

On the number of sum-free triplets of sets

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Abstract

We count the ordered sum-free triplets of subsets in the group $\mathbb{Z}/p\mathbb{Z}$, i.e., the triplets (A, B, C) of sets $A, B, C \subset \mathbb{Z}/p\mathbb{Z}$ for which the equation $a + b = c$ has no solution with $a \in A, b \in B$ and $c \in C$. Our main theorem improves on a recent result by Semchankau, Shabanov, and Shkredov using a different and simpler method. Our proof relates previous results on the number of independent sets of regular graphs by Kahn; Perarnau and Perkins; and Csikvári to produce explicit estimates on smaller order terms. We also obtain estimates for the number of sum-free triplets of subsets in a general abelian group.

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1 Introduction

Let p be a prime number and $\mathbb{Z}/p\mathbb{Z}$ the abelian (additive) group of integers modulo p . We estimate the number of ordered triplets $A, B, C \subseteq \mathbb{Z}/p\mathbb{Z}$ such that there is no triplet $a \in A, b \in B$ and $c \in C$ with $a + b = c$. We call such (A, B, C) a *sum-free triplet*. Consider the auxiliary 3-uniform 3-partite hypergraph \mathcal{H} with vertex set $X \cup Y \cup Z$, each of X, Y, Z being disjoint and a copy of $\mathbb{Z}/p\mathbb{Z}$. We have an edge $\{x, y, z\}$ for $x \in X, y \in Y, z \in Z$ when $x + y = z$. The number of ordered triplets $A, B, C \subseteq \mathbb{Z}/p\mathbb{Z}$ as above is the number of independent sets $A \cup B \cup C$ in \mathcal{H} , where $A \subset X, B \subset Y$, and $C \subset Z$. For simplicity, we denote by $i(H)$ the number of independent sets in the (hyper)graph H .

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As suggested by Semchankau, Shabanov, and Shkredov in Remark 19 of [9], we try to obtain a bound of the form

$$i(\mathcal{H}) = 3 \cdot 4^p + 3p \cdot 3^p + Q_2(p)\lambda_2^p + Q_3(p)\lambda_3^p + \cdots + O((\lambda_\ell - c_*)^p), \quad (1)$$

where $Q_2(p), \dots, Q_{\ell-1}(p)$ are some polynomials in p . The motivation for (1) comes from considering the separate cases when $\min\{|A|, |B|, |C|\} = k$. For this, we fix A with $|A| = k$ and count the ways of completing the sum-free triplet (A, B, C) . However, we do not give precise estimates for all possible choices of A when $k \geq 3$. Therefore, we only obtain the upper and lower bounds below.

Theorem 1. *The number of sum-free triplets $A, B, C \subseteq \mathbb{Z}/p\mathbb{Z}$ is at most*

$$3 \cdot 4^p + 3p \cdot 3^p + 3 \binom{p}{2} \left(\frac{1 + \sqrt{5}}{2} \right)^{2p} + Q_{3,1}(p) \cdot (1 + \sqrt{2})^p + Q_{3,2}(p) \cdot 458^{p/7} + 3 \binom{p}{4} (1 + o(1)) \cdot 31^{p/4},$$

and at least

$$3 \cdot 4^p + 3p \cdot 3^p + 3 \binom{p}{2} \left(\frac{1 + \sqrt{5}}{2} \right)^{2p} + Q_{3,1}(p) \cdot (1 + \sqrt{2})^p + Q_{3,2}(p) \cdot 2.38898^p,$$

where $Q_{3,1}(p) = 3 \cdot \frac{3}{p-2} \cdot \binom{p}{3} = 3 \cdot \binom{p}{2}$, and $Q_{3,2}(p) = 3 \cdot \frac{p-5}{p-2} \cdot \binom{p}{3}$.

Remark 2. The bases of the exponential terms on Theorem 1 are $4 > 3 > \left(\frac{1+\sqrt{5}}{2} \right)^2 \approx 2.6180 > 1 + \sqrt{2} \approx 2.4142 > 458^{1/7} \approx 2.3995 > 2.38898 > 31^{1/4} \approx 2.3596$.

Remark 3. The constant 2.38998 in Theorem 1 is a lower bound on the value $\alpha + \alpha^{-1} - 2$, where α is the unique real root of $\alpha(1 - \alpha)^2 = (1 - 2\alpha)^3$. See Theorem 14 for more details.

Remark 4. One could believe that, for each i , there is one λ_i as in (1) corresponding to the case $\min\{|A|, |B|, |C|\} = i$. The proof of Theorem 1 below contradicts this idea, by showing that when $\min\{|A|, |B|, |C|\} = 3$ we have at least two distinct exponential terms, all of them greater than the possible exponential terms for $\min\{|A|, |B|, |C|\} \geq 4$.

Remark 5. The proof of Theorem 1 uses the structure of $\mathbb{Z}/p\mathbb{Z}$. In Section 7, we obtain similar estimates for a general abelian group G , see Theorems 6 and 28.

Let $(G, +)$ be an additive abelian group of order N . Using the same methods as in the case $G = \mathbb{Z}/p\mathbb{Z}$, we get the following result.

Theorem 6. *Fix $\varepsilon \in (0, 1)$. Then for sufficiently large N the following holds. Let $(G, +)$ be an additive abelian group of order N . The number of ordered triplets $A, B, C \subseteq G$ such that there is no triplet $a \in A, b \in B$ and $c \in C$ with $a + b = c$ is at most*

$$3 \cdot 4^N + 3N \cdot 3^N + 2^{(3/2+3\varepsilon)N}.$$

Remark 7. In contrast with Theorem 1, in the proof of Theorem 6 we will see that we have more triplets corresponding to the case $k \geq \varepsilon N$ than from the case $k = 2$. Indeed, the number of triplets from the case $k = 2$ is bounded by $3\binom{N}{2}7^{N/2} \ll 2^{3N/2}$, since $7^{1/2} \approx 2.64575 < 2^{3/2} \approx 2.8284$.

Remark 8. Theorem 6 is sharp for $G = (\mathbb{Z}/2\mathbb{Z})^n$, when we have at least

$$3 \cdot 4^N + 3N \cdot 3^N + 3\binom{N}{2}7^{N/2} + (\log_2 N) \cdot 2^{3N/2} - O((\log_2 N)^2 \cdot 2^{3N/4})$$

triplets, where $N = 2^n$. The first three terms come from the cases of $\min(|A|, |B|, |C|) = 0, 1, 2$, respectively. The last terms come from, as in Remark 24, the n different groups of index 2 in $(\mathbb{Z}/2\mathbb{Z})^n$ and taking overcounting into consideration.

