$,*

(i) $\text{OPT}_r(P_3) = 12 \left(\frac{r-1}{r}\right)^3.$

(ii) *If n is sufficiently large, then P_3 is K_{r+1} -Turán good.*

Note that in [12], Gerbner and Palmer provided a proof of part (i) of Theorem 3. Part (ii) is an entirely new result. We will re-prove part (i) in the language of flag algebras, since we will require this proof to obtain part (ii).

In [11], Gerbner and Palmer proved that for two graphs T and F , where $\chi(F) = r$,

$$\text{ex}(n, T, F) \leq \text{ex}(n, T, K_r) + o(n^{|T|}).$$

Combined with Theorem 3, their theorem implies the following corollary.

Corollary 4. *For any graph F with chromatic number $r \geq 3$,*

$$\text{ex}(n, P_3, F) = \text{OPT}_r(P_3) \binom{n}{4} + o(n^4).$$

In the remainder of this section, we establish the conventions used thorough the paper, reference a few well-known results that will be of use throughout the proof, and then provide a brief introduction to the flag algebra method. Section 2 contains the flag algebra calculations we use to establish part (i) of Theorem 3. In Section 3 we establish a stability result, proving that near-extremal graphs have small edit distance from the Turán graph. Then in Section 4 we use that stability argument to show that the Turán graph is optimal for large enough n . We conclude in Section 5 with some thoughts on what this result means for Gerbner and Palmer's conjecture for general paths P_ℓ .

1.1 Background and Conventions

We use P_ℓ to denote the path graph with ℓ edges and $\ell + 1$ vertices. If a copy of P_3 in G is defined by the edges wx , xy and yz , then we will use $wxyz$ to denote it. Note that a set of four vertices in G will frequently give multiple distinct copies of P_3 . We use $wxyz$ for that specific ordering.



Figure 1: The path $wxyz$

We will need the following corollary of Theorem 1:

Corollary 5. *Let G be a K_{r+1} -free graph on n vertices. Then*

$$\nu(K_4, G) \leq \frac{r^3 - 6r^2 + 11r - 6}{r^3} \binom{n}{4} + o(n^4)$$

Proof. In the Turán graph $T_r(n)$, any set of four vertices inducing a copy of K_4 must come from four different partite sets. Thus there are

$$\binom{r}{4} \cdot \frac{n^4}{r^4} + o(n^4)$$

copies of K_4 in $T_r(n)$. The claim immediately follows. □

We will also need the following lemma from folklore characterizing multipartite graphs:

Lemma 6. *Define the co-cherry \overline{P}_2 to be the unique graph on three vertices with one edge. Then G is a complete multipartite graph if and only if it does not contain the co-cherry as an induced subgraph.*



Figure 2: The co-cherry

Proof. First, assume G is a complete multipartite graph and let $x, y, z \in V(G)$ such that x is adjacent to y but z is not adjacent to y . As G is complete multipartite, the only way z is not adjacent to y is if they are in the same vertex class. As x is adjacent to y , it must be in a different vertex class. Thus x and z do not share a vertex class and are adjacent, so $G[\{x, y, z\}]$ does not span a co-cherry.

Now let G be a graph that does not contain the co-cherry as an induced subgraph. Define a relation on $V(G)$ by $x \sim y$ if x is not adjacent to y . As G is simple, this relation is reflexive and symmetric, and if x is not adjacent to y and y is not adjacent to z , then x cannot be adjacent to z , as that would form an induced co-cherry, so the relation is transitive as well. Therefore this equivalence relation partitions the vertices of G into classes which contain no internal edges. Furthermore, two vertices from different classes are by definition adjacent and thus every edge between vertex classes is present. We conclude G is complete multipartite. \square

1.2 The Flag Algebra Method

Flag algebras were introduced by Razborov [24] as a tool to computationally solve problems in extremal combinatorics. In this section, we will introduce some main ideas necessary for our proof. For a complete overview see [24]. Flag algebras have been applied to study a variety of extremal problems on graphs [4, 5, 14, 17, 25] and hypergraphs [10, 13, 23], as well as oriented graphs [7, 18]. These only represent a handful of the many results in combinatorics which were obtained using flag algebras.

A *type* σ is a graph labeled by $[k]$. An *embedding* of σ into a graph F is an injective map $\theta : [k] \rightarrow V(F)$ so that $\text{im}(\theta)$ is isomorphic to σ . A σ -*flag* (F, θ) is a graph F together with an embedding θ of σ into $V(F)$. We will let \mathcal{F}^σ denote the set of all σ -flags up to isomorphism and \mathcal{F}_n^σ denote the associated subset containing all σ -flags on n vertices. If σ is the empty graph, then we will drop it from the notation and simply use \mathcal{F} to denote the set of all graphs, or \mathcal{F}_n to denote the set of all graphs on n vertices. As an example, if σ^* is the following labeled graph on two vertices,

$$\sigma^* = 2 \text{---} 1$$

then

$$\mathcal{F}_3^{\sigma^*} = \left\{ \begin{array}{c} \bullet \\ | \\ 1 \text{---} 2 \end{array}, \begin{array}{c} \bullet \\ / \backslash \\ 1 \text{---} 2 \end{array}, \begin{array}{c} \bullet \\ \backslash / \\ 1 \text{---} 2 \end{array}, \begin{array}{c} \bullet \\ / \backslash / \\ 1 \text{---} 2 \end{array} \right\}.$$

For a type σ labeled by $[k]$, two σ -flags (H, θ_1) and (G, θ_2) , and a set X_1 of size $|V(H)| - k$ selected uniformly at random from $V(G) \setminus \text{im}(\theta_2)$, $P((H, \theta_1), (G, \theta_2))$ is the probability that $X_1 \cup \text{im}(\theta_2)$ is isomorphic to (H, θ_1) . In the interests of completeness, if $|V(G)| < |V(H)|$, then we let $P(H, G) = 0$. If σ is the empty graph, then we will write $P(H, G)$ to mean $P((H, \theta_1), (G, \theta_2))$. In this case, the definition of $P(H, G)$ coincides with the standard notion of induced density. Using the same type σ^* from the previous example:

$$\text{If } H = \begin{array}{c} \bullet \\ \diagdown \\ \square_1 \text{---} \square_2 \end{array} \text{ and } G = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square_1 \text{---} \square_2 \end{array}, \text{ then } P((H, \theta_1), (G, \theta_2)) = \frac{1}{3}.$$

Now suppose that (J, θ_3) is another σ -flag. Let $X_1, X_2 \subseteq V(G)$ be two disjoint sets of size $|V(H)| - k$ and $|V(J)| - k$, respectively, selected uniformly at random from $V(G) \setminus \text{im}(\theta_2)$. Then $P((H, \theta_1), (J, \theta_3); (G, \theta_2))$ is the probability that $X_1 \cup \text{im}(\theta_2)$ is isomorphic to H and $X_2 \cup \text{im}(\theta_2)$ is isomorphic to J . Once again, if σ is empty, then we write $P(H, J; G)$ in place of $((H, \theta_1), (J, \theta_3); (G, \theta_2))$. Then (1) follows from the definition of $P((H, \theta_1), (J, \theta_3); (G, \theta_2))$.

$$|P((H, \theta_1), (J, \theta_3); (G, \theta_2)) - P((H, \theta_1), (G, \theta_2)) \cdot P((J, \theta_3), (G, \theta_2))| \leq O(|V(G)|^{-1}) \quad (1)$$

Thus, as the size of G tends toward infinity, we can assume that we select X_1 and X_2 independently.

Let $\mathbb{R}\mathcal{F}^\sigma$ be the set of all finite formal linear combinations of elements from \mathcal{F}^σ . For a given type σ , let \mathcal{K}^σ denote the linear subspace of $\mathbb{R}\mathcal{F}^\sigma$ generated by all elements of the form

$$F - \sum_{(H, \theta_2) \in \mathcal{F}_n^\sigma} P((F, \theta_1), (H, \theta_2)) \cdot (H, \theta_2)$$

where $|V(F)| < n$. Razborov showed that there exists an algebra $\mathcal{A}^\sigma = \mathbb{R}\mathcal{F}^\sigma / \mathcal{K}^\sigma$ with well-defined addition and multiplication. Addition is defined in the natural way by adding coefficients. For example, if $F_1, F_2 \in \mathcal{A}^{\sigma^*}$ such that

$$F_1 = 2 \cdot \begin{array}{c} \bullet \\ \diagdown \\ \square_1 \text{---} \square_2 \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \square_1 \text{---} \square_2 \end{array} \text{ and } F_2 = \begin{array}{c} \bullet \\ \diagdown \\ \square_1 \text{---} \square_2 \end{array} - \begin{array}{c} \bullet \\ \diagup \\ \square_1 \text{---} \square_2 \end{array},$$

then

$$F_1 + F_2 = \begin{array}{c} \bullet \\ \diagdown \\ \square_1 \text{---} \square_2 \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \square_1 \text{---} \square_2 \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \square_1 \text{---} \square_2 \end{array}.$$

For a fixed type σ of size k , if (F_1, θ_1) and (F_2, θ_2) are two elements in \mathcal{F}^σ such that

$$|V(F_1)| + |V(F_2)| - k = n,$$

then the product of F_1 and F_2 is defined as

$$(F_1, \theta_1) \cdot (F_2, \theta_2) = \sum_{(H, \theta_3) \in \mathcal{F}_n^\sigma} P((F_1, \theta_1), (F_2, \theta_2); (H, \theta_3)) \cdot (H, \theta_3).$$

For example, if

$$(F_1, \theta_1) = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square \\ \text{1} \quad \text{2} \end{array} \quad \text{and} \quad (F_2, \theta_2) = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \square \\ \text{1} \quad \text{2} \end{array}$$

then,

$$(F_1, \theta_1) \times (F_1, \theta_2) = \frac{1}{2} \cdot \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square \\ \text{1} \quad \text{2} \end{array} + \frac{1}{2} \cdot \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square \\ \text{1} \quad \text{2} \end{array}.$$

Observe that the set \mathcal{F}_4^σ contains more than just the two graphs pictured in the previous equation, but in all of these other graphs, $P((F_1, \theta_1), (F_2, \theta_2); (H, \theta_3)) = 0$. Multiplication in \mathcal{A}^σ is defined as an extension of multiplication in \mathcal{F}^σ .

