r-Critical numbers of natural intervals

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Abstract

The critical number $cr(r,n)$ of natural intervals $[r,n]$ was introduced by Herzog, Kaplan and Lev in 2014. The critical number $cr(r,n)$ is the smallest integer t satisfying the following conditions: (i) every sequence of integers $S = \{r_1 = r \leqslant r_2 \leqslant \cdots \leqslant r_k\}$ with $r_1+r_2+\cdots+r_k = n$ and $k \geqslant t$ has the following property: every integer between r and $n-r$ can be written as a sum of distinct elements of S, and (ii) there exists S with $k = t$, which satisfies that property. In this paper we study a variation of the critical number $cr(r,n)$ called the r-critical number $rcr(r,n)$. We determine the value of $rcr(r,n)$ for all r,n satisfying $r \mid n$.

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1 Introduction

Let r,n be positive integers satisfying $r \leqslant n$ and let $[r,n]$ denote the closed interval of integers between r and n. In [13], Herzog, Kaplan and Lev introduced the notion of the critical number $cr(r,n)$ of the interval $[r,n]$. We begin this paper with four definitions from [13], the last of which is the definition of $cr(r,n)$.

Definition 1. Let $R = \{r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k\}$ be a finite sequence of integers. The integer k is called the length of R and will be denoted by $|\boldsymbol{R}|$. Moreover, the sum of elements of R will be denoted by $\sigma(R)$ and the set of sums of elements of all subsequences of R will be denoted by $\boldsymbol{S_R}$. In other words,

$$\sigma(R) = r_1 + r_2 + \cdots + r_k$$

and

$$S_R = \{\sigma(B) \mid B \subseteq R,\ |B| > 0\}.$$

Definition 2. A sequence of integers

$$R = \{r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k\}$$

is called *an* (r,n)*-sequence* if $r_1 = r$ *and* $\sigma(R) = n$.
The set of all (r,n)-sequences will be denoted by $L(r,n)$.

Definition 3. Let R be an (r,n)-sequence. We say that R is *a spanning* (r,n)*-sequence* if *either* $n = r$ *or* $n \geqslant 2r$ *and*

$$S_R = [r, n-r] \cup \{n\}.$$

The set of spanning (r,n)-sequences will be denoted by $Sp(r,n)$.

If $n \geqslant 3r$, then a spanning (r,n)-sequence R must contain the subsequence $(r, r+1, r+2, \dots, 2r-1)$. This condition is incorporated in the following definition of the critical number $cr(r,n)$ of an interval $[r,n]$, which was studied in [13].

Definition 4. The *critical number* $cr(r,n)$ of an interval $[r,n]$ is the **smallest** positive integer t satisfying the following conditions:

1. There exists a spanning (r,n)-sequence R with $|R| = t$.
2. Each (r,n)-sequence D satisfying:
   (a) $|D| \geqslant t$, and
   (b) if $n \geqslant 3r$, then $(r, r+1, r+2, \dots, 2r-1) \subseteq D$,
   is a spanning (r,n)-sequence.

In [13] and in [16] the critical numbers are completely determined for the cases in which they exist.

The critical number $cr(1,n)$ may be viewed as a variation on the theme of $cr(G)$, when G is a cyclic group of order n. The critical number $cr(G)$ of the cyclic group G of prime order was first studied by P. Erdős and H. Heilbronn in [5] in 1964. It was defined as the smallest positive integer l such that every subset $S \subseteq G \setminus \{0\}$ with $|S| \geqslant l$ has the following property: every element of G can be written as a nonempty sum of distinct elements of S. Thirty years later, in 1994, A. Dias da Silva and Y. ould Hamidoune [3] proved that for odd primes p, $cr(\mathbb{Z}_p) \leqslant \sqrt{4p-7}$. This result is essentially best possible, the exact value of $cr(\mathbb{Z}_p)$ being $\lfloor 2\sqrt{p-2} \rfloor$ (see Theorem 1.2 in [6]).

The extended problem of the critical number of finite abelian groups was investigated by various authors (see [3, 4, 6, 8, 12, 17]).The last open case was settled in 2009 by M. Freeze, W. Gao and A. Geroldinger in [6], where the final result is stated in Theorem 1.2, and where more references may be found.

We mention also some recent results concerning the notion $c_0(G)$, which was introduced by J. E. Olson and then studied by C. Peng in 1987 (see [9, 10] for further details and references). A sequence S over an abelian group G is **regular** if for every subgroup $H \subseteq G$, S contains at most $|H|-1$ terms from H. Let $c_0(G)$ be the smallest integer t such that every regular sequence S over G of length $|S| \geqslant t$ forms an additive base of G (i.e., every element of G can be expressed as a sum over a nonempty subset of S). The value of $c_0(G)$ was determined for various families of abelian groups by W. Gao, D. Han, G. Qian, Y. Qu and H. Zhang in 2015 [9] and by W. Gao, Y. Qu and H. Zhang in 2020 [10].

When we look at the cyclic group G of order n, written additively as $(\mathbb{Z}_n, +)$, it is easily observed that the analogy between $cr(G)$ and $cr(1,n)$ is "relatively weak", due to different restrictions on the corresponding "Spanning sets". i.e., in the case of $cr(1,n)$ we consider:

1. A spanning sequence (and not just a set as in the case for $cr(G)$).
2. The sum of the elements of a spanning sequence equals n (and in particular is congruent to 0 modulo n), while the sum of the elements of a proper subsequence is less then n (and in particular not congruent to 0 modulo n).

These observations lead us to introduce the following two definitions.

Definition 5. Let $(G,+)$ be an abelian group and let $A = \{g_1, g_2, \dots, g_k\}$ be a sequence of elements of G. We say that A is *a minimal zero sum (MZS)-sequence* if the following holds:

1. $g_1 + g_2 + \cdots + g_k = 0$.
2. The sum of the elements of every proper subsequence of A is different from zero.

Definition 6. Let $(G,+)$ be an abelian group. The *minimal zero sum critical number of* G (the MZS-critical number of G), $zcr(G)$, is the smallest positive integer t for which the following two conditions hold:

1. There exists an MZS-sequence A of G such that $|A| = t$ and A spans G (i.e. for each $x \in G$ there exists a subsequence of A, such that the sum of its elements equals x).
2. Every MZS-sequence A of G with $|A| \geqslant t$ spans G.

Let M be a minimal zero-sum sequence of maximal length over a finite abelian group G. It is easily verified that M spans G. The maximal length of minimal zero-sum sequences over a finite abelian group G is called the **Davenport constant of** $\boldsymbol{G}$, denoted by $D(G)$ (see[7] or [11] Chapter 5). Therefore, we have $zcr(G) \leqslant D(G)$.

One easily observes that for cyclic groups the following holds:

Proposition 7. *If G is the cyclic group of order n, then $cr(1,n) \leqslant zcr(G)$.*

It is unclear to us whether the above lower bound is tight.

We consider now a variant of $zcr(G)$ referring to *subgroups* which is defined below (Definition 9).

Definition 8. An MZS-sequence of elements from the cyclic group of order n $(\mathbb{Z}_n, +)$ is called an *r-MZS-sequence* if $1, 2, \dots, r-1$ are not elements of the sequence and r is an element of the sequence.

Definition 9. Let $(G, +) = (\mathbb{Z}_n, +)$ be the cyclic group of order n, and let r be a positive integer such that $r \mid n$. Let $H_r = \{r, 2r, \dots, n\}$ be the subgroup of G of order $\frac{n}{r}$. The *critical number* $zcr(G, H_r)$ is the **smallest** positive integer t satisfying the following conditions:

1. There exists an r-MZS-sequence A of G such that $|A| = t$ and A spans H_r.
2. Every r-MZS-sequence A of G with $|A| \geqslant t$ spans H_r.