Remark 9. In [9], Semchankau, Shabanov, and Shkredov proved that the number of sum-free triplets is

$$3 \cdot 4^N + 3N \cdot 3^N + O((3 - c_*)^N),$$

for some absolute constant $c_* > 0$. Theorems 1 and 6 is an improvement on that estimates, as we can explicit compute the constant c_* and the base of the next exponential term.

2 Overview of the paper

We will prove Theorem 1 by considering the separate cases when $\min\{|A|, |B|, |C|\} = k$. For this, for various values of k , we fix A with $|A| = k$ and count the ways of completing the sum-free triplet (A, B, C) . The link graph of A , denoted by \mathcal{H}_A , is the bipartite graph with vertex set $V_1 \cup V_2$ and edge set the pairs $\{y, z\}$ where $x + y = z$ for some $x \in A$. For each fixed small A , we will count the independent sets $i(\mathcal{H}_A)$ in the link graph of A . After taking overcounting into consideration, we conclude that the number of sum-free triplets (A, B, C) with $\min\{|A|, |B|, |C|\} = k$ is $Q_k(p)i(\mathcal{H}_A) + O(p^{2k} \cdot 2^p)$ for some polynomial Q_k in p .

The first cases, when $k = 0$ or 1 , were dealt with in [9], and we include them here for completeness. In Section 3, we precisely (up to the first-order term) count the independent sets for each of the cases $k \leq 3$. From the cases $k \leq 2$, the number of sum-free triplets is

$$3 \cdot 4^p + 3p \cdot 3^p + 3\binom{p}{2} \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^{2p} + O(p^4 2^p). \quad (2)$$

For $k = 3$, we will see that the main term is $Q_{3,1}(p) \cdot (1 + \sqrt{2})^p$ for some polynomial $Q_{3,1}(p)$, and it comes from choices of A for which the link graph \mathcal{H}_A contains cycles C_4 as a subgraph. To show that, we apply a general result (namely, Theorem 12) for 3-regular graphs with girth at least 5 by Perarnau and Perkins [6] to upper bound the number of independent sets in the case the link graph does not contain C_4 as a subgraph. We obtain a lower bound on the number of independent sets by using a general result (namely, Theorem 14) for regular graphs by Csikvári [3].

In Section 4, we deal with the cases $4 \leq k \leq \varepsilon p$ for any fixed $\varepsilon < 1/10^3$. We make use of an upper bound for general regular graphs (see Theorem 17), by Kahn [5], to conclude that the number of sum-free triplets, in this case, is at most $3\binom{p}{4}(1+o(1)) \cdot 31^{p/4}$, lower than the number of sum-free triplets for $k = 3$.

In Section 5, we use the method of hypergraph containers to give an upper bound on the number of independent sets in the remaining case $k \geq \varepsilon p$. The idea is to use a supersaturation result for the number of sums $x + y = z$ with $x \in A$, $y \in B$, and $z \in C$ when A, B, C are large, together with the Hypergraph Container Lemma (Theorem 22) to obtain an upper bound on the number of independent sets of \mathcal{H} with a large intersection with each of the parts V_0 , V_1 and V_2 . Our method generalizes for other groups as well. Thus, we show the estimates for general abelian groups G in Section 5, and we briefly discuss the case when $G = \mathbb{Z}/p\mathbb{Z}$ in the beginning of Section 6, where we conclude the proof of Theorem 1.

Finally, in Section 7, we prove Theorem 6 and discuss the differences for the estimates for general abelian groups. We highlight that if one knows the number of elements of each order in G , a careful look at our method can lead us to precise estimates in the case $k = 2$. Subsequently, we examine the behavior of the estimates we obtain for different groups G , depending on the prime factorization of its order.

3 Smaller cases: $k \leq 3$

When $|A| = 0$, every choice of B and C works. In this case, $i(\mathcal{H}_A) = 4^p$. In total, we have $3 \cdot 4^p - 3 \cdot 2^p + 1$ triplets with at least one of the sets empty.

When $|A| = 1$, the link graph \mathcal{H}_A is a matching of size p , which has 3^p independent sets. In total, we have $3p \cdot (3^p - 1) - 3p^2 \cdot (2^{p-1} - 1) + p^2(p - 1)$ triplets with $\min\{|A|, |B|, |C|\} = 1$.

Note that, for each fixed k , we need to subtract a factor of $O(p^{2k} \cdot 2^p)$ from the counting of triplets (A, B, C) to avoid overcounting the cases when at least two of the sets A , B or C have at most k elements. For the sake of simplicity, from now on, we will focus only on counting $i(\mathcal{H}_A)$. Hence, we will obtain a bound of the form $Q_k(p)i(\mathcal{H}_A) + O(p^{2k} \cdot 2^p)$ for some polynomial Q_k in p . In what follows, $k = |A|$, and we count the independent sets $i(\mathcal{H}_A)$ in the link graph of A .

When $|A| = 2$, the link graph \mathcal{H}_A is a Hamiltonian cycle. We need the following easy claim.

Claim 10. *The number of independent sets in a cycle of length $2n$ is $\phi^{2n} + \phi^{-2n}$, where $\phi = \frac{1+\sqrt{5}}{2}$.*

Proof. Denote by F_{m+2} the number of independent sets in an m -vertex path. We break the counting into cases according to whether a fixed vertex is in the independent set or not. The number of independent sets in the cycle is then $F_{2n+1} + F_{2n-1}$. To count the independent sets in a path we repeat the procedure, breaking into cases of when an endpoint of the path is in the independent set or not. We obtain $F_{m+2} = F_{m+1} + F_m$, $F_2 = 1$, and $F_3 = 2$. Therefore, F_m is the m -th Fibonacci number.

In this case, $i(\mathcal{H}_A) = \frac{\phi^{2n+1} + \phi^{-2n-1}}{\sqrt{5}} + \frac{\phi^{2n-1} + \phi^{-2n+1}}{\sqrt{5}} = \phi^{2n} + \phi^{-2n}$, where we used $\phi + \phi^{-1} = \sqrt{5}$ for the last equality. \square

In total, we have $3\binom{p}{2} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{2p} + O(p^4 2^p)$ triplets with $\min\{|A|, |B|, |C|\} = 2$. This implies that $Q_2(p) = \frac{3p(p-1)}{2}$ and $\lambda_2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 \approx 2.6180$ on (1) above.