A sequence of graphs $(G_n)_{n \geq 1}$, where $|V(G_n)| = n$, is said to be *convergent* if for every finite graph H , the limit $\lim_{n \rightarrow \infty} P(H, G_n)$ exists. We prove here a proposition mentioned in [5].

Proposition 7. *Let $S = (G_n)_{n \geq 1}$ be any sequence of graphs with increasing orders. Then S contains a convergent subsequence $(G_{n_k})_{k \geq 1}$.*

Proof. Let $S_0 = S$ and enumerate the finite graphs H_1, H_2, \dots . Then $(P(H_1, G_n))_{n \geq 1}$ is a bounded sequence of real numbers and thus, by the Bolzano-Weierstrass theorem (see, e.g., Theorem 3.6(b) in [26]), contains a convergent subsequence, which induces a subsequence $S_1 = (G_{n_k})_{k \geq 1}$ of S_0 on which the density of H_1 converges. Then $(P(H_2, G_{n_k}))_{k \geq 1}$ is again a bounded sequence of real numbers, from which we find a subsequence S_2 of S_1 on which the densities of both H_1 and H_2 converge. Continuing in this way, for each i we generate a subsequence S_i on which the first i enumerated finite graphs have convergent densities. Let $T = (G_{n,n})_{n \geq 1}$ where $G_{n,n}$ is the n th term of S_n . Then for any finite graph H_j , as T is a subsequence of S_j after the first j elements, $\lim_{n \rightarrow \infty} P(H_j, G_{n,n})$ converges, so T is a convergent subsequence of S . \square

Let $\text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ denote the set of all homomorphisms from \mathcal{A}^σ to \mathbb{R} such that $\phi(F) \geq 0$ for each element $F \in \mathcal{F}^\sigma$. Razborov showed that functions $\phi \in \text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ correspond to convergent graph sequences $(G_n)_{n \geq 1}$; that is, the values of ϕ correspond to the limits of induced densities in $(G_n)_{n \geq 1}$. It is often more intuitive to think of addition and multiplication operations in \mathcal{A}^σ as representing induced densities of subgraphs in some very large graph G_{n_0} with an error term $O(n_0^{-1})$.

For each type σ labeled by $[k]$, Razborov also defined a function $\llbracket \cdot \rrbracket_\sigma : \mathbb{R}\mathcal{F}^\sigma \rightarrow \mathbb{R}\mathcal{F}$, which we will refer to as the *unlabelling operator*. For a σ -flag (F, θ) , let $q_\sigma(F)$ denote the probability that (F, θ') is isomorphic to F , where $\theta' : V(F) \rightarrow [k]$ is a randomly chosen injective mapping. Let F' denote the graph isomorphic to F when ignoring labels. Then

$$\llbracket F \rrbracket_\sigma = q_\theta(F)F'.$$

As an example,

$$\text{If, } F = \begin{array}{c} \square \\ \text{1} \quad \text{2} \end{array}, \text{ then } \llbracket F \rrbracket_\sigma = \frac{4}{6} \cdot \begin{array}{c} \square \\ \text{1} \quad \text{2} \end{array}.$$

Finally, it can be shown using the Cauchy-Schwarz inequality that if $\alpha \in \mathcal{A}^\sigma$ is some expression and $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$, then

$$\phi([\alpha^2]_\sigma) \geq 0. \tag{2}$$

2 Theorem 3 (i)

First we will prove a lower bound by counting the number of P_3 subgraphs in the Turán graph. After that, the remainder of the section will be devoted to proving the upper bound using flag algebras.

Lemma 8. *For all $r \geq 3$,*

$$12 \left(\frac{r-1}{r} \right)^3 \leq \text{OPT}_r(P_3)$$

Proof. We begin by counting the paths of length three in the Turán graph $T_r(n)$. To do so, we will first choose the central edge of the path and then select two additional vertices and describe how to attach them to the central edge.

As the Turán graph is multipartite, the central edge must fall between two of the r vertex classes. Assume for the moment that n is divisible by r . Then there are $\binom{r}{2} \left(\frac{n}{r}\right)^2$ choices for the central edge: first choose two vertex classes and select a vertex from each class.

Now we consider two cases. In the first case, the P_3 intersects exactly two of the vertex classes of $T_r(n)$. In this case, as we have already selected the central edge, the two vertex classes are already specified, so we need only select an additional vertex from each class. These vertices are each adjacent to a different vertex of our central edge and thus give a unique P_3 . There are $\left(\frac{n}{r} - 1\right)^2$ ways to choose these two vertices.

In the second case, the P_3 intersects at least three vertex classes of $T_r(n)$. (Note that as vertex classes contain no internal edges, the P_3 must contain vertices from more than one vertex class.) We first select this third vertex, for which there are $n - 2\left(\frac{n}{r}\right)$ choices, and then select a fourth unique vertex from the remaining $n - 3$ options. If the fourth vertex chosen happens to share a vertex class with either end of the central edge, then there is a unique P_3 containing the four vertices with the given central edge. Otherwise, there are two ways to connect the third and fourth vertices to the central edge. However, we also select pairs of vertices of this form twice as the fourth vertex we selected was an eligible choice when we selected the third in this case. Thus either way, this method produces

$$\left(n - 2 \left(\frac{n}{r} \right) \right) (n - 3)$$

unique copies of P_3 .

Putting all of our counts together, for all $r \geq 4$,

$$\nu(P_3, T_r(n)) = \binom{r}{2} \left(\frac{n}{r}\right)^2 \left(\left(\frac{n}{r} - 1\right)^2 + \left(n - 2\frac{n}{r} \right) (n - 3) \right) + o(n^4),$$

where the error terms accounts for the cases that n is not divisible by r . Factoring out leading terms gives

$$\nu(P_3, T_r(n)) = n^4 \cdot \frac{1}{2} \left(1 - \frac{1}{r}\right) \left(\left(\frac{1}{r} - \frac{1}{n}\right)^2 + \left(1 - \frac{2}{r}\right) \left(1 - \frac{3}{n}\right) \right) + o(n^4).$$

As $\binom{n}{4} = \frac{1}{24}n^4 + o(n^4)$, it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu(P_3, T_r(n)) \binom{n}{4}^{-1} &= \lim_{n \rightarrow \infty} \frac{n^4 \cdot \frac{1}{2} \left(1 - \frac{1}{r}\right) \left(\left(\frac{1}{r} - \frac{1}{n}\right)^2 + \left(1 - \frac{2}{r}\right) \left(1 - \frac{3}{n}\right) \right) + o(n^4)}{\frac{1}{24}n^4 + o(n^4)} \\ &= 12 \left(\frac{r-1}{r}\right)^3. \end{aligned}$$

Hence, $12 \left(\frac{r-1}{r}\right)^3 \leq \text{OPT}_r(P_3)$. □

We will now prove that $\text{OPT}_r(P_3) \leq 12 \left(\frac{r-1}{r}\right)^3$ using the flag algebra method. Unlike many proofs that employ this technique, ours does not require any computer assistance for verification. With that said, this section does require the multiplication and factoring of large polynomials. The authors have included a link to SageMath code used to verify these calculations in the appendix.

Proof of Theorem 3(i). Let $\mathcal{F}_4 = \{F_i\}_{i=0}^{10}$ denote the set of all unlabeled graphs on 4 vertices up to isomorphism, pictured below.

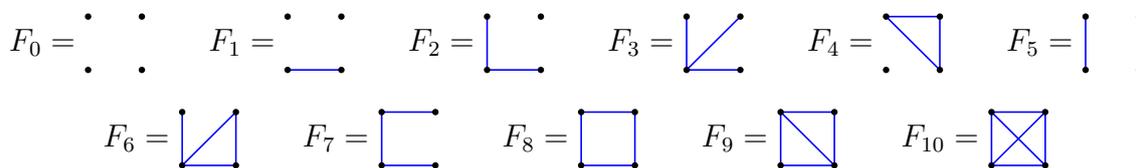


Figure 3: Enumeration of all graphs in \mathcal{F}_4 .