We remark that parallel problems of spanning a subgroup of finite groups G (not necessarily abelian) by subsets of G were investigated in various papers (see, for example, [1, 2, 14] and [15] for details and further references). This provides us a motivation to consider and investigate the notion of $rcr(r, n)$ which corresponds to the spanning of the set $\langle r \rangle_n = \{sr | s \in N \text{ and } sr \leqslant n\}$ (for $r \mid n$) by (r, n)-sequences (for the full definition, see Definition 12 below). Similarly to the analogy between $cr(r, n)$ and the problem of group covering, there is a clear analogy between $rcr(r, n)$ and the problem of subgroup covering.

In this paper we compute the value of $rcr(r, n)$ for every positive integer n and $r \geqslant 2$ which divides n. (see Theorem 14 below).

The connection between $rcr(r, n)$ and $zcr(G, H_r)$ (where G is the cyclic group of order n) is given by the following proposition.

Proposition 10. *Let G be the cyclic group of order n, and let $r \geqslant 2$ be such that $r \mid n$. Then $rcr(r, n) \leqslant zcr(G, H_r)$.*

As shown in Lemma 13(3), if R is an (r, n)-sequence and $\langle r \rangle_n \subseteq S_R$, then $r \mid n$. Therefore our new critical number is defined only if $r \mid n$, and for each such couple of integers, the (r, n)-sequence $R = (r, r, \dots, r)$ with $\frac{n}{r}$ elements satisfies our new spanning condition.

So here are our two new basic definitions.

Definition 11. Let R be an (r, n)-sequence with $r \mid n$. We say that R is an *r-spanning* (r, n)-sequence if

$$\langle r \rangle_n \subseteq S_R.$$

The set of r-spanning (r, n)-sequences will be denoted by $rSp(r, n)$.

Definition 12. For $r, n \in N$ with $r \mid n$, the r- *critical number* $rcr(r, n)$ of the interval $[r, n]$ is the **smallest** positive integer t satisfying the following conditions:

1. There exists an r-spanning (r,n)-sequence R with $|R| = t$.

2. Each (r,n)-sequence D satisfying $|D| \geqslant t$ is an r-spanning (r,n)-sequence.

Before stating our main result, we shall prove the following basic lemma.

Lemma 13. *Let R be an (r,n)-sequence. Then the following statements hold.*

1. *If $r = 1$ and $n \geqslant 2$, then $cr(1,n) = 1cr(1,n)$.*

2. *If $R \in Sp(r,n)$ and $r \mid n$, then $R \in rSp(r,n)$.*

3. *If $\langle r \rangle_n \subseteq S_R$, then $r|n$.*

4. *We have $|R| \leqslant \frac{n}{r}$. If $|R| \geqslant \frac{n}{r}$, then $|R| = \frac{n}{r}$, $r \mid n$, $R = (r, r, \ldots, r)$ and $R \in rSp(r,n)$.*

Proof. (1) If $R \in L(1,n)$ with $n \geqslant 2$, then $R \in Sp(1,n)$ if $S_R = [1, n-1] \cup \{n\} = [1,n]$ and $R \in 1Sp(1,n)$ if $\langle 1 \rangle_n = [1,n] \subseteq S_R$. Therefore $Sp(1,n) = 1Sp(1,n)$ and hence $cr(1,n) = 1cr(1,n)$.

(2) If $R \in Sp(r,n)$, then $S_R = [r, n-r] \cup \{n\}$. If $r \mid n$, then $\langle r \rangle_n \subseteq S_R$ and hence $R \in rSp(r,n)$.

(3) Suppose that $\langle r \rangle_n \subseteq S_R$ and $r \nmid n$. Then $r \geqslant 2$ and $n = tr + d$, where t and d are positive integers and $0 < d < r$. Since $tr < n$, $tr \in S_R$ and R contains a subsequence B with $\sigma(B) = tr$. Let $D = R \setminus B \subseteq R$. Then $\sigma(D) = d$ with $0 < d < r$, which contradicts our assumption that r is the least element in R.

(4) Since all elements of R are greater or equal to r, it follows that $|R| \leqslant \frac{n}{r}$. Hence if $|R| \geqslant \frac{n}{r}$, then $|R| = \frac{n}{r}$, $r \mid n$, $R = (r, r, \ldots, r)$ and $R \in rSp(r,n)$. $\square$

Since the values of $cr(1,n)$ were completely determined in [13], we shall restrict our discussion of $rcr(r,n)$ to the case $r \geqslant 2$.

Our main result is the following theorem.

Theorem 14. *If n and r are integers satisfying $r \mid n$ and $r \geqslant 2$, then $rcr(r,n)$ exists and*

$$rcr(r,n) = \left\lceil \frac{n+r-1}{r+1} \right\rceil,$$

with the following exceptions:

1. *$r \geqslant 2$ and $n = 3r$, in which case $rcr(r,n) = \frac{n-r}{r} = 2$.*

2. *$r = 2$, $3|n$ and $n \neq 6, 12$, in which case $rcr(r,n) = \frac{n}{r+1} = \frac{n}{3}$.*

3. *$r \geqslant 2$, $n > r^2 - r$ and $r + 1|n + r - 1$, in which case $rcr(r,n) = \frac{n+r-1}{r+1} + 1$.*

4. *$r > 2$, $n \neq 3r$ and $n = r^2 - \alpha r$ for an integer $2 \leqslant \alpha \leqslant r - 1$, in which case $rcr(r,n) = r - \alpha = \frac{n}{r} = \lceil \frac{n+r-1}{r+1} \rceil - 1$.*

Remark 15. Notice that there exist $A \in L(r,n)$ satisfying $A \in rSp(r,n)$ with $|A| < rcr(r,n)$. For example, $A = (5,5,10) \in L(5,20)$ satisfies $A \in 5Sp(5,20)$ and $|A| = 3$, while $n = r^2 - r$ and by Theorem 14 $rcr(5,20) = \lceil \frac{20+5-1}{6} \rceil = 4$. This happens because $rcr(r,n)$ denotes the minimal integer satisfying the property that if $A \in L(r,n)$ and $|A| \geqslant rcr(r,n)$, then $A \in rSp(r,n)$, and there exists another $B = (5,6,9) \in L(5,20)$ with $|B| = 3$ and $B \notin 5Sp(5,20)$.

Some preliminary results concerning the r-spanning (r,n)-sequences will be stated and proved in Section 2. Section 3 will be devoted to the proof of Theorem 14.

2 Preliminary results

From now on, we shall use the following notation, unless stated otherwise. If $R = (r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k)$ and $r_1 = f_1 < f_2 < \cdots < f_d$ are the distinct elements of R with the corresponding multiplicities $n_1, n_2, \ldots, n_d$, then we shall write $R = (f_1^{(n_1)}, f_2^{(n_2)}, \ldots, f_d^{(n_d)})$, with f_i for $f_i^{(1)}$. For example, $R = (2,2,2,3,3,5) = (2^{(3)}, 3^{(2)}, 5)$. Notice that if $R \in L(r,n)$, then n_1 is the multiplicity of r in R, $n_1 \geqslant 1$ and $\sum_{i=1}^{d} n_i f_i = n$.

We start with a basic result which was presented in [13] (see Proposition 1 in [13]). Here the multiplicity of $r+i$ in A is denoted by n_{r+i}. In particular, the multiplicity of r is denoted by n_r.