3.1 Case $k = 3$

When $|A| = 3$, we assume without loss of generality that $A = \{0, 1, x\}$ and break into cases depending on whether \mathcal{H}_A contains a C_4 or not. For this, let us first prove the following.

Claim 11. *Let $A = \{0, 1, x\}$. The link graph \mathcal{H}_A contains C_4 as a subgraph if and only if $x \in \{-1, 2, \frac{p+1}{2}\}$.*

Proof. We write (Y, t) for the vertex corresponding to $t \in \mathbb{Z}/p\mathbb{Z}$ in B and, similarly, (Z, t) for $t \in \mathbb{Z}/p\mathbb{Z}$ in C . Since the graph \mathcal{H}_A is vertex transitive, it is enough to check when $(Y, 1)$ is in a C_4 . The neighborhood of $(Y, 1)$ is $(Z, 1), (Z, 2), (Z, x+1)$. The second neighborhood of $(Y, 1)$ is $(Y, 0), (Y, 1-x), (Y, 2), (Y, 2-x), (Y, x), (Y, x+1)$. A vertex is contained in a C_4 if and only if two distinct neighbors have one extra common neighbor. Therefore, $(Y, 1)$ is contained in a C_4 if and only if two of the above second neighbors coincide. This is equivalent to one of the following equations to hold (we include trivial equations here only for the sake of completeness of the proof)

$$\begin{array}{lll} 0 = 2 & 1 - x = 2 & 2 = x \\ 0 = 2 - x & 1 - x = 2 - x & 2 = x + 1 \\ 0 = x & 1 - x = x & 2 - x = x \\ 0 = x + 1 & 1 - x = x + 1 & 2 - x = x + 1 \end{array}$$

Removing the trivial cases $x = 0$ and $x = 1$, the only solutions are $x \in \{-1, 2, \frac{p+1}{2}\}$. \square

Therefore, when $x \notin \{-1, 2, \frac{p+1}{2}\}$ we make use of the following result.

Theorem 12 (Perarnau-Perkins, Corollary 6 in [6]). *Let $H_{3,6}$ be the point-line incidence graph of the Fano plane, also called the Heawood graph. For any cubic n -vertex graph H of girth at least 5,*

$$i(H)^{1/n} \leq i(H_{3,6})^{1/14} = 458^{1/14},$$

with equality if and only if H is a union of copies of $H_{3,6}$.

By combining the previous two results, we obtain the following when \mathcal{H}_A is bipartite and C_4 -free.

Corollary 13. *If $A = \{0, 1, x\}$ and $x \notin \{-1, 2, \frac{p+1}{2}\}$, then*

$$i(\mathcal{H}_A) \leq 458^{p/7} \approx 2.3995^p.$$

The Heawood graph, the point-line incidence graph of the Fano plane, is the link graph \mathcal{H}_A when $p = 7$ and $x = 3$, which means we have equality in one case above. For larger values of p , the next result tells us we are not giving away too much by using this upper bound, as it implies $i(\mathcal{H}_A) \geq (2.38898\dots)^p$.

Theorem 14 below follows from Theorem 8.1 of [3], which follows from a result in [7]. Theorem 8.1 of [3] is more general as it states that the infinite regular tree minimizes the normalized independence polynomial over all regular graphs. Here, we only state its immediate consequence for the total number of independent sets.

Theorem 14. *Let α be the unique solution of*

$$\frac{\alpha}{1-\alpha} = \left(\frac{1-2\alpha}{1-\alpha} \right)^d$$

in the interval $[0, 1/2]$. Let H be a d -regular bipartite graph on n vertices. Then

$$i(H) \geq \left(\frac{(1-\alpha)^{d-1}}{\alpha} \right)^{n/2}.$$

Notice we make use of this result for the 3-regular graph \mathcal{H}_A . When $d = 3$, the unique real solution of $x(1-x)^2 = (1-2x)^3$ is $\alpha \approx 0.24109\dots$, which implies $i(\mathcal{H}_A) \geq (2.38898\dots)^n$.

Now, we restrict ourselves to the case $A = \{0, 1, x\}$ with $x \in \{-1, 2, \frac{p+1}{2}\}$. We will use the same notation as in Claim 11, i.e., we write (Y, t) for the vertex corresponding to $t \in \mathbb{Z}/p\mathbb{Z}$ in B and, similarly, (Z, t) for $t \in \mathbb{Z}/p\mathbb{Z}$ in C . In the next claims, we will see that the link graphs \mathcal{H}_A are isomorphic for $x \in \{-1, 2, \frac{p+1}{2}\}$ and then we may assume, without loss of generality, that we have $A = \{-1, 0, 1\}$ and count the independent sets $i(\mathcal{H}_A)$ of the link graph \mathcal{H}_A .

Claim 15. *The three graphs \mathcal{H}_A are isomorphic when $A = \{0, 1, x\}$ and $x \in \{-1, 2, \frac{p+1}{2}\}$.*

Proof. Consider the function φ given by

$$\varphi((X, j)) = \begin{cases} \left(Y, \frac{j(p+1)}{2} \right) & \text{if } X = Y \\ \left(Z, \frac{(j+1)(p+1)}{2} \right) & \text{if } X = Z \end{cases}$$

with the second coordinate taken modulo p .

The function φ is a graph isomorphism between the graph obtained for $x = -1$ and the one from $x = (p+1)/2$, since it sends (Y, j) to $(Y, j \cdot \frac{p+1}{2})$ and the neighbors $(Z, j-1), (Z, j), (Z, j+1)$ of (Y, j) for $x = -1$ to $(Z, j \cdot \frac{p+1}{2}), (Z, j \cdot \frac{p+1}{2} + \frac{p+1}{2}), (Z, j \cdot \frac{p+1}{2} + 1)$, neighbors of $(Y, j \cdot \frac{p+1}{2})$ for $x = \frac{p+1}{2}$.

Consider the function θ given by

$$\theta((X, j)) = \begin{cases} (Y, j) & \text{if } X = Y \\ (Z, j+1) & \text{if } X = Z \end{cases}$$

with the second coordinate taken modulo p . Similarly, θ is a graph isomorphism between the graph obtained for $x = -1$ and the one from $x = 2$. \square

Similarly to Claim 10, we can use recursion to count the independent sets in the graph \mathcal{H}_A when $A = \{-1, 0, 1\}$. We obtain the following Claim, which we prove, using generating function, in the Appendix.