Throughout this section, we will be working with the induced densities of subgraphs in a convergent sequence of K_{r+1} -free graphs $(G_n)_{n \geq 1}$. In order to simplify notation we will let $P(F) = \lim_{n \rightarrow \infty} P(F, G_n)$ and similarly $d(F) = \lim_{n \rightarrow \infty} d(F, G_n)$. Summing over all of the graphs on \mathcal{F}_4 , we observe the following:

$$\sum_{i=0}^{10} P(F_i) = 1. \tag{3}$$

In order to make expressions like this easier to visualize, we will often use a drawing of F in place of $P(F)$ in our computations. For example, if $(G_n)_{n \geq 1}$ was the sequence of

complete graphs on n vertices, then $P(K_4) = \lim_{n \rightarrow \infty} P(K_4, G_n) = 1$. Using a drawing of K_4 in order to represent this density, we would write:

$$\begin{array}{c} \diagup \\ \square \\ \diagdown \end{array} = P(K_4) = 1.$$

Fix $r \geq 4$ and let $(G_n)_{n \geq 1}$ be an arbitrary convergent sequence of K_{r+1} -free graphs. By the law of total probability, the (non-induced) density of the path P_3 can be expressed as the sum of induced densities of graphs on four vertices in the following way,

$$d(P_3) = \sum_{i=0}^{10} P(F_i) \cdot \nu(P_3, F_i). \tag{4}$$

This expression can be simplified, however, as over half of the graphs in \mathcal{F}_4 do not contain a P_3 subgraph.

$$d(P_3) = \begin{array}{c} \square \\ \square \\ \square \end{array} + 2 \cdot \begin{array}{c} \diagup \\ \square \\ \diagdown \end{array} + 4 \cdot \begin{array}{c} \square \\ \square \\ \square \end{array} + 6 \cdot \begin{array}{c} \square \\ \diagdown \\ \square \end{array} + 12 \cdot \begin{array}{c} \diagup \\ \square \\ \diagdown \end{array}.$$

From Corollary 5 we obtain the following upper bound on $P(K_4)$ in $(G_n)_{n \geq 1}$.

$$\begin{array}{c} \diagup \\ \square \\ \diagdown \end{array} \leq \frac{r^3 - 6r^2 + 11r - 6}{r^3}. \tag{5}$$

Note that

$$\begin{aligned} P_0(r) &:= \sum_{i=0}^9 \left(\frac{r^3 - 6r^2 + 11r - 6}{r^3} \right) \cdot P(F_i) + \begin{array}{c} \diagup \\ \square \\ \diagdown \end{array} \frac{-6r^2 + 11r - 6}{r^3} \\ &= \left(\frac{r^3 - 6r^2 + 11r - 6}{r^3} \right) - \begin{array}{c} \diagup \\ \square \\ \diagdown \end{array} \tag{3} \\ &\geq 0 \tag{5} \end{aligned}$$

In the following computations, we will use two sets of labeled flags $\mathcal{F}_3^{\sigma_1}$ and $\mathcal{F}_3^{\sigma_2}$, where

$$\begin{aligned} \sigma_1 &= \begin{array}{c} \square \\ \square \\ \square \end{array}, \\ \sigma_2 &= \begin{array}{c} \square \\ \square \\ \square \end{array}. \end{aligned}$$

By the Cauchy-Schwarz inequality, each of the following three expressions is nonnegative for all $r \geq 4$.

$$\begin{aligned} 1. \quad P_1(r) &= 6 \cdot \left[\left((r-1) \begin{array}{c} \cdot \\ \square \\ \square \end{array} - \begin{array}{c} \cdot \\ \diagup \\ \square \\ \diagdown \end{array} \right)^2 \right]_{\sigma_1} = \\ &= (6r^2 - 12r + 6) \cdot \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + (r^2 - 2r + 1) \cdot \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + (1-r) \cdot \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + (3-3r) \cdot \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + 2 \cdot \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} + \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \end{aligned}$$

$$2. P_2(r) = 6 \cdot \left[\left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right)^2 \right]_{\sigma_2} =$$

$$3 \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} - 4 \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

$$3. P_3(r) = 6 \cdot \left[\left((r-2) \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} + (r-2) \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} - 2 \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \quad \diagup \\ \text{---} \end{array} \right)^2 \right]_{\sigma_2} =$$

$$(3r^2 - 12r + 12) \cdot \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} + (r^2 - 8r + 12) \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + (r^2 - 6r + 12) \cdot \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} + (4r^2 - 16r + 16) \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \\ + (20 - 8r) \cdot \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \quad \diagup \\ \text{---} \end{array} + 24 \cdot \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \quad \diagup \\ \text{---} \end{array}$$

Moreover, it can be quickly verified that for all $r \geq 4$, the following polynomials are all nonnegative.

$$1. p_0(r) = \frac{18(r^2 - 2r + 1)}{3r^2 - 11r + 9}$$

$$2. p_1(r) = \frac{3r^3 - 10r^2 + 7r}{3r^5 - 11r^4 + 9r^3}$$

$$3. p_2(r) = \frac{9r^5 - 32r^4 + 25r^3}{4(3r^5 - 11r^4 + 9r^3)}$$

$$4. p_3(r) = \frac{15r^3 - 24r^2 + 7r}{4(3r^5 - 11r^4 + 9r^3)}$$

We can add the sum $\sum_{j=0}^3 p_j(r)P_j(r)$ to (4) to obtain the following upper bound on $d(P_3)$.

$$d(P_3) \leq \sum_{i=0}^{10} P(F_i) \cdot \nu(P_3, F_i) + \sum_{j=0}^3 p_j(r)P_j(r). \quad (6)$$

For each $F_i \in \mathcal{F}_4$, let C_{F_i} denote the coefficient of the graph F_i after combining like-terms in (6). This gives the following, simplified upper bound on $d(P_3)$.

$$d(P_3) \leq \sum_{i=0}^{10} C_{F_i} P(F_i).$$

Since $\sum_{i=0}^{10} P(F_i) = 1$, it follows that

$$d(P_3) \leq \max\{C_{F_i} : F_i \in \mathcal{F}_4\}. \quad (7)$$

The following are the exact values of each C_{F_i} .

- $C_{F_0} = C_{F_3} = C_{F_8} = C_{F_9} = C_{F_{10}} =$

$$12 \left(\frac{r-1}{r} \right)^3$$

- $C_{F_1} =$

$$\frac{(21r^2 - 97r + 108)(r-1)^3}{3r^5 - 11r^4 + 9r^3}$$

- $C_{F_2} =$

$$\frac{(18r^3 - 111r^2 + 205r - 108)(r-1)^2}{3r^5 - 11r^4 + 9r^3}$$

- $C_{F_4} = C_{F_5} =$

$$\frac{18(r-1)^3(r-2)(r-3)}{3r^5 - 11r^4 + 9r^3}$$

- $C_{F_6} =$

$$\frac{45r^5 - 351r^4 + 1035r^3 - 1389r^2 + 870r - 216}{2(3r^5 - 11r^4 + 9r^3)}$$

- $C_{F_7} =$

$$\frac{(30r^4 - 180r^3 + 371r^2 - 327r + 108)(r-1)}{3r^5 - 11r^4 + 9r^3}$$

By examining leading coefficients and factoring, it is clear that for all $r > 1000$,

$$\max\{C_{F_i} : F_i \in \mathcal{F}_4\} = 12 \left(\frac{r-1}{r} \right)^3. \tag{8}$$

We have provided a link in the appendix for SageMath code which can be used to verify (8) for $4 \leq r \leq 1000$. This fact, together with (7) are enough to show that

$$\text{OPT}_r(P_3) \leq 12 \left(\frac{r-1}{r} \right)^3.$$

Along with Lemma 8, this completes the proof of Theorem 3(i). □

3 Stability

For two graphs G and H of the same order, the *edit distance* between G and H , denoted $\text{Dist}(G, H)$, is the minimum number of adjacencies one needs to add or remove in order to change G into a graph isomorphic to H . Our goal in this section is to prove that graphs with P_3 density approaching $\text{OPT}_r(P_3)$ are close in structure to the Turán graph $T_r(n)$. Specifically, we prove the following lemma:

Lemma 9. For every $\varepsilon > 0$, there exists an n_0 and $\delta > 0$ such that for every K_{r+1} -free graph G of order $n \geq n_0$, if $d(P_3, G) \geq \text{OPT}_r(n) - \delta$, then $\text{Dist}(G, T_r(n)) \leq \varepsilon n^2$.

We prepare for the proof of Lemma 9 with a collection of lemmas. Several of these lemmas use the epsilon-delta paradigm, and so in the interest of legibility we have labeled the lemmas in this section by letter. We adopt the convention that ε_A , for example, will always refer to the ε in Lemma A. The exception to this rule is Lemma 9 which uses unadorned variables.

The first lemma is the Induced Removal Lemma, proved by Alon, Fischer, Krivelevich and Szegedy [1].

Lemma 10 (Lemma A, Induced Removal Lemma). Let \mathcal{F} be a set of graphs. For each $\varepsilon_A > 0$, there exist η_A and $\delta_A > 0$ such that for every graph G of order $n \geq \eta_A$, if G contains at most $\delta_A n^{|V(H)|}$ induced copies of H for every $H \in \mathcal{F}$, then G can be made \mathcal{F} -free by removing or adding at most $\varepsilon_A n^2$ edges from G .