Proposition 16. *Let*

$$A = (r^{(n_r)}, (r+1)^{(n_{r+1})}, (r+2)^{(n_{r+2})}, \ldots, (r+j)^{(n_{r+j})}) \in L(r,m),$$

where $r \in N$, $j \in N \cup \{0\}$, $m = \sum_{i=0}^{j} n_{r+i}(r+i)$, $n_r, n_{r+1}, \ldots, n_{r+j} \geqslant 1$, *and either* $j \geqslant r$ *or* $j = r-1$ *and* $n_r \geqslant 2$. *Then* $A \in Sp(r,m)$.

Notice that if $r \mid m$, then by Lemma 13(2), Proposition 16 implies that $A \in rSp(r,m)$.

Let $r \mid n$. In the following lemma we determine conditions under which the existence of a subsequence B of $A \in L(r,n)$, satisfying $B \in rSp(r,k)$ for certain $k < n$, implies that $A \in rSp(r,n)$.

Lemma 17. *Let* $A \in L(r,n)$ *with* $r|n$ *and let* k *be an integer satisfying*

$$\frac{n-r}{2} \leqslant k < n.$$

If there exists a subsequence B *of* A *satisfying* $B \in L(r,k)$, *and* $\langle r \rangle_k \subseteq S_B$ *then* $A \in rSp(r,n)$.

Proof. Since B is a subsequence of A satisfying $B \in rSp(r,k)$, it follows that $S_B \subseteq S_A$ and if $x \leqslant k$ is an integer divisible by r, then $x \in S_B \subseteq S_A$. Hence it remains only to prove that if x is an integer satisfying $r \mid x$ and $k < x \leqslant n-r$, then $x \in S_A$. So let x be such an integer. Since $\frac{n-r}{2} \leqslant k$, it follows that

$$r \leqslant n - x < n - k \leqslant k + r.$$

But $r \mid n-x$ and by Lemma 13(3) also $r \mid k$, so the above inequalities imply that $r \leqslant n-x \leqslant k$. Hence $n-x \in S_B \subseteq S_A$, so there exists a subsequence U of A with $\sigma(U) = n-x$. But $\sigma(A) = n$, so $\sigma(A \setminus U) = x$, and since $A \setminus U \subseteq A$, it follows that $x \in S_A$, as required. □

Lemma 17 yields the following corollary. Recall that if $R \in L(r,n)$, then n_1 denotes the multiplicity of r in R.

Corollary 18. *If $A \in L(r,n)$ with $r|n$ and $n_1 r \geqslant \frac{n-r}{2}$, then $A \in rSp(r,n)$.*

Proof. Let $k = n_1 r$ and $B = (r^{(n_1)})$. Clearly B is a subsequence of A and $B \in rSp(r,k)$. If $k = n$, then $A \in rSp(r,n)$. So assume that $k < n$. Then $\frac{n-r}{2} \leqslant k < n$ and by Lemma 17 $A \in rSp(r,n)$. □

In the next proposition we determine conditions under which $A \in L(r,n)$ with $r|n$ and $|A| \geqslant \frac{n+r-1}{r+1}$ satisfies $A \in rSp(r,n)$.

Proposition 19. *Let $A \in L(r,n)$ and suppose that $r|n$, $n_1 \geqslant r > 1$ and $|A| \geqslant \frac{n+r-1}{r+1}$. Then $A \in rSp(r,n)$.*

Proof. We begin this proof with a general remark concerning subsequences of a sequence of integers. Given a sequence $S = (a_1 \leqslant a_2 \leqslant \ldots \leqslant a_r)$ of r integers, we can always find a non-empty subsequence S_1 of S satisfying $r|\sigma(S_1)$. Indeed, among the r partial sums $a_1, a_1+a_2, a_1+a_2+a_3, \ldots, a_1+a_2+\cdots+a_r$ either at least one is divisible by r, or there are two distinct sums whose difference is divisible by r. Hence there exists a non-empty subsequence S_1 of S satisfying $r|\sigma(S_1)$, as claimed. This is a classical result in zero-sum theory: $D(C_r) = r$, where $D(C_r)$ is the Davenport constant of C_r (see [7] or [11] Chapter 5).

We proceed with the proof of Proposition 19. Assume, by way of contradiction, that Proposition 19 does not hold and let n be the least integer for which there exists $A \in L(r,n)$ satisfying our assumptions, but not belonging to $rSp(r,n)$.

We consider first the case when

$$|A| - n_1 \geqslant r.$$

Then A contains at least r elements greater than r and therefore, by the opening remark, there exists a non-empty subsequence S of A consisting of elements of greater than r and with $x = \sigma(S)$ satisfying $r|x$. Choosing a minimal such S, we may assume that $|S| \leqslant r$ and since the elements of S are all greater than r, we also have $|S| \leqslant \frac{x}{r+1}$. Let $D = A \setminus S$. Then $D \in L(r, n-x)$ and

$$|D| = |A| - |S| \geqslant \frac{n+r-1}{r+1} - \frac{x}{r+1} = \frac{(n-x)+r-1}{r+1}.$$

Since D contains n_1 r-elements, it satisfies the assumptions of Proposition 19, and it follows by the minimality of n that $D \in rSp(r, n-x)$. If $n - x \geqslant \frac{n-r}{2}$, then by Lemma

17 $A \in rSp(r,n)$, a contradiction. So we must have $2(n-x) < n-r$. Since both sides of this inequality are divisible by r, it follows that $2(n-x) \leqslant n-2r$, so $n \leqslant 2x-2r$.

Let B be the subsequence of A obtained by adjoining to S the n_1 r's of A. Then $B \in L(r, n_1r+x)$ and $\sigma(A \setminus B) = n-x-n_1r$. Since $A \setminus B$ contains only elements greater than r, it follows that $|A \setminus B| \leqslant \frac{n-x-n_1r}{r+1}$ and hence

$$|B| = |A| - |A \setminus B| \geqslant \frac{n+r-1}{r+1} - \frac{n-x-n_1r}{r+1} = \frac{x+n_1r+r-1}{r+1}.$$

Thus B satisfies the conditions of Proposition 19 and if $B \neq A$, then it follows by the minimality of n that $B \in rSp(r, x+n_1r)$. Since $x+n_1r > x \geqslant \frac{n+2r}{2} \geqslant \frac{n-r}{2}$, Lemma 17 implies that $A \in rSp(r,n)$, a contradiction. So assume that $B = A$. Then $n_1r + x = n$, yielding $n_1 = \frac{n-x}{r}$ and since $2(n-x) \leqslant n-2r$, it follows that $n-x \leqslant \frac{n-2r}{2}$ and $n_1 \leqslant \frac{n-2r}{2r}$. Thus we obtain

$$\frac{n+r-1}{r+1} \leqslant |A| = n_1 + |S| \leqslant \frac{n-2r}{2r} + r. \tag{1}$$

We shall show now that the case $|A| - n_1 \leqslant r-1$ leads to the same inequality. Having done that, we shall complete the proof by showing that inequality (1) leads to a contradiction.

So suppose that $|A| - n_1 \leqslant r-1$. Then

$$n_1 \geqslant |A| - r + 1 \geqslant \frac{n+r-1}{r+1} - r + 1.$$

If $n_1r \geqslant \frac{n}{2}$, then $n_1r \geqslant \frac{n-r}{2}$ and $A \in rSp(r,n)$ by Corollary 18, a contradiction. So we may assume that $n_1 \leqslant \frac{n}{2r}$, which implies that $\frac{n+r-1}{r+1} - r + 1 \leqslant \frac{n}{2r}$, which is identical to inequality (1).