Claim 16. *Let $A = \{-1, 0, 1\}$. For $p \geq 3$, the link graph \mathcal{H}_A has*

$$i(\mathcal{H}_A) = (1 + \sqrt{2})^p + (1 - \sqrt{2})^p + 1.$$

For the polynomial terms $Q_{3,1}(p)$ and $Q_{3,2}(p)$ in Theorem 1, note that if we choose 2 elements in A we have 3 possible options for the third element to form a graph isomorphic to the one with $A = \{-1, 0, 1\}$. We count each triplet 3 times this way, so we get $Q_{3,1}(p) = 3\binom{p}{2} = 3 \cdot \frac{3}{p-2} \cdot \binom{p}{3}$ possible sets corresponding to link graphs isomorphic to \mathcal{H}_A with $A = \{-1, 0, 1\}$. The number of remaining 3-sets is $Q_{3,2}(p) = 3\binom{p}{3} - Q_{3,1}(p) = 3 \cdot \frac{p-5}{p-2} \cdot \binom{p}{3}$.

In total, the number of triplets with $\min\{|A|, |B|, |C|\} = 3$ is bounded from above by

$$Q_{3,1}(p) \cdot (\sqrt{2} + 1)^p + Q_{3,2}(p) \cdot 458^{p/7} + O(p^6 2^p), \quad (3)$$

and from below by

$$Q_{3,1}(p) \cdot (\sqrt{2} + 1)^p + Q_{3,2}(p) \cdot (2.38898 \dots)^p + O(p^6 2^p). \quad (4)$$

4 Small case: $4 \leq k \leq \varepsilon p$

To estimate the triplets with $\min\{|A|, |B|, |C|\} = k$ for $k \geq 4$ we use the following result.

Theorem 17 (Kahn, [5]). *If H is a bipartite d -regular graph on n vertices, then*

$$i(H) \leq i(K_{d,d})^{n/2d} = (2^{d+1} - 1)^{n/2d}.$$

Hence, the number of independent sets in a 4-regular bipartite graph on $2p$ vertices is at most $31^{p/4}$, and in a k -regular bipartite graph on $2p$ vertices for $k \geq 5$ is at most $63^{p/5}$.

Therefore, the number of triplets for which $4 \leq k \leq \varepsilon p$ is at most

$$3\binom{p}{4} 31^{p/4} + \sum_{5 \leq k \leq \varepsilon p} 3\binom{p}{k} \cdot 63^{p/5} \leq 3\binom{p}{4} 31^{p/4} + 3 \cdot 63^{p/5} \cdot \sum_{k \leq \varepsilon p} \binom{p}{k}.$$

For the sum $\sum_{k \leq \varepsilon p} \binom{p}{k}$, we use the standard binomial estimates $\binom{n}{k} \leq (en/k)^k$ to obtain

$$\sum_{k \leq \varepsilon p} \binom{p}{k} \leq 2 \cdot (e^{\varepsilon(1+\log(1/\varepsilon))})^p.$$

We conclude that, for $\varepsilon < 1/10^3$, the number of triplets with $4 \leq \min\{|A|, |B|, |C|\} \leq \varepsilon p$ is at most

$$\left(3\binom{p}{4} + o(1)\right) 31^{p/4}. \quad (5)$$

5 Big case: $k \geq \varepsilon p$

To count the triplets (A, B, C) with $|A|, |B|, |C| \geq \varepsilon p$, we use the method of hypergraph containers developed by Balogh, Morris, and Samotij [2], and, independently, by Saxton and Thomasson [8]. Since its development, the method has found several applications, in particular for counting sum-free sets [1]. We highlight that a container theorem for sum-free subsets of abelian groups was found earlier by Green and Ruzsa (see Proposition 2.1 of [4]) using Fourier analysis. Here, we need a 3-partite variant of their result, hence we make use of the container lemma for general hypergraphs.

Throughout this section, we will state and prove the results for a general abelian group G and at the end of the section, we discuss the differences when $G = \mathbb{Z}/p\mathbb{Z}$.

5.1 Supersaturation

To use the container method, we need the supersaturation result (Proposition 20) below. We highlight that Propositions 20 and 25 are very similar in essence to Remark 20 of [9].

Definition 18. Given $A, B \subseteq G$ with G an abelian group and given $\varepsilon \in (0, 1)$, we write $A +_\varepsilon B$ for the set of $x \in G$ having at least $\varepsilon|G|$ representations as a sum $a + b = x$ with $a \in A$ and $b \in B$.

As in [9], we obtain a supersaturation result as a corollary of the following Lemma.

Lemma 19 (Theorem 6 in [9]). *Let G be an abelian group, and $A, B \subset G$ be sets. Let $\varepsilon \in (0, 1)$ be such that $\sqrt{\varepsilon}|G| < |A|, |B|$. Let H be a maximal proper subgroup of G . Then*

$$|A +_\varepsilon B| \geq \min\{|G|, |A| + |B| - |H|\} - 3\sqrt{\varepsilon}|G|.$$

Proposition 20. *Fix $\varepsilon \in (0, 1)$. Then for sufficiently large N the following holds. Let G be an abelian group of order N and let $A, B, C \subset G$ be sets. There exists $\sigma > 0$ such that if $|A|, |B|, |C| \geq \varepsilon N$ and $|A| + |B| + |C| \geq \left(\frac{3}{2} + \varepsilon\right)N$ then there are at least σN^2 sums $x + y = z$ with $x \in A$, $y \in B$, and $z \in C$.*

Proof. We know that $|A|, |B| \geq \varepsilon N > (\varepsilon/4)N$. By Lemma 19, we get

$$|A +_{\varepsilon^2/16} B| \geq |A| + |B| - |H| - \left(\frac{3\varepsilon}{4}\right)N.$$

Since H is a proper subgroup, we have $|H| \leq N/2$ and then

$$|A +_{\varepsilon^2/16} B| \geq |A| + |B| - \frac{N}{2} - \left(\frac{3\varepsilon}{4}\right)N.$$

We obtain

$$\begin{aligned} |C \cap (A +_{\varepsilon^2/16} B)| &\geq |C| + |(A +_{\varepsilon^2/16} B)| - N \geq \\ &\left(\left(\frac{3}{2} + \varepsilon\right)N - |A| - |B|\right) + \left(|A| + |B| - \frac{N}{2} - \left(\frac{3\varepsilon}{4}\right)N\right) - N = \left(\frac{\varepsilon}{4}\right)N. \end{aligned}$$

Then there are at least $(\varepsilon^3/64)N^2$ sums of the form $x + y = z$ with $x \in A$, $y \in B$ and $z \in C$. This concludes the proof of the proposition with $\sigma = \varepsilon^3/64$. \square

5.2 Containers

Now, we state the Hypergraph Container Lemma (Theorem 22) and obtain the upper bound on the number of triplets (A, B, C) with $|A|, |B|, |C| \geq \varepsilon p$. First, we need some definitions.