We define the set T to contain each of the graphs $F \in \mathcal{F}_4$ for which $c_F = \text{OPT}_r(P_3)$ in the proof of Theorem 3.

$$T = \left\{ \begin{array}{c} \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \end{array}, \begin{array}{c} \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \end{array}, \begin{array}{c} \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \end{array}, \begin{array}{c} \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \end{array}, \begin{array}{c} \cdot \\ \cdot \end{array} \begin{array}{c} \cdot \\ \cdot \end{array} \right\}.$$

The following is a restatement of Lemma 2.4.3 appearing in [3]. For completeness, we will provide a short proof.

Lemma 11. [3] Let $(G_n)_{n \geq 1}$ be a sequence of K_{r+1} -free graphs of increasing order such that

$$\lim_{n \rightarrow \infty} d(P_3, G_n) = \lim_{n \rightarrow \infty} \sum_{i=0}^{10} C_{F_i} \cdot P(F_i, G_n) = \text{OPT}_r(P_3),$$

where $F_i \in \mathcal{F}_4$ for all $i = 0, \dots, 10$. Then for all $F \in \mathcal{F}_4$, $\lim_{n \rightarrow \infty} P(F, G_n) > 0$ implies that $F \in T$.

Proof. Let \mathcal{F}_4^* denote the set of graphs F in \mathcal{F}_4 for which $\lim_{n \rightarrow \infty} P(F, G_n) > 0$. Then $\lim_{n \rightarrow \infty} \sum_{F \in \mathcal{F}_4^*} P(F, G_n) = 1$, implying from Theorem 3(i) that

$$\lim_{n \rightarrow \infty} \sum_{F \in \mathcal{F}_4^*} C_F \cdot P(F, G_n) = \text{OPT}_r(P_3).$$

For each graph $H \in \mathcal{F}_4 \setminus T$, we know from the proof of Theorem 3(i) that $C_H < \text{OPT}_r(P_3)$. Thus, $H \notin \mathcal{F}_4^*$ as otherwise $\lim_{n \rightarrow \infty} \sum_{F \in \mathcal{F}_4^*} C_F \cdot P(F, G_n) < \text{OPT}_r(P_3)$. \square

Note that the original statement of Lemma 11 required that $(G_n)_{n \geq 1}$ be convergent. Proposition 7 permits us to apply the lemma with the less stringent restriction that the sequence have increasing order.

Given the fact that only those graphs in T can appear with positive density in the limit of any extremal sequence, we can now prove the following lemma.

Lemma 12 (Lemma B). *For each $\varepsilon_B > 0$, there exists a η_B and $\delta_B > 0$ such that any K_{r+1} -free graph G of order $n \geq \eta_B$ satisfying $d(P_3, G) \geq \text{OPT}_r(P_3) - \delta_B$ contains at most $\varepsilon_B n^3$ copies of $\overline{P_2}$.*

Proof. Assume the contrary; that is, there is $\varepsilon_B > 0$ such that for every η_B and δ_B , there is a K_{r+1} -free graph G of order $n \geq \eta_B$ satisfying $d(P_3, G) \geq \text{OPT}_r(P_3) - \delta_B$ containing at least $\varepsilon_B n^3$ copies of $\overline{P_2}$. Let $G_0 = \overline{P_2}$. Then, for each $n \in \mathbb{N}$, let G_n have order at least $|V(G_{n-1})|$ and $d(P_3, G_n) \geq \text{OPT}_r - \frac{1}{n}$ with at least $\varepsilon_B n^3$ copies of $\overline{P_2}$. We have

$$\lim_{n \rightarrow \infty} d(P_3, G_n) = \text{OPT}_r(P_3).$$

By inspection, none of the graphs in T contain $\overline{P_2}$ as a subgraph. Thus from Lemma 11,

$$\lim_{n \rightarrow \infty} d(\overline{P_2}, G_n) = 0.$$

This is a contradiction as

$$\lim_{n \rightarrow \infty} d(\overline{P_2}, G_n) = \lim_{n \rightarrow \infty} \nu(\overline{P_2}, G_n) \binom{n}{3}^{-1} \geq \lim_{n \rightarrow \infty} \varepsilon_B n^3 \binom{n}{3}^{-1} = 6\varepsilon_B > 0. \quad \square$$

Next we prove that among all complete r -partite graphs on at least four vertices, the Turán graph $T_r(n)$ contains the most P_3 subgraphs.

Lemma 13. *For $n \geq 4$ and $r \geq 4$, if G is any complete r -partite graph on n vertices then $\nu(P_3, G) \leq \nu(P_3, T_r(n))$.*

Proof. We count the number of P_3 subgraphs in a complete multipartite graph using a similar approach to that in the proof of Theorem 8. We sum over each edge and count the number of P_3 with that edge as the center. If $e = xy$ is an edge in the center of P_3 with x in vertex class V_x and y in vertex class V_y , let the other edges of the P_3 be wx and yz . We classify the P_3 into one of four types depending on the location of w and z .

- There are $(|V_x| - 1)(|V_y| - 1)$ such P_3 with $w \in V_y$ and $z \in V_x$ as we may not reselect x or y .
- When $w \in V_y$ but $z \notin V_x$, there are $(|V_y| - 1)(n - |V_x| - |V_y|)$ choices for the P_3 as z falls in some vertex class other than V_x or V_y .
- Similarly, when $w \notin V_y$ and $z \in V_x$, there are $(n - |V_x| - |V_y|)(|V_x| - 1)$ many such P_3 .
- Finally, if $w \notin V_y$ and $z \notin V_x$, then we must take care to select them uniquely. Choosing w first and then z gives $(n - |V_x| - |V_y|)(n - |V_x| - |V_y| - 1)$ many such P_3 .

Thus in total, for complete multipartite graphs G ,

$$\begin{aligned} \nu(P_3, G) &= \sum_{e=xy} (|V_x| - 1)(|V_y| - 1) + (|V_y| - 1)(n - |V_x| - |V_y|) + \\ &\quad + (n - |V_x| - |V_y|)(|V_x| - 1) + (n - |V_x| - |V_y|)(n - |V_x| - |V_y| - 1) \\ &= \sum_{e=xy} (|V_x| - 1)(|V_y| - 1) + (n - |V_x| - |V_y|)(n - 3) \end{aligned}$$

Now suppose that G has r parts V_1, \dots, V_r . There are $|V_i||V_j|$ edges between parts V_i and V_j , each of which contributes the same term in the sum above. Thus we may also write

$$\nu(P_3, G) = \sum_{1 \leq i < j \leq r} |V_i||V_j|((|V_i| - 1)(|V_j| - 1) + (n - |V_i| - |V_j|)(n - 3)). \quad (9)$$

Let G be a complete r -partite graph on n vertices with parts V_1, \dots, V_r such that $|V_1| \geq |V_2| + 2$. If G has no edges but at least four vertices, it cannot be extremal, so assume G contains at least one edge. Define G' to be the complete multipartite graph on n vertices with parts V'_1, V'_2, \dots, V'_r where $|V'_1| = |V_1| - 1$, $|V'_2| = |V_2| + 1$, and $|V'_i| = |V_i|$ for $i \geq 3$.

After straightforward, if tedious, calculation, we use (9) to see $\nu(P_3, G') - \nu(P_3, G) = \Delta_{P_3}$ where

$$\Delta_{P_3} = (|V_1| - |V_2| - 1)((n - |V_1| - |V_2|)(n - 3) + 2(|V_1| - 1)|V_2| + \sum_{j=3}^r |V_j|(n - 2 - |V_j|))$$

Note that by assumption $|V_1| \geq |V_2| + 2$ and $n \geq 4$, that $n \geq |V_1| + |V_2|$ as $V_1, V_2 \subseteq V(G)$ and that $n - 2 \geq |V_j|$ for $j \geq 3$ because $|V_j| \leq n - |V_1|$ and V_1 must have at least two vertices to satisfy $|V_1| \geq |V_2| + 2$. Thus $|V_1| - |V_2| - 1$ is strictly positive and $(n - |V_1| - |V_2|)(n - 3)$, $2(|V_1| - 1)|V_2|$, and $\sum_{j=3}^r |V_j|(n - 2 - |V_j|)$ are each nonnegative. If $\Delta_{P_3} = 0$, then each term must be exactly zero. This means $n = |V_1| + |V_2|$ and $|V_2| = 0$. But then $n = |V_1|$, so all of the vertices of G are in one part which contradicts that G has at least one edge. We conclude $\Delta_{P_3} > 0$ and thus G' contains more P_3 than G .

Thus we see G was not extremal and therefore the Turán graph, the unique complete r -partite graph in which no pair of vertex classes differs in size by more than one, is the complete r -partite graph with the greatest number of P_3 . \square

In the next lemma, we prove that if G has large P_3 -density, it is close in edit distance to a nearly balanced complete r -partite graph.