So suppose that inequality (1) holds. Then

$$\frac{n+r-1}{r+1} \leqslant \frac{n-2r+2r^2}{2r},$$

which implies that $n \leqslant 2r^2 \leqslant 2n_1r$. Hence $A \in rSp(r,n)$ by Corollary 18, a final contradiction. $\square$

Our final proposition in this section determines upper bounds for $rcr(r,n)$ and lists some important extreme cases. These results will be used for the evaluation of $rcr(r,n)$ in the proof of Theorem 14.

Proposition 20. *Let n, r be positive integers satisfying $r \geqslant 2$ and $r \mid n$, and let $A \in L(r,n)$. Denote $\Pi = \lceil \frac{n+r-1}{r+1} \rceil$. Then the following statements hold.*

1. *If $r \geqslant 2$, $n = 3r$ and $|A| \geqslant \frac{n-r}{r} = 2$, then $A \in rSp(r,n)$.*

2. *If $r = 2$, $3|n$, $n \neq 6$ and $|A| \geqslant \frac{n}{r+1} = \frac{n}{3} = \Pi - 1$, then $A \in 2Sp(2,n)$, with the following exception: $r = 2$, $n = 12$, $|A| = \frac{n}{3} = 4 = \Pi - 1$ and $A = (2,2,3,5) \notin 2Sp(2,12)$.*

3. *If* $r \geqslant 2$, $r \neq 4$, $n = r^2 - r$ *and* $|A| \geqslant \Pi = \frac{n}{r}$, *then* $|A| = \Pi$ *and* $A = (r^{(\frac{n}{r})}) \in rSp(r,n)$.

4. *If* $r \geqslant 2$, $n > r^2 - r$, $3 \nmid n$ *if* $r = 2$, $n \neq 9$ *if* $r = 3$ *and* $|A| \geqslant \Pi$, *then* $A \in rSp(r,n)$, *with the following exceptions:* $r+1 \mid n+r-1$, $|A| = \frac{n+r-1}{r+1} = \Pi$ *and* $A = (r^{(r-1)}, (r+1)^{(\frac{n+r-r^2}{r+1})}) \notin rSp(r,n)$.

5. *If* $r > 2$, $n \neq 3r$, $n = r^2 - \alpha r$ *with* $2 \leqslant \alpha \leqslant r-1$ *and* $|A| \geqslant \Pi - 1 = \frac{n}{r}$, *then* $|A| = \Pi - 1$ *and* $A = (r^{(\frac{n}{r})}) \in rSp(r,n)$.

Before starting with the proof of Proposition 20, we shall prove the following auxiliary Lemma 21. In the proofs of Lemma 21 and of item (2) in Proposition 20, we shall change our notation and we shall denote by n_i the multiplicity of i in A. In particular, n_2 will denote the multiplicity of 2 in A.

Lemma 21. 1. *Let* $A \in L(2,n)$ *with* $|A| \geqslant \frac{n}{3}$ *and let* $b_1, b_2, \ldots, b_k$ *be the elements of* A *that are greater than* 4. *Then* $n_2 \geqslant \sum_{i=1}^{k}(b_i - 3)$. *In other words, each such* b_i *must be accompanied by* $b_i - 3$ *elements* 2.

2. *Let* $A \in L(2,n)$ *with* $6 \mid n$ *and* $n \geqslant 12$, *and suppose that* A *satisfies the following conditions:* $|A| \geqslant \frac{n}{3}$ *and* $n_2 = 1$. *Then* $A = (2, 3^{(n_3)}, 4^{(n_4)})$ *with* $n_3, n_4 > 0$ *and* $A \in 2Sp(2,n)$.

3. *Let* r *and* n *be integers satisfying* $r \geqslant 2$ *and* $r \mid n$ *and let* $t = \lceil \frac{n+r-1}{r+1} \rceil$. *Then the following statements hold.*

   (i) If $n < r^2 - r$, *then* $n = (t-1)r$ *and if* $n \geqslant r^2 - r$, *then* $n \geqslant tr$.

   (ii) $t = 1 \iff (r,n) = (2,2)$.

   (iii) $t \leqslant 2 \iff$ *either* $n = r \geqslant 2$ *or* $(r,n) \in \{(2,4),(3,6)\}$.

Proof of Lemma 21. (1) Let $A \in L(2,n)$ with $|A| \geqslant \frac{n}{3}$ and let $b_1, b_2, \ldots, b_k$ be the elements of A that are greater than 4. Then

$$n - \sum_{i=1}^{k} b_i - 2n_2 \geqslant (|A| - k - n_2)3 \geqslant n - 3k - 3n_2,$$

which implies that $n_2 \geqslant \sum_{i=1}^{k}(b_i - 3)$.

(2) Suppose that $6|n$, $n \geqslant 12$ and $A \in L(2,n)$ satisfies $|A| \geqslant \frac{n}{3}$ and $n_2 = 1$. Since $n_2 = 1$, it follows by (1) that $A = (2, 3^{(n_3)}, 4^{(n_4)})$. Thus $n = 2 + 3n_3 + 4n_4$ and since $3|n$, we must have $n_4 \neq 0$. If $n_3 = 0$, then $|A| = 1 + n_4$ and $n = 2 + 4n_4 \leqslant 3|A| = 3 + 3n_4$, implying that $n_4 \leqslant 1$ and $n \leqslant 6$, a contradiction. Hence also $n_3 > 0$ and by Proposition 16 $A \in Sp(2,n)$. Since $2|n$, Lemma 13(2) implies that $A \in 2Sp(2,n)$. as required.

(3) Let n and r be integers and suppose that $r \geqslant 2$ and $r \mid n$. Denote $k = \frac{n}{r}$ and let $t = \lceil \frac{n+r-1}{r+1} \rceil$. We have

$$t = \left\lceil \frac{kr + r - 1}{r+1} \right\rceil = \left\lceil k + 1 + \frac{-k - r - 1 + r - 1}{r+1} \right\rceil = \left\lceil k + 1 - \frac{k+2}{r+1} \right\rceil.$$

Therefore if $1 \leqslant k < r-1$, then $t = k+1$ and if $k \geqslant r-1$, then $t \leqslant k$. Thus

(i) If $n < r^2 - r$ then $n = (t-1)r$ and if $n \geqslant r^2 - r$ then $n \geqslant tr$.

We always have $t \geqslant 1$ and $t = 1$ if and only if $n + r - 1 \leqslant r + 1$, hence if and only if $n \leqslant 2$. Thus $t = 1$ if and only if $(r, n) = (2, 2)$.

Moreover, $t = 2$ if and only if $r + 1 < n + r - 1 \leqslant 2r + 2$, hence if and only if $2 < n \leqslant r + 3$. Since $r|n$, it follows that $t = 2$ if and only if one of the following holds: (1) $n = r > 2$, (2) $r = 2$, $n = 4$, and (3) $r = 3$, $n = 6$. Thus

(ii) $t = 1$ if and only if $(r, n) = (2, 2)$, and

(iii) $t \leqslant 2$ if and only if either $n = r \geqslant 2$ or $(r, n) \in \{(2, 4), (3, 6)\}$. □

We continue now with the proof of Proposition 20. By (i) we shall denote item (i) in Proposition 20.

Proof of Proposition 20. (1) Since $L(r, 3r) = \{(r, 2r), (r^{(3)})\}$, with both these sequences belonging to $rSp(r, n)$, it follows that if $A \in L(r, 3r)$ and $|A| \geqslant 2$, then $A \in rSp(r, n)$.

(2) Recall that in this item n_i denotes the multiplicity of i in A. Let $A \in L(2, n)$ with $6|n$, $n \geqslant 12$ and $|A| \geqslant \frac{n}{3} = \Pi - 1$. Our aim is to prove that $A \in 2Sp(2, n)$, with the following exception: $r = 2$, $n = 12$, $|A| = \frac{n}{3} = 4$ and $A = (2, 2, 3, 5) \notin 2Sp(2, n)$. The case $n = 6$ was excluded since it belongs to item (1).