Definition 21. Fix a r -regular hypergraph \mathcal{G} with average degree d ; fix $v \in V(\mathcal{G})$; fix $2 \leq j \leq |V(\mathcal{G})|$ and $\tau > 0$. Define

$$d^{(j)}(v) := \max\{|S| : v \in A \subseteq V(\mathcal{G}), |S| = j\}.$$

Define δ_j with the equation

$$\delta_j \tau^{j-1} d |V(\mathcal{G})| = \sum_v d^{(j)}(v).$$

Finally, define the *co-degree* function $\delta(\mathcal{G}, \tau)$ with

$$\delta(\mathcal{G}, \tau) := 2^{\binom{r}{2}-1} \sum_{j=2}^r 2^{-\binom{j-1}{2}} \delta_j.$$

If $d = 0$ define $\delta(\mathcal{G}, \tau) = 0$.

Theorem 22 (Corollary 3.6 in [8]). *Let \mathcal{G} be an r -uniform hypergraph with vertex set $[N]$. Let $0 < \varepsilon, \tau < 1/2$. Suppose that $\tau < 1/(200 \cdot r \cdot r!^2)$ and $\delta(\mathcal{G}, \tau) \leq \varepsilon/(12r!)$. Then there exists $c = c(r) \leq 1000 \cdot r \cdot r!^3$ and a collection \mathcal{C} of vertex subsets such that*

- (i) *every independent set in \mathcal{G} is a subset of some $D \in \mathcal{C}$;*
- (ii) *for every $D \in \mathcal{C}$, $e(\mathcal{G}[D]) \leq \varepsilon e(\mathcal{G})$;*
- (iii) *$\log |\mathcal{C}| \leq cN\tau \cdot \log(1/\varepsilon) \cdot \log(1/\tau)$.*

Claim 23. *Fix $\varepsilon \in (0, 1)$. Then for sufficiently large N , the number of triplets $A, B, C \subset G$ with $\min\{|A|, |B|, |C|\} \geq \varepsilon N$ is at most $2^{\left(\frac{3}{2}+2\varepsilon\right)N}$.*

Proof. Fix $\varepsilon \in (0, 1)$. Using Proposition 20, we obtain a $\sigma > 0$. We choose $\varepsilon_0 > 0$ sufficiently small, in particular with $\varepsilon_0 < \sigma$. We apply Theorem 22 to the 3-uniform hypergraph \mathcal{H} . First, realize that every vertex in \mathcal{H} has degree N , hence the average degree of \mathcal{H} is N . For any $v \in V(\mathcal{H})$, we have $d^{(2)}(v) = d^{(3)}(v) = 1$. Therefore,

$$\delta_2 \tau^1 \cdot 3N^2 = \sum_v d^{(2)}(v) = 3N \quad \Rightarrow \quad \delta_2 = 1/N\tau,$$

$$\delta_3 \tau^2 \cdot 3N^2 = \sum_v d^{(3)}(v) = 3N \quad \Rightarrow \quad \delta_3 = 1/N\tau^2.$$

We obtain that

$$\delta(\mathcal{H}, \tau) = \frac{4}{N\tau} + \frac{2}{N\tau^2}.$$

Take $\tau = 400\varepsilon_0^{-1}N^{-1/2}$, so that $\delta(\mathcal{H}, \tau) = \frac{4}{N\tau} + \frac{2}{N\tau^2} \leq \frac{\varepsilon_0}{72}$. We can use Theorem 22 with the given τ and ε_0 to get a collection of containers \mathcal{C} with size

$$\log |\mathcal{C}| \leq c \cdot \varepsilon_0^{-1} N^{1/2} \cdot \log(1/\varepsilon_0) \cdot \log(1/\tau) \leq C_{\varepsilon_0} N^{1/2} \log(N),$$

with C_{ε_0} some positive constant depending on ε_0 .

Now, suppose we fix a container $D \in \mathcal{C}$ with $D \cap X = A^*$, $D \cap Y = B^*$ and $D \cap Z = C^*$. Note that the independent sets contained in containers with $\min\{|A^*|, |B^*|, |C^*|\} < \varepsilon|G|$ are not included in the count for this claim, we already counted them in the previous sections. Assume, then, that $|A^*|, |B^*|, |C^*| \geq \varepsilon N$.

If $|A^*| + |B^*| + |C^*| \geq (3/2 + \varepsilon)N$, then by Proposition 20, we get a contradiction with $e(\mathcal{H}[D]) \geq \sigma N^2$ and $e(\mathcal{H}[D]) \leq \varepsilon_0 e(\mathcal{H}) < \sigma N^2$. Then $|A^*| + |B^*| + |C^*| \leq (\frac{3}{2} + \varepsilon)N$, and the number of independent sets contained in D is at most the number of subsets in it, $2^{(3/2 + \varepsilon)N}$. Therefore, for sufficiently small ε_0 , we have at most

$$2^{(3/2 + \varepsilon)N + C_{\varepsilon_0} N^{1/2} \log(N)} < 2^{(3/2 + 2\varepsilon)N}$$

independent sets. □

Remark 24. As mentioned in Remark 22 of [9], the exponent $3/2$ in Claim 23 is best possible. If H is a subgroup of index 2, $A, B \subset H$ and $C \subset G \setminus H$, then (A, B, C) is a sum-free triple. Therefore, in the case G has a subgroup of index 2, there are at least $2^{3N/2}$ triplets.

6 Proof of Theorem 1

When $G = \mathbb{Z}/p\mathbb{Z}$, there is only one proper subgroup H for which $|H| = 1$. Hence, adapting the proof of Proposition 20 we obtain the following.

Proposition 25. *Fix $\varepsilon \in (0, 1)$. Then for sufficiently large p the following holds. Let $A, B, C \subset \mathbb{Z}/p\mathbb{Z}$ be sets. There exists $\sigma > 0$ such that if $\varepsilon p < |A|, |B|, |C|$ and $|A| + |B| + |C| \geq (1 + \varepsilon)p$ then there are at least σp^2 sums $x + y = z$ with $x \in A$, $y \in B$, and $z \in C$.*

An alternative way to prove Proposition 25 directly is by using the following Lemma.