Lemma 14 (Lemma C). *For any two independent parameters $\varepsilon_C > 0$ and $\gamma_C > 0$ there are η_C and $\delta_C > 0$ such that if G is a K_{r+1} -free graph with order $n \geq \eta_C$ satisfying $d(P_3, G) > \text{OPT}_r(P_3) - \delta_C$, then there is a complete r -partite graph G' with parts X_1, \dots, X_r satisfying $\text{Dist}(G, G') \leq \gamma_C n^2$ and, for each $1 \leq i \leq r$,*

$$\frac{1 - \varepsilon_C}{r} n \leq |X_i| \leq \frac{1 + \varepsilon_C}{r} n.$$

Proof. Let $\varepsilon_C, \gamma_C > 0$ be given. We require a $\gamma'_C > 0$ but defer its exact definition until later. Take η_A and δ_A to be as in Lemma 10 so that any graph G of order $n \geq \eta_A$ containing at most $\delta_A n^3$ copies of $\overline{P_2}$ can be made $\overline{P_2}$ -free by editing at most $\gamma'_C n^2$ edges. Then take η_B and δ_B to be as in Lemma 12 so that for any graph G of order $n \geq \eta_B$ which satisfies $d(P_3, G) \geq \text{OPT}_r(P_3) - \delta_B$ contains at most $\delta_A n^3$ copies of $\overline{P_2}$ (that is, apply Lemma 12 with $\varepsilon_B = \delta_A$).

Though we are not ready to define them yet, we will ensure $\eta_C \geq \max(\eta_A, \eta_B)$ and $\delta_C \leq \min(\delta_A, \delta_B)$. Let G be a graph of order $n \geq \eta_C$ satisfying $d(P_3, G) \geq \text{OPT}_r(n) - \delta_C$. By Lemma 12, G has at most $\delta_A n^3$ copies of $\overline{P_2}$ and thus by Lemma 10 we may edit at most $\gamma'_C n^2$ edges of G to get a $\overline{P_2}$ -free graph, G' . It follows from Lemma 6 that G' is a complete r -partite graph as it is both K_{r+1} -free and $\overline{P_2}$ -free. Let X_1, \dots, X_r denote the partite sets of G' . We complete the proof by demonstrating these partite sets all have size nearly $\frac{n}{r}$.

There is a constant $c > 0$ such that each edge removed from G is contained in at most cn^2 copies of P_3 . (The constant c counts the number of ways to extend an edge and two other vertices into a copy of P_3 .) Thus

$$d(P_3, G') \geq \text{OPT}_r(P_3) - \delta_C - c\gamma'_C$$

as the P_3 -density of the removed edges is at most

$$\frac{\gamma'_C n^2 \cdot cn^2}{n^{|V(P_3)|}} = c\gamma'_C.$$

To prove that the partite sets have bounded size, we will show that if they do not, we may alter G' to increase its P_3 density beyond $\text{OPT}_r(P_3)$. As $\text{OPT}_r(P_3)$ is, by definition, a limit, we can, for large enough η_C , get upper bounds on the P_3 -density of such graphs that are as close to $\text{OPT}_r(P_3)$ as necessary to arrive at a contradiction.

We require a partial result from the proof of Lemma 13. Recall that when moving one vertex from vertex class V_1 to vertex class V_2 the change in the number of P_3 subgraphs was

$$\Delta_{P_3} = (|V_1| - |V_2| - 1)((n - |V_1| - |V_2|)(n - 3) + 2(|V_1| - 1)|V_2| + \sum_{j=3}^r |V_j|(n - 2 - |V_j|))$$

Assume first that there is a partite set that is too large. Specifically, assume, without loss of generality, that $|X_1| > \frac{1+\varepsilon_C}{r}n$. We consider two cases.

First, assume

$$\frac{1 + \varepsilon_C}{r}n < |X_1| \leq \frac{n}{2}.$$

There must be a partite set of G' , say X_2 , that satisfies $|X_2| < \frac{n}{r}$; if not,

$$n = \sum_{i=1}^r |X_i| \geq \frac{1 + \varepsilon_C}{r}n + (r - 1)\frac{n}{r} = n + \frac{\varepsilon_C}{r}n$$

is a contradiction. Consider the process of moving one vertex from X_1 to X_2 repeated $\frac{\varepsilon_C}{3r}n$ times. At each step of this process,

$$|X_1| - |X_2| > \left(\frac{1 + \varepsilon_C}{r}n - \frac{\varepsilon_C}{3r}n \right) - \left(\frac{1}{r}n + \frac{\varepsilon_C}{3r}n \right) = \frac{\varepsilon_C}{3r}n.$$

We take η_C large enough that this value is always at least 2 so that number of P_3 subgraphs increases at every step. In particular, as $|X_1| + |X_2|$ stays constant and

$$|X_1| + |X_2| < \frac{n}{2} + \frac{n}{r} \leq \frac{3}{4}n$$

we have

$$\Delta_{P_3} \geq (|X_1| - |X_2| - 1)((n - |X_1| - |X_2|)(n - 3)) \geq \left(\frac{\varepsilon}{3r}n - 1 \right) \left((n - \frac{3}{4}n)(n - 3) \right).$$

Take η_C large enough so that $n \geq \eta_C$ implies $n - 3 \geq \frac{n}{2}$ and $\frac{\varepsilon_C}{3r}n - 1 \geq \frac{\varepsilon_C}{4r}n$, giving

$$\Delta_{P_3} \geq \frac{\varepsilon_C}{4r}n \cdot \frac{n}{4} \cdot \frac{n}{2} = \frac{\varepsilon_C}{32r}n^3.$$

Now, as we repeat this process $\frac{\varepsilon_C}{3r}n$ times, the total increase in the number of copies of P_3 is at least

$$\frac{\varepsilon_C}{3r}n \cdot \frac{\varepsilon_C}{32r}n^3 = \frac{\varepsilon_C^2}{96r^2}n^4.$$

As $|V(P_3)| = 4$, this increases the P_3 density of G' by at least $\frac{\varepsilon_C^2}{96r^2}$. By choosing δ_C and γ'_C such that

$$\delta_C + c\gamma'_C < \frac{\varepsilon_C^2}{96r^2}$$

we arrive at a graph G'' with

$$d(P_3, G'') \geq d(P_3, G') + \frac{\varepsilon_C^2}{96r^2} \geq \text{OPT}_r(P_3) - \delta_C - c\gamma'_C + \frac{\varepsilon_C^2}{96r^2} > \text{OPT}_r(P_3),$$

a contradiction for large enough η_C .

Otherwise, we have $|X_1| \geq \frac{n}{2}$. We wish to use a similar approach to the first case, but we must assure that the lower bound on Δ_{P_3} is cubic in n at each step of the process. There must be a partite set of G' , say X_2 , that satisfies $|X_2| \leq \frac{1}{2(r-1)}n \leq \frac{1}{6}n$ (recall $r \geq 4$); if not,

$$n = |X_1| + \sum_{i=2}^r |X_i| > \frac{n}{2} + (r-1)\frac{1}{2(r-1)}n = n,$$

a contradiction. We start by moving $\frac{n}{12}$ vertices from X_1 to X_2 . These moves increase the number of copies of P_3 , but we disregard those increases. After these moves we have $|X_1| > \frac{n}{2} - \frac{n}{12} = \frac{5}{12}n$ and

$$\frac{n}{12} \leq |X_2| \leq \frac{n}{6} + \frac{n}{12} = \frac{3}{12}n.$$

Starting from this modified graph we can move $\frac{n}{24}$ additional vertices from X_1 to X_2 . For each such move, we have

$$(|X_1| - |X_2| - 1) \geq \left(\frac{5}{12}n - \frac{1}{24}n\right) - \left(\frac{3}{12}n + \frac{1}{24}n\right) - 1 = \frac{n}{12} - 1 \geq \frac{n}{13}$$

by choosing η_C large enough, and

$$2(|X_1| - 1)|X_2| \geq 2\left(\frac{5}{12}n - 1\right)\left(\frac{n}{12}\right) > \frac{n^2}{18},$$

again with η_C large enough. Thus

$$\Delta_{P_3} \geq \frac{n}{13} \cdot \frac{n^2}{18} = \frac{n^3}{234}$$

and repeating this process $\frac{n}{24}$ times increases the total number of P_3 subgraphs by at least $\frac{n^4}{5616}$, increasing the P_3 density of G' by $\frac{1}{5616}$. By taking $\delta_C + c\eta'_C < \frac{1}{5616}$ we again get a graph with P_3 density larger than the optimal density, a contradiction when η_C is sufficiently large.

Finally, we now assume for contradiction that $|X_1| < \frac{1-\varepsilon_C}{r}n$. If $|X_1| < \frac{1-(r-1)\varepsilon_C}{r}n$, then there must be another partite set X_i with $|X_i| > \frac{1+\varepsilon_C}{r}n$ as otherwise

$$n = \sum_{i=1}^r |X_i| < \frac{1-(r-1)\varepsilon_C}{r}n + (r-1)\frac{1+\varepsilon_C}{r}n = n$$

is a contradiction. As we have already handled cases with a too large part, we may assume

$$\frac{1-(r-1)\varepsilon_C}{r}n \leq |X_1| < \frac{1-\varepsilon_C}{r}n.$$

There must be a partite set X_i with $|X_i| > \frac{n}{r}$, again because otherwise the parts combined cannot contain n vertices. Then we move a vertex from X_i to X_1 and repeat the move $\frac{\varepsilon}{3r}n$ times. Then as before at every step of the process

$$|X_i| - |X_1| \geq \frac{1-\varepsilon}{3r}n > 0$$

and, using very rough bounds,

$$|X_1| + |X_i| \leq \frac{1-\varepsilon_C}{r}n + \frac{1+\varepsilon_C}{r}n < \frac{n}{r} + \frac{n}{2} \leq \frac{3}{4}n.$$

Therefore this process also increases the P_3 density of G' by at least $\frac{\varepsilon_C^2}{96r^2}$, a contradiction for δ_C small enough. We conclude each partite set X_1, \dots, X_r must be within the specified bounds.