We first notice that if $A \in L(2, n)$, $6|n$, $n \geqslant 12$, $|A| \geqslant \frac{n}{3}$ and $n_2 = 1$, then $A \in 2Sp(2, n)$ by Lemma 21(2). This statement will be referred to as "our first result".

So we may assume, from now on, that $n \geqslant 12$ and $n_2 \geqslant 2$.

Suppose, first, that $n = 12$ and $|A| \geqslant 4 = \Pi - 1$. If $n_2 \geqslant 3$, then $2n_2 \geqslant \frac{12-2}{2}$ and $A \in 2Sp(2, n)$ by Corollary 18. If $n_2 = 2$, then either $A = (2, 2, 3, 5)$, the excluded sequence satisfying $A \notin 2Sp(2, 12)$, or $A = (2, 2, 4, 4)$, which belongs to $2Sp(2, 12)$, as required. So we may assume, from now on, that $n \geqslant 18$ and $n_2 \geqslant 2$.

Suppose, next, that $n = 18$, $A \in L(2, 18)$ and $|A| \geqslant \frac{n}{3} = 6$. If $n_2 \geqslant 4$, then $2n_2 \geqslant \frac{18-2}{2}$ and $A \in 2Sp(2, 18)$ by Corollary 18. So assume that $2 \leqslant n_2 \leqslant 3$. Since $|A| \geqslant \frac{n}{3} = 6$, it follows by Lemma 21(1) that there is at most one element $b \in A$ satisfying $b \geqslant 5$, and such b must satisfy $5 \leqslant b \leqslant 6$. Consider the integer

$$m = 2n_2 + 3n_3 + 4n_4 \geqslant 12 > \frac{18-2}{2},$$

and let $B = (2^{(n_2)}, 3^{(n_3)}, 4^{(n_4)})$, allowing $n_3 = 0$ or $n_4 = 0$. Then $\sigma(B) = m$ and B is a subsequence of A. If $n_3 \neq 0$, then Proposition 16 implies that $2j \in S_B$ for all $2 \leqslant 2j \leqslant m - 2$ and the same conclusion certainly holds if $n_3 = 0$. If m is even, then $\langle 2 \rangle_m \subseteq S_B$ and since $m \geqslant \frac{18-2}{2}$, it follows by Lemma 17 that $A \in 2Sp(2, 18)$, as required. If m is odd, then $b = 5$ is an element in A and $m = 13$. Since m is odd, it follows that n_3 is odd, and in particular $n_3 \geqslant 1$. Let $C = (2^{(n_2)}, 3^{(n_3-1)}, 4^{(n_4)})$. Then $\sigma(C) = 10$ and it follows using the above arguments that $\langle 2 \rangle_{10} \subseteq S_C$. Since $10 \geqslant \frac{18-2}{2}$, it follows by Lemma 17 that in this case also $A \in 2Sp(2, 18)$. Thus $A \in 2Sp(2, 18)$ in all cases.

Suppose, finally, that $n \geqslant 24$, $6|n$, $n_2 \geqslant 2$, $|A| \geqslant \frac{n}{3}$ and (2) holds for all appropriate $l < n$. If one of $(2, 4)$, (6) and $(3, 3)$ is a subsequence of A denoted by B, then $C = A \setminus B \in L(2, n-6)$ and $|C| = |A| - |B| \geqslant \frac{n}{3} - 2 = \frac{n-6}{3}$. Notice that $n - 6 > 12$. If n_2 of

C equals 1, then $C \in 2Sp(2, n-6)$ by "our first result", and the same holds by induction if that n_2 satisfies $n_2 \geqslant 2$. Since $n - 6 \geqslant \frac{n-2}{2}$, Lemma 17 implies that $A \in 2Sp(2, n)$.

So we may assume, from now on, that $n \geqslant 24$, $6 \mid n$, $n_2 \geqslant 2$, $|A| \geqslant \frac{n}{3}$, $n_3 \leqslant 1$, $n_4 = 0$ and $n_6 = 0$.

Suppose that $n_5 \geqslant 2$. Since $|A| \geqslant \frac{n}{3}$, it follows by Lemma 21(1) that $n_2 \geqslant 4$. Hence $B = (2^{(4)}, 5^{(2)})$ is a subsequence of A and by Corollary 18, $B \in 2Sp(2, 18)$, since $4{\cdot}2 \geqslant \frac{18-2}{2}$. If $n \leqslant 38 = 2 \cdot 18 + 2$, then it follows by Lemma 17 that $A \in 2Sp(2, n)$, as required. So we may assume that $n > 38$.

If $n_2 > 4$, then $C = A \setminus B \in L(2, n-18)$ and $|C| \geqslant \frac{n-18}{3}$. Since $n - 18 > 12$, it follows by "our first result" if n_2 of C equals 1 and by induction if that n_2 satisfies $n_2 \geqslant 2$, that $C \in 2Sp(2, n-18)$. Since $n - 18 \geqslant \frac{n-2}{2}$, Lemma 17 implies that $A \in 2Sp(2, n)$.

If $n_2 = 4$, then $n_5 = 2$ and $n_b = 0$ if $b > 5$. Thus $n = 2n_2 + 3n_3 + 4n_4 + 5n_5 \leqslant 8 + 3 + 0 + 10 = 21$, a contradiction.

So we may assume, from now on, that $n_5 \leqslant 1$.

Suppose, first, that $n_b > 0$ for some even $b > 6$. Then, by Lemma 21(1), $B = (2^{(b-3)}, b) \in L(2, m)$ is a subsequence of A and since b is even, $m = 2(b-3) + b = 3b - 6$ is divisible by 6. As $|B| = b - 2 = \frac{m}{3}$ and $2(b-3) > \frac{m-2}{2} = \frac{3b-8}{2}$, it follows by Corollary 18 that $B \in 2Sp(2, m)$ and if $m \geqslant \frac{n-2}{2}$, then $A \in 2Sp(2, n)$ by Lemma 17. So suppose that $m < \frac{n-2}{2}$, which implies that $n - m > \frac{n+2}{2} > 12$. If $n_2 > b - 3$, then $C = A \setminus B \in L(2, n-m)$ satisfies $|C| = |A| - |B| \geqslant \frac{n}{3} - \frac{m}{3} = \frac{n-m}{3}$ and it follows by "our first result" if n_2 of C equals 1 and by induction if that n_2 satisfies $n_2 \geqslant 2$, that $C \in 2Sp(2, n-m)$. Hence it follows by Lemma 17 that $A \in 2Sp(2, n)$, since $n - m > \frac{n+2}{2}$. Finally, if n_2 of A satisfies $n_2 = b - 3$, then $n_b = 1$ and $n_d = 0$ for $\{d > 3 | d \neq b\}$. Thus $n \leqslant 2(b-3) + 3 + b = 3b - 3$ and $2n_2 = 2b - 6 \geqslant \frac{3b-3-2}{2} \geqslant \frac{n-2}{2}$, implying by Corollary 18 that $A \in 2Sp(2, n)$.

So we may also assume that $n_b = 0$ for all even $b \geqslant 4$.