Lemma 26 (Theorem 5 in [9]). *Let $A, B \subset \mathbb{Z}/p\mathbb{Z}$ be sets and $\varepsilon \in (0, 1)$ be a real number, such that $\sqrt{\varepsilon}p < |A|, |B|$. Then*

$$|A +_{\varepsilon} B| \geq \min\{p, |A| + |B|\} - 2p\sqrt{\varepsilon}.$$

Similarly to the proof of Claim 23, but now using Proposition 25, we obtain the following.

Claim 27. Fix $\varepsilon \in (0, 1)$. Then for sufficiently large p , the number of sum-free triplets $A, B, C \subset \mathbb{Z}/p\mathbb{Z}$ with $\min\{|A|, |B|, |C|\} \geq \varepsilon p$ is at most

$$2^{(1+2\varepsilon)p}. \quad (6)$$

Now that we covered all the possible cases for k , we can proceed to prove Theorem 1.

Proof of Theorem 1. Using the same setting described in the introduction, we count the ordered sum-free triplets (A, B, C) such that $\min\{|A|, |B|, |C|\} = k$. We have an upper bound for the number of triplets in the cases $k = 0, 1, 2$ on (2); for $k = 3$ on (3); for $4 \leq k \leq \varepsilon N$ on (5) and finally for $k \geq \varepsilon N$ on (6). Summing up the upper bounds, for $\varepsilon < 1/10^3$, we obtain that there are at most

$$3 \cdot 4^p + 3p \cdot 3^p + 3 \binom{p}{2} \left(\frac{1 + \sqrt{5}}{2} \right)^{2p} + Q_{3,1}(p) \cdot (1 + \sqrt{2})^p + Q_{3,2}(p) \cdot 458^{p/7} + 3 \binom{p}{4} (1 + o(1)) \cdot 31^{p/4}$$

many triplets, where we add the missing terms $2^{(1+2\varepsilon)p}$ and $O(p^6 2^p)$ to the $o(31^{p/4})$ term.

Now, for the lower bound, the triplets for the cases $k = 0, 1, 2$ are counted on (2). For the triplets in the case $k = 3$, we obtained a lower bound on (4). The amount of triplets that are overcounted in both calculations is upper bounded by $9 \binom{p}{3} 2^p$. This error term is omitted in the final result, since we truncate the value of $(1 - \alpha)^2 / \alpha$ and write 2.38898 instead. The sum of the two lower bounds shows there are at least

$$3 \cdot 4^p + 3p \cdot 3^p + 3 \binom{p}{2} \left(\frac{1 + \sqrt{5}}{2} \right)^{2p} + Q_{3,1}(p) \cdot (1 + \sqrt{2})^p + Q_{3,2}(p) \cdot 2.38898^p$$

many triplets. □

7 General abelian groups

Let $(G, +)$ be an additive abelian group of order N . Using the same methods as in the case $G = \mathbb{Z}/p\mathbb{Z}$, we now prove the following similar result.

Theorem 6. Fix $\varepsilon \in (0, 1)$. Then for sufficiently large N the following holds. Let $(G, +)$ be an additive abelian group of order N . The number of ordered triplets $A, B, C \subseteq G$ such that there is no triplet $a \in A$, $b \in B$ and $c \in C$ with $a + b = c$ is at most

$$3 \cdot 4^N + 3N \cdot 3^N + 2^{(3/2+3\varepsilon)N}.$$

Proof. Assume that for a fixed integer k we have $\min\{|A|, |B|, |C|\} = k$ and $|A| = k$. When $k = 0$ and $k = 1$ we obtain the same number of triplets A, B, C as in $G = \mathbb{Z}/p\mathbb{Z}$. We get exactly

$$3 \cdot 4^N - 3 \cdot 2^N + 1 \quad (7)$$

triplets for $k = 0$ and

$$3N \cdot (3^N - 1) - 3N^2 \cdot (2^{N-1} - 1) + N^2(N - 1) \quad (8)$$

triplets for $k = 1$.

As in Section 4, we use Theorem 17 to upper bound the triplets with $2 \leq k \leq \varepsilon N$. The number of independent sets in a 2-regular bipartite graph on $2N$ vertices is at most $7^{N/2}$, and in a k -regular bipartite graph on $2N$ vertices for $k \geq 3$ is at most $15^{N/3}$. Therefore, for sufficiently small ε , the number of triplets with $2 \leq \min\{|A|, |B|, |C|\} \leq \varepsilon N$ is at most

$$\left(3 \binom{N}{2} + o(1)\right) 7^{N/2}. \quad (9)$$

For $k \geq \varepsilon N$, by Claim 23, we obtain that the number of triplets is at most

$$2^{(3/2+2\varepsilon)N}. \quad (10)$$

The upper bounds in (7), (8), (9) and (10) give together the desired upper bound for the total number of triplets. Note that even that we obtained the bound (9) only for small ε , the final result for all $\varepsilon \in (0, 1)$ follows from it. \square

7.1 More precise computations in the general case

Write $N = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ where $p_1 < \cdots < p_t$ are prime numbers and $\alpha_i \geq 1$. Then, we can write $G = G_1 \times \cdots \times G_t$, where $|G_i| = p_i^{\alpha_i}$. Given p_1 , we can get better estimates for the upper and lower bounds in the number of ordered triplets (A, B, C) . We highlight that the following Theorem implies Theorem 6.

Theorem 28. *Fix $\varepsilon \in (0, 1)$. Then for sufficiently large N the following holds. Let $(G, +)$ be an additive abelian group of order $N = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$. Let $\varepsilon \in (0, 1)$. The number of ordered triplets $A, B, C \subseteq G$ such that there is no triplet $a \in A$, $b \in B$ and $c \in C$ with $a + b = c$ is at most*

$$3 \cdot 4^N + 3N \cdot 3^N + 2^{(1+1/p_1+2\varepsilon)N} + 3 \binom{N}{2} (\phi^{2p_1} + \phi^{-2p_1})^{N/p_1} + 3 \binom{N}{3} (1 + o(1)) 15^{N/3},$$

and at least

$$3 \cdot 4^N + 3N \cdot 3^N + 2^{(1+1/p_1)N} + \frac{3N}{2} (\phi^{2p_1} + \phi^{-2p_1})^{N/p_1}.$$

Proof. We proceed as in the proof of Theorem 6 and obtain the same bounds when $k = 0, 1$. When $k = 2$, we write $A = \{x, y\}$. In this case, we have that \mathcal{H}_A is the disjoint union of cycles of length $2 \cdot o(x - y)$, where $o(x - y)$ denotes the order of the element $x - y$ in G . We count the independent sets in \mathcal{H}_A using Claim 10.