For completeness, we explicitly specify our choices of η_C, δ_C , and γ'_C . We set

$$\begin{aligned}\delta_C &= \min\left(\delta_A, \delta_B, \frac{1}{20000}, \frac{\varepsilon_C^2}{200r^2}\right) \\ \gamma'_C &= \min\left(\gamma_C, \frac{1}{20000c}, \frac{\varepsilon_C^2}{200cr^2}\right) \\ \eta_C &\geq \max\left(\eta_A, \eta_B, \frac{12r}{\varepsilon_C}, 144\right)\end{aligned}$$

where η_C is also large enough to guarantee all graphs of this form are sufficiently close to $\text{OPT}_r(P_3)$.

These choices assure that we can combine Lemmas 10 and 12 to produce a G' with $\text{Dist}(G, G') \leq \gamma'_C n^2 \leq \gamma_C n^2$ also that

$$\delta_C + c\gamma'_C \leq \frac{1}{20000} + \frac{c}{20000c} = \frac{1}{10000} < \frac{1}{5616}$$

and

$$\delta_C + c\gamma'_C \leq \frac{\varepsilon_C^2}{200r^2} + \frac{c\varepsilon_C^2}{200cr^2} = \frac{\varepsilon_C^2}{100r^2} < \frac{\varepsilon_C^2}{96r^2},$$

as well as the bounds we use on n , all hold. □

We are now ready to prove Lemma 9.

Proof of Lemma 9. Let $\varepsilon > 0$ be given. Set $n_0 = \eta_C$ and $\delta = \delta_C$ from Lemma 14 with $\gamma_C = \varepsilon/2$ and $\varepsilon_C = \varepsilon/2r$. For any G of order $n \geq n_0$ such that $d(P_3, G) \geq \text{OPT}_r(P_3) - \delta$, we get a complete r -partite graph G' satisfying $\text{Dist}(G, G') \leq \frac{\varepsilon}{2}n^2$ and with parts X_1, \dots, X_r satisfying

$$\frac{1 - \frac{\varepsilon}{2r}}{r}n \leq |X_i| \leq \frac{1 + \frac{\varepsilon}{2r}}{r}n.$$

We claim $\text{Dist}(G', T_r(n)) \leq \frac{\varepsilon}{2}n^2$. From each of the r parts, at most $\frac{\varepsilon}{2r}n$ vertices must be added to or removed from that part. Thus in total, $\frac{\varepsilon}{2}n$ vertices are altered. Each vertex requires changing at most n adjacencies, so the total edit distance is bounded above by $\frac{\varepsilon}{2}n^2$.

Finally, by first making the at most $\frac{\varepsilon}{2}n^2$ edits to change G into G' and then making the at most $\frac{\varepsilon}{2}n^2$ edits to change G' into $T_r(n)$, we have demonstrated $\text{Dist}(G, T_r(n)) \leq \varepsilon n^2$, completing the proof. □

4 Exact Result

In this section we will prove Theorem 3(ii). We now know that for large enough n , if G is an n -vertex K_{r+1} -free graph that is close to being extremal, then G is close in edit-distance to $T_r(n)$. As we will show in this section, the process of adding or removing the necessary edges in order to transform G into $T_r(n)$ must increase the number of P_3 -subgraphs in G . First we need the following proposition, which shows that in any extremal graph each

pair of vertices must be contained in approximately the same number of P_3 -subgraphs. We define $\nu_G(v, T)$ as the number of (not necessarily induced) subgraphs of a graph G isomorphic to T containing v .

Proposition 15. *Fix $r \geq 4$. Then there exists an $n_0 = n_0(r)$ such that if G a K_{r+1} -free graph on $n \geq n_0$ vertices for which $\nu(P_3, G) = ex(n, P_3, K_{r+1})$, then for every vertex $v \in V(G)$*

$$\nu_G(v, P_3) \geq \left(\text{OPT}_r(P_3) - \frac{1}{r^{10}} \right) \binom{n-1}{3} - \frac{1}{r^4} n^3.$$

Proof. From the proof of Theorem 3(i), there must exist some n_0 such that

$$\nu(P_3, G) \geq \left(\text{OPT}_r(P_3) - \frac{1}{r^{10}} \right) \binom{n}{4}$$

for every extremal graph G on $n \geq n_0$ vertices. Suppose that G is such a graph on $n \geq \max\{n_0, 2r^4\}$ vertices. We count the copies of P_3 in G in two ways to see

$$\sum_{v \in V(G)} \nu_G(v, P_3) = 4\nu(P_3, G) \geq 4 \left(\text{OPT}_r(P_3) - \frac{1}{r^{10}} \right) \binom{n}{4}.$$

Thus, by averaging there must exist some vertex $u \in V(G)$ for which

$$\nu_G(u, P_3) \geq \left(\text{OPT}_r(P_3) - \frac{1}{r^{10}} \right) \binom{n-1}{3}.$$

Suppose for contradiction that for some vertex $v \in V(G)$,

$$\nu_G(v, P_3) < \left(\text{OPT}_r(P_3) - \frac{1}{r^{10}} \right) \binom{n-1}{3} - \frac{1}{r^4} n^3.$$

Let G' be the graph obtained from G by deleting v and replacing it with a vertex u' so that $N(u') = N(u)$. We claim that G' is K_{r+1} -free. Suppose for contradiction that it is not. Then u' must be contained in every copy of K_{r+1} in G' . As u is not adjacent to u' , none of these K_{r+1} contain u . However, since $N(u) = N(u')$, this implies that we can replace u' with u in each $(r+1)$ -clique. Since $V(G') - \{u'\} = V(G) - \{v\}$, this implies the existence of an $(r+1)$ -clique in G , which is a contradiction.

Let $\nu_G(u, v, P_3)$ denote the number of P_3 subgraphs containing both u and v in G . Then since $\nu_G(u', P_3) = \nu_G(u, P_3)$, we have added at least $\nu_G(u, P_3) - \nu_G(u, v, P_3)$ subgraphs and removed at most $\nu_G(v, P_3)$ subgraphs. Hence,

$$\nu(P_3, G') = \nu(P_3, G) + \nu_G(u, P_3) - \nu_G(u, v, P_3) - \nu_G(v, P_3)$$

Since $\nu_G(u, v, P_3) \leq 2n^2$,

$$\nu(P_3, G') > \nu(P_3, G) + \frac{1}{r^4} n^3 - 2n^2.$$

By assumption, $\frac{1}{r^4} n^3 - 2n^2 > 0$. This would imply that $\nu(P_3, G') > \nu(P_3, G)$ which contradicts the assumption that G was extremal, completing the proof. \square

We will also require the following proposition much later in the proof of Theorem 3(ii), where we will provide more explanation of why it is required. For completeness, we will state it here.

Proposition 16. *For all integers $r \geq 4$, there exists an $n_0 = n_0(r)$ such that for all $n \geq n_0$,*

$$(i) \text{ OPT}_r(P_3) \binom{n-1}{3} - \delta_1(r) \frac{n^3}{6} \geq \left(\frac{9}{r} - \frac{39}{2r^2} \right) n^3, \text{ where}$$

$$\delta_1(r) = 12 - \frac{45}{r} + \frac{111}{2r^2} - \frac{27}{2r^3} - \frac{21}{r^4} + \frac{24}{r^5} + \frac{3}{2r^6} - \frac{3}{2r^7}.$$

$$(ii) \text{ OPT}_r(P_3) \binom{n-1}{3} - \delta_2(r) \frac{n^3}{6} \geq \left(\frac{18}{r} - \frac{42}{r^2} - \frac{12}{r^3} \right) n^3, \text{ where}$$

$$\delta_2(r) = 12 - \frac{54}{r} + \frac{78}{r^2} - \frac{96}{r^4} + \frac{72}{r^5} + \frac{24}{r^6} - \frac{24}{r^7}$$

Proof. Part (i) immediately follows from the inequality below, which is true for all $r \geq 4$.

$$\text{OPT}_r(P_3) - \delta_1(r) = \frac{9}{r} - \frac{39}{2r^2} + \frac{3}{2r^3} + \frac{21}{r^4} - \frac{24}{r^5} - \frac{3}{2r^6} + \frac{3}{2r^7} > \frac{9}{r} - \frac{39}{2r^2},$$

In an identical manner, part (ii) is implied from the following, which is true for all $r \geq 4$.

$$\text{OPT}_r(P_3) - \delta_2(r) = \frac{18}{r} - \frac{42}{r^2} - \frac{12}{r^3} + \frac{96}{r^4} - \frac{72}{r^5} - \frac{24}{r^6} + \frac{24}{r^7} > \frac{18}{r} - \frac{42}{r^2} - \frac{12}{r^3},$$

completing the proof. □

After assuming that G is an extremal graph, and therefore close in edit-distance to $T_r(n)$, we will show that most vertices in G must closely resemble a vertex appearing in the Turán graph. Given this fact, we will use Proposition 15 to show that any vertex that does not look like this cannot be contained in enough copies of P_3 to justify G being extremal. This will ultimately show that G must be isomorphic to $T_r(n)$, since the removal/duplication process described in the proof of Proposition 15 would otherwise increase the number of P_3 copies in G .