Suppose that A has two odd elements $b, c \geqslant 5$, where $b \leqslant c$. It follows from $n_5 \leqslant 1$ that either $b = 5 < c$ or $7 \leqslant b \leqslant c$. In the first case, $B = (2^{(2+(c-3))}, 5, c) \in L(2, m)$ is a subsequence of A, where $m = 2[2+(c-3)]+5+c = 3c+3$ is divisible by 6. If $m = n$, then $A = B \in 2Sp(2, n)$ by Corollary 18, since $2(2+(c-3)) = 2c-2 \geqslant \frac{m-2}{2} = \frac{3c+1}{2}$. So suppose that $m < n$. Since $m > 12$ and $|B| = c+1 = \frac{m}{3}$, it follows by induction that $B \in 2Sp(2, m)$ and if $m \geqslant \frac{n-2}{2}$, then $A \in 2Sp(2, n)$ by Lemma 17. So suppose that $m < \frac{n-2}{2}$. Then $n - m > \frac{n+2}{2} > 12$ and $C = A \setminus B$ satisfies $|C| = |A| - |B| \geqslant \frac{n}{3} - \frac{m}{3} = \frac{n-m}{3}$. If $n_2 > 2+(c-3)$, then it follows by "our first result" if n_2 of C equals 1 and by induction if that n_2 satisfies $n_2 \geqslant 2$, that $C \in 2Sp(2, n-m)$, which implies that $A \in 2Sp(2, n)$ by Lemma 17, since $n - m > \frac{n+2}{2}$. Finally, if $n_2 = 2 + (c-3)$, then $n_d = 0$ for $\{d > 3 \mid d \neq 5, c\}$ and $n_5 = n_c = 1$. Hence $A = (2^{(c-1)}, 3^{(n_3)}, 5, c)$ and since n is even and c is odd, $n_3 \leqslant 1$ implies that $n_3 = 0$ and $A = (2^{(c-1)}, 5, c)$. Thus $n = 3c + 3$ and $2n_2 = 2(c-1) \geqslant \frac{n-2}{2} = \frac{3c+1}{2}$, which implies by Corollary 18 that $A \in 2Sp(2, n)$ and we are finished with the first case.

So suppose that the second case holds: $7 \leqslant b \leqslant c$. Suppose, first, that $b = c$. Then by Lemma 21(1) $B = (2^{(2(b-3))}, b^{(2)}) \in L(2, m)$ is a subsequence of A, where $m = 2[2(b-3)] + 2b = 6b - 12$ is divisible by 6. If $n = m$, then $A \in 2Sp(2, n)$ by Corollary 18, since $2[2(b-3)] \geqslant \frac{n-2}{2} = 3b - 7$.

So suppose that $m < n$. Since $m > 12$ and $|B| = 2b - 4 = \frac{m}{3}$, it follows by induction that $B \in 2Sp(2,m)$ and if $m \geqslant \frac{n-2}{2}$, then $A \in 2Sp(2,n)$ by Lemma 17. So suppose that $m < \frac{n-2}{2}$. Then $n-m > \frac{n+2}{2} > 12$ and $C = A \setminus B$ satisfies $|C| = |A| - |B| \geqslant \frac{n}{3} - \frac{m}{3} = \frac{n-m}{3}$. If $n_2 > 2(b-3)$, then it follows by "our first result" if n_2 of C equals 1 and by induction if that n_2 satisfies $n_2 \geqslant 2$, that $C \in 2Sp(2, n-m)$, which implies that $A \in 2Sp(2,n)$ by Lemma 17, since $n - m > \frac{n+2}{2}$. Finally, if $n_2 = 2(b-3)$, then $n_d = 0$ for $\{d > 3 \mid d \neq b\}$ and $n_b = 2$. Hence $A = (2^{(2b-6)}, 3^{(n_3)}, b^{(2)})$ and since n is even, $n_3 \leqslant 1$ implies that $n_3 = 0$ and $A = (2^{(2b-6)}, b^{(2)})$. Thus $n = 6b - 12$ and $2n_2 = 2(2b-6) \geqslant \frac{n-2}{2} = 3b - 7$, which implies by Corollary 18 that $A \in 2Sp(2,n)$.

It remains to deal with the case: $7 \leqslant b < c$ and both b and c are odd. Then by Lemma 21(1) $B = (2^{[(b-3)+(c-3)]}, b, c) \in L(2,m)$ is a subsequence of A, where $m = 2[(b+c)-6] + b + c = 3(b+c) - 12$ is divisible by 6. If $n = m$, then $A \in 2Sp(2,n)$ by Corollary 18, since $2[(b+c)-6]) \geqslant \frac{n-2}{2} = \frac{3(b+c)-14}{2}$.

So suppose that $m < n$. Since $m > 12$ and $|B| = (b+c) - 4 = \frac{m}{3}$, it follows by induction that $B \in 2Sp(2,m)$ and by Lemma 17 $A \in 2Sp(2,n)$ if $m \geqslant \frac{n-2}{2}$. So suppose that $m < \frac{n-2}{2}$. Then $n - m > \frac{n+2}{2} > 12$ and $C = A \setminus B \in L(2, n-m)$ satisfies $|C| = |A| - |B| \geqslant \frac{n}{3} - \frac{m}{3} = \frac{n-m}{3}$. If $n_2 > b + c - 6$, then it follows by "our first result" if n_2 of C equals 1 and by induction if that n_2 satisfies $n_2 \geqslant 2$, that $C \in 2Sp(2, n-m)$, which implies that $A \in 2Sp(2,n)$ by Lemma 17, since $n - m > \frac{n+2}{2}$. Finally, if $n_2 = b + c - 6$, then $n_d = 0$ for $\{d > 3 \mid d \neq b, c\}$ and $n_b = n_c = 1$. Hence $A = (2^{(b+c-6)}, 3^{(n_3)}, b, c)$ and since n is even and b and c are odd, $n_3 \leqslant 1$ implies that $n_3 = 0$, $A = (2^{(b+c-6)}, b, c)$ and $n = 3(b+c) - 12$. Since

$$2n_2 = 2(b+c-6) = 2(b+c) - 12 \geqslant \frac{n-2}{2} = \frac{3(b+c)-14}{2},$$

Corollary 18 implies that $A \in 2Sp(2,n)$.

So we may assume from now on that A contains at most one element $b > 4$, which needs to be an odd integer. If A contains no such element, then since n is even and $n_3 \leqslant 1$, it follows that $A = (2^{(n_2)}) \in 2Sp(2,n)$. So assume, finally, that such an element b does exist. Since n is even, b is odd and $n_3 \leqslant 1$, it follows that $A = (2^{(n_2)}, 3, b)$. Moreover, since $|A| \geqslant \frac{n}{3}$, it follows that $n_2 + 2 \geqslant \frac{2n_2+3+b}{3}$, implying that $b \leqslant n_2 + 3$. If $2n_2 \geqslant \frac{n-2}{2}$, then $A \in 2Sp(2,n)$ by Corollary 18. So suppose that $2n_2 = n - 3 - b < \frac{n-2}{2}$. Then $n_2 < \frac{n-2}{4}$, implying that $b < \frac{n-2}{4} + 3$ and

$$n < \frac{n-2}{2} + 3 + b < \frac{n-2}{2} + 3 + \frac{n-2}{4} + 3.$$

Hence $n < 18$, a final contradiction. The proof of item (2) of Proposition 20 is now complete.

(3) Notice that if $n = r^2 - r$ and $r = 2$, then $n = 2 = r$. Moreover, if $r = 4$, then $n = 12 = 3r$, and this case was excluded since it belongs to item (1).

Now for all $r \geqslant 2$ we have $\Pi = \lceil \frac{r^2-r+r-1}{r+1} \rceil = r - 1 = \frac{n}{r}$. Hence by Lemma 13(4), if $A \in L(r,n)$ satisfies $|A| \geqslant \Pi$, then $|A| = \Pi$ and $A = (r^{(\frac{n}{r})}) \in rSp(r,n)$.