We conclude that there are

$$\sum_{t=2}^N \frac{a_t}{N-1} \cdot 3 \binom{N}{2} (\phi^{2t} + \phi^{-2t})^{N/t} = \frac{3N}{2} \phi^{2N} \sum_{t=2}^N a_t \cdot (1 + \phi^{-4t})^{N/t} \quad (11)$$

triplets with $\min\{|A|, |B|, |C|\} = 2$, where a_t is the number of elements of order t in G . As $(1 + \phi^{-4x})^{N/x}$ is a decreasing function in x and $a_t = 0$ for $1 < t < p_1$, the maximum value of the sum in (11) is obtained when $a_{p_1} = N - 1$. Thus, for $k = 2$, the number of triplets is at most

$$3 \binom{N}{2} (\phi^{2p_1} + \phi^{-2p_1})^{N/p_1}. \quad (12)$$

Moreover, as p_1 divides $|G|$, there is at least one element of order p_1 in G . Hence, we have at least

$$\frac{3N}{2} (\phi^{2p_1} + \phi^{-2p_1})^{N/p_1} \quad (13)$$

many triplets.

When $3 \leq k \leq \varepsilon N$ and ε is small, we follow the same method as in Section 4, which gives that there are at most

$$\left(3 \binom{N}{3} + o(1) \right) 15^{N/3} \quad (14)$$

many triplets.

For the case $k \geq \varepsilon N$, we note that the maximum order of a proper subgroup is N/p_1 . Similarly to the proof of Claim 23, by Lemma 19 with H of order $|H| = N/p_1$, we obtain that there are at most

$$2^{(1+1/p_1+2\varepsilon)N} \quad (15)$$

many triplets.

For a lower bound, notice that if H is a subgroup of index p_1 , $A, B \subset H$ and $C \subset G \setminus H$, then (A, B, C) is a sum-free triple. Therefore there are at least

$$2^{(1+1/p_1)N} \quad (16)$$

many triplets.

Similarly to the proof of Theorem 6, we conclude by adding the bounds for all possible k . From (7), (8), (12), (14) and (15), we obtain that for small enough ε there are at most

$$3 \cdot 4^N + 3N \cdot 3^N + 2^{(1+1/p_1+2\varepsilon)N} + 3 \binom{N}{2} (\phi^{2p_1} + \phi^{-2p_1})^{N/p_1} + 3 \binom{N}{3} (1 + o(1)) 15^{N/3}$$

triplets, and from (7), (8), (13) and (16), we obtain that there are at least

$$3 \cdot 4^N + 3N \cdot 3^N + 2^{(1+1/p_1)N} + \frac{3N}{2} (\phi^{2p_1} + \phi^{-2p_1})^{N/p_1}$$

many triplets. Finally, note that the upper bound for all $\varepsilon \in (0, 1)$ follows from the upper bound for small ε . \square

At last, we discuss the estimates we obtain from Theorem 28 for small values of p_1 . For $p_1 = 2$, we have more triplets coming from the $k \geq \varepsilon N$ case, since

$$2^{1+1/2} = 8^{1/2} > 7^{1/2} = (\phi^4 + \phi^{-4})^{1/2} \approx 2.64575 > 15^{1/3} \approx 2.466.$$

For $p_1 = 3$, we have more triplets coming from $k = 2$ case, then $k \geq \varepsilon N$, then $k = 3$, since

$$(\phi^6 + \phi^{-6})^{1/3} = 18^{1/3} > 16^{1/3} = 2^{1+1/3} > 15^{1/3}.$$

For $p_1 = 5$, we have more triplets coming from $k = 2$ case, then $k = 3$, then $k \geq \varepsilon N$, since

$$(\phi^{10} + \phi^{-10})^{1/5} = 123^{1/5} \approx 2.618 > 15^{1/3} \approx 2.466 > 2^{1+1/5} \approx 2.297.$$

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A Proof of Claim 16

Claim 16. Let $A = \{-1, 0, 1\}$. For $p \geq 3$, the link graph \mathcal{H}_A has

$$i(\mathcal{H}_A) = (1 + \sqrt{2})^p + (1 - \sqrt{2})^p + 1.$$

Proof. We count the independent sets in \mathcal{H}_A recursively. First, assume $(Y, 0)$ is part of the independent set. So $(Z, -1)$, $(Z, 0)$ and $(Z, 1)$ are not in the independent set I . Let G_p denote the graph obtained by removing $(Y, 0)$ and $N((Y, 0))$ from the original graph \mathcal{H}_A . Let a_p denote the number of independent sets on G_p .

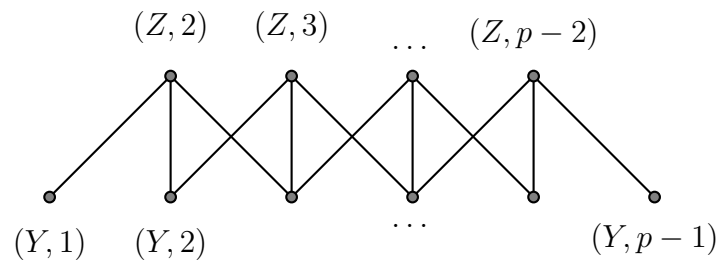


Figure 1: The graph G_p .

From here we will calculate a_p using recurrence. Assume $(Y, 1)$ is in the independent set. The graph obtained by erasing $(Y, 1)$ and $N((Y, 1))$ from G_p is isomorphic to G_{p-1} . If neither $(Y, 1)$ and $(Z, 2)$ are part of the independent set, erasing them we get again a graph isomorphic to G_{p-1} .

The remaining case is when $(Z, 2)$ belongs to the independent set. Let H_p be the graph obtained after erasing $(Z, 2)$ and its neighbors from G_p , and let b_p be the number of independent sets in H_p .