Proof of Theorem 3(ii). Let $r \geq 4$. Fix $\varepsilon > 0$ and assume that $n_0 = n_0(r)$ is large enough to satisfy the following conditions.

- (i) Any K_{r+1} -free graph G on $n \geq n_0$ vertices with

$$d(P_3, G) > \text{OPT}_r(P_3) - \varepsilon$$

must satisfy $\text{Dist}(G, T_r(n)) \leq \frac{2}{r^{10}} n^2$.

- (ii) $n_0 \geq 2r^4$ and is large enough to satisfy the conditions of Proposition 15.

- (iii) n_0 is large enough to satisfy the conditions of Proposition 16.

Let G be an extremal graph on $n \geq n_0$ vertices. Recall that (i) means that we can transform G into $T_r(n)$ by changing at most $\frac{2}{r^{10}}n^2$ adjacencies. We will call each edge removed in the process of transforming G into $T_r(n)$ a *surplus edge*, and each added edge a *missing edge*. Let $b(v)$ denote the total number of surplus edges and missing edges incident with a vertex v . If v is a vertex for which $b(v) > \frac{1}{r^5}n$, then we say that v is a *bad vertex*.

Partition the vertex set of G into sets X_1, X_2, \dots, X_r so that after changing all required adjacencies in G the sets X_1, X_2, \dots, X_r are the partite sets of $T_r(n)$. For the moment, move each bad vertex from its original set and place it into a new set X_0 .

Claim 17. $|X_0| \leq \frac{1}{r^5}n$.

Proof. Since $\text{Dist}(G, T_r(n)) \leq \frac{2}{r^{10}}n^2$ and each vertex $v \in X_0$ satisfies $b(v) > \frac{1}{r^5}n$,

$$|X_0| \cdot \frac{1}{r^5}n \leq \frac{1}{r^{10}}n^2.$$

Claim 17 follows immediately. □

Now we will show that all surplus edges must be incident with at least one vertex in X_0 . This will allow us to focus only on the bad vertices. For a finite collection of vertices $x_1, x_2, \dots, x_\ell \in V(G)$ let $N(x_1, x_2, \dots, x_\ell)$ denote the *common neighborhood* of x_1, x_2, \dots, x_ℓ , which is the set of vertices in $V(G)$ adjacent to each of x_1, x_2, \dots, x_ℓ .

Claim 18. *There are no surplus edges in $V(G) \setminus X_0$.*

Proof. Suppose for contradiction that for two vertices u and v in $X_j \setminus X_0$ are adjacent for some integer $j \in [r]$. By symmetry we may assume that $j = 1$. Since neither vertex is contained in X_0 , both u and v are incident with at most $\frac{1}{r^5}n$ missing edges in $X_2 \setminus X_0$. This implies that there are at most $\frac{2}{r^5}n$ vertices in $X_2 \setminus X_0$ not contained in $N(u, v)$. Since Claim 17 implies that we have moved at most $\frac{1}{r^5}n$ vertices from X_2 to X_0 ,

$$|(N(u, v) \cap X_2) \setminus X_0| \geq \left\lfloor \frac{n}{r} \right\rfloor - \left\lfloor \frac{3n}{r^5} \right\rfloor > 0.$$

Let w_2 be one of the vertices contained in the set $(N(u, v) \cap X_2) \setminus X_0$. Then uvw_2 induces a triangle in G . Since w_2 is also only incident with $\frac{1}{r^5}n$ missing edges, we can apply an identical argument using u, v, w_2 and the set X_3 to show:

$$|(N(u, v, w_2) \cap X_3) \setminus X_0| \geq \left\lfloor \frac{n}{r} \right\rfloor - \left\lfloor \frac{4n}{r^5} \right\rfloor > 0,$$

implying that we can find some $w_3 \in X_3$ such that uvw_2w_3 induces a K_4 in G . Continuing this process for each $j \in \{4, \dots, r\}$, we can always select one vertex $w_j \in X_j$ in an identical manner so that $uvw_2 \dots w_j$ induces a copy of K_{j+1} in G . This is possible since

$$|(N(u, v, w_2, \dots, w_{j-1}) \cap X_j) \setminus X_0| \geq \left\lfloor \frac{n}{r} \right\rfloor - \left\lfloor \frac{(j+1)n}{r^5} \right\rfloor > 0$$

for each j . This, however, would imply that after selecting vertices $u, v, w_2, \dots, w_{r-1}$ that induce a copy of K_r ,

$$|(N(u, v, w_2, \dots, w_{r-1}) \cap X_r) \setminus X_0| \geq \left\lfloor \frac{n}{r} \right\rfloor - \left\lfloor \frac{(r+1)n}{r^5} \right\rfloor > 0.$$

Thus, we can select a vertex in X_r that is adjacent to each of $u, v, w_2, \dots, w_{r-1}$. This, however, induces a copy of K_{r+1} in G which is a contradiction. \square

For each $i \in [r]$, let $d_i(v) = |(N(v) \cap X_i) \setminus X_0|$. We say that $v \in X_0$ is a *type 2 vertex* if $d_i(v) > 0$ for all $i = 1, \dots, r$. Otherwise, if there exists some $i \in [r]$ for which $d_i(v) = 0$, then v is a *type 1 vertex*.

Claim 19. *If v is a type 2 vertex, then there exist $i, j \in [r]$ for which*

$$1 \leq d_i(v) \leq d_j(v) \leq \frac{1}{r^3}n.$$

Proof. Suppose for contradiction that for all $i \in [r]$, $d_i(v) > \frac{1}{r^3}n$. By symmetry, we may assume that

$$\frac{1}{r^3}n < d_1(v) \leq d_2(v) \leq \dots \leq d_r(v).$$

Let $w_1 \in X_1$ be a neighbor of v . Then $d_i(w_1) \geq \frac{n}{r} - \frac{n}{r^5}$ for all integers $i \geq 2$ since $w_1 \notin X_0$. This, along with Claim 17, implies that

$$|(N(v, w_1) \cap X_2) \setminus X_0| \geq \left\lfloor \frac{n}{r^3} \right\rfloor - \left\lfloor \frac{2n}{r^5} \right\rfloor > 0.$$

Using an argument identical to that in Claim 18, we continue selecting vertices $w_j \in X_j$ for each $j \in \{3, \dots, r\}$ so that $vw_1w_2\dots w_j$ induces a copy of K_{j+1} . This is possible since for each $j \in \{3, \dots, r-1\}$,

$$|(N(v, w_1, w_2, \dots, w_{j-1}) \cap X_j) \setminus X_0| \geq \left\lfloor \frac{n}{r^3} \right\rfloor - \left\lfloor \frac{jn}{r^5} \right\rfloor > 0.$$

This would imply, however, that $vw_1\dots w_r$ induces a copy of K_{r+1} . Since the above argument only relied on $d_1(v)$ being nonzero, and v is a type 2 vertex, this implies that $d_2(v) < \frac{1}{r^3}n$ completing the proof of Claim 19. \square

Given $v \in G$ and a copy $P = vxyz$ or $P = xvyz$ of P_3 containing v , we say that P is *v-good* if none x, y , or z is contained in X_0 . The next claim will show that a type 2 vertex in G would not be contained in enough copies of P_3 to justify G being extremal.

Claim 20. *G does not contain any type 2 vertices.*

Proof. Suppose that $v \in X_0$ is a type 2 vertex. Then by symmetry, $d_1(v) \leq d_2(v) < \frac{1}{r^3}n$. Let $vu_1u_2u_3$ be a v -good path. We can count the number of these paths by considering the possible locations of u_1 . The following list will provide the location of u_1 , followed by the maximum number of paths of the form $vu_1u_2u_3$.

1. If $u_1 \in X_1$ or $u_1 \in X_2$, then there are at most $\frac{2n}{r^3}$ ways to select u_1 . Otherwise, there are at most $\frac{r-2}{r}n$ ways to select u_1 . There are $\frac{(r-1)^2}{r^2}n^2$ ways to select u_2 and u_3 since the only requirement is that each vertex cannot be in the same set as its predecessor. This gives

$$\left(\frac{2}{r^3} + \frac{r-2}{r}\right) \cdot \frac{(r-1)^2}{r^2}n^3$$

v -good copies of P_3 where v is an end point.

Next suppose that $u_1vu_2u_3$ is a v -good path. The maximum number of such paths can be counted by considering the locations of u_1 and u_2 . In each case below, we give the location of u_1 and u_2 , followed by the corresponding maximum number of P_3 subgraphs.