(4) Let $A \in L(r,n)$ and suppose that $n > r^2 - r$, $3 \nmid n$ if $r = 2$, $n \neq 9$ if $r = 3$ and $|A| \geqslant \lceil \frac{n+r-1}{r+1} \rceil$. Cases $(r,n) = (2,n)$ with $3 \mid n$ were excluded, since they belong to item (2) and the case $(r,n) = (3,9)$ was excluded since it belongs to item (1).

Our aim is to prove that $A \in rSp(r,n)$, with the following exceptions: $r+1 \mid n+r-1$, $|A| = \frac{n+r-1}{r+1}$ and $A = (r^{(r-1)}, (r+1)^{(\frac{n+r-r^2}{r+1})})$.

Denote $A = (f_1^{(n_1)}, f_2^{(n_2)}, \ldots, f_d^{(n_d)})$ with $f_1 = r < f_2 < \cdots < f_d$ and $\sum_{i=1}^{d} n_i f_i = n$. In addition to our previous assumptions, assume that $n_1 = r - j \leqslant r-1$ for some j satisfying $1 \leqslant j \leqslant r-1$ and $A \notin rSp(r,n)$. Since $A \notin rSp(r,n)$, we must have $d \geqslant 2$ and $f_2 \geqslant r+1$. But then

$$\frac{n+r-1}{r+1} \leqslant |A| \leqslant r - j + \frac{n-(r-j)r}{r+1} = \frac{n+r-j}{r+1}$$

which implies that $j = 1$, $|A| = \frac{n+r-1}{r+1}$, $r+1 \mid n+r-1$, $d = 2$, $f_2 = r+1$ and $n_1 = r-1$. Hence A is the excluded sequence $A = (r^{(r-1)}, (r+1)^{(\frac{n+r-r^2}{r+1})})$.

So we may assume, from now on, that $n_1 \geqslant r > 1$. Since $|A| \geqslant \frac{n+r-1}{r+1}$, it follows by Proposition 19 that $A \in rSp(r,n)$.

(5) If $r > 2$, $n \neq 3r$ and $n = r^2 - \alpha r$, with $2 \leqslant \alpha \leqslant r-1$, then

$$\begin{aligned}\Pi &= \left\lceil \frac{r^2 - \alpha r + r - 1}{r+1} \right\rceil = \left\lceil \frac{(r-\alpha)(r+1) - r + \alpha + r - 1}{r+1} \right\rceil \\ &= \left\lceil r - \alpha + \frac{\alpha - 1}{r+1} \right\rceil = r - \alpha + 1 = \frac{n}{r} + 1.\end{aligned}$$

It follows by Lemma 13(4) that if $|A| \geqslant \Pi - 1 = \frac{n}{r}$, then $|A| = \Pi - 1$ and $A = (r^{(\frac{n}{r})}) \in rSp(r,n)$. Cases $n = 3r$ were excluded, since they belong to item (1).

The proof of Proposition 20 is now complete. $\square$

3 Proof of Theorem 14

In this section we prove Theorem 14.

Proof of Theorem 14. Let n and r be positive integers satisfying $r \geqslant 2$ and $r|n$. In Proposition 20, each such couple (r,n) was considered once and only once, and for each such couple we determined an integral function $f(r,n)$ such that if $A \in L(r,n)$ and $|A| \geqslant f(r,n)$, then $A \in rSp(r,n)$.

Theorem 14 claims that for each $r \geqslant 2$ and each n divisible by r, the function $f(r,n)$ is equal to $rcr(r,n)$.

In order to prove Theorem 14, it is necessary to show that for each such r and n, there exists $A \in L(r,n)$ with $|A| = f(r,n)$, and on the other hand to show that if $A \in L(r,n)$ and $|A| = f(r,n) - 1$, then either A does not exist or there exists such an A which does not belong to $rSp(r,n)$.

We shall perform these two tasks in two propositions. First we shall establish the existence of $A \in L(r,n)$ with $|A| = f(r,n)$ for each of the five items of Proposition 20.

Proposition 22. *For each set of values of the couples (r,n) in the items of Proposition 20, there exists $A \in L(r,n)$ satisfying $|A| = f(r,n)$.*

Proof. Given the couple (r,n) with $r \geqslant 2$ and $r|n$, let $t = \lceil \frac{n+r-1}{r+1} \rceil$. We shall go over the items of Proposition 20, denoted by 20(i) with $i \in \{1,2,3,4,5\}$, and for each (r,n) we shall present $A \in L(r,n)$ satisfying $|A| = f(r,n)$.

In 20(1), $r \geqslant 2$, $n = 3r$ and $f(r,3r) = 2$. Then $A = (r, 2r) \in L(r,3r)$ satisfy $|A| = 2 = f(r,n)$.

In 20(2), $r = 2$, $6|n$, $n \neq 6$ and $f(r,n) = t-1$, unless $(r,n) = (2,12)$, in which case $f(r,n) = t = 5$. Suppose, first, that $(r,n) \neq (2,12)$. Since $n > r^2 - r$, it follows by Lemma 21(3(i)) that $n \geqslant tr$, and Lemma 21(3(iii)) implies that $t \nleq 2$, so $t \geqslant 3$. Hence $n = (t-2)r + b$, with $t - 2 > 0$, $b \geqslant 2r$ and $A = (r^{(t-2)}, b) \in L(r,n)$ satisfy $|A| = t - 1 = f(r,n)$.

Suppose, now, that $(r,n) = (2,12)$. Then $A = (2^{(4)}, 4) \in L(2,12)$ and $|A| = 5 = f(r,n)$.

In 20(3), $r \geqslant 2$, $r \neq 4$, $n = r^2 - r$ and $f(r,n) = t = \frac{n}{r}$. Then $A = (r^{(\frac{n}{r})})$ satisfies $|A| = \frac{n}{r} = f(r,n)$.

In 20(4), $r \geqslant 2$, $n > r^2 - r$, $3 \nmid n$ if $r = 2$, $(r,n) \neq (3,9)$ and $f(r,n) = t$, unless $r+1 \mid n+r-1$, in which case $f(r,n) = t+1$. Suppose, first, that $r+1 \nmid n+r-1$. Since $n > r^2 - r$, it follows by Lemma 21(3(i)) that $n \geqslant tr$, and Lemma 21(3(ii)) implies that $t \neq 1$, so $t \geqslant 2$. Hence $n = (t-1)r + a$ with $t - 1 > 0$, $a \geqslant r$ and $A = (r^{(t-1)}, a) \in L(r,n)$ satisfies $|A| = t = f(r,n)$.

Suppose, now, that $r+1|n+r-1$. Recall that $k = \frac{n}{r} > r-1$, so $\frac{k+2}{r+1} > 1$. Now $t = \frac{n+r-1}{r+1} = k + 1 - \frac{k+2}{r+1}$, so $t \leqslant k-1$. Hence $k \geqslant t+1$, $n \geqslant (t+1)r$ and $n = tr + a$, with $t > 0$ and $a \geqslant r$. It follows that $A = (r^{(t)}, a) \in L(r,n)$ with $|A| = t+1 = f(r,n)$.

In 20(5), $n < r^2 - r$, $n \neq 3r$ and $f(r,n) = t - 1 = \frac{n}{r}$. Hence $A = (r^{(\frac{n}{r})}) \in L(r,n)$ and $|A| = \frac{n}{r} = f(r,n)$. ☐

Our final result deals with sequences satisfying $|A| = f(r,n) - 1$. We shall prove the following proposition.