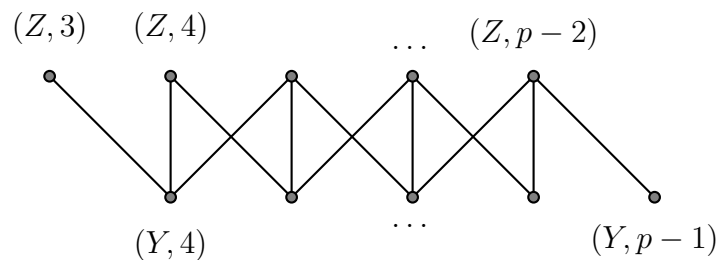


Figure 2: The graph H_p .

Then we have

$$a_p = 2a_{p-1} + b_p.$$

Now we proceed with the same recursive method for b_p . If $(Z, 3)$ is part of the independent set, by erasing it and its neighbors we get a graph isomorphic to H_{p-1} . If neither $(Z, 3)$ nor $(Y, 4)$ belong to I , by erasing them we get a graph isomorphic to H_{p-1} .

Finally, if $(Y, 4)$ belongs to the independent set we get a graph isomorphic to G_{p-4} , so

$$b_p = 2b_{p-1} + a_{p-4}.$$

We can calculate $a_3 = 4, a_4 = 9$ and $b_4 = 1, b_5 = 3, b_6 = 8$. Define $a_2 := 2$, so the equation $b_6 = 2b_5 + a_2$ holds. Let $F(x) = \sum_{p \geq 2} a_p x^p$ and $G(x) = \sum_{p \geq 4} b_p x^p$.

The equation $b_p = 2b_{p-1} + a_{p-4}$ holds for $p \geq 6$. The equation $a_p = 2a_{p-1} + b_p$ holds for $p \geq 4$.

From the equation $b_p = 2b_{p-1} + a_{p-4}$ and the first terms, we have $F(x) = 2xF(x) + G(x) + 2x^2$. So

$$G(x) = F(x) - 2xF(x) - 2x^2.$$

From the equation $b_p = 2b_{p-1} + a_{p-4}$ and the first terms, we have

$$G(x) = 2xG(x) + x^4F(x) + x^4 + x^5.$$

Substituting, we get

$$F(x) - 2xF(x) - 2x^2 = 2xF(x) - 4x^2F(x) - 4x^3 + x^4F(x) + x^4 + x^5$$

$$F(x) (1 - 4x + 4x^2 - x^4) = x^4 + x^5 - 4x^3 + 2x^2$$

$$F(x) = -\frac{x^4 + x^5 - 4x^3 + 2x^2}{(x-1)^2(x+1+\sqrt{2})(x+1-\sqrt{2})}.$$

$$F(x) = -x^2 \frac{x^2 + 2x - 2}{(x-1)(x+1+\sqrt{2})(x+1-\sqrt{2})}.$$

From here we can use partial fractions, the fraction is irreducible. We get

$$a_p = \frac{1}{4}(\sqrt{2} + 1)^p + \frac{1}{4}(1 - \sqrt{2})^p + \frac{1}{2} \text{ for } p \geq 2.$$

We return to the initial setting. Assume $(Z, 0)$ is in the independent set. After erasing $(Z, 0)$ and its neighbors from \mathcal{H}_A , we get a graph isomorphic to G_p and it has a_p independent sets.

Assume neither $(Y, 0)$ or $(Z, 0)$ belong to the independent set. Let G_p^* be the graph obtained after deleting $(Y, 0)$ and $(Z, 0)$ from \mathcal{H}_A . And let c_p be the number of independent sets in G_p^* .

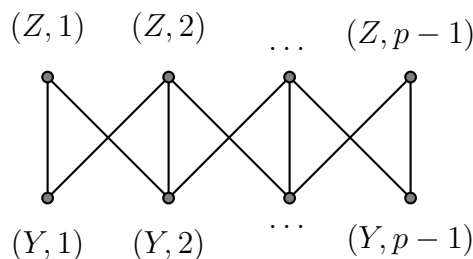


Figure 3: The graph G_p^* .

If $(Y, 1)$ is on the independent set, we erase it and its two neighbors so we get a graph and call it H_p^* . If $(Z, 1)$ is on the independent set, we erase it we get a graph isomorphic to H_p^* . Call d_p the number of independent sets in H_p^* .

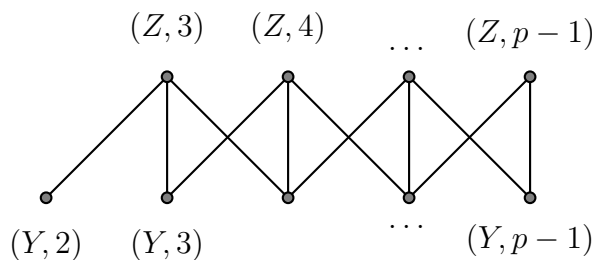


Figure 4: The graph H_p^* .

Finally, if neither of $(Y, 1)$ or $(Z, 1)$ is on the independent set, we get a graph isomorphic to G_{p-1}^* . We obtained the recursion

$$c_p = 2d_p + c_{p-1}.$$

Now, to count how many independent sets there are in H_p^* , we have that if $(Y, 2)$ is not on I then erasing it gives a graph isomorphic to G_{p-2}^* . If $(Y, 2)$ is on I , erasing it and its neighbor we get a graph isomorphic to H_{p-1}^* . Therefore

$$d_p = c_{p-2} + d_{p-1}.$$

We realize $c_2 = 3, c_3 = 7$ and $d_3 = 2, d_4 = 5$. Let $F^*(x) = \sum_{p \geq 2} c_p x^p$ and $G^*(x) = \sum_{p \geq 3} d_p x^p$. The recursion $c_p = 2d_p + c_{p-1}$ works for $p \geq 3$. In generating function form, the first recursion becomes

$$F^*(x) = 2G^*(x) + xF^*(x) + 3x^2.$$

$$G^*(x) = (F^*(x) - xF^*(x) - 3x^2)/2.$$

The recursion $d_p = c_{p-2} + d_{p-1}$ holds for $p \geq 4$ and it becomes

$$G^*(x) = x^2 F(x) + xG(x) + 2x^3.$$

After substituting we obtain

$$F^*(x) = -\frac{x^3 + 3x^2}{-1 + 2x + x^2}.$$

Again using partial fractions, we get

$$c_p = \frac{1}{2}(\sqrt{2} + 1)^p + \frac{1}{2}(1 - \sqrt{2})^p \text{ for } p \geq 3.$$

We have that the total number of independent sets is

$$2a_p + c_p = (\sqrt{2} + 1)^p + (1 - \sqrt{2})^p + 1. \quad \square$$