1. If $u_1, u_2 \in X_1$ or $u_1, u_2 \in X_2$, then there are at most $\frac{n^2}{r^6}$ ways to select each of u_1 and u_2 from either of the two sets. There are at most $\frac{r-1}{r}n$ ways to select u_3 . If $u_1 \in X_1$ and $u_2 \in X_2$ or $u_1 \in X_2$ and $u_2 \in X_1$, then there are at most then there are at most $\frac{n^2}{r^6}$ ways to select u_1 and u_2 from each of their given sets. Again, there are at most $\frac{r-1}{r}n$ ways to select u_3 . This accounts for at most

$$\frac{4}{r^6} \cdot \frac{(r-1)}{r} \cdot n^3.$$

copies of P_3 .

2. If exactly one of u_1 or u_2 is contained in $X_1 \cup X_2$, then there are at most $\frac{n}{r^3}$ ways to select that particular vertex. The vertex not in $X_1 \cup X_2$ can be selected from $(r-2)$ possible sets. Thus, there are $\frac{4(r-2)n^2}{r^4}$ ways to select u_1 and u_2 . Finally, there are at most $\frac{r-1}{r}n$ ways to select u_3 . This accounts for at most

$$\frac{4}{r^3} \cdot \frac{(r-1)(r-2)}{r^2} \cdot n^3$$

copies of P_3 .

3. If $u_1, u_2 \notin X_1 \cup X_2$, then there are at most $\frac{(r-2)(r-3)}{r^2}n^2$ ways to choose u_1 and u_2 if they are in different sets, and $\frac{(r-2)}{r^2}n^2$ ways to choose u_1 and u_2 if they are in the same set. As there are at most $\frac{r-1}{r}n$ ways to select u_3 , this accounts for at most

$$\left(\frac{(r-1)(r-2)(r-3)}{r^3} + \frac{(r-1)(r-2)}{r^3}\right) n^3$$

copies of P_3 .

There are at most $\frac{2}{r^3}n^3$ subgraphs containing v and at least one other vertex in X_0 . Thus, combining each of the terms we have calculated, we get the following upper bound:

$$\nu(v, P_3) \leq \delta_2(r) \frac{n^3}{6}.$$

where $\delta_2(r)$ is taken from Proposition 16, which then implies the following:

$$\text{OPT}_r(P_3) \binom{n-1}{3} - \nu(v, P_3) \geq \left(\frac{18}{r} - \frac{42}{r^2} - \frac{12}{r^3} \right) n^3. \quad (10)$$

It is straightforward to verify that for all $r \geq 4$,

$$\frac{18}{r} - \frac{42}{r^2} - \frac{12}{r^3} > \frac{1}{r^4}.$$

Since (10) must be true of each type 2 vertex and G is assumed to be an extremal graph, Proposition 15 implies that G cannot contain any type 2 vertices. \square

By Claim 20, each $v \in X_0$ is a type 1 vertex. We will now show that if u and v are two type 1 vertices for which $d_i(v) = d_i(u) = 0$, then u and v cannot be adjacent. Specifically, we will prove that if u and v are adjacent, then one or the other is not contained in sufficiently many P_3 subgraphs to justify G being extremal. Note this is slightly different from our approach to type 2 vertices, as we will not disprove the existence of type 1 vertices.

Claim 21. *Suppose that u and v are adjacent type 1 vertices for which $d_i(v) = d_i(u) = 0$. Then there exists some index $j \neq i$ for which*

$$|N(u, v) \cap (X_j \setminus X_0)| \leq \frac{1}{r^3} n.$$

Proof. By symmetry, we may assume that $i = 1$. Suppose, for contradiction, that $|N(u, v) \cap (X_j \setminus X_0)| > \frac{1}{r^3} n$ for all $j \in \{2, \dots, r\}$. Using an argument identical to those in Claims 18 and 19, select one vertex w_j in X_j for all $j = 2, \dots, r$, starting with X_2 , so that $w_j \in N(u, v, w_2, \dots, w_{j-1}) \cap X_j$. This is possible since

$$|(N(u, v, w_2, \dots, w_{j-1}) \cap X_j) \setminus X_0| \geq \left\lfloor \frac{n}{r^3} \right\rfloor - \left\lfloor \frac{(j-1)n}{r^5} \right\rfloor > 0$$

for all $j \in \{2, \dots, r-1\}$. After selecting vertices w_2, \dots, w_r in this way, we once again obtain a copy of K_{r+1} in G which is a contradiction. \square

Claim 22. *If u and v are two type one vertices for which $d_i(v) = d_i(u) = 0$, then u and v are not adjacent.*

Proof. By symmetry, Claim 21 implies that

$$|N(u, v) \cap (X_2 \setminus X_0)| \leq \frac{1}{r^3} n.$$

Therefore without loss of generality,

$$|(N(v) \cap (X_2 \setminus X_0)) \setminus N(u)| \leq \frac{r^2 - 1}{2r^3} n.$$

Hence,

$$d_2(v) \leq \frac{r^2 + 1}{2r^3}n.$$

Suppose that $vu_1u_2u_3$ is a v -good path. Similar to Claim 20 we can count the number of such paths by considering the location of u_1 .

1. If $u_1 \in X_2$ then there are $\frac{r^2+1}{2r^3}n$ ways to choose u_1 . Otherwise, there are $\frac{r-2}{r}n$ ways to choose u_1 . Similar to before, there are $\frac{(r-1)^2}{r^2}n^2$ ways to choose u_2 and u_3 . This accounts for at most

$$\left(\frac{r^2 + 1}{2r^3} + \frac{r - 2}{r} \right) \frac{(r - 1)^2}{r^2} n^3$$

copies of P_3 where v is an end-vertex.

Next we can count the number of v -good paths of the form $u_1vu_2u_3$ by considering the locations of u_1 and u_2 .

1. If $u_1, u_2 \in X_2$, then there are at most $\left(\frac{r^2+1}{2r^3}n \right)^2$ ways to select u_1 and u_2 . There are $\frac{(r-1)}{r}n$ ways to select u_3 . This gives an upper bound of

$$\left(\frac{r^2 + 1}{2r^3} \right)^2 \cdot \frac{(r - 1)}{r} \cdot n^3$$

copies of P_3 .

2. If exactly one of u_1 or u_2 is contained in X_2 , then there are $\frac{r^2+1}{2r^3}n$ to choose that specific vertex. Since neither of the remaining vertices can be contained in the same set as its neighbors, there are at most $\frac{(r-1)(r-2)}{r^2}n^2$ ways to choose the remaining vertices on the path. This gives at most

$$2 \cdot \frac{r^2 + 1}{2r^3} \cdot \frac{(r - 1)(r - 2)}{r^2} \cdot n^3$$

copies of P_3 .

3. If $u_1, u_2 \notin X_2$, then there are at most $\frac{(r-2)(r-3)}{r^2}n^2$ ways to choose u_1 and u_2 if they are in a different set and $\frac{(r-2)}{r^2}n^2$ ways if they are in the same set. There are $\frac{(r-1)}{r}n$ ways to select u_3 , giving an upper bound of

$$\left(\frac{(r - 1)(r - 2)(r - 3)}{r^3} + \frac{(r - 1)(r - 2)}{r^3} \right) n^3$$

copies of P_3 .

Since there are at most $\frac{2}{r^5}n^3$ copies of P_3 containing v and at least one other vertex in X_0 ,

$$\nu(v, P_3) \leq \delta_1(r) \frac{n^3}{6}.$$

Where $\delta_1(r)$ is taken from Proposition 16, which implies the following:

$$\text{OPT}_r(P_3) \binom{n-1}{3} - \nu(v, P_3) \geq \left(\frac{9}{r} - \frac{39}{2r^2} \right) n^3.$$

It is straightforward to verify that for all $r \geq 4$,

$$\frac{9}{r} - \frac{39}{2r^2} > \frac{1}{r^4}.$$

Thus, by Proposition 15, vertex v cannot exist in G under the assumption that G is extremal. Since u and v were arbitrarily chosen, this completes the proof of Claim 22. \square

Proof of Theorem 1.3(ii), continued. From Claim 22, if two vertices u and v in X_0 have the property that $d_i(u) = d_i(v) = 0$ for some $i \in [j]$, then u and v cannot be adjacent. Thus, we can take each vertex in X_0 (since each vertex is a type 1 vertex) and place it in some partite set so that G is an r -partite graph. Adding the necessary edges to make G a complete r -partite graph, however, would increase the number of P_3 subgraphs in G . As we have already shown by Proposition 13 that the Turán graph is best possible among all complete r -partite graphs, this completes the proof of Theorem 3(ii). \square

5 Concluding Remarks

The main result in this paper follows a similar approach to that used in [19], which determined that the five cycle C_5 is also K_{r+1} -Turán-good for $r \geq 3$. It is likely that this method could be applied to other graphs, perhaps including P_4 or C_6 . However, as the number of vertices in the target graph increases, the number of graphs considered in the flag algebra step grow exponentially and the number of cases in the stability result increase as well. Therefore, the authors believe a different method will need to be used to investigate the conjecture of Gerbner and Palmer that P_ℓ is K_{r+1} -Turán-good for all values of ℓ .

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6 Appendix

Link to SageMath code that can be used to verify (8) in Theorem 3(i):

<https://www.combinatorics.org/ojs/index.php/eljc/article/view/v28i4p34/data>