Proposition 23. *Let n and r be positive integers satisfying $r \geqslant 2$ and $r|n$ and suppose that $A \in L(r,n)$ satisfies one of the following assumptions:*

1. $r \geqslant 2$, $n = 3r$ *and* $|A| = 1 \leqslant \lceil \frac{n+r-1}{r+1} \rceil - 2$.

2. $r = 2$, $3|n$, $n \neq 6, 12$ *and* $|A| = \frac{n}{3} - 1 = \lceil \frac{n+r-1}{r+1} \rceil - 2$.

3. $r \geqslant 2$, $n > r^2 - r$, $r+1 \mid n+r-1$ *and* $|A| = \frac{n+r-1}{r+1}$.

4. $r > 2$, $n \neq 3r$, $n = r^2 - \alpha r$ *with* $2 \leqslant \alpha \leqslant r-1$, *and* $|A| = r - \alpha - 1 = \lceil \frac{n+r-1}{r+1} \rceil - 2$.

5. *In all other cases,* $|A| = \lceil \frac{n+r-1}{r+1} \rceil - 1$.

Then either such sequence A does not exist or there exists such sequence A which does not belong to $rSp(r,n)$.

Proof of Proposition 23. If $n = r = 2$, then $A \in (5)$ and $|A| = \lceil \frac{n+r-1}{r+1} \rceil - 1 = \lceil \frac{2+2-1}{2+1} \rceil - 1 = 0$, so A does not exist. If $n = r > 2$, then $A \in (4)$ with $\alpha = r - 1$ and again $|A| = r - (r-1) - 1 = 0$, so A does not exist.

So we may assume that $n > r$ and $|A| \geqslant 2$. Since $r \mid n$, we must have $n \geqslant 2r$. If $n = 2r$ and $r = 2$, then $A \in (5)$ and $|A| = \lceil \frac{4+2-1}{2+1} \rceil - 1 = 1$, so A does not exist. Finally, if $n = 2r$ and $r \geqslant 3$, then $n = r^2 - (r-2)r$, $A \in (4)$ and $|A| = r - (r-2) - 1 = 1$, and as before A does not exist. So we may assume that $n > 2r$, hence $n \geqslant 3r$. If $n = 3r$ and $r \geqslant 2$, then $A \in (1)$ and $|A| = 1$, so again A does not exist. So we may assume that $n \geqslant 4r$. If $|A| = 2$, then $A = (r, n-r)$ and $A \notin rSp(r,n)$ since $2r \notin S_A$. So we may assume from now on that $n \geqslant 4r$ and $|A| \geqslant 3$.

Assume that $r = 2$. If $3|n$ and $n \neq 6, 12$, then $A \in (2)$ and $|A| = \frac{n}{3} - 1$. As $n \geqslant 4r = 8$ and $6|n$, we must have $n \geqslant 18$ and $|A| \geqslant 5$. Then $n = 3|A| + 1 = 2 + 3(|A|-2) + 5$ and $A = (2, 3^{(|A|-2)}, 5) \in L(2,n) \setminus 2Sp(2,n)$, since $4 \notin S_A$. If $n = 12$, then $A \in (5)$, $|A| = \lceil \frac{12+1}{3} \rceil - 1 = 4$ and $A = (2,2,3,5) \in L(2,12)$ satisfies $|A| = 4$ and it does not belong to $2Sp(2,12)$ since $6 \notin S_A$. Finally, suppose that $3 \nmid n$. If $3 \mid n+2-1$, then $A \in (3)$ and $|A| = \frac{n+1}{3}$. Thus $n = 3|A| - 1 = 2 + 3(|A|-1)$ and $A = (2, 3^{(|A|-1)}) \in L(2,n) \backslash 2Sp(2,n)$ since $4 \notin S_A$. If $3 \nmid n + 2 - 1 = n + 1$, then $A \in (5)$ and since $3 \nmid n, n+1$, it follows that $|A| = \lceil \frac{n+1}{3} \rceil - 1 = \frac{n+2}{3} - 1$. Thus $n = 3(|A|+1) - 2 = 2 + 3(|A|-2) + 5$ and $A = (2, 3^{(|A|-2)}, 5) \in L(2,n) \setminus 2Sp(2,n)$ since $4 \notin S_A$. So the proposition also holds for $r = 2$.

So assume that $n \geqslant 4r$, $r > 2$ and $|A| \geqslant 3$. Since for all A satisfying our assumptions we have $|A| \leqslant \frac{n+r-1}{r+1}$, it follows that $n \geqslant (r+1)|A| - r + 1$ and therefore the integer $s = n - (r+1)(|A|-2) - r$ satisfies $s \geqslant 3$. If $s \geqslant r+1$ and $s \neq 2r$, then let $A = (r, (r+1)^{(|A|-2)}, s)$ if $s > r+1$ and $A = (r, (r+1)^{(|A|-1)})$ if $s = r+1$. In both cases $A \in L(r,n)$ and $A \notin rSp(r,n)$ since $2r \notin S_A$. If $s = 2r$, then let $A = (r, (r+1)^{(|A|-3)}, r+2, 2r-1)$ if $r > 3$ and let $A = (r, (r+1)^{(|A|-3)}, (2r-1)^{(2)})$ if $r = 3$. In both cases $A \in L(r,n)$ and $A \notin rSp(r,n)$ since $2r \notin S_A$. If $s = r$, then let $A = (r^{(2)}, (r+1)^{(|A|-2)})$. Then $A \in L(r,n)$ and $3r \notin S_A$ since $r \geqslant 3$. Hence $A \notin rSp(r,n)$.

By the previous paragraph, we may assume that $3 \leqslant s < r$. Recall that $(|A|-2)(r+1) = n - s - r$, which implies by Lemma 13(4) that $(|A|-1)r + |A| - 2 + s = n \geqslant |A|r$. It follows that $|A| - 2 \geqslant r - s$ and $r \mid |A| - 2 + s$ since $r|n$.

Assume, first, that $|A| - 2 > r - s$. Then $|A| \geqslant 3 + r - s$ and by Lemma 13(4) $n \geqslant |A|r \geqslant r(3+r-s)$. Let $A = (r^{(2+r-s)}, (r+1)^{(|A|+s-r-2)})$. Notice that $A \in L(r,n)$, and since $r > s \geqslant 3$ we have $2 < 2 + r - s \leqslant r - 1$. Moreover, since $|A| - 2 > r - s$ and $r \mid |A| - 2 + s$, we have $|A| + s - r - 2 \geqslant r$. Now $3 < 3 + r - s \leqslant r$ and hence $3r < r(3+r-s) < r(r+1)$, which implies that $r(3+r-s) \notin S_A$. Since $n > r(3+r-s)$, it follows that $A \notin rSp(r,n)$.

Assume, finally, that $|A| - 2 = r - s$. Then $n = (r+1)(|A|-2) + r + s = |A|r$, since $s = r - |A| + 2$ in this case. As $|A| \leqslant \frac{n+r-1}{r+1}$ and $n = |A|r$, we have $|A| \leqslant r - 1$ and $n \leqslant r^2 - r$. As $A \notin (3)$, it follows by our assumptions that $|A| < \frac{n+r-1}{r+1}$, which implies that $|A| < r - 1$ and $n < r^2 - r$. Since $r|n$, we may conclude that $n = r^2 - \alpha r$ for an integer α satisfying $2 \leqslant \alpha \leqslant r - 1$. By our assumptions $n \geqslant 4r$, so $A \in (4)$ and $|A| = \frac{n}{r} = r - \alpha$, contradicting our assumption that $|A| = r - \alpha - 1$ in this case. Hence A does not exist, and the proof of Proposition 23 is now complete. □

As mentioned above, it follows from Propositions 22 and 23 that Theorem 14 is correct. □

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